An Extension of the Kreiss Stability Theorem to Families of Matrices of Unbounded Order

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ABSTRACT

The Kreiss matrix theorem asserts three necessary and sufficient conditions for a family of matrices of fixed finite order to be L_2-stable: a resolvent condition (R), a triangularization condition (S) and a Hermitian norm condition (H). We extend the Kreiss theorem to families of matrices of finite but unbounded order with the restriction that the degrees of the minimal polynomials of all matrices in the family are less than a fixed constant. For such matrix families, we show that (R) and (H) remain necessary and sufficient for L_2-stability, while (S) must be replaced by a somewhat stronger "block triangularization" condition (S′). This extended Kreiss theorem permits a corresponding extension of the Buchanan stability theorem.

1. INTRODUCTION AND STATEMENT OF MAIN THEOREM

The Kreiss matrix theorem [1] gives four conditons equivalent to (spectral norm) stability for a family G of square complex matrices, all of the same order p:

(A) [Definition of stability.] There exists a constant C such that for all matrices A in G and all positive integers n, $\|A^n\| \leqslant C$.

(R) There exists a constant C_R such that for all A in G and all complex numbers z with $|z|>1$, the matrix $(A-zI)^{-1}$ exists and

$$\|(A-zI)^{-1}\| \leqslant C_R\,(|z|-1)^{-1}. \tag{1.1}$$

(S) There exist constants C_S and C_B and for each A in G a non-singular matrix S such that (i) $\|S\|, \|S^{-1}\| \leqslant C_S$; and (ii) $B = SAS^{-1}$ is upper triangular:

$$B = \begin{bmatrix} x_1 & B_{12} & B_{13} & \cdot & \cdot & \cdot & B_{1p} \\ & x_2 & B_{23} & \cdot & \cdot & \cdot & B_{2p} \\ & & x_3 & \cdot & \cdot & \cdot & B_{3p} \\ & & & \cdot & & & \cdot \\ & 0 & & & \cdot & & \cdot \\ & & & & & \cdot & \cdot \\ & & & & & & x_p \end{bmatrix} \tag{1.2}$$

and its off-diagonal elements satisfy

$$|B_{ij}| \leqslant C_B \min(1-|x_i|, 1-|x_j|), \tag{1.3}$$

where x_i are the diagonal elements of B, i.e., the eigenvalues of A and B (note that this implies $|x_i| \leqslant 1$).

(H) There are a constant $C_H > 0$ and, for each A in G, a positive definite Hermitian matrix H such that

$$C_H^{-1} I \leqslant H \leqslant C_H I \tag{1.4a}$$

and

$$A^* H A \leqslant H. \tag{1.4b}$$

The matrix norm used in conditions (A), (R) and (S) is the *spectral norm*

$$\|A\| = \sqrt{\rho(A^*A)} = \sup_{u \neq 0} \frac{\|Au\|}{\|u\|}, \tag{1.5}$$

where ρ denotes the spectral radius, while the vector norm $\|u\|$ is the usual Euclidean norm

$$\|u\| = \|\langle u_1, \ldots, u_p \rangle\| = \left(|u_1|^2 + \cdots + |u_p|^2\right)^{1/2} = (u^* u)^{1/2}. \tag{1.6}$$

The matrix inequalities in condition (H) refer to the order relationship among Hermitian matrices induced by positive definiteness: $A \leqslant B$ means $B - A \geqslant 0$, while $M \geqslant 0$ means that $u^* M u \geqslant 0$ for all vectors u, or alternatively that all eigenvalues of M are non-negative.

Since the Kreiss matrix theorem refers to matrices of a fixed finite order, it has generally been applied in numerical analysis to amplification matrices, i.e., to the Fourier transforms of solution operators of finite difference equations. When the difference equations have variable coefficients, however, the amplification matrices are only distantly related to the solution operator, and the crucial question of concern is the stability of the solution operator itself. In many cases (e.g., for equations of evolution on a compact domain with homogeneous boundary conditions) the solution operator can be represented as a family of finite matrices, but the order of these matrices is not fixed. Instead, the order increases without bound as the mesh size of the difference net is allowed to approach zero.

It therefore seems useful to extend the Kreiss matrix theorem to a family F of square matrices which are not all of the same order, and whose orders, though all finite, have no fixed upper bound. We perform such an extension in this paper. The extension cannot be completely unrestricted, since McCarthy and Schwartz [8] have given an example of a family of unbounded order which satisfies condition (R) but not (A). Here we shall impose upon F the restriction that

$$\deg m(A) \leqslant K, \qquad \text{all} \quad A \in F, \tag{1.7}$$

where $m(A)$ denotes the minimal polynomial of the matrix A and deg means "the degree of". This restriction implies, among other things, that each matrix of F has no more than K distinct eigenvalues. The results we obtain are thus too narrow to be immediately applicable to the solution operators of any interesting problems. The authors hope to relax this restriction in a later paper.

