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Smoothness of Solutions for Delay-Difference Equations

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Abstract—For the delay-difference equation, $x(n+1) = f(n, x(n), x(n-\varphi(n)))$, satisfying $x(j) = \tau(j)$, j belonging to an initial segment, smooth dependence of a solution, $x(n, \tau)$, with respect to τ, is considered.

Keywords—Difference equation, Delay, Initial data.

1. INTRODUCTION

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, and $\mathbb{Z}^+$ the nonnegative integers. Given $a < b$ in $\mathbb{Z}$, let $[a, b] = \{a, a+1, \dots, b\}$, $[a, b) = \{a, \dots, b-1\}$, with $(-\infty, a)$, $[a, +\infty)$, etc., denoting the analogous discrete sets.

For finite difference equations, recent attention has been given to smoothness of solutions (i.e., continuity and differentiability), with respect to initial and boundary data specified; for example see [1–5], among others. In this paper, we consider the dependence of solutions of delay-difference equations upon initial data. That is, for the finite delay-difference equation,

$$x(n+1) = f(n, x(n), x(n-\varphi(n))), \qquad n \in \mathbb{Z}^+, \tag{1}$$

satisfying the initial conditions,

$$x(j) = \tau(j), \qquad j \in [-r, 0], \tag{2}$$

where $f(n, u_1, u_2) : \mathbb{Z}^+ \times \mathbb{R}^2 \to \mathbb{R}$ and $f_{u_i}(n, u_1, u_2) : \mathbb{Z}^+ \times \mathbb{R}^2 \to \mathbb{R}, i = 1, 2$, are continuous, $r \in \mathbb{Z}^+, \varphi : \mathbb{Z}^+ \to [0, r]$, and $\tau : [-r, 0] \to \mathbb{R}$, we wish to determine the smoothness dependence of a solution, x, upon the initial function τ.

For this analysis, given a solution $x(n)$ of (1), we will make use of the *variational equation along* $x(n)$ given by

$$z(n+1) = f_{u_1}(n, x(n), x(n-\varphi(n)))z(n) + f_{u_2}(n, x(n), x(n-\varphi(n)))z(n-\varphi(n)). \tag{3}$$

The results obtained in this work can be considered as analogues of theorems on the continuous dependence of solutions, with respect to initial conditions, for functional differential equations by Driver [6], Hale [7], Lakshmikantham and Leela [8], and Melvin [9], as well as theorems on smooth dependence for solutions of functional differential equations by Hale [7]. In fact, the main motivations for this paper are the recent result by Hale and Ladeira [10] on differentiability of solutions, with respect to delays, for delay-differential equations, and the generalizations in [11,12] to differentiability of solutions, with respect to delays and boundary data, for boundary value problems for functional differential equations.

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2. SMOOTHNESS OF SOLUTIONS

Given $\tau : [-r, 0] \to \mathbb{R}$, let $x(n, \tau)$ denote a solution of (1), (2) corresponding to τ. It follows immediately from the equation that $x(n, \tau)$ is the unique solution of (1), (2) on $[-r, \infty)$, in the obvious sense. Moreover, from the continuity of f and this uniqueness of solutions "to the right," we readily obtain our first result on the continuity of $x(n, \tau)$ with respect to τ; see [4] for a typical argument.

THEOREM 1. *Let $\tau : [-r, 0] \to \mathbb{R}$. Given $\epsilon > 0$ and $k \in \mathbb{Z}^+$, there exists a $\delta(\epsilon, \tau, k) > 0$ such that, if $\sigma : [-r, 0] \to \mathbb{R}$ and*

$$|\tau(j) - \sigma(j)| < \delta, \qquad j \in [-r, 0],$$

then the unique solution, $x(n, \sigma)$, of (1), (2) corresponding to σ satisfies

$$|x(n, \tau) - x(n, \sigma)| < \epsilon, \qquad \text{on } [-r, k].$$

For the main theorem of the paper, given an initial function $\tau : [-r, 0] \to \mathbb{R}$, we shall identify it with its values by

$$\tau = (\tau(-r), \dots, \tau(0)) = (\tau_{-r}, \dots, \tau_0).$$

Having established the continuous dependence of $x(n, \tau)$ with respect to τ, we now show that dependence to be smooth, and we also characterize the resulting partial derivatives.

THEOREM 2. *Let $x(n) = x(n, \tau)$ denote the solution of (1), (2) on $\mathbb{Z}^+$ corresponding to $\tau : [-r, 0] \to \mathbb{R}$. Then, for $-r \le i \le 0$, $\dfrac{\partial x}{\partial \tau_i}$ exists on $[-r, \infty)$, and $z_i(n) = \dfrac{\partial x}{\partial \tau_i}(n)$ is the solution of the variational equation (3) along $x(n, \tau)$ on $\mathbb{Z}^+$ and satisfies the initial value,*

$$z_i(j) = \begin{cases} 0, & j \in [-r, 0] \setminus \{i\}, \\ 1, & j = i. \end{cases}$$

PROOF. For $-r \le i \le 0$, let $e_i = (\delta_{-ri}, \dots, \delta_{0i})$, where δ_{pq} denotes the Kronecker delta. Let $\epsilon > 0$ and $k \in \mathbb{N}$ be given, and let $\delta > 0$ be as in Theorem 1. For $0 < |h| < \delta$, consider the difference quotient,

$$z_{ih}(n) = \frac{1}{h}[x(n, \tau + he_i) - x(n, \tau)], \qquad n \in [-r, k].$$

