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GENERALIZED FIBONACCI MAXIMUM PATH GRAPHS

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We investigate the following problem: Given integers m and n, find an acyclic directed graph with m edges and n vertices and two distinguished vertices s and t such that the number of distinct paths from s to t (not necessarily disjoint) is maximized. It is shown that there exists such a graph containing a Hamiltonian path, and its structure is investigated.

We give a complete solution to the cases (i) $m \leq 2n-3$ and (ii) $m = kn - \frac{1}{2}k(k+1) + r$ for $k = 1, 2, \ldots, n-1$ and $r = 0, 1, 2$.

1. Introduction

A digraph $G = (V, E)$ consists of a finite set V of *vertices* and a collection E of ordered pairs of distinct vertices called *edges*. A sequence of vertices $[v_0, v_1, v_2, \ldots, v_k]$, where $(v_{i-1}, v_i) \in E$ for $i = 1, 2, \ldots, k$ is called a *path* from v_0 to v_k. Two paths are *distinct* if they are not identical sequences of vertices. The digraph G is *acyclic* if there is no closed path $[v_0, v_1, \ldots, v_k, v_0]$ in G. Finally, G is *connected* if for every pair of distinct vertices x and y there is a *chain* $[\![x = v_0, v_1, v_2, \ldots, v_k = y]\!]$ where either $(v_{i-1}, v_i) \in E$ or $(v_i, v_{i-1}) \in E$ for $i = 1, 2, \ldots, k$. We say that an edge (x, y) emanates from x and goes to y, or alternatively, it exits from x and it enters y. A vertex into which no edge enters is called a *source*; a vertex from which no edge emanates is called a *sink*; a vertex with no edge adjacent to it is called isolated. In this paper we deal with *acyclic digraphs without isolated vertices* exclusively.

Let G contain two distinguished vertices s and t. Define $N(G)$ to be the number of distinct paths in G from s to t. Consider the following problem: Given integers m and n, find an acyclic digraph G with m edges and n vertices

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maximizing the number $N(G)$ of distinct paths from s to t. We call G a *maximum path graph* with parameters m and n, and we define $N_{n,m} = N(G)$. Clearly, $N_{n,m}$ is defined only in the domain $n-1 \leq m \leq \frac{1}{2}n(n-1)$. The numbers $N_{n,m}$ are of interest in the complexity analysis of algorithms for generating all simple paths between two distinguished verticies in a graph, [1, 7]. Another application follows in Section 4.

Example 1. The digraph $\mathscr{F}_n$ has vertices $\{1, 2, 3, \ldots, n\}$ and edges

$$\{(i, i+1) \mid i = 1, 2, \ldots, n-1\} \cup \{(i, i+2) \mid i = 1, 2, \ldots, n-2\},$$

(see Fig. 1a). Let $s = 1$ and $t = n$. What is $N(\mathscr{F}_n)$? Every path from 1 to n consists of either

(i) a path from 1 to $n-2$ followed by the edge $(n-2, n)$, or
(ii) a path from 1 to $n-1$ followed by the edge $(n-1, n)$.

There are $N(\mathscr{F}_{n-2})$ of type (i) and $N(\mathscr{F}_{n-1})$ of type (ii); therefore,

$$N(\mathscr{F}_n) = N(\mathscr{F}_{n-2}) + N(\mathscr{F}_{n-1}).$$

Adding the trivial cases $N(\mathscr{F}_1) = 1$ and $N(\mathscr{F}_2) = 1$, we obtain the result,

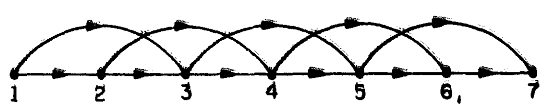
$$N(\mathscr{F}_n) = F_n$$

where F_n is the nth Fibonacci number. We will call $\mathscr{F}_n$ the nth *Fibonacci graph*. It has $m = 2n-3$ edges.

