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Existence and boundedness of solutions for a singular phase field system

Elena Bonetti^a, Pierluigi Colli^{a,*}, Mauro Fabrizio^b, Gianni Gilardi^a

^a Dipartimento di Matematica "F. Casorati", Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy

^b Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy

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ABSTRACT

This paper is devoted to the mathematical analysis of a thermo-mechanical model describing phase transitions in terms of the entropy and order structure balance law. We consider a macroscopic description of the phenomenon and make a presentation of the model. Then, the initial and boundary value problem is addressed for the related PDE system, which contains some nonlinear and singular terms with respect to the temperature variable. Existence of the solution is shown along with the boundedness of both phase variable χ and absolute temperature ϑ . Finally, uniqueness is proved in the framework of a source term depending Lipschitz continuously on ϑ .

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1. Introduction

This paper deals with phenomena of phase transitions, of first and second order, in binary systems (cf., e.g., [19,20,24]). In a first order phase transition phenomenon, as in the solid–liquid or liquid–vapor phase change, the phase transition occurs at a critical temperature, say ϑ_c : if the absolute temperature ϑ in the body is strictly greater than the critical temperature ϑ_c , then the minimum of the energy potential is attained in one of the pure phases, while if $\vartheta < \vartheta_c$ the minimum is attained in the other phase. In the case when $\vartheta = \vartheta_c$ the energy potential has two minima attained for the two phases, that is phase change may occur. On the other hand, in the case of second order phase transitions, the system behaves differently provided ϑ is greater or less than the critical tempera-

* Corresponding author.

E-mail addresses: elena.bonetti@unipv.it (E. Bonetti), pierluigi.colli@unipv.it (P. Colli), fabrizio@dm.unibo.it (M. Fabrizio), gianni.gilardi@unipv.it (G. Gilardi).

ture ϑ_c . Indeed, for high temperatures the energy potential has only one minimum, while for $\vartheta < \vartheta_c$ two minima are attained with the same values. This second behaviour is characteristic, for instance, of some solid–solid phase transitions, ferromagnetism, and superconductivity.

We are going to investigate a model describing these phenomena by use of phase-field theories, in terms of temperature ϑ and a phase parameter χ , that includes the effects of micro-motions and micro-forces responsible for the phase transition (cf. [22] and [25]). Indeed, it is known that phase transitions are caused by changes occurring at a microscopic level in the (atomic and/or crystal) structure of the system. These changes are the effects of micro-forces and motions, which have to be included in the balance of the energy of the whole system, even if we are providing a macroscopic description of the phenomenon. We follow the suggestion of M. Frémond, who proposed a new balance law for phase transitions (cf. [22] and [16]). Let us quote a fairly similar theory devised by Gurtin [25] for Ginzburg–Landau and Cahn–Hilliard equations.

Hence, we combine this theory with a model, recently introduced, based on a *reduced energy balance* equation, subsequently termed as *entropy equation* since entropy is involved. We mainly refer to [8] and especially [5] for the derivation of the model and related analytical results. Note that in [5] also thermal memory effects are accounted for (according to the theory of [26]), while in the present contribution we are neglecting them. The main advantage of the model itself is that, once the problem is solved in a suitable sense, one can obtain directly the positivity of the temperature, mainly due to the presence of a logarithmic nonlinearity in the resulting system of partial differential equations. This avoids the application of maximum principle arguments, which are difficult to set in a number of interesting situations. This fact is pointed out in [5], where the model has been introduced in a general setting and a global existence result is proved for a weak formulation. For the sake of completeness, we quote a recent contribution [9], where a more general (non-smooth) relation between the entropy and the absolute temperature is considered. Also in this case the physical constraint on the positivity of the absolute temperature is ensured by the model itself. In [9] the model is recovered by writing the first principle in a dual formulation (in the sense of the convex analysis) using the entropy in place of the temperature as state variable. The authors of [9] prove existence of solutions (ϑ, χ) and provide a characterization of the long-time behaviour of the solution trajectories (see also [4,10]). However, uniqueness is an open problem in both [5] and [9]: this is mainly due to the lack of regularity on the ϑ component of the solution. In our approach, we can prove uniqueness on the basis that a time derivative of ϑ is present in our entropy equation. Indeed, we make a different choice for the free energy with respect to [5] and [8,9], consisting in a second order approximation of the caloric term of the free energy in [5]. Let us also mention [6,7] where a suitable choice (far from the present approach) of the heat flux law leads to a linear operator acting on the temperature, which is of some help in showing existence, uniqueness, and regularity of the solution. Also the large time behaviour and the ω -limit set are investigated in [7].

1.1. The model

In the actual contribution, we mainly proceed according to [5]. Indeed, we consider a two-phases system, located in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$, and look at its evolution during a finite time interval $(0, T)$. We denote by Γ the boundary $\partial\Omega$. The thermo-mechanical equilibrium of the system is described in terms of state variables and governed by the free energy, while the dynamics ensues from the presence of a pseudo-potential of dissipation (depending on dissipative variables). Let us point out that the properties of the pseudo-potential of dissipation ensure thermodynamical consistency of the model.