Our precise result is as follows:

THEOREM 1. *Let F be a family of square matrices of various (not necessarily bounded) orders. Suppose that F satisfies* (1.7). *Then the stability of F is equivalent to each of the conditions* (A), (R), (S′) *and* (H). *Here* (A), (R) *and* (H) *are exactly as given above, with each matrix endowed with the spectral norm appropriate for its order. Condition* (S′) *is the same statement as* (S) *except that in* (1.2) *B is a block upper triangular matrix with p^2 blocks ($p \leqslant K$) of possibly varying sizes, so that x_i is replaced by the scalar matrix $x_i I$ of appropriate order d_i, and in* (1.3) *the absolute value $|B_{ij}|$ is replaced by the standard operator norm of the $d_i \times d_j$ matrix B_{ij}:*

$$\|B_{ij}\| = \sup_{u \neq 0} \frac{\|B_{ij} u\|_{d_i}}{\|u\|_{d_j}}, \tag{1.8}$$

where $\|\cdot\|_d$ denotes the Euclidean vector norm (1.6) *in a space of d dimensions.*

The proof of Theorem 1 is essentially an adaptation of the proof of Kreiss's theorem given in Richtmyer and Morton [2]. The detailed arguments are set forth in Sec. 5. As a preparatory step, we derive (in Sec. 2) a special triangular form under unitary equivalence for matrices satisfying (1.7). This new triangular form can also be used to extend previous results of the first author, in particular his "block Buchanan condition" [3, Theorem IV.1], to general families satisfying (1.7). This is discussed in an appendix.

For the proof of Theorem 1 we also need an extension to block matrices of Gerschgorin's theorem [4] on the location of eigenvalues; this is given in Sec. 4. The authors are indebted to Alan Hoffman for the information that a somewhat stronger extension of Gerschgorin's theorem can be found in Feingold and Varga [5]. Thanks are also due to Hans Schneider for two valuable clarifications.

2. REDUCTION TO BLOCK TRIANGULAR FORM

In this section we lay the foundation for our later consideration of condition (S′). We recall that the rationale for a condition such as (S) comes from Schur's theorem [6, p. 75]: an arbitrary square matrix is unitarily equivalent (similar under a unitary similarity transformation) to an upper triangular matrix. Unitary equivalence does not affect stability, since $\|U\| = \|U^{-1}\| = 1$ for a unitary matrix U and $(U^{-1}AU)^n = U^{-1}A^nU$. Triangular matrices, however, are easier to handle, since they display their eigenvalues explicitly on the main diagonal. Moreover, these eigenvalues can be made to appear in any arbitrary order [2, p. 77].

Our restriction (1.7) has, as already observed, the consequence that the set of distinct eigenvalues of any matrix of our family F can number at most K. It seems natural, therefore, to arrange our Schur triangularization so that equal eigenvalues occur at adjacent positions along the main diagonal (Gorelick [3] calls this *semi-nesting*). The result will be a block matrix, with at most K diagonal blocks, and with each block along the main diagonal having the form $\lambda I + N$, where λ is some eigenvalue of the original matrix and N is a *strictly* upper triangular (hence nilpotent) matrix:

$$\begin{bmatrix} \lambda & & & & & N \\ & \lambda & & & & \\ & & \lambda & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & & \lambda \end{bmatrix}.$$

The key fact which enables us to use condition (S′) is that we can actually carry the reduction one step further than this. For a suitably chosen partition, perhaps finer than the one just described but nevertheless containing at most K diagonal blocks, we can make these diagonal blocks scalar and reduce the N to zero. This is the content of the following theorem.

THEOREM 2. *Let A be a square complex matrix whose minimal polynomial has degree p:*

$$m(A)=P(x)=(x-\lambda_1)^{r_1}\cdots(x-\lambda_q)^{r_q}, \qquad \sum_{\nu=1}^{q} r_\nu = p, \tag{2.1}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$. Then A is unitarily equivalent to a partitioned matrix B of the form

$$B=(B_{ij}), \qquad i,j=1,\ldots,p, \tag{2.2}$$

where the B_{ij} are matrix blocks of size $d_i \times d_j$ (with all $d_i \geqslant 1$), and

$$B_{ij}=0, \qquad j<i, \tag{2.3}$$

while

$$B_{ii}=x_i I, \tag{2.4}$$

$$B_{i,i+1} \neq 0 \qquad \text{if} \quad x_i = x_{i+1}. \tag{2.5}$$

The scalars x_i are the eigenvalues of A ordered as follows. The first r_1 of the x_i are equal to λ_1, the next r_2 are equal to λ_2, and so on. Precisely, let $\sum_{\nu=1}^{k} r_\nu = s_k$; then $x_i=\lambda_k$ for $s_{k-1}<i\leqslant s_k$.

