The dimension and basis of spaces of multivariate splines *

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Abstract: The purpose of this paper is to study the recent development of certain aspects of multivariate spline spaces. More specifically, we will survey the existence of bivariate spline, the dimension and basis of the bivariate spline spaces $S_k(\Delta)$ with various partitions $\Delta$. Furthermore, we will also introduce some results on the higher dimensional splines.

Keywords: Smoothing cofactor, global conformality condition, existence, dimension, basis, B-spline, truncated power, non-uniform, higher-dimension.

AMS(MOS) Subject Classification: 41A15, 41A25, 41A63, 65D05, 65D07, 65D15.

As we know that Dahmen and Micchelli have made a fairly complete survey concerning the recent progress in multivariate splines (cf. [20]). Here we will try to introduce some results on the dimension, basis, and related problems of multivariate spline spaces. It can be seen that many results shown in this paper are published for the first time.

Because of the incompleteness of this survey and the list of references the reader is referred to the following articles [1,20] and their bibliographies.

1. On the existence for bivariate splines

Let $D$ be a domain in $\mathbb{R}^2$, and $\Delta$ a grid of irreducible algebraic curves that divide $D$ into a finite number of cells. Each boundary curve segment that separates two adjacent cells will be called a grid-segment, the two endpoints of a grid segment will be called grid-point, and the grid-point inside of $D$ will be called interior grid-point. Let $P_k$ denote the collection of all polynomials with real coefficients and total degree $k$, that is, each $p \in P_k$ has the form

$$p(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} c_{ij} x^i y^j,$$

where $c_{ij}$ are real numbers. A function $s$ in $C^\mu(D)$ will be called a bivariate spline function of degree $(k, \mu)$ on a grid partition $\Delta$, if the restriction of $s$ to each cell of this partition is in $P_k$. The collection of all these bivariate spline functions will be denoted by $S_k^\mu(\Delta) = S_k^\mu(\Delta; D)$.

Using Bezout’s theorem in Algebraic geometry, the following result was pointed out in [28].

* Project supported by the Science Fund of the Chinese Academy of Sciences.

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Lemma 1.1. Let $D_i$ and $D_j$ be two adjacent cells of a partition $\Delta$ with common grid-segment $\Gamma_{ij}$: $l_{ij}(x, y) = 0$, and the representation of function $s(x, y)$ on $D_i$ and $D_j$ be polynomials in $P_k$ given by $p_i(x, y)$ and $p_j(x, y)$ respectively. Then $s(x, y) \in S^k(\Delta; D_i \cup D_j)$ if and only if

$$p_i(x, y) - p_j(x, y) = [l_{ij}(x, y)]^{a+1} \cdot Q_{ij}(x, y),$$

where $Q_{ij}(x, y) \in P_{k-(\mu+1)}$. The above polynomial $Q_{ij}(x, y)$ is called the smoothing cofactor of the function $s(x, y)$ across $c_j$ from $D_i$ to $D_j$ (cf. [7,28]). Note that $\Gamma_{ij} = \Gamma_{ji}$ (or $l_{ij}(x, y) = l_{ji}(x, y)$), therefore $-Q_{ij}(x, y) = Q_{ji}(x, y)$.

An immediate corollary of Lemma 1.1 is the following result [28].

Corollary 1.1. Let $\Delta$ be a partition that all grid segments are given by $\Gamma_1, \ldots, \Gamma_N$:

$$\Gamma_i: \quad l_i(x, y) = 0, \quad i = 1, \ldots, N,$$

where $l_i(x, y)$ is an irreducible polynomial of degree $n_i$ ($i = 1, \ldots, N$). In order to exist a $s(x, y) \in S^k(\Delta) \setminus P_k$, the integers $k$ and $\mu$ have to satisfy

$$k \geq (\mu + 1) \cdot \min\{n_1, \ldots, n_N\}.$$  

Let $\Gamma_1, \ldots, \Gamma_{N(\Delta)}$ be all grid segments which are passing through the interior grid point $A$, where $\Gamma_i: l_i(x, y) = 0, i = 1, \ldots, N(\Delta)$. Denote by $Q_i(x, y)$ the smoothing cofactor of function $s(x, y)$ across $\Gamma_i$ travelling in the counter-clockwise direction around $A$. We call ([28], also cf. [7])

$$\sum_{i=1}^{n(\Delta)} [l_i(x, y)]^{a+1} \cdot Q_i(x, y) = 0,$$

the conformality condition at grid-point $A$ of the partition $\Delta$. It is well known that Strang has already displayed this idea for $C^1$ piecewise polynomials on triangulation [27].

Let $A1, \ldots, AM$ be all interior grid-points of a given partition $\Delta$, we call [28]

$$\sum_{i=1}^{N(A)} [l_i(x, y)]^{a+1} \cdot Q_i(x, y) = 0, \quad r = 1, \ldots, M$$

(1.4)

the global conformality condition of the partition $\Delta$.

We have the following an existence theorem of bivariate splines [28].

Theorem 1.1. Let $\Delta$ be any partition of $D$. The function $s(x, y)$ is a bivariate spline belonging to $S^k(\Delta)$ if and only if for any grid-segment $\Gamma_{ij}$, there exists a smoothing cofactor $Q_{ij}(x, y)$ of the function $s(x, y)$, and the function $s(x, y)$ satisfies the global conformality condition (1.4).

As in [28], a function $s(x, y) \in S^k(\Delta)$ is called the non-degenerate bivariate spline, if $s(x, y)$ has at least one non-zero smoothing cofactor. It is obvious that the following three statements are equivalent:

(i) the function $s(x, y)$ is a non-degenerate bivariate spline;
(ii) $s(x, y)$ belongs to $S^k(\Delta) \setminus P_k$;
(iii) the linear equations corresponding to the global conformality condition (1.4) of $s(x, y)$ has non-trivial solution.
Theorem 1.1 shows that the existence and related properties of bivariate splines are depended on the partition $\Delta$ of $D$. It is important for applications to know how many dimensions the space $S_k^e(\Delta)$ have.

