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Brauer characters and the Harris–Knörr correspondence in p-solvable groups

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ABSTRACT

If b is a p-block of a normal subgroup N of a p-solvable group G and b^* is its Brauer correspondent in $N_N(L)$, where L is a defect group of b, then Harris–Knörr correspondents over b and b^* contain equal numbers of irreducible Brauer characters over height zero Brauer characters.

1. Introduction

Fix a prime p and let N be a normal subgroup of a finite p-solvable group G. Let b be a p-block of N with defect group L. If b^* is the Brauer correspondent of b in $N_N(L)$, then the main result of [2] tells us that the Brauer correspondence gives a defect group preserving bijection of the set $\mathrm{Bl}(N_G(L)|b^*)$ of p-blocks of $N_G(L)$ which cover b^* onto the set $\mathrm{Bl}(G|b)$ of p-blocks of G which cover b.

Now denote by $\mathrm{IBr}_0(b)$, the set of irreducible Brauer characters in b of height zero, and for a p-block $B \in \mathrm{Bl}(G|b)$, let

$$\mathrm{IBr}\big(B\big|\mathrm{IBr}_0(b)\big) = \big\{\psi \in \mathrm{IBr}(B)\colon\ \psi \text{ lies over some } \varphi \in \mathrm{IBr}_0(b)\big\}.$$

The main objective of this paper is to prove the following result.

Theorem. *Let N be a normal subgroup of a p-solvable group G, and let B and b be p-blocks of G and N respectively such that B covers b. Suppose L is a defect group of b and let b^* be the Brauer correspondent of b in $N_N(L)$. Then if $B^* \in \mathrm{Bl}(N_G(L)|b^*)$ is the Harris–Knörr correspondent of B in $N_G(L)$,*

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$$\left|\mathrm{IBr}\left(B\middle|\mathrm{IBr}_0(b)\right)\right| = \left|\mathrm{IBr}\left(B^*\middle|\mathrm{IBr}_0\left(b^*\right)\right)\right|.$$

Consequently, $|\mathrm{IBr}(B)| = |\mathrm{IBr}(B^*)|$ *if and only if every Brauer character in* $\mathrm{IBr}(b)$ *is of height zero.*

Since the irreducible Brauer characters in b^* are all of height zero, note that the right side of the above equation is just $|\mathrm{IBr}(B^*)|$.

The last assertion of our theorem follows from the well-known fact that each member of $\mathrm{IBr}(B)$ lies over some element in $\mathrm{IBr}(b)$ and each member of $\mathrm{IBr}(b)$ lies under some element in $\mathrm{IBr}(B)$.

In case $N = G$, then Theorem is just [7, Theorem 4.1], which is the modular analogue of the Alperin–McKay conjecture for p-solvable groups.

We should finally mention that (non-p-solvable) examples exist in which our main theorem fails. Convincingly, let $G = GL(3,2)$ and let B be the principal 2-block of G. Next take $N = G$, and so $b = B$. Now B has three irreducible Brauer characters, all of height zero. On the other hand, if P is a Sylow 2-subgroup of G, then $N_G(P) = P$ and so the principal 2-block B^* of $N_G(P)$ has a unique irreducible Brauer character, namely the trivial one.

2. Proof of Theorem

In order to prove the main theorem, a series of preliminary results is needed. The proofs of some of these results as well as the proof of the main result itself rely on the Glauberman correspondence. This correspondence is defined whenever a solvable group L acts on some group H under the assumption that $(|H|, |L|) = 1$. The correspondence is a naturally defined bijection $\chi \mapsto \chi^*$ from the set $\mathrm{Irr}_L(H)$ of L-invariant irreducible characters of H onto $\mathrm{Irr}(C_H(L))$. One of the properties of this correspondence worth mentioning is invariance with respect to group automorphisms. More explicitly, for any $\chi \in \mathrm{Irr}_L(H)$ and any automorphism τ of the semidirect product HL such that $L^\tau = L$, one has $(\chi^*)^\tau = (\chi^\tau)^*$. (For a full description of the Glauberman correspondence, the reader is kindly referred to Chapter 13 of [3].)

From now to the end of this section, a prime p is fixed. Our first lemma involves vertices of irreducible Brauer characters. By a vertex of an irreducible p-Brauer character φ of an arbitrary (finite) group G, we mean any vertex of the simple G-module (in characteristic p) corresponding to φ. Let now H be a normal p'-subgroup of the group G and let $\mu \in \mathrm{Irr}(H)$. For p-subgroups Q of G, we write $n(G, \mu, Q)$ to denote the number of irreducible Brauer characters of G lying over μ and having vertex Q.

Lemma 2.1. *Let H be a normal p'-subgroup of a p-solvable group G and let L and Q be p-subgroups of G such that $N_G(Q) \subseteq N_G(L)$. Write $K = N_G(L) \cap H$, so that $K = C_H(L)$. Suppose $\mu \in \mathrm{Irr}(H)$ is G-invariant and let $\nu \in \mathrm{Irr}(K)$ be the Glauberman correspondent of μ with respect to the action of L on H, then*

$$n(G, \mu, Q) = n\left(N_G(L), \nu, Q\right).$$

Proof. Write $E = N_G(Q) \cap H$, so that $E = C_H(Q)$. Now let $\xi \in \mathrm{Irr}(E)$ be the Glauberman correspondent of μ with respect to the action of Q on H.