Proof. Schur's theorem tells us that A is unitarily equivalent to a matrix

$$C=(C_{kl}), \qquad k,l=1,\ldots,q, \tag{2.6}$$

with C_{kl} of size $\delta_k \times \delta_l$, with $C_{kl}=0$ for $l<k$, and with

$$C_{kk}=\lambda_k I + N_k, \qquad k=1,\ldots,q, \tag{2.7}$$

where N_k is a nilpotent (strictly upper triangular) matrix. We define the *index of nilpotence* of a nilpotent matrix N to be the *smallest* positive integer r for which $N^r=0$. We state two lemmas concerning this index. ∎

LEMMA 2.1. *The index of nilpotence of N_k is exactly equal to r_k.*

LEMMA 2.2 *Let $\tilde{C}$ be a square matrix of the form*

$$\tilde{C}=\lambda I+N, \tag{2.8}$$

where N is nilpotent of index r. Then $\tilde{C}$ is unitarily equivalent to an $r\times r$ block matrix of the form

$$G=\begin{bmatrix} \lambda I & G_{12} & \cdot & \cdot & \cdot & G_{1r} \\ & \lambda I & \cdot & \cdot & \cdot & G_{2r} \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & 0 & & & \cdot & \cdot \\ & & & & & \lambda I \end{bmatrix} \tag{2.9}$$

with all $G_{i,i+1}\neq 0$.

From these lemmas, Theorem 2 would follow, for (2.7) and the lemmas assert the existence for each k of a $\delta_k\times\delta_k$ unitary transformation U_k which reduces C_{kk} to the form (2.9) with $r=r_k$. Then the unitary transformation

$$U=\begin{bmatrix} U_1 & & & & \\ & U_2 & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & U_q \end{bmatrix} \tag{2.10}$$

will reduce C to the form (2.2) with entries satisfying (2.3), (2.4) and (2.5). In particular, the kth block of (2.6) will break up via (2.9) into the $(s_{k-1}+1)$th through s_kth blocks of (2.2).

Thus it remains only to prove Lemmas 2.1 and 2.2. The first of these has already been proved by Gorelick [3], but we give a slightly simplified version of the proof here for the reader's convenience. The second lemma is believed to be new.

Proof of Lemma 2.1. Since A and C are similar matrices, they have the same minimal polynomial $P(x)$. Thus from (2.1) we have

$$0=P(C)=\prod_{l=1}^{q}(C-\lambda_l I)^{r_l}. \tag{2.11}$$

Looking at the main (block) diagonal of $P(C)$, we find

$$0=[P(C)]_{kk}=\prod_{l=1}^{q}(C_{kk}-\lambda_l I)^{r_l}=\prod_{l=1}^{q}[(\lambda_k-\lambda_l)I+N_k]^{r_l}$$

$$=N_k^{r_k}\prod_{l\neq k}[(\lambda_k-\lambda_l)I+N_k]^{r_l}, \tag{2.12}$$

using (2.7). But $\lambda_k-\lambda_l\neq 0$ and N_k is strictly upper triangular, so that every factor under the final product sign of (2.12) is non-singular. Hence

$$N_k^{r_k}=0, \tag{2.13}$$

and the index of nilpotence of N_k is at most r_k.

Conversely, suppose $N_k^{t_k}=0$ for some set of positive integers t_k, and put

$$Q(x)=\prod_{k=1}^{q}(x-\lambda_k)^{t_k}. \tag{2.14}$$

We shall show that $Q(C)=0$, which would imply that $P(x)$ divides $Q(x)$ and therefore $r_k\leqslant t_k$ for all k. This together with (2.13) would prove the Lemma.

We demonstrate that $Q(C)=0$ by induction on q. For $q=1$, we have $C=\lambda_1 I+N_1$ and $Q(C)=N_1^{t_1}=0$. Now for a general q we assume the result for matrices with $q-1$ or fewer distinct eigenvalues, and break up C in the form

$$C=\begin{pmatrix} C' & D \\ 0 & \lambda_q I+N_q \end{pmatrix}.$$

Then

$$Q(C)=\prod_{k=1}^{q-1}(C-\lambda_k I)^{t_k}(C-\lambda_q I)^{t_q}$$

$$=\begin{pmatrix} Q_1(C') & X \\ 0 & Y \end{pmatrix}\begin{pmatrix} (C'-\lambda_q I)^{t_q} & Z \\ 0 & N_q^{t_q} \end{pmatrix},$$