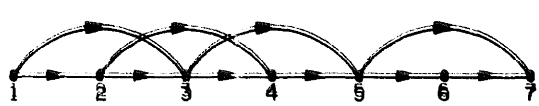
Example 2. The digraph $\mathscr{A}_n$ has vertices $\{1, 2, 3, \ldots, n\}$ and edges

$$\{(i, i+1) \mid i = 1, 2, \ldots, n-1\}$$
$$\cup \{(i, i+2) \mid i = 1, 2, \ldots, n-4 \text{ or } i = n-2\},$$

(see Fig. 1b). We will call $\mathscr{A}_n$ the nth *almost Fibonacci graph* since it is constructed from $\mathscr{F}_n$ by deleting the edge $(n-3, n-1)$. It is straight-forward to



(a) The Fibonacci graph $\mathscr{F}_7$



(b) The almost Fibonacci graph $\mathscr{A}_7$

Fig. 1.

show that,

$$N(\mathscr{A}_n) = 2N(\mathscr{F}_{n-2}) = 2F_{n-2}.$$

In Perl [4] we investigate several cases of maximum path graphs given only the number of edges m. It is shown there that for acyclic digraphs without parallel edges the (almost) Fibonacci graphs are maximum path graphs for (even) odd number of edges. Hence $N_{n,2n-3} = N(\mathscr{F}_n) = F_n$ and $N_{n,2n-4} = N(\mathscr{A}_n) = 2F_{n-2}$. In this paper we will investigate $N_{n,m}$ and give a complete solution to the cases (i) $m \leqslant 2n-3$ and (ii) $m = kn - \frac{1}{2}k(k+1) + r$ for $k = 1, 2, \ldots, n-1$ and $r = 0, 1, 2$.

2. The structure of maximum path graphs

We can make certain assumptions about the structure of a maximum path graph G which will be useful in calculating $N(G)$. First, we may assume that

$$\text{every vertex } x \text{ lies on some path from } s \text{ to } t, \tag{1}$$

for otherwise we could rename t as t' and replace x and the k edges ($k \geqslant 1$) incident on x with a new distinguished vertex t and k new edges from t' and any $k-1$ other vertices to t, thus obtaining a graph G' with $N(G') \geqslant N(G)$. In particular, *we may assume that G is connected, has only one source s and only one sink t.* The second and more powerful assumption is that G can be chosen to have a *Hamiltonian path*, that is, a path which contains every vertex.

Theorem 1. *For all n and m, there exists a maximum path graph G which contains a Hamiltonian path, thus uniquely ordering the vertices.*

Proof. Let $G = (V, E)$ be a maximum path graph satisfying (1) with parameters n and m, and let $P = [v_1, v_2, \ldots, v_k]$ be a path of maximum length in G. (Note that $s = v_1$ and $t = v_k$.) Suppose $k < n$, then we will construct a graph G' containing a path of length $k+1$ such that $N(G') \geqslant N(G)$.

Every path which passes through a vertex not on P is of the form

$$Q = [v_1, v_2, \ldots, v_i, x, \ldots, t]$$

where $i \geqslant 1$ and x does not lie on P. Choose Q to be the longest such path. (See Fig. 2.) The maximality of P implies that $(x, v_{i+1}) \notin E$, and the maximality of Q implies that there is no path (and hence no edge) in G from v_{i+1} to x. Define

$$E' = E - \{(v_i, v_{i+1})\} + \{(x, v_{i+1})\}.$$

The acyclic graph $G' = (V, E')$ has a path of length $k+1$, and $N(G') \geqslant N(G)$ since every path in G which used (v_i, v_{i+1}) can be rerouted using the new segment $v_i \Rightarrow x \Rightarrow v_{i+1}$. This procedure, carried out $n-k$ times, will give a maximum path

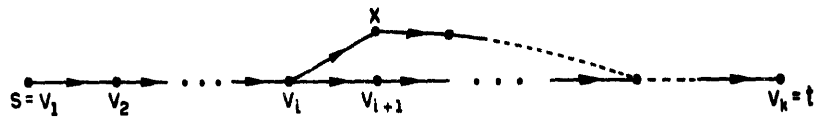


Fig. 2.

graph with a Hamiltonian path. The Hamiltonian path uniquely orders the vertices, since the graph is acyclic.