Note at first that,

$$z_{ih}(j) = \begin{cases} 0, & j \in [-r, 0] \setminus \{i\}, \\ 1, & j = i. \end{cases} \tag{4}$$

Next, we consider the delay-difference equation satisfied by $z_{ih}(n)$ on $[0, k]$. Using a telescoping sum, we have

$$\begin{aligned} z_{ih}(n+1) &= \frac{1}{h}\Big[f(n, x(n, \tau + he_i), x(n - \varphi(n), \tau + he_i)) - f(n, x(n, \tau), x(n - \varphi(n), \tau))\Big] \\ &= \frac{1}{h}\Big[f(n, x(n, \tau + he_i), x(n - \varphi(n), \tau + he_i)) - f(n, x(n, \tau), x(n - \varphi(n), \tau + he_i)) \\ &\qquad + f(n, x(n, \tau), x(n - \varphi(n), \tau + he_i)) - f(n, x(n, \tau), x(n - \varphi(n), \tau))\Big]. \end{aligned}$$

Applying the Mean Value Theorem, we obtain

$$\begin{aligned} z_{ih}(n+1) &= \int_0^1 f_{u_1}\Big(n, sx(n, \tau + he_i) + (1-s)x(n, \tau), x(n - \varphi(n), \tau + he_i)\Big)\, ds \\ &\quad \times \left(\frac{1}{h}\big[x(n, \tau + he_i) - x(n, \tau)\big]\right) \\ &\quad + \int_0^1 f_{u_2}\Big(n, x(n, \tau), sx(n - \varphi(n), \tau + he_i) + (1-s)x(n - \varphi(n), \tau)\Big)\, ds \\ &\quad \times \left(\frac{1}{h}\big[x(n - \varphi(n), \tau + he_i) - x(n - \varphi(n), \tau)\big]\right) \\ &= A_{1h}(n, \tau) z_{ih}(n) + A_{2h}(n, \tau) z_{ih}(n - \varphi(n)), \end{aligned} \tag{5}$$

where A_{1h} and A_{2h} represent the first and second integrals, respectively, in (5). The continuity of $f_{u_\ell}, \ell = 1, 2$, implies that,

$$\lim_{h\to 0} A_{\ell h}(n,\tau)w = f_{u_\ell}(n, x(n,\tau), x(n-\varphi(n),\tau))w,$$

uniformly on compact subsets of $[0,k] \times \mathbb{R}$.

In conjunction with (4), if we mimic continuous dependence arguments for initial value problems for functional differential equations, as in [7], we obtain that

$$\lim_{h\to 0} z_{ih}(n) = z_i(n)$$

on $[-r,k]$, where $z_i(n)$ is the solution of

$$z_i(n+1) = f_{u_1}(n, x(n,\tau), x(n-\varphi(n),\tau))z_i(n) + f_{u_2}(n, x(n,\tau), x(n-\varphi(n),\tau))z_i(n-\varphi(n)),$$

on $[0,k]$, and satisfies

$$z_i(j) = \begin{cases} 0, & j \in [-r,0] \setminus \{i\}, \\ 1, & j = i. \end{cases}$$

From the definition of $z_{ih}(n)$, we conclude $\dfrac{\partial x}{\partial \tau_i}(n)$ exists, and $z_i(n) = \dfrac{\partial x}{\partial \tau_i}(n)$ on $[-r,k]$. But $k \in \mathbb{N}$ was arbitrary, and so $\dfrac{\partial x}{\partial \tau_i}(n)$ exists on $[-r,\infty)$ and is the solution of the asserted problem. The proof is complete.

REFERENCES

1. A. Datta and J. Henderson, Differentiation of solutions of difference equations with respect to right focal boundary values, *Pan Am. Math. J.* **2**, 1–16 (1992).
2. D. Hankerson, An existence and uniqueness theorem for difference equations, *SIAM J. Math. Anal.* **20**, 1208–1217 (1989).
3. J. Henderson, M. Horn and L. Howard, Differentiation of solutions of difference equations with respect to boundary values and parameters, (preprint).
4. J. Henderson and L. Lee, Continuous dependence and differentiation of solutions of finite difference equations, *Int. J. Math. and Math. Sci.* **14**, 747–756 (1991).
5. A.C. Peterson, Existence and uniqueness theorems for nonlinear difference equations, *J. Math. Anal. Appl.* **125**, 185–191 (1987).
6. R.D. Driver, Existence and continuous dependence of solutions of a neutral functional-differential equation, *Arch. Rat. Mech. Anal.* **19**, 149–166 (1965).
7. J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, (1977).
8. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Academic Press, New York, (1969).
9. W.R. Melvin, Topologies for neutral functional differential equations, *J. Diff. Eqs.* **13**, 24–34 (1973).
10. J.K. Hale and L.A.C. Ladeira, Differentiability with respect to delays, *J. Diff. Eqs.* **92**, 14–26 (1991).
11. J. Ehme and J. Henderson, Functional boundary value problems and smoothness of solutions, (preprint).
12. J. Ehme, P.W. Eloe and J. Henderson, Differentiability with respect to boundary conditions and deviating argument for functional differential systems, *Diff. Eqs. and Dyn. Systems* (preprint).