where X, Y, Z are block matrices whose detailed forms are irrelevant, and

$$Q_1(C')=\prod_{k=1}^{q-1}(C'-\lambda_k I)^{t_k}=0$$

by the induction hypothesis. Moreover, $N_q^{t_q}=0$. Hence

$$Q(C)=\begin{pmatrix}0 & X\\ 0 & Y\end{pmatrix}\begin{pmatrix}W & Z\\ 0 & 0\end{pmatrix}=0,$$

completing the induction and the proof of Lemma 2.1. ■

Proof of Lemma 2.2. We may assume without loss of generality that $\lambda=0$, since the addition of a scalar matrix does not affect unitary equivalence:

$$U^*(\lambda I+N)U=\lambda I+U^*NU$$

for any unitary matrix U. We need thus only demonstrate

LEMMA 2.3. *Let N be a nilpotent square matrix of index r. Then $N=UTU^*$, where U is unitary and T has the block representation*

$$T=\begin{bmatrix}0 & T_{12} & \cdot & \cdot & \cdot & T_{1r}\\ \cdot & 0 & \cdot & \cdot & \cdot & T_{2r}\\ \cdot & & \cdot & & & \cdot\\ \cdot & & & \cdot & & \cdot\\ \cdot & & & & \cdot & \cdot\\ 0 & \cdot & \cdot & \cdot & \cdot & 0\end{bmatrix} \tag{2.15}$$

with all $T_{i,i+1}\neq 0$.

Proof of Lemma 2.3. We obtain the representation T by induction on r. For $r=1$ we have $N=N^1=0=T$ and $U=I$. Now assume the result for $r-1$. Since $N^r=N\cdot N^{r-1}=0$, the range of N^{r-1} is contained in the null space S of N. Note that $\dim S=d\geqslant 1$, since N, being nilpotent, is singular. Write δ for the order of N. Let $\{v_1,\dots,v_d\}$ be an orthonormal basis for S, and complete this in an arbitrary fashion to an orthonormal basis $\{v_1,\dots,v_d,v_{d+1},\dots,v_\delta\}$ for the Euclidean δ-space in which N acts. Transform both N and N^{r-1} to this basis by a single unitary transformation. Since the first d basis vectors lie in S, the first d columns of N will become zero, while the last $\delta-d$ rows of N^{r-1} will vanish because the range of N^{r-1} is spanned by $\{v_1,\dots,v_d\}$. Thus, with partitioning into blocks of size d and $\delta-d$, we have for some unitary matrix U_1,

$$N=U_1\begin{pmatrix}0 & M_1\\ 0 & M_2\end{pmatrix}U_1^* \tag{2.16}$$

and

$$N^{r-1} = U_1 \begin{pmatrix} M_3 & M_4 \\ 0 & 0 \end{pmatrix} U_1^*. \tag{2.17}$$

But from (2.16) we find by direct computation that

$$N^{r-1} = U_1 \begin{pmatrix} 0 & M_1 M_2^{r-2} \\ 0 & M_2^{r-1} \end{pmatrix} U_1^*. \tag{2.18}$$

Comparing (2.18) and (2.17) yields $M_2^{r-1} = 0$. Thus by the induction hypothesis M_2 is unitarily equivalent (say via U_2) to an $(r-1)\times(r-1)$ block strictly upper triangular matrix. Setting

$$U_3 = \begin{pmatrix} I & 0 \\ 0 & U_2 \end{pmatrix},$$

we see from (2.16) that

$$N = (U_1 U_3) T (U_1 U_3)^*, \tag{2.19}$$

where T has the form (2.15). This completes the induction.

To establish that all $T_{i,i+1} \neq 0$, observe that otherwise T could be rewritten in the same form but with $r-1$ or fewer blocks. But then $T^{r-1} = 0$ and therefore $N^{r-1} = 0$, contradicting the assumed index r of nilpotence. The proof of Theorem 2 is complete. ■

3. PROPERTIES OF THE BLOCK NORM.

Condition (S′) includes a definition (1.8) of the norm of a rectangular matrix block A_{ij} which is embedded in a larger square matrix A. In this section, we collect for ready reference some of the properties of this "block norm".

First of all, we note that the block norm possesses all properties shared by operator norms in general. In particular, the following equations and inequalities are valid for the block norm whenever the indicated sums and/or products are defined:

$$\|\alpha A_{ij}\| = |\alpha| \cdot \|A_{ij}\| \qquad \text{for scalar } \alpha; \tag{3.1}$$

$$\|A_{ij}\| = 1 \qquad \text{when } A_{ij} \text{ is an identity matrix;} \tag{3.2}$$

$$\|A_{ij} + B_{ij}\| \leqslant \|A_{ij}\| + \|B_{ij}\|; \tag{3.3}$$

$$\|A_{ij} u\| \leqslant \|A_{ij}\| \cdot \|u\| \qquad \text{for vector } u; \tag{3.4}$$

$$\|A_{ij} \cdot B_{kl}\| \leqslant \|A_{ij}\| \cdot \|B_{kl}\|. \tag{3.5}$$

Next, we state two useful inequalities connecting the block norm with the norm of the matrix in which the block is embedded. The proofs have been omitted.