By means of Theorem 1.1, we can obtain [31]:

**Theorem 1.2.** Let $\Delta$ be a partition of $D$, and the degrees of all grid segments are $n_1, n_2, \ldots, n_N$ respectively. Then

$$\dim S_k^e(\Delta) = \left( \frac{k + 2}{2} \right) + \sum_{i=1}^{N} \left( \frac{k - (\mu + 1)n_i + 2}{2} \right) - \tau,$$

where $\tau$ is the rank of coefficient matrix $[GCC]$ of linear equations determined by the global conformality condition of the partition $\Delta$.

In view of Theorem 1.2, to determine the dimension of $S_k^e(\Delta)$, it is sufficient to calculate the rank of matrix $[GCC]$.

**Corollary 1.2.** Let $n_1 = n_2 = \cdots = n_N = 1$. Then

$$\dim S_k^e(\Delta) = \left( \frac{k + 2}{2} \right) + N \cdot \left( \frac{k - \mu + 1}{2} \right) - \tau,$$

where $\tau$ is the same as stated above [29].

By Lemma 1.1, it is easy to see that the bivariate spline possesses a property of 'piecewise extension'. Using this property, it is not hard to show the representation formulas for multivariate splines.

Let a partition $\Delta$ divide the domain $D$ into finite cells $D_1, D_2, \ldots, D_N$. Without loss of generality, we will take the cell $D_1$ as a 'source cell'. Let $\bar{c}$ be any given flow curve from the source cell $D_1$ into all the other cells of this partition $\Delta$, and satisfying the following conditions:

1. For every cell, $\bar{c}$ is to flow in and out one time, respectively;
2. $\bar{c}$ does not pass through any grid-point;
3. The number of times that $\bar{c}$ crosses over every gridsegment does not more than one.

Let $\bar{c}$ be any given flow curve, and $\Gamma_{ij}: l_{ij}(x, y) = 0$ be any grid-segment of the partition $\Delta$. When the $\Gamma_{ij}$ is crossed by the flow $\bar{c}$, we denote by $U(\Gamma_{ij}^+)$ the union of closed cells that the flow $\bar{c}$ will reach after it crosses $\Gamma_{ij}$, and denote by $U(\Gamma_{ij}^-)$ the union of closed cells that the flow $\bar{c}$ has reached before it arrives at $\Gamma_{ij}$. We introduce the following notation:

$$\left[ l_{ij}(x, y) \right]_* = \begin{cases} l_{ij}(x, y), & \text{if } (x, y) \in U(\Gamma_{ij}^+) \setminus U(\Gamma_{ij}^-), \\ 0, & \text{otherwise, or when } \bar{c} \text{ does not cross } \Gamma_{ij}. \end{cases}$$

According to these definitions, the following representation formula of bivariate splines holds [29]:

**Theorem 1.3.** For every $s(x, y) \in S_k^e(\Delta)$, we have

$$s(x, y) = p(x, y) + \sum_{\bar{c}} \left[ l_{ij}(x, y) \right]_{\bar{c}}^{\alpha+1} \cdot Q_{ij}(x, y),$$

where $\alpha$ is the multiplicity of the knot $\bar{c}$.
where \( p(x, y) \in \mathbb{P}_k \) is the representation of \( s(x, y) \) in the source cell \( D_i \), \( \sum \bar{e} \) denotes the summation over all grid-segment crossed by \( \bar{e} \), \( Q_{ij}(x, y) \) is the smoothing cofactor across \( \Gamma_{ij} \); \( l_{ij}(x, y) = 0 \), and \( \bar{e} \) crosses \( \Gamma_{ij} \) from \( D_j \) to \( D_i \).

Observe that for any given partition \( \Delta \) of \( D \) and certain flow curve \( \bar{e} \), in order that \( s(x, y) \in S^{k}_{\mathbb{P}}(\Delta) \), if and only if both the representation formula (1.8) and the global conformality condition (1.4) hold.

From now on, we will suppose that the partition \( \Delta \) consists of line segments. \( \Delta \) is called the cross-cut partition, if \( \Delta \) consists of some straight lines which are cross over the \( D \) [7,28]. The following results can be obtained [28]:

**Corollary 1.3.** Let \( \Delta \) be a cross-cut partition, and \( 0 \leq \mu \leq k - 1 \). Then the non-degenerate bivariate spline \( s(x, y) \in S^{k}_{\mathbb{P}}(\Delta) \setminus \mathbb{P}_k \) does always exist.

**Corollary 1.4.** Let \( \Delta \) be any given partition of \( D \), and \( \mu \) be any given non-negative integer. Then there exists a large enough positive integer \( k \), such that the non-degenerate \( s(x, y) \in S^{k}_{\mathbb{P}}(\Delta) \) does exist.

In [29], we used Theorem 1.3 as a basic tool for establishing the general theory of interpolation for bivariate splines under general partitions.

### 2. The dimension and basis of some bivariate spline spaces

It is important for theories and applications to study how we can determine the dimension, and obtain a basis of space \( S^{k}_{\mathbb{P}}(\Delta) \).

It is well known that the dimension problem was first formally proposed by Strang in [27], where several conjectures were made.

According to Theorem 1.2 or Corollary 1.2, to determine the dimension of \( S^{k}_{\mathbb{P}}(\Delta) \), it is sufficient to calculate the rank \( \tau \) of matrix \([GCC]\) corresponding to the global conformality condition (1.4).

There exist some cases, such that the rank \( \tau \) of the matrix \([GCC]\) will be not hard to calculate. As we will see later, for example, when the partition is the cross-cut partition or quasi cross-cut partition, the rank \( \tau \) can be calculated by means of the ‘isolated’ property of all interior gridpoints.

Let \( D \) be a simply connected domain in \( \mathbb{R}^2 \). next we will study the dimension of \( S^{k}_{\mathbb{P}}(\Delta) \), where \( \Delta \) is an arbitrary cross-cut partition of \( D \).