For the remainder of this paper, we will choose our maximum path graph G to have vertices named $1, 2, 3, \ldots, n$ and edges set which includes the edges $(i, i+1)$ for $i = 1, 2, \ldots, n-1$. Such a graph will be called a *Hamiltonian maximum path graph*; it is both a maximum path graph and contains a Hamiltonian path. We denote by $P(i, j)$ the number of distinct paths in G from i to j. Thus, $P(1, n) = N(G)$, $P(i, i) = 1$, and $P(i, i+1) = 1$. An edge $(i, i+k)$ is said to be of *level* k. We will say that *level* k *is full* in G if $(i, i+k)$ is an edge of G for $i = 1, 2, \ldots, n-k$. By our assumption, level 1 is always full. Thus, our problem becomes the following: After using $n-1$ of our m edges to build the Hamiltonian path, how shall we spend the remainder of the edges in order to maximize the number of paths from vertex 1 to vertex n? We begin to answer this question in the next section.

3. Generalized Fibonacci graphs

Let G be a Hamiltonian maximum path graph. We say that an edge $e = (i, j)$ *properly covers* an edge (or nonedge) $e' = (i', j')$ if $i \leq i' < j' \leq j$ and $e \neq e'$. The distance between two vertices x and y $(x \leq y)$ is denoted by $\Delta(x, y) = y - x$.

The generalized Fibonacci number $F_n^{(j)}$ is defined recursively as follows:

$$F_1^{(j)} = 1,$$

$$F_h^{(j)} = \sum_{i=1}^{h-1} F_i^{(j)} \qquad (h \leq j),$$

$$F_h^{(j)} = \sum_{i=h-j}^{h-1} F_i^{(j)} \qquad (h > j).$$

The digraph $\mathscr{F}_n^{(k)}$ has vertices $\{1, 2, 3, \ldots, n\}$ and edges

$$\{(i, j) \mid 1 \leq i < j \leq n \text{ and } |j - i| \leq k\}.$$

It is straightforward to show that

$$N(\mathscr{F}_n^{(k)}) = F_n^{(k)}.$$

We call $\mathscr{F}_n^{(k)}$ the nth *generalized k-Fibonacci graph*.

Theorem 2. *For all parameters n and m, there exists a Hamiltonian maximum path graph G satisfying the following property: If G has an edge of level k, then for each $l<k$, level l is full.*

Proof. By Theorem 1, there exists a Hamiltonian maximum path graph G. If G has an edge e which properly covers a nonedge e', then replacing one with the other does not decrease $P(1,n)$, since any path formerly using e can be rerouted in at least one way via e'. Thus, we may assume that no edge of G properly covers a nonedge. Let l_x denote the smallest (leftmost) ingoing neighbor of x. Then

$$P(1,x)=\sum_{i=l_x}^{x-1} P(1,i), \tag{2}$$

and

$$w\leq x \quad \Rightarrow \quad l_w\leq l_x. \tag{3}$$

Suppose G has an edge whose level is strictly higher than some nonedges. We will show that an interchange can be made that does not decrease the number of paths.

Choose a nonedge (a,c) of level j and an edge (b,d) of level k with $j<k$ and c to the left of d (by reversing all edges of G and renaming the vertices if necessary) such that the distance $\Delta(c,d)$ between their right endpoints is smallest possible. (See Fig. 3.) The minimality of $\Delta(c,d)$ implies that

$$l_x=x-j \qquad (c<x<d), \tag{4}$$

since an *edge* (w,x) of level $>j$ would be "closer" than $\Delta(c,d)$ to (a,c) and a *nonedge* (w,x) of level $\leq j$ would be "closer" to (b,d). Furthermore, (4) and (3) imply that