THEOREM 3.2. *Let A denote a partitioned matrix of the form*

$$A = (A_{ij}), \qquad i = 1, \ldots, k, \quad j = 1, \ldots, l.$$

Then for each i and j

$$\|A_{ij}\| \leqslant \|A\|, \tag{3.6}$$

while

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \|A_{ij}\| \geqslant \|A\|. \tag{3.7}$$

Finally, we assert an almost obvious result on the block norm of the adjoint (conjugate transpose) matrix.

THEOREM 3.3. $\|A_{ij}^*\| = \|A_{ij}\|$.

4. A GERSCHGORIN THEOREM FOR PARTITIONED MATRICES

In establishing the sufficiency of the Kreiss condition (S), Richtmyer and Morton [2, p. 76] employ at a crucial point the Gerschgorin circle theorem on the location in the complex plane of the eigenvalues of a matrix. At a corresponding step, we shall need an extension of Gerschgorin's theorem to partitioned matrices in order to establish the sufficiency of our condition (S′). We derive the needed extension in this section.

THEOREM 4. *Let*

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdot & \cdot & \cdot & M_{1n} \\ M_{21} & M_{22} & \cdot & \cdot & \cdot & M_{2n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ M_{n1} & M_{n2} & \cdot & \cdot & \cdot & M_{nn} \end{bmatrix}$$

be a matrix of arbitrary order such that M_{ii} is a square matrix of order d_i and M_{ij} is a $d_i \times d_j$ matrix. Put $M_{ii} = \Lambda_i + E_i$, where Λ_i is a scalar matrix of the form $\lambda_i I_{d_i}$, and $E_i = M_{ii} - \Lambda_i$. Then every eigenvalue λ of M must lie in the union of the n circles

$$|z - \lambda_i| \leqslant \|E_i\| + \sum_{j \neq i} \|M_{ij}\|, \qquad i = 1, \ldots, n, \tag{4.1}$$

where the norms are defined by (1.8).

Proof. Let v be an eigenvector of M corresponding to λ, so that $Mv = \lambda v$. Write $v = (v_1, v_2, \ldots, v_n)^T$, where v_j is a vector of d_j components. Let

$$\|v_i\| = \max_{1 \leqslant j \leqslant n} \|v_j\|, \tag{4.2}$$

where $\|v_j\|$ denotes the Euclidean d_j-norm.

We set $Mv = [(Mv)_1, (Mv)_2, \ldots, (Mv)_n]^T$, where $(Mv)_j$ is also a vector of d_j components. Since $Mv = \lambda v$, we have

$$(Mv)_k = \lambda v_k \tag{4.3}$$

Furthermore,

$$(Mv)_k = \sum_{j=1}^{n} M_{kj} v_j, \tag{4.4}$$

so that

$$\lambda v_k = \sum_{j=1}^{n} M_{kj} v_j. \tag{4.5}$$

Putting $k = i$ and $M_{ii} = \Lambda_i + E_i$, we find that

$$(\lambda - \lambda_i) v_i = E_i v_i + \sum_{j \neq i} M_{ij} v_j. \tag{4.6}$$

The relation (3.4) and the vector triangle inequality then yield

$$|\lambda - \lambda_i| \cdot \|v_i\| \leqslant \|E_i\| \cdot \|v_i\| + \sum_{j \neq i} \|M_{ij}\| \cdot \|v_j\|. \tag{4.7}$$

Dividing both sides of (4.7) by $\|v_i\|$ and using (4.2), we get

$$|\lambda-\lambda_i| \leqslant \|E_i\| + \sum_{j\neq i} \|M_{ij}\|. \tag{4.8}$$

This proves the theorem. ∎

5. PROOF OF MAIN THEOREM

In this section we give the detailed proof of Theorem 1. We shall follow the proof of the standard Kreiss matrix theorem as given in Richtmyer and Morton [2]: we show that (A)→(R)→(S′)→(H)→(A).

(H) *implies* (A); (A) *implies* (R). The standard proofs of these statements [2, pp. 76, 77] do not depend on the order of the matrices involved, and so carry over word for word to the present situation. For completeness, we repeat them here.