Since μ is invariant in G, then ν is invariant in $N_G(L)$. Furthermore, as $Q \subseteq N_G(Q) \subseteq N_G(L)$, then Q acts on K and $\nu \in \mathrm{Irr}_Q(K)$. Next we have

$$C_K(Q) = N_G(Q) \cap K = N_G(Q) \cap N_G(L) \cap H = N_G(Q) \cap H = E.$$

Let $\rho \in \mathrm{Irr}(E)$ be the Glauberman correspondent of ν with respect to the action of Q on K. We claim that $\rho = \xi$.

By [3, Theorem 13.1(c)], ν is the unique irreducible constituent of μ_K such that $p \nmid [\mu_K, \nu]$, and ρ is the unique irreducible constituent of ν_E such that $p \nmid [\nu_E, \rho]$. It follows that ρ is the unique irreducible constituent of μ_E satisfying $p \nmid [\mu_E, \rho]$. Since by [3, Theorem 13.1(c)] again, ξ is the unique irreducible constituent of μ_E such that $p \nmid [\mu_E, \xi]$, we must have $\rho = \xi$, as claimed.

By [4, Theorem 6.3], we have $n(G, \mu, Q) = n(HN_G(Q), \mu, Q)$ and

$$n\big(N_G(L), \nu, Q\big) = n\big(KN_G(Q), \nu, Q\big).$$

Also, a double application of Proposition 6.4 in [4] yields

$$n\big(HN_G(Q), \mu, Q\big) = n\big(N_G(Q), \xi, Q\big) = n\big(KN_G(Q), \nu, Q\big).$$

It follows that $n(G, \mu, Q) = n(N_G(L), \nu, Q)$, as needed to be shown. $\square$

Let B be a p-block of a p-solvable group G. If H is a normal p'-subgroup of G, then there exists a (uniquely determined up to G-conjugacy) character $\mu \in \mathrm{Irr}(H)$ such that the members of $\mathrm{Irr}(B) \cup \mathrm{IBr}(B)$ all lie over μ. Accordingly, B is said to lie over μ. We mention that in the special case where $H = O_{p'}(G)$ and μ is G-invariant, the set $\mathrm{Irr}(B)$ (resp. $\mathrm{IBr}(B)$) equals the full set $\mathrm{Irr}(G|\mu)$ (resp. $\mathrm{IBr}(G|\mu)$) of ordinary irreducible (resp. irreducible Brauer) characters of G that lie over μ. (See, for instance, [8, Theorem 2.8].)

Let now P be a p-subgroup of G and suppose β is a p-block of $N_G(P)$ such that $\beta^G = B$. Let $H = O_{p'}(G)$ and $K = N_G(P) \cap H$, so that $K = C_H(P)$. Next choose $\nu \in \mathrm{Irr}(K)$ over which β lies. By the discussion in the beginning of Section 3 of [9] and Theorem 3.2 of the same paper, we can choose μ to lie inside $\mathrm{Irr}_P(H)$ and so that ν is its Glauberman correspondent with respect to the action of P on H.

The following well-known result is frequently used in the sequel.

Theorem 2.2. *Let H be a normal p'-subgroup of a p-solvable group G, and let $\mu \in \mathrm{Irr}(H)$ with $T = I_G(\mu)$, the inertial group of μ in G.*

(i) *Block induction defines a bijection from the set of p-blocks of T over μ onto the set of p-blocks of G over μ.*
(ii) *If β is a p-block of T over μ, then any defect group of β is also a defect group for β^G.*
(iii) *Let β be a p-block of T over μ. Then the map $\theta \mapsto \theta^G$ defines a bijection of $\mathrm{Irr}(\beta)$ onto $\mathrm{Irr}(\beta^G)$. The corresponding statement for $\mathrm{IBr}(\beta)$ and $\mathrm{IBr}(\beta^G)$ also holds.*

Proof. Let b be the block of H to which μ belongs. Then $\mathrm{Irr}(b) = \{\mu\}$, so that T is the stabilizer of b in G. All statements are then immediate from Theorem 5.5.10 in [6]. $\square$

Let B be a block of a p-solvable group G. For p-subgroups Q of G, we denote by $\mathrm{IBr}(B|Q)$ the set of all irreducible Brauer characters belonging to B and having vertex Q.