$$(d-1)-j=l_{d-1}\leq l_d\leq b=d-k\leq d-(j+1),$$

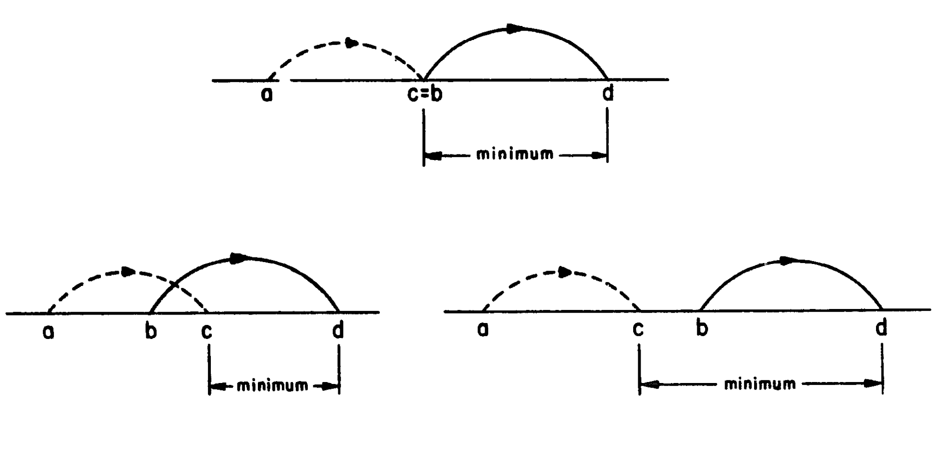


Fig. 3. The three possible configurations. A broken line indicates a nonedge.

so equality holds throughout; hence $k = j+1$ and

$$l_d = d-(j+1). \tag{5}$$

Consider the graph G' obtained from G by removing (b, d) and adding (a, c). By (4) and (5),

$$P'(c, x) = F^{(j)}_{\Delta(c,x)+1} \qquad (c \leqslant x \leqslant d),$$

where $P'(v, w)$ denotes the number of paths in G' from v to w. Therefore,

$$\begin{aligned} P'(1, x) &= P(1, x) && (0 \leqslant x < c),\\ P'(1, x) &= P(1, x) + P(1, a)F^{(j)}_{\Delta(c,x)+1} && (c \leqslant x < d),\\ P'(1, d) &= \underbrace{P(1, d)}_{\text{Old Paths}} + \underbrace{P(1, a)F^{(j)}_{\Delta(c,d)+1}}_{\text{New Paths}} - \underbrace{P(1, b).}_{\text{Destroyed Paths}} \end{aligned}$$

The number of paths from 1 to any vertex to the left of d has not been decreased. If we can say the same for d, then clearly $P'(1, y)$ will be greater than or equal to $P(1, y)$ for each vertex y, which will prove the theorem. Thus, since

$$1 = k - j = \Delta(b, d) - \Delta(a, c) = \Delta(c, d) - \Delta(a, b),$$

it suffices to show that $P(1, b) \leqslant P(1, a)F^{(j)}_{\Delta(a,b)+2}$. We will prove the following stronger claim:

$$P(1, x) \leqslant P(1, a)\cdot F^{(j)}_{\Delta(a,x)+2} \qquad (a \leqslant x < d). \tag{6}$$

Certainly (6) is true for $x = a$. Using (6) inductively, we have for $a < x < c$,

$$\begin{aligned} P(1, x) &\leqslant \sum_{i=l_a}^{x-1} P(1, i) && \text{[by (2) and (3)]},\\ &= P(1, a) + \sum_{i=a}^{x-1} P(1, i) && \text{[by (2)]},\\ &\leqslant P(1, a)\left[1 + \sum_{i=a}^{x-1} F^{(j)}_{\Delta(a,i)+2}\right] && \text{[by induction]},\\ &= P(1, a)F^{(j)}_{\Delta(a,x)+2}. \end{aligned}$$

We must still show that the claim is true for $c \leqslant x < d$. Using (4) and the fact that $l_c > c - j$ we have for $c \leqslant x < d$,

$$\begin{aligned} P(1, x) &= \sum_{i=l_x}^{x-1} P(1, i) \leqslant \sum_{i=x-j}^{x-1} P(1, i)\\ &\leqslant P(1, a) \sum_{i=x-j}^{x-1} F^{(j)}_{\Delta(a,i)+2} && \text{[by induction]},\\ &= P(1, a)F^{(j)}_{\Delta(a,x)+2}. \end{aligned}$$

This concludes the proof of Theorem 2.

Theorem 2 considerably reduces the problem of constructing a maximum path graph. We can write m uniquely in the form:

$$m = r + \sum_{i=1}^{k} (n-i) = r + kn - \tfrac{1}{2}k(k+1)$$

where $1 \leqslant k \leqslant n-1$ and $0 \leqslant r < n-k$. We then fill levels 1 through k and (somehow optimally) distribute the remaining r edges in level $k+1$. Exactly how to spend these extra r edges has yet to be shown, however, if $r=0$, then Theorem 2 does give the answer. We state the result, in this special case, in the following corollary.