If (H) is satisfied, consider the iteration

$$w_\nu = Aw_{\nu-1} = A^\nu w_0.$$

Then

$$w_\nu^* Hw_\nu = w_{\nu-1}^* A^* HAw_{\nu-1} \leqslant w_{\nu-1}^* Hw_{\nu-1} \leqslant \cdots \leqslant w_0^* Hw_0,$$

and so, from (1.4a), $\|w_\nu\|^2 \leqslant C_H^2 \|w_0\|^2$, i.e., $\|A^\nu\| \leqslant C_H$, which is (A).

If (A) is true, the eigenvalues x_i of A lie within the closed unit disk, and therefore $(A-zI)^{-1}$ exists for $|z|>1$. Moreover,

$$\|(A-zI)^{-1}\| = \left\| \sum_{\nu=0}^{\infty} A^\nu z^{-\nu-1} \right\| \leqslant C \sum_{\nu=0}^{\infty} |z|^{-\nu-1} = C(|z|-1)^{-1},$$

so that (R) is satisfied with $C_R \leqslant C$.

(S′) *implies* (H). Introduce the real diagonal matrix

$$D = \begin{bmatrix} \Delta I & & & \\ & \Delta^2 I & & 0 \\ & & \ddots & \\ & 0 & & \Delta^p I \end{bmatrix}, \tag{5.1}$$

where Δ is a constant greater than 1, and the coefficient I of Δ^i is an identity matrix of the same order d_i as the corresponding block in B [cf. (1.2)]. We claim that Δ can be chosen so large that

$$D - B^* DB \geqslant 0, \tag{5.2}$$

i.e.,

$$M = I - (D^{-1/2} B^* D^{1/2})(D^{1/2} B D^{-1/2}) \geqslant 0.$$

To show this, we apply Theorem 4 to $M = (M_{ij})$. The diagonal block of M is seen from (1.2) to be

$$\begin{aligned} M_{ii} &= I_{d_i} - \sum_{k=1}^{p} \Delta^{k-i} B_{ki}^* B_{ki} \\ &= (1 - |x_i|^2) I_{d_i} - \sum_{k=1}^{i-1} \Delta^{k-i} B_{ki}^* B_{ki} \\ &= (1 - |x_i|^2) I_{d_i} + \varepsilon_i, \end{aligned} \tag{5.3}$$

where we can estimate ε_i by (1.3), (3.5) and Theorem 3.3:

$$\|\varepsilon_i\| \leqslant C_B^2 \frac{(1 - |x_i|)^2}{\Delta - 1}. \tag{5.4}$$

The off-diagonal blocks of M have the form

$$M_{ij} = - \sum_{k=1}^{\min(i,j)} \Delta^{(2k-i-j)/2} B_{ki}^* B_{kj}, \tag{5.5}$$

so that

$$\|M_{ij}\| \leqslant C_B^2 \frac{1 - |x_i|}{\Delta^{1/2} - 1},$$

and the sum δ_i of the norms of the off-diagonal blocks is estimated by

$$\delta_i = \sum_{j \neq i} \|M_{ij}\| \leqslant (K-1) C_B^2 \frac{1 - |x_i|}{\Delta^{1/2} - 1}, \tag{5.6}$$

where K is the bound assumed in (1.7). Hence for sufficiently large Δ depending only on C_B and K, we have

$$\delta_i + \|\varepsilon_i\| \leqslant 1 - |x_i| \leqslant 1 - |x_i|^2. \tag{5.7}$$

Thus, by (5.3) and Theorem 4, every eigenvalue of M lies in the right half plane. Since M is Hermitian, its eigenvalues are real, hence non-negative, and therefore $M \geqslant 0$. This establishes (5.2).

We thus have $B^*DB \leqslant D$. Since $B = SAS^{-1}$, and matrix inequalities are unaffected by pre-multiplying by S^* and simultaneously post-multiplying by S, we find

$$A^*S^*DSA \leqslant S^*DS. \tag{5.8}$$

Setting $H = S^*DS$ yields (1.4b), while (1.4a) is satisfied with $C_H = \Delta^K C_S^2$. Thus (S′) implies (H).

(R) *implies* (S′). Using Theorem 2, we may first transform A by a unitary transformation to $p \times p$ block upper triangular form (2.2) or (1.2) with $p \leqslant K$. This leaves the resolvent condition unaffected. Then, following [2], we shall obtain the similarity transformation which yields the required result by working on each upper block diagonal of A successively. In other words, the inequality

$$\|B_{ij}\| \leqslant C_B \min(1 - |x_i|, 1 - |x_j|) \tag{5.9}$$

will be obtained in turn for $j - i = 1, 2, \ldots, p-1$.