Lemma 2.3. *Let G be a p-solvable group and let B be a p-block of G with defect group D. Suppose L is a subgroup of D such that $N_G(D) \subseteq N_G(L)$, and let B^* be the unique p-block of $N_G(L)$ with defect group D such that $(B^*)^G = B$. If Q is a p-subgroup of G with $N_G(Q) \subseteq N_G(L)$, then $|\mathrm{IBr}(B|Q)| \geqslant |\mathrm{IBr}(B^*|Q)|$.*

Proof. Set $H = O_{p'}(G)$ and choose $\mu \in \mathrm{Irr}(H)$ under B. Next let T be the inertial group of μ in G, and write β for the unique block of T over μ corresponding to B via Theorem 2.2(i). In view of Theorem 2.2(ii), since B lies over all G-conjugates of μ, we may assume that D is a defect group for β. We have now $L \subseteq D \subseteq T$ and thus $N_T(D) = T \cap N_G(D) \subseteq T \cap N_G(L) = N_T(L)$. Write β^* for the unique block of $N_T(L)$ with defect group D such that $(\beta^*)^T = \beta$.

Next let $K = N_G(L) \cap H$. Then $K = C_H(L)$, and by [9, Lemma 3.7], we also have $K = O_{p'}(N_G(L))$. Let $\mu^* \in \mathrm{Irr}(K)$ be the Glauberman correspondent of μ with respect to the action of L on H. Then $I_{N_G(L)}(\mu^*) = N_G(L) \cap T = N_T(L)$. Next, we want to show that μ^* lies under β^*. So choose $\nu \in \mathrm{Irr}(K)$ under β^*, and let $\rho \in \mathrm{Irr}_L(H)$ be the Glauberman correspondent of ν with respect to the action of L on H. Then [9, Theorem 4.6(c)] (with T, L, H, ν in place of G, D, M and α, respectively) implies that

$(\beta^*)^T$ lies over ρ. But $(\beta^*)^T = \beta$ lies over μ and T stabilizes μ. It follows that $\rho = \mu$ and hence $\nu = \mu^*$. Thus β^* lies over μ^*, as wanted.

Suppose $T = G$ (so that μ is invariant in G). Then μ^* is invariant in $N_G(L)$. In this situation, $\mathrm{IBr}(B) = \mathrm{IBr}(G|\mu)$ and $\mathrm{IBr}(B^*) = \mathrm{IBr}(N_G(L)|\mu^*)$. Now in view of Lemma 2.1, we get $|\mathrm{IBr}(B|Q)| = |\mathrm{IBr}(B^*|Q)|$, and the result clearly holds in this case.

Now assume $T < G$. Since the result is trivial if $|\mathrm{IBr}(B^*|Q)| = 0$, we can assume that $\mathrm{IBr}(B^*|Q) \neq \emptyset$.

Since β^* lies over μ^* and has defect group D, Theorem 2.2 tells us that $(\beta^*)^{N_G(L)}$ is defined and has D as a defect group. Next we know that $(\beta^*)^T = \beta$ and $\beta^G = B$. So by [6, Lemma 5.3.4], $(\beta^*)^G$ is defined and equals B. Then again by Lemma 5.3.4 of [6], we deduce that $((\beta^*)^{N_G(L)})^G$ is defined and equals B. Now by the uniqueness of the block B^*, we must have $(\beta^*)^{N_G(L)} = B^*$. Theorem 2.2(iii) then tells us that Brauer character induction gives rise to a bijection from $\mathrm{IBr}(\beta^*)$ onto $\mathrm{IBr}(B^*)$.

For any $\psi \in \mathrm{IBr}(B^*|Q)$, write ψ' for the unique member of $\mathrm{IBr}(\beta^*)$ such that $\psi = (\psi')^{N_G(L)}$. It is clear that ψ' has an $N_G(L)$-conjugate of Q as a vertex. We may then choose a minimal subset S of $N_G(L)$ such that for each $\psi \in \mathrm{IBr}(B^*|Q)$, there is a unique $s \in S$ such that ψ' has vertex Q^s.

Let $s \in S$. Then $N_G(Q^s) = (N_G(Q))^s \subseteq N_G(L)$ (as $s \in N_G(L)$), and hence $N_T(Q^s) \subseteq N_T(L)$. Now by induction we have

$$\bigl|\mathrm{IBr}\bigl(\beta\bigm|Q^s\bigr)\bigr| \geqslant \bigl|\mathrm{IBr}\bigl(\beta^*\bigm|Q^s\bigr)\bigr|. \tag{1}$$

Next by our choice of the set S, it is clear that

$$\left|\bigcup_{s\in S}\mathrm{IBr}\bigl(\beta^*\bigm|Q^s\bigr)\right| = \bigl|\mathrm{IBr}\bigl(B^*\bigm|Q\bigr)\bigr|,$$

and that the sets $\mathrm{IBr}(\beta^*|Q^s)$, $s \in S$, are mutually disjoint. Therefore

$$\bigl|\mathrm{IBr}\bigl(B^*\bigm|Q\bigr)\bigr| = \sum_{s\in S}\bigl|\mathrm{IBr}\bigl(\beta^*\bigm|Q^s\bigr)\bigr|. \tag{2}$$

Next we claim that the sets $\mathrm{IBr}(\beta|Q^s)$, $s \in S$, are mutually disjoint. So let $s, s' \in S$ and suppose that $\mathrm{IBr}(\beta|Q^s) \cap \mathrm{IBr}(\beta|Q^{s'}) \neq \emptyset$. Then $Q^{s'} = (Q^s)^t$ for some $t \in T$, and hence $st(s')^{-1} \in N_G(Q)$. Since $N_G(Q) \subseteq N_G(L)$, and $s, s' \in N_G(L)$, we get $t \in N_G(L)$. Thus $t \in N_T(L)$ (as $t \in T$). Then by the choice of S, we must have $s = s'$. This clearly proves our claim.