Let \( (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \) be pairwise linearly independent ordered pairs; that is, \( \alpha_i \beta_j \neq \alpha_j \beta_i \) for \( i \neq j \), \( i, j = 1, \ldots, n \), and \( V_n \) be the vector space corresponding to the conformality condition at the origin \( O \):

\[
V_n = \left\{ (q_1, \ldots, q_n) : \sum_{i=1}^{n} q_i(x, y)(\alpha_i x + \beta_i y)^{\mu-1} \equiv 0, \quad q_1, \ldots, q_n \in \mathbb{P}_{k-\mu-1} \right\},
\]

where \( 0 \leq \mu \leq k - 1 \).

Denote by \( d^{k}_{\mathbb{P}}(n) \) the dimension of \( V_n \). In [25], Schumaker gave a formula for \( d^{k}_{\mathbb{P}}(n) \). The following is a different formulation of Schumaker's result (cf. [12]):
Lemma 2.1.

\[ d_k^p(n) = \frac{1}{2} \left( k - \mu - \left[ \frac{\mu + 1}{n - 1} \right] \right) + \left( n - 1 \right) k - (n + 1) \mu + (n - 3) + (n - 1) \left[ \frac{\mu + 1}{n - 1} \right], \quad (2.2) \]

where \([x]\) is the integer part of \(x\).

The following theorem was pointed out in [12].

Theorem 2.1. Let \(D\) be a simply connected domain in \(\mathbb{R}^2\) and \(\Delta_c\) a cross-cut partition of \(D\) with \(L\) cross-cuts and \(V\) interior grid-points \(A_1, \ldots, A_V\) in \(D\) such that \(n_i\) cross-cuts intersect at \(A_i, i = 1, \ldots, V\). Then the dimension of the bivariate spline space \(S^p_k(\Delta_c)\), \(0 < \mu < k - 1\), is

\[ \dim S^p_k(\Delta_c) = \left( k + 2 \right) + L \cdot \left( k - \mu + 1 \right) + \sum_{i=1}^{V} d_k^p(n_i). \quad (2.3) \]

In order to give a basis of \(S^p_k(\Delta_c)\), we have to make the use of the following notations. Let \(\Gamma_1, \ldots, \Gamma_L\) be the cross-cuts of the partition \(\Delta_c\), and let \(\Gamma_i\) lie on the line \(\gamma_i: a_i x + b_i y + c_i = 0\), \(i = 1, \ldots, L\).

Let \(D_*\) be a source cell in this partition such that a part of the boundary of \(D_*\) lies on \(\partial D\). Since each \(\Gamma_i\) is a cross-cut of \(D\) and \(D\) is simply connected, it separates \(D\) into two cells \(D_i\) and \(D'_i\) where \(D_* \subset D'_i\). We define a bivariate spline function \((\Gamma_i)_\sharp\) by

\[ (\Gamma_i)_\sharp(x, y) = \begin{cases} a_i x + b_i y + c_i, & \text{if } (x, y) \in D_i, \\ 0, & \text{if } (x, y) \in D'_i \cup \Gamma_i, \end{cases} \quad (2.4) \]

and define \((\Gamma_i)^{\mu+1}_\sharp\) by

\[ (\Gamma_i)^{\mu+1}_\sharp(x, y) = (\left((\Gamma_i)_\sharp(x, y)\right)^{\mu+1}. \quad (2.5) \]

Using the fundamental solution space of the conformality condition at each interior grid-point, we may define cone spline functions \(s_{i,t}(x, y), i = 1, \ldots, V, t = 1, \ldots, n_i\) (cf. [12]). We have:

Theorem 2.2. The collection of bivariate splines

\[ \mathcal{A} = \left\{ x^a y^b, x^a y^d(\Gamma_u)^{\mu+1}_\sharp(x, y), s_{i,t}(x, y) : 0 \leq a + b \leq k, 0 \leq c + d \leq k - \mu - 1, u = 1, \ldots, L, i = 1, \ldots, V \text{ and } t = 1, \ldots, n_i \right\} \]

is a basis of \(S^p_k(\Delta_c)\).

A grid partition \(\Delta_{qc}\) of a simply connected domain \(D\) will be called a quasi-cross-cut partition if each of its grid segments lies either on a cross-cut of \(D\) or on a ray in \(D\) that starts at a interior grid-point and terminates on \(\partial D\). The following result holds [12].

Theorem 2.3. Let \(D\) be a simply connected domain, and \(\Delta_{qc}\) a quasi-cross-cut partition of \(D\) consisting of \(L\) cross-cuts, a finite number of rays, and \(V\) interior gridpoints \(A_1, \ldots, A_V\) such that the total number of cross-cuts and rays sharing the interior grid-point \(A_i\) is \(N_i, i = 1, \ldots, V\). Then the dimension of \(S^p_k(\Delta_{qc})\), \(0 \leq \mu < k - 1\), is also given by (2.3).
Recently, a basis of $S_k^p(\Delta_{qc})$ has been found [32]. Of course, some of the basis functions will be somewhat implicit. A cross-cut partition is said to be a simple cross-cut grid partition if no more than two cross-cuts meet at an interior grid-point. We can now at once obtain the following [7]

**Corollary 2.1.** Let $\Delta_{sc}$ be a simple cross-cut partition of a simply connected domain $D$. Then

$$\dim S_k^p(\Delta_{sc}) = \left(\frac{k + 2}{2}\right) + L\left(\frac{k - \mu + 1}{2}\right) + Vd_k^p(2),$$

(2.6)

and a basis of $S_k^p(\Delta_{sc})$ is given by

$$\mathcal{B} = \left\{ x^a y^b, x^c y^d (\Gamma_i)_x^{u+1} (x, y), x^e y^f (\Gamma_i)_y^{u+1} (x, y); u \neq v, \Gamma_u \cap \Gamma_v \cap D \neq 0, \right\},$$

where $0 \leq a + b \leq k, 0 \leq c + d \leq k - 1, 0 \leq f + g \leq k - 2\mu - 2, i, u, v = 1, \ldots, L$, $L$ is the number of all cross-cuts, $V$ is the number of all interior grid-points, and the functions $x^c y^d (\Gamma_i)_x^{u+1} (x, y), x^e y^f (\Gamma_i)_y^{u+1} (x, y)$ are to be deleted if $k - 2\mu + 2$.