The first observation is that each corner (upper left or lower right principal sub-matrix) of A satisfies (R) with the same constant C_R. This follows immediately from the triangularity of A. Furthermore, each block on the first upper diagonal of A lies in a block 2×2 corner of such a corner, namely

$$Q = \begin{pmatrix} x_i I & A_{i,i+1} \\ 0 & x_{i+1} I \end{pmatrix}. \tag{5.10}$$

Thus $\|(Q - zI)^{-1}\| \leqslant C_R(|z| - 1)^{-1}$, and by (3.6) the upper right-hand corner of $(Q - zI)^{-1}$, which is

$$-(x_i - z)^{-1}(x_{i+1} - z)^{-1} A_{i,i+1},$$

satisfies this same inequality. Thus we get

$$\|A_{i,i+1}\| \leqslant \frac{C_R|z-x_i|\cdot|z-x_{i+1}|}{|z|-1}. \tag{5.11}$$

From here, we may obtain the more useful form

$$\|A_{i,i+1}\| \leqslant 16C_R \max(\gamma_{i,i+1}, |x_i - x_{i+1}|), \tag{5.12}$$

where

$$\gamma_{ij} = \min(1-|x_i|, 1-|x_j|), \tag{5.13}$$

by making use of the freedom of choice of z just as on p. 78 of [2].

We now show that inequality (5.9) can be achieved for each block $B_{i,i+1}$ of the first upper diagonal, with $C_B \leqslant 16C_R$. If $\gamma_{i,i+1} \geqslant |x_i - x_{i+1}|$, this inequality is already true for $A_{i,i+1}$ and no transformation is required; otherwise, $A_{i,i+1}$ can be annihilated by a bounded similarity transformation. For in general, the block A_{ij} with $j > i$ is annihilated by the transformation $S_{ij}AS_{ij}^{-1}$, where

$$S_{ij} = I + T_{ij} \tag{5.14}$$

and T_{ij} is a block matrix all of whose blocks are zero except the (i,j)th, which is $(x_i - x_j)^{-1}A_{ij}$. Thus, when the transformation is needed, (5.12) provides a bound for it and its inverse $S_{ij}^{-1} = I - T_{ij}$, the bound being the same for all A in the family F. By composing at most $K-1$ such transformations we fulfill the requirements of condition (S′) for the first upper diagonal, with $C_S \leqslant 1 + 16C_R$.

To continue this process we need the following key lemma.

LEMMA 5.1. *If the family of block $m \times m$ upper triangular matrices A satisfy the resolvent condition* (R) *with a constant C_1, and if all their off-diagonal blocks except A_{1m} satisfy the inequality*

$$\|A_{ij}\| \leqslant C_2\gamma_{ij}, \tag{5.15}$$

then

$$\|A_{1m}\| \leqslant 16C_1\left[1+(m-2)C_2^2\right]^{1/2} \max(\gamma_{1m}, |x_1 - x_m|). \tag{5.16}$$

Assuming the lemma, the proof of the statement (S′) is straightforward. When the $(m-1)$th upper diagonal has been made to satisfy (5.9), each block of the mth is the top right block of a block matrix to which the lemma can be applied. Then, by use of the similarity transformations $S_{i,i+m}$, as defined in (5.14), and using (5.16) in place of (5.12), the inequality (5.9) is extended to this upper diagonal. This process succeeds because each similarity transformation $S_{i,i+m}$, apart from effecting the desired annihilations, changes only elements in diagonals yet to be considered, and leaves the resolvent condition unaffected except for multiplication of the constant by $\|S_{i,i+m}\|^2$. Hence Theorem 1 will be fully proved when we have established Lemma 5.1.

Proof of Lemma 5.1. *We reduce the general case to the case* $m=2$, *which has already been proved above* [*cf.* (5.12)]. *We permute the rows and columns of* $A-zI$ *so that the second row (and column) in block form is interchanged with the mth row (and column). It is easily seen that this leaves the resolvent condition unchanged. Then we partition the result E in the following way*:

$$E=\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}=\left[\begin{array}{cc|cccccc} \zeta_1 I & A_{1m} & A_{13} & A_{14} & \cdot & \cdot & \cdot & A_{1,m-1} & A_{12} \\ 0 & \zeta_m I & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \hline 0 & A_{3m} & & & & & & & \\ 0 & A_{4m} & & & & & & & \\ \cdot & \cdot & & & & & & & \\ \cdot & \cdot & & & & E_4 & & & \\ \cdot & \cdot & & & & & & & \\ 0 & A_{m-1,m} & & & & & & & \\ 0 & A_{2m} & & & & & & & \end{array}\right],$$

where $\zeta_i = x_i - z$ *and the detailed form of* E_4 *is not required. Clearly* $E_3E_2=0$, *and indeed* $E_3E_1^{-1}E_2=0$. *Hence, if we perform a triangular decomposition of E into* $E=LU$, *we have*

$$L=\begin{bmatrix} I & 0 \\ E_3E_1^{-1} & I \end{bmatrix}, \qquad U=\begin{bmatrix} E_1 & E_2 \\ 0 & E_4 \end{bmatrix}. \tag{5.17}$$