Let now $\varphi \in \bigcup_{s\in S}\mathrm{IBr}(\beta|Q^s)$. Then $\varphi^G \in \mathrm{IBr}(B)$ by Theorem 2.2(iii), and Q is clearly a vertex for φ^G. It follows (using Theorem 2.2(iii) once again) that

$$\bigl|\mathrm{IBr}(B|Q)\bigr| \geqslant \sum_{s\in S}\bigl|\mathrm{IBr}\bigl(\beta\bigm|Q^s\bigr)\bigr|.$$

Now in view of (1) and (2), we conclude that $|\mathrm{IBr}(B|Q)| \geqslant |\mathrm{IBr}(B^*|Q)|$, which clearly ends the proof of the lemma. □

Lemma 2.4. *Let $N \lhd G$, where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Let L be a defect group of b and write b^* for the Brauer correspondent of b in $N_N(L)$. Suppose b lies over a G-invariant character $\mu \in \mathrm{Irr}(O_{p'}(N))$. If Q is a p-subgroup of G such that $Q \cap N = L$ and $B^* \in \mathrm{Bl}(N_G(L)|b^*)$ is the Harris–Knörr correspondent of B in $N_G(L)$, then $|\mathrm{IBr}(B|Q)| = |\mathrm{IBr}(B^*|Q)|$.*

Proof. Let $H = O_{p'}(N)$ and $K = N_G(L) \cap H (= N_N(L) \cap H)$. Then $K = C_H(L)$, and we let $\mu^* \in \mathrm{Irr}(K)$ be the Glauberman correspondent of μ with respect to the action of L on H. Since μ is N-invariant, then μ^* is $N_N(L)$-invariant. Moreover, as b lies over μ, the block b^* lies over μ^* (see the discussion preceding Theorem 2.2). Also by [9, Lemma 3.7], note that $K = O_{p'}(N_N(L))$. Then $\mathrm{Irr}(b) = \mathrm{Irr}(N|\mu)$ and $\mathrm{Irr}(b^*) = \mathrm{Irr}(N_N(L)|\mu^*)$.

As $Q \cap N = L$, we have $N_G(Q) \subseteq N_G(L)$. Lemma 2.1 then says that

$$n(G, \mu, Q) = n\big(N_G(L), \mu^*, Q\big).$$

Now let $B_1(= B), \dots, B_m$ be all the (distinct) blocks of G covering b, and for each i, let $B_i^* \in \mathrm{Bl}(N_G(L)|b^*)$ be the Harris–Knörr correspondent of B_i in $N_G(L)$. Then $B_1^*, \dots, B_m^*$ are all the (distinct) blocks of $N_G(L)$ that cover b^*. Since $\mathrm{Irr}(b) = \mathrm{Irr}(N|\mu)$, we have $\mathrm{Bl}(G|b) = \mathrm{Bl}(G|\mu)$, the set of all blocks of G lying over μ. Similarly, $\mathrm{Bl}(N_G(L)|b^*) = \mathrm{Bl}(N_G(L)|\mu^*)$, as $\mathrm{Irr}(b^*) = \mathrm{Irr}(N_N(L)|\mu^*)$. Then $\bigcup_{i=1}^m \mathrm{IBr}(B_i|Q)$ (resp. $\bigcup_{i=1}^m \mathrm{IBr}(B_i^*|Q)$) is the set of irreducible Brauer characters of G (resp. $N_G(L)$) lying over μ (resp. μ^*) and having vertex Q. It follows that

$$\sum_{i=1}^m \big|\mathrm{IBr}(B_i|Q)\big| = \sum_{i=1}^m \big|\mathrm{IBr}\big(B_i^*\big|Q\big)\big|. \qquad (*)$$

For each i, choose a defect group D_i for B_i^*. Then B_i has defect group D_i, and as $L \lhd N_G(L)$, $L \subseteq D_i$ by [6, Theorem 5.2.8]. Next since B_i^* covers $b^*$and $D_i \subseteq N_G(L)$, then using [5, Proposition 4.2] yields $L = D_i \cap N_N(L) = D_i \cap N$. Hence $N_G(D_i) \subseteq N_G(L)$. Now Lemma 2.3 tells us that $|\mathrm{IBr}(B_i|Q)| \geqslant |\mathrm{IBr}(B_i^*|Q)|$, for each i. Taking into account $(*)$, we are forced to have $|\mathrm{IBr}(B_i|Q)| = |\mathrm{IBr}(B_i^*|Q)|$ for each i, and in particular $|\mathrm{IBr}(B|Q)| = |\mathrm{IBr}(B^*|Q)|$, as needed to be proved. □

Let B be a block of a p-solvable group G and suppose B covers a block b of a normal subgroup N. Let $\psi \in \mathrm{IBr}(B)$ with vertex Q. If ψ lies over $\varphi \in \mathrm{IBr}(b)$, then $|Q \cap N|$ is the order of any vertex of φ, as implied by Corollary 3 in [10]. It follows by [1, Theorem 2.1] that $\psi \in \mathrm{IBr}(B|\mathrm{IBr}_0(b))$ if and only if $|Q \cap N| = p^{d(b)}$, where $d(b)$ is the defect of b.