Because the rectangular grid partition is a special case of the simple cross-cut grid partition, the result corresponding to Corollary 2.1 will hold (cf. [7]).

We know that Dahmen and Micchelli have already obtained a spanning set of space $S_k^p(\Delta_{qc})$. Their main tool for giving this spanning set is bivariate truncated power functions (cf. [18,19,20]).

A bivariate spline function $s(x, y)$ is said to be a B-spline, if it identically vanishes outside its supporting Jordan curve, and is strictly positive inside this curve. Using (2.2), we have [3,12,18,20]:

**Proposition 2.1.** Let $s(x, y)$ be a bivariate B-spline in $S_k^p(\Delta)$, $0 \leq \mu \leq k - 1$, whose supporting curve is a convex polygon $F$ with vertices $A_i$. For each $i$, let $N_i$ be the number of grid-segments on or inside $F$ with $A_i$ as the common interior grid-point. Then

$$N_i > \left(\frac{k + 1}{k - \mu}\right).$$

Let $D$ be the rectangle $D = [a, b] \otimes [c, d]$. We use the horizontal and vertical lines $x = x_i = 0$ and $y = y_j = 0$, $i = 1, \ldots, m - 1, j = 1, \ldots, n - 1$ to partition $D$ into $mn$ rectangular cells $D_{ij} = [x_i, x_{i+1}] \otimes [y_j, y_{j+1}]$. By drawing the diagonal with positive slope in each $D_{ij}$, we obtain a uni-diagonal triangulation $\Delta_{mn}^{(1)}$; and by drawing in both diagonals of each $D_{ij}$, we obtain a criss-cross triangulation $\Delta_{mn}^{(2)}$. $\Delta_{mn}^{(1)}$ and $\Delta_{mn}^{(2)}$ are sometimes called the type-1 and type-2 triangulation respectively. A triangulation $\Delta_{mn}^{(i)}$ is said to be uniform if the original rectangular grid partition is uniform. Hence, a uniform uni-diagonal triangulation gives a three directional mesh, while a uniform criss-cross triangulation yields a four directional one. These two partitions are very important in finite element methods and smooth surface fitting. In [12], we obtained:

**Corollary 2.2.** Let $\Delta_{mn}^{(1)}$ and $\Delta_{mn}^{(2)}$ be uniform grid partition of $D$ as described. Then the dimension of the spaces $S_k^p(\Delta_{mn}^{(i)}), i = 1, 2$ are

$$\dim S_k^p(\Delta_{mn}^{(i)}) = \left(\frac{k + 2}{2}\right) + (2m + 2n - 3)\left(\frac{k - \mu + 1}{2}\right) + (m - 1)(n - 1)d_k^p(3)$$

(2.7)
and
\[
\dim S^\mu_k(\Delta_{mn}^{(2)}) = \left( \frac{k + 2}{2} \right) + (3m + 3n - 4)\left( \frac{k - \mu + 1}{2} \right) + mn\left( \frac{k - 2\mu}{2} \right) + (m - 1)(n - 1)d^\mu_k(4) \tag{2.8}
\]
respectively, where
\[
d^\mu_k(3) = \left( k - \mu - \left[ \frac{\mu + 1}{2} \right] \right) \left( k - 2\mu + \left[ \frac{\mu + 1}{2} \right] \right),
\]
\[
d^\mu_k(4) = \frac{1}{3} \left( k - \mu - \left[ \frac{\mu + 1}{3} \right] \right) \left( 3k - 5\mu + 1 + 3 \left[ \frac{\mu + 1}{3} \right] \right). \tag{2.9}
\]

Note that Schumaker [25] showed \( \dim S^\mu_k(\Delta_{mn}^{(1)}) \) for \( 1 \leq \mu \leq k - 1 \), where \( 2 \leq k \leq 4 \) and \( \dim S^k(\Delta_{mn}^{(2)}) \), where \( 2 \leq k \leq 4 \) by using different method.

Suppose that \( \Delta_{mn}^{(1)} \) and \( \Delta_{mn}^{(2)} \) are non-uniform grid partitions of \( D \), then Corollary 2.2 does not hold. We note, however, that Schumaker [26] showed the dimensions of \( S^k(\Delta_{mn}^{(1)}) \) and \( S^k(\Delta_{mn}^{(2)}) \) \( (0 \leq \mu \leq 2) \) do not change with the uniformity of the grid partitions. It was pointed out in [31], where \( \Delta_{mn}^{(i)}, i = 1, 2 \) are non-uniform grid partitions, the dimensions of spaces \( S^k(\Delta_{mn}^{(1)}) (\mu \geq 2) \), and \( S^k(\Delta_{mn}^{(2)}) (\mu \geq 3) \) will be different from those formulas corresponding to uniform grid partitions. We now turn to this. We need the following notations. Denote by \( N_r \) the number of interior grid-points which are passed through \( r \) lines with different slopes, and let
\[
\sigma(n, \mu) = \frac{1}{2} \left[ \frac{\mu + 1}{n - 1} \right] \left( 2\mu + 3 - n - (n - 1)\left[ \frac{\mu + 1}{n - 1} \right] \right). \tag{2.10}
\]

According to (2.2), if \( k > \mu + (\mu + 1)/(n - 1) \), then
\[
d^\mu_k(n) = (n - 1)\left( \frac{k - \mu + 1}{2} \right) - (\mu + 1)(k - \mu) + \sigma(n, \mu), \tag{2.11}
\]
and if \( k \leq \mu + (\mu + 1)/(n - 1) \), then \( d^\mu_k(n) = 0 \).

Let us label all interior grid-points in order of the unicursal, and use (2.11) to calculate successively the dimensions of solution space of linear equations defined by the conformality condition at each interior grid-point. The following two theorems hold [31].

**Theorem 2.5.** For any given grid partition \( \Delta_{mn}^{(1)} \) of domain \( D \), the following formula always holds
\[
\dim S^\mu_k(\Delta_{mn}^{(1)}) = \left( \frac{k + 1}{2} \right) + (2mn - 1)\left( \frac{k - \mu + 1}{2} \right) - (m - 1)(n - 1)(\mu + 1)(k - \mu) + N_3\sigma(3, \mu) + N_4\sigma(4, \mu), \tag{2.12}
\]
when \( k > \mu + \frac{1}{2}(\mu + 1) \).