The resolvent condition therefore gives, for any vector u,

$$\|E^{-1}u\|^2 \leqslant \left(\frac{C_1}{\zeta}\right)^2 \|u\|^2, \tag{5.18}$$

where $\zeta = |z| - 1$. *Putting* $u = Lv$, *where*

$$v = (v_1, v_2, 0, \ldots, 0) = (w, 0), \tag{5.19}$$

with $\dim v_1 = d_1$ *and* $\dim v_2 = d_m = \operatorname{ord}(\zeta_m I)$, gives

$$\|E_1^{-1}w\|^2 \leqslant \left(\frac{C_1}{\zeta}\right)^2 \|Lv\|^2. \tag{5.20}$$

But

$$\|Lv\|^2 = \|w\|^2 + \|E_3 E_1^{-1} w\|^2 \tag{5.21}$$

and

$$E_3 E_1^{-1} w = \zeta_m^{-1}(A_{3m}v_2, A_{4m}v_2, \ldots, A_{m-1,m}v_2, A_{2m}v_2), \tag{5.22}$$

so that

$$\begin{aligned} \|E_3 E_1^{-1} w\|^2 &\leqslant |\zeta_m|^{-2} \|v_2\|^2 \sum_{i=2}^{m-1} \|A_{im}\|^2 \\ &\leqslant |x_m - z|^{-2} \|w\|^2 (m-2) C_2^2 \gamma_{im}^2 \\ &\leqslant \frac{(1-|x_m|)^2}{|z - x_m|^2} C_2^2 (m-2) \|w\|^2 \leqslant (m-2) C_2^2 \|w\|^2 \end{aligned} \tag{5.23}$$

by (5.15), (5.13) *and the geometrical positions of* z *and* x_m *in the complex plane. Putting these inequalities together, we see that*

$$\|E_1^{-1}\| \leqslant C_1 \left[1 + (m-2) C_2^2\right]^{1/2} / \zeta. \tag{5.24}$$

Then the 2×2 *result previously proved yields the inequality* (5.16) *for* A_{1m}. *This establishes the lemma and completes the proof of Theorem* 1.

APPENDIX. AN EXTENSION OF THE BLOCK BUCHANAN THEOREM

In a previous paper [3], the first author has given a "block" version of the Buchanan necessary and sufficient condition for stability of a family of matrices. The original Buchanan condition ([7]; see also Section 4.10 of [2]) refers to families of $p \times p$ matrices which are already in upper triangular form

$$A = \begin{bmatrix} x_1 & A_{12} & \cdot & \cdot & \cdot & A_{1p} \\ & \cdot & & & & \cdot \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & 0 & & & \cdot & \cdot \\ & & & & & x_p \end{bmatrix} \tag{A.1}$$

with "uniformly nested" eigenvalues:

$$|x_r - x_s| \leqslant C_N |x_i - x_j|, \qquad i \leqslant r \leqslant s \leqslant j, \tag{A.2}$$

C_N a constant independent of A. It asserts that such a family is stable iff the von Neumann condition

$$|x_i| \leqslant 1, \qquad 1 \leqslant i \leqslant p \tag{A.3a}$$

holds and the off-diagonal elements satisfy

$$|A_{ij}| \leqslant K_2 \max(1 - |x_i|, 1 - |x_j|, |x_i - x_j|) \tag{A.3b}$$

for some constant K_2 independent of A.

The block Buchanan condition proved in [3] is valid for families F of diagonalizable matrices of possibly unbounded order, provided F satisfies the minimal polynomial condition (1.7). Each A in F can be written in $p \times p$ block form

$$A = \begin{bmatrix} x_1 I & A_{12} & \cdot & \cdot & \cdot & A_{1p} \\ & x_2 I & \cdot & \cdot & \cdot & A_{2p} \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & 0 & & & \cdot & \cdot \\ & & & & & x_p I \end{bmatrix} \tag{A.4}$$

with $p \leqslant K$, K fixed. For such families the conditions (A.3) with $|A_{ij}|$ replaced by $\|A_{ij}\|$ are necessary and sufficient for stability whenever (A.2) holds.

Our Theorem 2 allows us now to extend the block Buchanan condition to families F of (not necessarily diagonalizable) matrices of possibly unbounded order provided that F satisfies the minimal polynomial condition (1.7). For any such family F can be brought to form (A.4) by unitary transformations.

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Received 14 December 1973; revised 19 July 1975