Corollary 3. *Let n, m, and k be postive integers satisfying $m = kn + k(k+1)/2$, then the nth generalized k-Fibonacci graph $\mathcal{F}_n^{(k)}$ is a maximum path graph, that is,*

$$N(\mathcal{F}_n^{(k)}) = N_{n,m}.$$

If $r \neq 0$, then the situation becomes complicated. We shall show that the best place to put one extra edge is at the extreme left [or right], i.e., we add the edge $(1, k+2)$. Let G and G' be the graphs obtained by adding the edges $(1, k+2)$ and $(l, l+k+1)$, respectively, to the graph $\mathcal{F}_n^{(k)}$, where $1 < l < n-k-1$. By symmetry, $P_{l+k+1} = P'_{l+k+1}$. By the contribution of the extra edge in G, we obtain

$$P_i = P'_i + F^{(k)}_{i-k-1} \qquad (k+1 \leqslant i \leqslant l+k).$$

Hence,

$$P_i > P'_i \qquad (l+k+2 \leqslant i \leqslant n).$$

Furthermore,

$$N(G) = F_n^{(k)} + F^{(k)}_{n-k-1}.$$

In the case $r=2$ it can be shown similarly that the best place to put the two extra edges is one at the extreme right and one at the extreme left and, for such a graph G, we obtain

$$N(G) = F_n^{(k)} + 2F^{(k)}_{n-k-1} + F^{(k)}_{n-2k-2}.$$

Let G be a (Hamiltonian) maximum path graph with n vertices and $m = (n-1)+r$ edges where $0 < r \leqslant n-2$, i.e., G has r edges of level 2 and has level 1 full. It is easy to describe the structure of G. If $t = n-1-r$, then G is constructed by concatenating in series exactly t Fibonacci graphs $\mathcal{F}_{s_1}, \mathcal{F}_{s_2}, \ldots, \mathcal{F}_{s_t}$. It follows that

$$N(G) = F_{s_1} F_{s_2} \cdots F_{s_t} \quad \text{and} \quad \sum_{i=1}^{t} s_i = n+t-1.$$

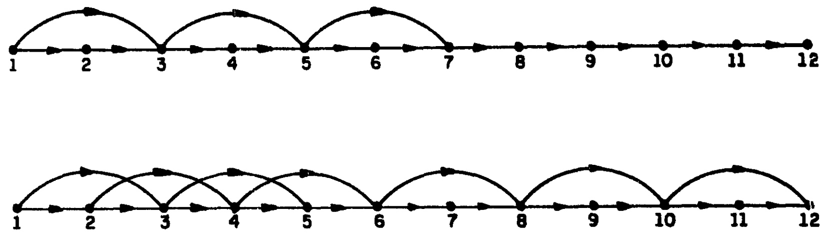


Fig. 4. Some maximum path graphs of level 2.

The order in which the $\mathcal{F}_{s_i}$ occur is immaterial. The following result can be shown [2]:

Theorem 4. *Let m and n be positive integers such that* $m=(n-1)+r$ *where* $0\leq r\leq n-2$. *Then,*

$$N_{n,m}=\begin{cases}2^r & \text{if } r\leq\frac{1}{2}(n-1),\\ 2^{n-2-r}F_{2r-n+4} & \text{if } r>\frac{1}{2}(n-1).\end{cases}$$

Figure 4 shows the maximum path graphs for $n=12$, $m=14$ and for $n=12$, $m=18$.

4. An application

Consider a network with two cost functions for the edges, for example, distance and fare. The *length* of a path is an ordered pair representing the respective sums of the two cost functions along the path. A path connecting two distinguished vertices is a *shortest path* if there exists no other such path with one component strictly smaller and the other smaller or equal. For algorithms for finding shortest paths in a network with two cost functions see [3, 5, 8].

Suppose the edges of the Fibonacci graph $\mathcal{F}_n$ are assigned two costs as follows: an edge $(i, i+1)$, $i=1,2,\ldots,n-1$, is assigned $(1+2^{-2i+1}, 1-2^{-2i+1})$ and an edge $(i, i+2)$, $i=1,2,\ldots,n-2$, is assigned $(2+2^{-2i}, 2-2^{-2i})$. Then clearly every path in $\mathcal{F}_n$ connecting 1 and n is a shortest path. Hence the maximum number of paths is a bound for the number of shortest paths in such networks, and this is required in the complexity analysis of the algorithms described above.

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