**Theorem 2.6.** For any given grid partition \( \Delta_{mn}^{(2)} \) of domain \( D \), the following formula always holds
\[
\dim S^\mu_k(\Delta_{mn}^{(2)}) = \left( \frac{k + 2}{2} \right) + (3mn - 1)\left( \frac{k - \mu + 1}{2} \right) - (m - 1)(n - 1)(\mu + 1)(k - \mu) + mn\left( \frac{k - 2\mu}{2} \right) + \sum_{i=4}^{6} N_i\sigma(i, \mu), \tag{2.13}
\]
when $k > \mu + \frac{1}{3}(\mu + 1)$.

It is clear that $\dim S_k^2(\Delta_{mn}^{(1)})$ is dependent on the uniformity of partition $\Delta_{mn}^{(1)}$. In fact, we have ($k \geq 4$)

$$\dim S_k^2(\Delta_{mn}^{(1)}) = mnk^2 - 3(2mn - m - n)k + 8mn - 6(m + n) + 6 + N_j,$$

where $L$ is the number of cross-cuts.

As for the space $S_2^2(\Delta_{mn}^{(2)})$, we have

$$\dim S_2^2(\Delta_{mn}^{(2)}) = 10 + L,$$

where ($k \geq 4$)

$$\dim S_k^2(\Delta_{mn}^{(2)}) = mnk^2 - 3(2mn - m - n)k + 8mn - 6(m + n) + 6 + N_j,$$

As pointed out by Morgan and Scott [23] (also cf. [12,25]), the dimension of $S_j^2(\Delta)$ depends very heavily on the geometry of the partition $\Delta$.

We next discuss the function with minimum support in $S_j^2(\Delta_{mn}^{(2)})$ where the criss-cross triangulation $\Delta_{mn}^{(2)}$ is not necessarily uniform triangulation. We use the notation: $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, $A_i = h_i/(h_i + h_{i+1})$, $B_j = k_j/(k_j + k_{j+1})$, $A'_i = 1 - A_i$, $B'_j = 1 - B_j$. The B-spline with support which is shown by Fig. 1 can be obtained by solving fairly complicated linear equations that arise from the global conformality condition, so that the information on minimum

![Fig. 1.](image-url)
support can be obtained simultaneously [15,16]. We will denote this B-spline by $B_{ij}$. Since a bivariate quadratic polynomial on a triangle is uniquely determined by its values at the three vertices and the mid-points of the sides of the triangle, only these values are given in Fig. 1.

Some of the important properties of $B_{ij}$ are included in the following [15,16].

**Theorem 2.7.** Let $x_2 < x_1 < a = x_0 < \cdots < x_m = b < x_{m+1} < x_{m+2}$ and $y_2 < y_1 < c = y_0 < \cdots < y_n - d < y_{n+1} < y_{n+2}$. Then

(a) for $-1 \leq i \leq m$ and $-1 \leq j \leq n$, $B_{ij}$ is unique and strictly positive inside its support, and every function in $S^1_2(\Delta_m^2)$ supported by the support of $B_{ij}$ is a constant multiple of $B_{ij}$;

(b) for all $(x, y) \in D = [a, b] \otimes [c, d]$,
\[
\sum_{i=-1}^{m} \sum_{j=-1}^{n} B_{ij}(x, y) = 1;
\]

(c) for all $(x, y) \in D$,
\[
\sum_{i=-1}^{m} \sum_{j=-1}^{n} (-1)^{i+j} h_{i+1} k_{j+1} B_{ij}(x, y) = 0;
\]

(d) for each $(i_0, j_0)$, $-1 \leq i_0 \leq m$ and $-1 \leq j_0 \leq n$, the collection
\[
\{ B_{ij} : (i, j) \neq (i_0, j_0), -1 \leq i \leq m, -1 \leq j \leq n \}
\]
is a basis of $S^1_2(\Delta_m^2)$.

When $\Delta_m^m$ is a uniform criss-cross triangulation, the B-spline $B_{ij}$ was first constructed in a different form in [32] and its algebraic and approximation properties were obtained in [10].

For uniform crisscross triangulations $\Delta_m^2$ Dahmen and Micchelli already showed in [21] that whenever
\[
\sum_{i, j \in \mathbb{Z}} c_{ij} B(x - i, y - j | X_M) = 0, \quad (x, y) \in \mathbb{R}^2,
\]
then $c_{ij} = (-1)^{i+j} p(i, j)$ where for some $a_{ir}$
\[
p(x, y) = \sum_{i=0}^{m_1-1} \sum_{r=0}^{m_2-1} a_{ir} (x+y)^i (y-x)^r,
\]
$B(x | X)$ is the box spline defined by [2,3], and
\[
X_M = \left\{ e^1, \ldots, e_1^1, e^2, \ldots, e_2^1, e^{e+1}, \ldots, e^{e+1}, e^{e+1}, \ldots, e^{e+1} \right\},
\]
where $e^1 = (1,0), e^2(0,1), \ldots, e^{m_3}$. Thus discarding any $m_3 m_4$ of the box splines $B(\cdot - i, \cdot - j | X_M)$ whose support intersects a given domain $\Omega \subset \mathbb{R}^2$ the remaining translates form a basis for
\[
\text{span} \left\{ B(\cdot - i, \cdot - j | X_M) | \Omega : \text{span} B(\cdot - i, \cdot - j | X_M) \cap \Omega \neq \emptyset \right\}.
\]

Denote by $S^\delta_k$ the space $S^\delta_k(h \Delta)$ that the triangulation is determined by multi-integer translates of the four lines $x = 0, y = 0, x - y = 0$ and $x + y = 0$. It was pointed out in [21], the approximation properties of $S^\delta_k$ are completely governed by those of the space spanned by the translated of all box splines contained in $S^\delta_k$. 
It is well known that Morgan and Scott have already shown the exact dimension of $S_k^1(\Delta)$ $(k \geq 5)$ for which the partition $\Delta$ may be any given triangulation [24].