Now the following special case of Theorem is an essential step toward the proof of that result.

Proposition 2.5. *Let $N \lhd G$ where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Let L be a defect group of b and let b^* be the Brauer correspondent of b in $N_N(L)$. Suppose b lies over a G-invariant character $\mu \in \mathrm{Irr}(O_{p'}(N))$. If $B^* \in \mathrm{Bl}(N_G(L)|b^*)$ is the Harris–Knörr correspondent of B in $N_G(L)$, then*

$$\big|\mathrm{IBr}\big(B\big|\mathrm{IBr}_0(b)\big)\big| = \big|\mathrm{IBr}\big(B^*\big)\big|.$$

Proof. Fix a defect group D for B^*, and let

$$\Gamma = \big\{Q \leqslant G\colon\ Q \cap N = L \text{ and } Q \subseteq D^g \text{ for some } g \in N_G(L)\big\}.$$

Note that $\Gamma \neq \emptyset$ as $L \in \Gamma$. Also, $N_G(L)$ clearly acts on Γ by conjugation. Now choose $\Omega = \{Q_1, \dots, Q_m\}$, a complete set of representatives for the orbits of this action such that $Q_i \subseteq D$ for every i.

Let $\xi \in \mathrm{IBr}(B^*)$. Then ξ has vertex $P \subseteq D$. Since $L \lhd N_G(L)$, we have $D \cap N_N(L) = L$ from [5, Proposition 4.2] and $L \subseteq P$. It follows that $P \cap N = P \cap N_N(L) = L$. So $P \in \Gamma$, and hence ξ has vertex Q_i for some i. Consequently $\mathrm{IBr}(B^*) = \bigcup_{i=1}^m \mathrm{IBr}(B^*|Q_i)$, and it follows by the choice of the set Ω that

$$\big|\mathrm{IBr}\big(B^*\big)\big| = \sum_{i=1}^m \big|\mathrm{IBr}\big(B^*\big|Q_i\big)\big|. \qquad (*)$$

Next let $\psi \in \mathrm{IBr}(B|\mathrm{IBr}_0(b))$. As B has defect group D, we may choose a vertex R for ψ with $R \subseteq D$. Also, since every element in $\mathrm{IBr}_0(b)$ has vertex L (by [6, Theorem 5.1.9(ii)]), Corollary 3 in [10] implies that $R \cap N = L^x$ for some $x \in G$. Furthermore, as $D \subseteq N_G(L)$, we have $D \cap N = D \cap N_N(L) = L$. It follows that $R \cap N = L$. Hence $R \in \Gamma$ and so ψ has vertex Q_i for some i. Consequently

$$\operatorname{IBr}\bigl(B \bigm| \operatorname{IBr}_0(b)\bigr) \subseteq \bigcup_{i=1}^{m} \operatorname{IBr}(B|Q_i).$$

Conversely, suppose $\omega \in \operatorname{IBr}(B|Q_i)$ for some i. As $Q_i \cap N = L$, we have $\omega \in \operatorname{IBr}(B|\operatorname{IBr}_0(b))$ (see the comments preceding the proposition). Thus

$$\operatorname{IBr}\bigl(B \bigm| \operatorname{IBr}_0(b)\bigr) = \bigcup_{i=1}^{m} \operatorname{IBr}(B|Q_i).$$

Next, we want to show that the sets $\operatorname{IBr}(B|Q_i)$ are mutually disjoint. So let $i, j \in \{1, \dots, m\}$ with $i \neq j$, and suppose, on the contrary, that

$$\operatorname{IBr}(B|Q_i) \cap \operatorname{IBr}(B|Q_j) \neq \emptyset.$$

Then $Q_i = (Q_j)^y$ for some $y \in G$. Now $L = Q_i \cap N = (Q_j \cap N)^y = L^y$, and so $y \in N_G(L)$, contradicting our choice of the set Ω.

We now have $|\operatorname{IBr}(B|\operatorname{IBr}_0(b))| = \sum_{i=1}^{m} |\operatorname{IBr}(B|Q_i)|$. By Lemma 2.4, for each i, we have $|\operatorname{IBr}(B|Q_i)| = |\operatorname{IBr}(B^*|Q_i)|$. Then, in view of $(*)$, it follows that

$$\bigl|\operatorname{IBr}\bigl(B \bigm| \operatorname{IBr}_0(b)\bigr)\bigr| = \bigl|\operatorname{IBr}\bigl(B^*\bigr)\bigr|.$$

This completes the proof of the proposition. □

The above proposition takes care of a special case of the main theorem. Now to prove the theorem in its full generality, we need a couple of more preliminary results. Our first result, which should be well known, is a quite general fact about the Clifford correspondence needed for the proof of the second result.