**Theorem 2.8.** For any given triangulation $\Delta$ of $D$, and $k \geq 5$, the following formula holds

$$\dim S_k^1(\Delta) = \frac{1}{2}(k+1)(k+2)T - (2k+1)E_0 + 3V_0 + \Omega,$$

where $T$ is the number of triangles, $E_0$ is the number of grid segments, $V_0$ is the number of interior grid-points, and $\Omega$ is the number of so-called ‘singular’ interior grid-points for which the adjacent grid segments of each grid segment are collinear.

Morgan and Scott [24] also described the structure of $S_k^1(\Delta)$ completely for $k > 5$ by exhibiting a nodal basis for $S_k^1(\Delta)$.

So far the following question is still open: For any given grid partition $\Delta$ of domain $D$, what is the exact dimension of $S_k^p(\Delta)$? An affirmative answer to this question would be very useful.

As an immediate consequence of Corollary 2.1, we can decide how ‘well’ the bivariate splines in $S_k^p(\Delta) = \bigcup_c S_k^p(\Delta, c)$ approximate, where $N$ is the number of different directions of all simple cross-cuts, and $c$ denote a collection of all such as above simple cross-cut partitions. That is, we have the following [2,7]

**Corollary 2.3.** The closure of $S_k^p(\Delta)$ in the topology of uniform convergence on compact subsets of $D$ is $C(D)$ if and only if $p < \frac{1}{2}(k-2)$. If $p > \frac{1}{2}(k-2)$, then the closure of $S_k^p(\Delta)$ is the space of continuous functions of the form

$$p_k(x, y) + \sum_{i=1}^N q_i(x, y)f_i(a_i x + b_i y),$$

where $p_k \in P_k$, $q_i \in P_{k-\mu-1}$, $i = 1, \ldots, N$, each $f_i$ is continuous function of univariate, and $(a_i, b_i)$. $i = 1, \ldots, N$ are those different directions of cross-cuts.

Furthermore, for uniform uni-diagonal triangulation $\Delta_n$ of the whole $\mathbb{R}^2$, de Boor and DeVore [2] showed an interesting result as follows

**Theorem 2.9.** $\bigcup_n S_k^p(\Delta_n)$ is dense in $C(\mathbb{R}^2)$ if and only if $p \leq \frac{1}{2}(2k-2)$. If $p > \frac{1}{2}(2k-2)$, then every $s(x, y) \in S_k^p(\Delta_n)$ can be represented on $\mathbb{R}^2_+$ as a linear combination of the truncated powers in

$$\{ x^py^q, (x-x_i)_+^p y^q, x^p(y-y_j)_+^q, (x-y-x_i)_+^p (x+y)_+^q : p, q \geq 0; p + q \leq k;$$

$$p, q \geq \mu + 1, \text{ whenever they exponentiate a truncated function} \}.$$

In the following paragraph we briefly describe some bases for spaces of bivariate cubic and quartic splines on type-1 uniform triangulations. Let $D = [0, m] \times [0, n]$, where $m$ and $n$ are positive integers. The partition $A_{mn}^{(1)}$: $x = i$, $y = j$, $x - y = k$; $i = 1, \ldots, m - 1$, $j = 1, \ldots, n - 1$, $k = -n + 1, \ldots, m - 1$. By virtue of Proposition 2.1, a necessary condition for the existence of a B-spline in $S_k^p(A_{mn}^{(1)})$ is that $k$ and $\mu$ must satisfy

$$k \geq \frac{1}{2}(3\mu + 2). \quad (2.18)$$

In practice, spline spaces in $C^p(D)$ with the lowest possible degrees are more useful. So the
important spaces to study are $S^0_0(\Delta_{mn})$, $S^1_1(\Delta_{mn})$, $S^2_2(\Delta_{mn})$, $S^3_3(\Delta_{mn})$, and so on.

Fredrickson [22] gave a cubic B-spline $B^1$. The boundary of the support of $B^1$ is a polygon with vertices $(0, 1), (-1, 0), (-1, -2), (0, -2), (2, 0),$ and $(2, 1)$. Using area coordinates, the polynomial pieces of $B^1$ on some triangles are: $Q_{ABF} = a^3$, $Q_{ABC} = a^2(a + 3c)$, $Q_{ADC} = 3(a + d)^2$, $Q_{AEF} = 1 + 3(a + d) - 3(a^2 + d^2) - 3ad(a + d)$, where $A = (1, 0)$, $B = (2, 1)$, $C = (1, 1)$, $D = (0, 0)$, $E = (0, -1)$, and $F = (2, 1)$.

From $B^1$, we may define $B^2$ by $B^2(x, y) = B^1(-x, -y)$. According to Corollary 2.2, $\dim S^1_1(\Delta_{mn}) = 2(m + 2)(n + 2) - 5$. Let $B^p_{ij}(x, y) = B^p(x - i, y - j)$, $p = 1, 2$, and introduce the index sets $\Omega_p = \{(i, j) : B^p_{ij} does not vanish identically on D\}$, and $\mathcal{U}_p(i_1, j_1; \ldots ; i_q, j_q) = \Omega_p \setminus \{(i_1, j_1), \ldots , (i_q, j_q)\}$. The following theorem holds [14]:

**Theorem 2.10.** The collection
\[ \mathcal{B} = \{B^1_{ij}, B^2_{ij}, (i, j) \in \Omega_1(m, n + 1), (s, t) \in \Omega_2(m + 1, n; m + 1, n - 1)\} \]
is a basis of $S^1_1(\Delta_{mn})$.

We gave a general criteria for choosing a basis of the space $S^1_1(\Delta_{mn})$ (cf. [14]).