Lemma 2.6. *Let $M \lhd G$ where G is an arbitrary finite group and let N be a subgroup of G containing M. Let $\mu \in \operatorname{Irr}(M)$ with inertial group G_0 in G and suppose $\chi \in \operatorname{Irr}(G|\mu)$ and $\theta \in \operatorname{Irr}(N|\mu)$. Then if $\chi_0 \in \operatorname{Irr}(G_0|\mu)$ (resp. $\theta_0 \in \operatorname{Irr}(G_0 \cap N|\mu)$) is the Clifford correspondent of χ (resp. θ), we have $[\chi_N, \theta] = [(\chi_0)_{G_0 \cap N}, \theta_0]$.*

Proof. Write $\chi_N = \sum_{i=1}^{m} a_i\theta_i + \alpha$, where $\operatorname{Irr}(N|\mu) = \{\theta_1, \dots, \theta_m\}$, every a_i is a nonnegative integer and α is a character of N such that $[\alpha_M, \mu] = 0$. Next for each $i \in \{1, \dots, m\}$, let $\hat{\theta}_i \in \operatorname{Irr}(G_0 \cap N|\mu)$ be the Clifford correspondent of θ_i. Since $\operatorname{Irr}(G_0 \cap N|\mu) = \{\hat{\theta}_1, \dots, \hat{\theta}_m\}$, we may write $(\chi_0)_{G_0 \cap N} = \sum_{i=1}^{m} b_i\hat{\theta}_i$, where every b_i is a nonnegative integer. Now to prove the lemma, it suffices to show that $a_i = b_i$ for each i.

Since $[\chi_M, \mu] = [(\chi_0)_M, \mu]$, we have $\chi_{G_0} = \chi_0 + \zeta$, where ζ is a character of G_0 such that $[\zeta_M, \mu] = 0$. Similarly, as $[(\theta_i)_M, \mu] = [(\hat{\theta}_i)_M, \mu]$, we have $(\theta_i)_{G_0 \cap N} = \hat{\theta}_i + \gamma_i$, where γ_i is some character of $G_0 \cap N$ with $[(\gamma_i)_M, \mu] = 0$. It follows that

$$\chi_{G_0 \cap N} = (\chi_N)_{G_0 \cap N} = \sum_{i=1}^{m} a_i(\theta_i)_{G_0 \cap N} + \alpha_{G_0 \cap N} = \sum_{i=1}^{m} a_i\hat{\theta}_i + \delta,$$

where δ is a character of $G_0 \cap N$ satisfying $[\delta_M, \mu] = 0$. On the other hand, we have

$$\chi_{G_0 \cap N} = (\chi_{G_0})_{G_0 \cap N} = (\chi_0)_{G_0 \cap N} + \zeta_{G_0 \cap N} = \sum_{i=1}^{m} b_i \hat{\theta}_i + \zeta_{G_0 \cap N}.$$

Since $[\zeta_M, \mu] = 0$, we get $a_i = b_i$ for every $i \in \{1, \ldots, m\}$. □

The next lemma will help us settle the case of the main theorem in which b is assumed to be stable.

Lemma 2.7. *Let $N \lhd G$ where G is p-solvable, and let b be a G-stable p-block of N. Fix a character $\mu \in \mathrm{Irr}(O_{p'}(N))$ that lies under b, and let b_0 be the p-block of the inertial group N_0 of μ in N corresponding to b via Theorem* 2.2(i)*. Then if G_0 is the stabilizer of (N_0, b_0) in G, the following hold.*

(i) *$G_0 = I_G(\mu)$, the inertial group of μ in G.*
(ii) *Block induction defines a bijection of $\mathrm{Bl}(G_0|b_0)$ onto $\mathrm{Bl}(G|b)$. Furthermore, every defect group of $B_0 \in \mathrm{Bl}(G_0|b_0)$ is also a defect group for $(B_0)^G$.*
(iii) *If $B_0 \in \mathrm{Bl}(G_0|b_0)$, then the map $\varphi \mapsto \varphi^G$ defines a bijection of $\mathrm{IBr}(B_0)$ onto $\mathrm{IBr}((B_0)^G)$, and restricts to a bijection from $\mathrm{IBr}(B_0|\mathrm{IBr}_0(b_0))$ onto $\mathrm{IBr}((B_0)^G|\mathrm{IBr}_0(b))$.*

Proof. (i) First assume $x \in G_0$. Then $(b_0)^x = b_0$ and so b_0 lies over μ^x. Since μ is N_0-stable, it follows that $\mu^x = \mu$. Therefore $x \in I_G(\mu)$. Next assume $y \in I_G(\mu)$. As $N \lhd G$, we have $(N_0)^y = I_N(\mu^y) = I_N(\mu) = N_0$. We deduce then that $(b_0)^y$ is a block of N_0 lying over μ. Let now $\theta \in \mathrm{Irr}(b_0)$. Then $\theta^y \in \mathrm{Irr}((b_0)^y)$ and $(\theta^y)^N = (\theta^N)^y \in \mathrm{Irr}(b^y)$ (as $\theta^N \in \mathrm{Irr}(b)$ by Theorem 2.2(iii)). But $b^y = b$ as b is G-stable. Now in view of Theorem 2.2, we are forced to have $(b_0)^y = b_0$. Thus $y \in G_0$, and the proof of (i) is complete.

(ii) Since $G_0 = I_G(\mu)$ by (i), Theorem 2.2(i) tells us that the correspondence $B_0 \mapsto (B_0)^G$ defines a bijection from the set of blocks of G_0 over μ onto the set of blocks of G over μ. Therefore, to prove the first assertion, it is enough to show that $B_0 \in \mathrm{Bl}(G_0|b_0)$ if and only if $(B_0)^G \in \mathrm{Bl}(G|b)$ (note that since b_0 and b lie over μ, every block in $\mathrm{Bl}(G_0|b_0)$ or in $\mathrm{Bl}(G|b)$ must lie over μ).

Let $\gamma \in \mathrm{Irr}(G_0|\mu)$ and $\delta \in \mathrm{Irr}(N_0|\mu)$. Then Lemma 2.6 implies that γ lies over δ if and only if $\chi = \gamma^G \in \mathrm{Irr}(G|\mu)$ lies over $\zeta = \delta^N \in \mathrm{Irr}(N|\mu)$. In particular, by Theorem 2.2(iii), $\gamma \in \mathrm{Irr}(B_0)$ and $\delta \in \mathrm{Irr}(b_0)$ if and only if $\chi \in \mathrm{Irr}((B_0)^G)$ and $\zeta \in \mathrm{Irr}(b)$, and it follows that B_0 covers b_0 if and only if $(B_0)^G$ covers b, as needed to be shown.

The remaining assertion on defect groups is immediate from Theorem 2.2(ii).

(iii) Let $B_0 \in \mathrm{Bl}(G_0|b_0)$. Since B_0 lies over μ, the correspondence $\varphi \mapsto \varphi^G$ defines a bijection from $\mathrm{IBr}(B_0)$ onto $\mathrm{IBr}((B_0)^G)$ by Theorem 2.2(iii). Next to prove the remaining assertion, it suffices to show that $\varphi \in \mathrm{IBr}(B_0)$ lies over some height-zero irreducible Brauer character of b_0 if and only if φ^G lies over some height-zero irreducible Brauer character of b. Now since $\varphi \in \mathrm{IBr}(B_0)$ and φ^G have a common vertex Q ($\subseteq G_0$), the blocks b_0 and b have equal defects (from Theorem 2.2(ii)), and $Q \cap N = (Q \cap G_0) \cap N = Q \cap N_0$, we have the result (by the comments preceding Proposition 2.5). □

We also need the following easy lemma.

Lemma 2.8. *Let $N \lhd G$ where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. If B' is the Fong–Reynolds correspondent of B in the inertial group T of b in G, then*

$$\left|\mathrm{IBr}\left(B' \middle| \mathrm{IBr}_0(b)\right)\right| = \left|\mathrm{IBr}\left(B \middle| \mathrm{IBr}_0(b)\right)\right|.$$

Proof. By [6, Theorem 5.5.10(ii)], the map $\varphi \mapsto \varphi^G$ defines a bijection from $\mathrm{IBr}(B')$ onto $\mathrm{IBr}(B)$. Since $\varphi \in \mathrm{IBr}(B')$ and φ^G both lie over some $\omega \in \mathrm{IBr}(b)$, the above map restricts to a bijection from $\mathrm{IBr}(B'|\mathrm{IBr}_0(b))$ onto $\mathrm{IBr}(B|\mathrm{IBr}_0(b))$. The result then immediately follows. □

Finally, we are ready to prove our main result.

Proof of Theorem. Step 1. Suppose b is G-stable. Our objective here is to prove the theorem in this case. We will proceed by induction on $|G|$.

Let $H = O_{p'}(N)$ and $K = N_G(L) \cap H (= N_N(L) \cap H)$. Then $K = C_H(L)$. Choose a character $\nu \in \mathrm{Irr}(K)$ lying under b^*, and let $\mu \in \mathrm{Irr}_L(H)$ be the character that corresponds to ν under the Glauberman correspondence with respect to the action of L on H. Then b lies over μ (by the discussion preceding Theorem 2.2). Next set $G_0 = I_G(\mu)$ and $N_0 = I_N(\mu)$.