Denote by $IS^2_2(\Delta_{mn})$ the space of all functions in $S^2_2(\Delta_{mn})$ which are locally supported relative to $D$ (cf. [14]). As had been pointed out in [14],
\[ IS^2_2(\Delta_{mn}) = \text{span}\{\text{Fredrickson's quartic B-spline and its translation}\}, \]
and
\[ S^2_2(\Delta_{mn}) = IS^2_2(\Delta_{mn}) \oplus \text{span}\{\text{some of truncated powers}\}. \]

By means of the box splines, de Boor and Höllig [3] had completely solved the problem of basis for every space $S^p_{k,\mu}(\Delta_{mn})$, $\mu = 0, 1, 2, 3, \ldots$, where $k(\mu) = \frac{1}{2}(3\mu + 1) + 1$. They showed a most important property: the approximation properties of $S^p_{k,\mu}$ are totally determined by the space spanned by the translates of all box splines contained in $S^p_{k,\mu}$, where $S^p_{k,\mu}$ denotes the space of all $p$ times continuously differentiable piecewise polynomials on the scaled triangulation $h\Delta$ (cf. [3]).

We now turn to describe some problems concerning bivariate splines on the rectangle $D = [a, b] \times [c, d]$ which satisfy certain boundary conditions. When partition $\Delta$ is a triangulation, some important results were described by Dahmen and Micchelli in [20] (cf. [4]). Here we do not repeat those results.

Let the partition $\Delta$ be either $\Delta_{mn}^{(1)}$ or $\Delta_{mn}^{(2)}$. Define
\[ S^{p,\alpha}_{k,\mu}(\Delta) = \{s \in S^p_{k,\mu}(\Delta) : D_x^is|_{x=a} = D_x^is|_{x-b} = D_y^js|_{y=c} = D_y^js|_{y-d} = 0, i = 0, \ldots , \alpha\}, \]
where $D_x = \partial / \partial x$, and $D_y = \partial / \partial y$. Such spaces will be of special interest in surface fitting, finite element, and Ritz–Galerkin's methods.

For uniform type-1 triangulation $\Delta_{mn}^{(1)}$, using special cone splines functions, we can obtain the following theorem [5].

**Theorem 2.11.** For all $m, n \geq 1$, we have
\[ \dim S^{0,0}_{3,0}(\Delta_{mn}^{(1)}) = 2mn - 1. \]
Using the B-spline basis of $S^1_2(\Delta^{(1)}_{mn})$ shown by Theorem 2.10, we can obtain a number of local support splines which belong to the spaces $S^1_3(\Delta^{(1)}_{mn})$ for all $m, n \geq 3$. All of these B-splines will have supports which are smaller than 3 squares by 3 squares (cf. [5]). It was pointed out in [5], however, that any basis for the spline space $S^1_3(\Delta^{(1)}_{mn})$ must include at least one basis element which is global in the following sense.

A spline $s \in S^1_3(\Delta^{(1)}_{mn})$ is called global provided it satisfies

(i) $D_s a(a, \cdot) \neq 0$ and $D_s b(b, \cdot) \neq 0$, or
(ii) $D_s c(c, \cdot) \neq 0$ and $D_s d(d, \cdot) \neq 0$.

For the space $S^1_3(\Delta^{(1)}_{mn})$, we have [5]:

**Theorem 2.12.** For all $m, n \geq 1$,

$$\dim S^1_3(\Delta^{(1)}_{mn}) = 2(m - 2) + (n - 2)_+,$$

and a basis for this space is given by the B-splines

$$\left\{ B^1_{i,j+1}, B^2_{i+1,j} \right\}_{i=1,j=1}^{m-2,n-2},$$

where $B^1_{i,j}$ and $B^2_{i,j}$ are defined by Theorem 2.10.

For the uniform type-2 triangulation $\Delta^{(2)}_{mn}$, using the B-spline basis of $S^1_2(\Delta^{(2)}_{mn})$ given by Theorem 2.7, we obtain [6]:

**Theorem 2.13.** (i) For all $m, n \geq 2$,

$$\dim S^1_2(\Delta^{(2)}_{mn}) = mn - 1,$$

and a basis for the space $S^1_2(\Delta^{(2)}_{mn})$ may be given by some of the linear combinations of B-splines of space $S^1_2(\Delta^{(2)}_{mn})$ (cf. Theorem 2.7).

(ii) For all $m, n \geq 1$,

$$\dim S^1_2(\Delta^{(2)}_{mn}) = (m - 2)_+ (n - 2)_+,$$

and a basis for this space is given by the B-splines of

$$\left\{ B_{ij} : \text{support} \left[ B_{ij} \right] \setminus D = \emptyset \right\},$$

where B-spline $B_{ij}$ are defined by Theorem 2.7.

For non-uniform type-2 triangulation $\Delta^{(2)}_{mn}$, the spaces $S^1_2(\Delta^{(2)}_{mn})$ and $S^1_2(\Delta^{(2)}_{mn})$ were studied in [32].

3. On the space of higher dimensional splines

In order to give as clear a presentation as possible, only spline functions of three variables will be discussed. It will be clear that the following results can be generalized to an arbitrary higher dimensional setting.

Let $D$ be a domain in $\mathbb{R}^3$, and $T$ a lattice of planes that divide $D$ into a finite number of cells. Each boundary face that separates two adjacent cells will be called an interior face, the edges of
intersection of the partition $T$ that lie in $D$ are called interior edges, and the points of intersection of some of interior edges are called interior lattice points. Let $P_k = P_k(x, y, z)$ denote the collection of all 3-dimensional polynomials with real coefficients and total degree $k$ as follows:

$$p(x, y, z) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \sum_{l=0}^{k-i-j} c_{ijl} x^i y^j z^l,$$

where $c_{ijl}$ are real numbers. A function $s$ in $C^\mu(D)$ will be called a 3-dimensional spline function of degree $(k, \mu)$ on the the partition $T$ if the restriction of $s$ to each cell of this partition is in $P_k(x, y, z)$. The collection of all these 3-dimensional spline functions will be denoted by $S_{k,3}(T) = S_{k,3}(T; D)$. The following result can be obtained [30]:

**Lemma 3.1.** Let $D_i$ and $D_j$ be two adjacent cells of the partition $T$ with common interior face $\Pi_{ij}$: $a_{ij}x + b_{ij}y + c_{ij}z + d_{ij} = 0$, and the representation of function $s(x, y, z)$ on $D_i$ and $D_j$ be polynomials in $P_k(x, y, z)$ given by $p_i(x, y, z)$ and $p_j(x, y, z)$ respectively. Then $s(x, y, z) \in S_{k,3}(T, D, U)$ if and only if

$$p_i(x, y, z) - p_j(x, y, z) = (l_{ij}(x, y, z))^{\mu+1} \cdot Q_{ij}(x, y, z), \quad (3.1)$$

where $l_{ij} = a_{ij}x + b_{ij}y + c_{ij}z + d_{ij}$, and $Q_{ij}(x, y, z) \in P_{k-\mu-1}$.