If $G_0 = G$, then we have the result by Proposition 2.5. We can assume therefore that $G_0 < G$. We have $N_{N_0}(L) = N_0 \cap N_N(L) = I_{N_N(L)}(\nu)$ (see the comments in the beginning of Section 2). Let now b_0 be the block of N_0 over μ that corresponds to b via Theorem 2.2(i). Similarly, let b_0^* be the block of $N_{N_0}(L)$ over ν corresponding to b^* (via the same theorem). Then Theorem 4.6 in [9] says that $(b_0^*)^{N_0} = b_0$. Next, since b_0^* and b^* have a common defect group by Theorem 2.2(ii) and L is a defect group for b^*, the block b_0^* must have L as a defect group. Then by [6, Theorem 5.3.8], L is a defect group of b_0.

Notice that $N_{G_0}(L) = G_0 \cap N_G(L) = I_{N_G(L)}(\nu)$. Also, in view of [9, Lemma 3.7], we have $O_{p'}(N_N(L)) = H \cap N_N(L) = K$.

Next we claim that b^* is $N_G(L)$-stable. So let $g \in N_G(L)$. Then $(b^*)^g$ is a block of $N_N(L)$ (as $N_N(L) \lhd N_G(L)$). Moreover, $((b^*)^g)^N = ((b^*)^N)^g = b^g = b$, as b is G-stable. Since b^* is the unique block of $N_N(L)$ such that $(b^*)^N = b$, we are forced to have $(b^*)^g = b^*$. This proves our claim.

Let now B_0^* be the block in $\mathrm{Bl}(N_{G_0}(L)|b_0^*)$ corresponding to B^* via Lemma 2.7(ii). Then $(B_0^*)^{G_0}$ is defined. We claim that $(B_0^*)^{G_0}$ equals the block B_0 of G_0 covering b_0 that corresponds to B via Lemma 2.7(ii).

As b_0^* lies over ν, then so does B_0^*. Therefore B_0^* is the block of $N_{G_0}(L)$ over ν that corresponds to B^* through Theorem 2.2(i). Similarly, since b_0 lies over μ, then so does B_0. Hence B_0 is the block of G_0 over μ corresponding to B through Theorem 2.2(i). Now by Theorem 4.6 of [9], we have $(B_0^*)^{G_0} = B_0$, as we claimed.

Since $G_0 < G$, the inductive hypothesis guarantees that $|\mathrm{IBr}(B_0|\mathrm{IBr}_0(b_0))| = |\mathrm{IBr}(B_0^*)|$. Also, Lemma 2.7(iii) implies that $|\mathrm{IBr}(B|\mathrm{IBr}_0(b))| = |\mathrm{IBr}(B_0|\mathrm{IBr}_0(b_0))|$ and $|\mathrm{IBr}(B^*)| = |\mathrm{IBr}(B_0^*)|$. It follows that $|\mathrm{IBr}(B|\mathrm{IBr}_0(b))| = |\mathrm{IBr}(B^*)|$. This proves the theorem in this case.

Step 2. We will prove the theorem here for any b (not necessarily G-stable).

Let T be the inertial group of b in G and let B' be the Fong–Reynolds correspondent of B in T. First we show that $N_T(L) = I_{N_G(L)}(b^*)$, the inertial group of b^* in $N_G(L)$.

Suppose $x \in N_T(L)$. Then $(b^*)^x$ is a block of $N_N(L)$ and $((b^*)^x)^N = ((b^*)^N)^x = b^x = b$. Therefore $(b^*)^x = b^*$ (as b^* is the unique block of $N_N(L)$ with $(b^*)^N = b$), and hence $x \in I_{N_G(L)}(b^*)$. Next assume $y \in I_{N_G(L)}(b^*)$. Then $b^y = ((b^*)^y)^N = (b^*)^N = b$, and so $y \in T \cap N_G(L) = N_T(L)$. We have thus shown that $N_T(L) = I_{N_G(L)}(b^*)$, as needed.

Now let β be the Fong–Reynolds correspondent of B^* in $N_T(L)$. Since $\beta^{N_G(L)} = B^*$ and $(B^*)^G = B$, Lemma 5.3.4 in [6] tells us that β^G is defined and equals B. Also, as β^T is defined, then by using the same lemma, we get that $(\beta^T)^G$ is defined and equals B. Next the Harris–Knörr theorem implies that β^T covers b (as β covers b^*). It follows that β^T must be the Fong–Reynolds correspondent B' of B in T.

By Step 1, we have $|\mathrm{IBr}(B'|\mathrm{IBr}_0(b))| = |\mathrm{IBr}(\beta)|$. Next, [6, Theorem 5.5.10(ii)] implies that $|\mathrm{IBr}(B^*)| = |\mathrm{IBr}(\beta)|$ and by Lemma 2.8,

$$\left|\mathrm{IBr}\left(B\middle|\mathrm{IBr}_0(b)\right)\right| = \left|\mathrm{IBr}\left(B'\middle|\mathrm{IBr}_0(b)\right)\right|.$$

It follows that $|\mathrm{IBr}(B|\mathrm{IBr}_0(b))| = |\mathrm{IBr}(B^*)|$. The proof of the theorem is now complete. □

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