The polynomial $Q_{ij}(x, y, z)$ in (3.1) is called the smoothing cofactor of the function $s(x, y, z)$ across $\Pi_{ij}$ from $D_j$ to $D_i$. We call the following condition the edge conformality condition at interior edge $L$ of the partition $T$ [30]

$$\sum_L [l_{ij}(x, y, z)]^{\mu+1} Q_{ij}(x, y, z) = 0, \quad (3.2)$$

where $\Sigma_L$ denotes the sum for all interior faces which are passing through the interior edge $L$, these interior faces $\Pi_{ij}$ from $D_j$ to $D_i$. We call the following condition the global conformality condition of the partition $T$ [30]

$$\sum_{L_r} [l_{ijr}(x, y, z)]^{\mu+1} Q_{ijr}(x, y, z) = 0, \quad r = 1, \ldots, M \quad (3.3)$$

the global conformality condition of the partition $T$, where for each $r$, the mean of (3.3) is the same as stated in (3.2).

We now turn to give a theorem of existence as follows [30]:

**Theorem 3.1.** Let $T$ be any partition of $D$. The function $s(x, y, z)$ is a 3-dimensional spline belonging to $S_{k,3}(T)$, if and only if for any interior face $\Pi_{ij}$, there exists a smoothing cofactor $Q_{ij}(x, y, z)$ of the function $s(x, y, z)$, and the global conformality condition (3.3) is satisfied.

It is not difficult to verify the following [31]:

**Theorem 3.2.**

$$\dim S_{k,3}(T) = \left(\frac{k + 3}{3}\right) + \Pi \left(\frac{k - \mu + 2}{3}\right) - \tau, \quad (3.4)$$
where $\Pi$ is the number of all interior faces, and $\tau$ is the rank of coefficient matrix of linear equations determined by the global conformity condition (3.3).

Let $D = [a_1, b_1] \otimes [a_2, b_2] \otimes [a_3, b_3]$, and $T_{mnl}$ denote a rectangular parallelepiped partition consisting of the planes $x = x_i$ ($i = 1, \ldots, m$), $y = y_j$ ($j = 1, \ldots, n$), and $z = z_r$ ($r = 1, \ldots, l$), where $a_1 < x_1 < \cdots < x_m < b_1$, $a_2 < y_1 < \cdots < y_n < b_2$, and $a_3 < z_1 < \cdots < z_l < b_3$. We have [33]:

**Theorem 3.3.** Let $T_{mnl}$ be a rectangular parallelepiped partition of $D$ as stated above. Then

$$\dim S^\mu_{k,3}(T_{mnl}) = \left( \frac{k+3}{3} \right) + (m+n+l-3) \left( \frac{k-\mu+2}{3} \right)$$

$$+ (mn+ml+nl-2m-2n-2l+3) \left( \frac{k-2\mu+1}{3} \right)$$

$$+ (mnl-mn-ml-nl+m+n+l-1) \left( \frac{k-3\mu}{3} \right).$$

and a basic of $S^\mu_{k,3}(T_{mnl})$ is given by

$$B = \left\{ x^a y^b z^c, x^d y^e z^f (x-x_i)^{\mu+1}, x^d y^e z^f (y-y_j)^{\mu+1}, x^d y^e z^f (z-z_r)^{\mu+1}, \right.$$ 

$$x^h y^p (x-x_i)^{\mu+1} (y-y_j)^{\mu+1}, x^h y^p (y-y_j)^{\mu+1} (z-z_r)^{\mu+1},$$

$$x^h y^p (z-z_r)^{\mu+1} (x-x_i)^{\mu+1}, x^h y^p (x-x_i)^{\mu+1} (y-y_j)^{\mu+1} (z-z_r)^{\mu+1},$$

$$0 < a + b + c < k, 0 \leq d + e + f < k - \mu - 1,$$

$$0 \leq g + h + p < k - 2\mu - 2, 0 \leq u + v + w < k - 3\mu - 3,$$

$$i = 1, \ldots, m-1, j = 1, \ldots, n-1, r = 1, \ldots, l-1 \}. \quad (3.6)$$

Let $T$ be a collection of all such as above rectangular parallelepiped partitions of $D$, and $S^\mu_{k,3}(T) = \{ S^\mu_{k,3}(T_{mnl}) : T_{mnl} \in T \}$. It can be seen from Theorem 3.3 that the following corollary holds [33].

**Corollary 3.1.** The closure of $S^\mu_{k,3}(T)$ in the topology of uniform convergence on compact subjects of $D$ is $C(D)$ if and only if $\mu \leq \frac{1}{3}(k-3)$. If $\frac{1}{3}(k-3) < \mu \leq \frac{1}{2}(k-2)$, then this closure is the space consisting of all functions

$$p_k(x, y, z) + q_1(x, y, z)f(x) + q_2(x, y, z)g(y) + q_3(x, y, z)h(z)$$

$$+ t_1(x, y, z)u(x, y) + t_2(x, y, z)v(y, z) + t_3(x, y, z)w(z, x);$$

if $\mu > \frac{1}{2}(k-2)$, then this closure is the space consisting of all functions

$$p_k(x, y, z) + q_1(x, y, z)f(x) + q_2(x, y, z)g(y) + q_3(x, y, z)h(z),$$

where $p_k \in P_k$, $q_1, q_2, q_3 \in P_{k-\mu-1}$, $t_1, t_2, t_3 \in P_{k-2\mu-2}$, and $f, g, h, u, v, w \in C$.

**References**