We do not consider mechanical effects, so that the variables of the system are just the absolute temperature ϑ and a phase parameter χ , related to the proportion of one phase with respect to the other. In general, χ attains its physical admissible values in a range $[\chi_*, \chi^*]$ (e.g., $\chi \in [0, 1]$) and this physical constraint has to be ensured by the model itself. Hence, we derive the equations of the system by thermo-mechanical laws. More precisely, we use the approach followed by Frémond (yielding the evolution of the phase parameter), and the first and second principle of Thermodynamics (from which we recover a balance equation ruling the evolution of the temperature).

We first discuss, with some details, the derivation of the evolution equation for the phase parameter χ . It is known that the phase transition can be described as a change in the order structure of the thermo-mechanical system we are considering, so that the phase parameter χ can be interpreted also as an “order parameter”. More precisely, below a critical temperature it is observed that the structure order of many materials is greater than above. For instance, in the solid–liquid phase transition the solid phase has a greater order structure due to the crystal symmetry group. Analogously, we may think to ferromagnetism where the magnetic moments are aligned below the Curie temperature. Obviously, order change in the structure of the system occurs at a microscopic level. However, the parameter χ is a macroscopic parameter whose evolution is governed by a balance law on the order structure, responsible for the phase transition at a microscopic level. Thus, the evolution of the order parameter χ can be derived from the balance conditions

$$B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega \times (0, T), \quad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0, T) \quad (1.1)$$

\mathbf{n} being the outward normal to the boundary Γ , where the scalar quantity B can be interpreted as an internal order structure density per units of concentration χ , and \mathbf{H} represents an order structure flux vector. Note that here we are assuming that no external action is provided to the system.

As far as the description of the evolution of the temperature is concerned, we use a balance law in which higher order dissipative contributions are neglected by means of the small perturbations assumption (cf. [23]). The resulting equation rules the evolution of the entropy s of the system in terms of the entropy flux \mathbf{Q} and an external source R , possibly depending on the state field, that is

$$s_t + \operatorname{div} \mathbf{Q} = R \quad \text{in } \Omega \times (0, T). \quad (1.2)$$

Then, (1.2) is combined with boundary conditions on the entropy flux, e.g., if no flux is assumed through the boundary, then one states that $\mathbf{Q} \cdot \mathbf{n} = 0$ on $\Gamma \times (0, T)$. Another possibility, actually followed by us in the analysis, is to prescribe the value of the temperature on the boundary.

Referring to [5] and [9], we prefer not to detail here the derivation of the model. However, for the sake of completeness, we just recall that (1.2) can be obtained, in the framework of small perturbations assumptions, dividing by $\vartheta > 0$ the energy balance

$$e_t + \operatorname{div} \mathbf{q} = r + B\chi_t + \mathbf{H} \cdot \nabla \chi_t. \quad (1.3)$$

In (1.3) e denotes the internal energy of the system, $\mathbf{q} = \vartheta \mathbf{Q}$ represents the heat flux, r stands for an external source, while $B\chi_t + \mathbf{H} \cdot \nabla \chi_t$ is internal and comes from the order structure. We will add some comments below on the relation between the two equations (1.2) and (1.3).

1.2. Free energy

We specify the involved physical quantities with the help of two energy functionals: the free energy Ψ , depending on the state variables and accounting for the thermo-mechanical equilibrium of the system, and the pseudo-potential of dissipation Φ (see [28]), defined for dissipative variables and responsible of the evolution of the system. More precisely, we consider as state variables the absolute temperature ϑ , the order parameter χ , and its gradient $\nabla \chi$. It is known from thermodynamics that the free energy is a concave function with respect to the temperature, while there are not constraints concerning the dependence on the other variables. In the present contribution, we choose the functional Ψ of the following form

$$\Psi(\vartheta, \chi, \nabla \chi) := -\frac{c_0}{2} \vartheta^2 + F(\chi) \vartheta_c + G(\chi) \vartheta + \frac{\nu}{2} |\nabla \chi|^2 \quad (1.4)$$

the constants c_0, ν being positive and $\vartheta_c > 0$ representing the critical value of the temperature for the phase transition. Note that the purely caloric part in the free energy $-(c_0/2)\vartheta^2$ is, in fact, concave

with respect to the temperature. Then, the functions F and G characterize the behaviour of the phase transition. For instance, in a first order phase transition, as vapor–liquid or liquid–solid, they can be prescribed as follows

$$F(\chi) = \frac{\chi^4}{4} - \frac{\chi^3}{3}, \quad G(\chi) = \frac{\chi^4}{4} - \frac{2\chi^3}{3} + \frac{\chi^2}{2} \quad (1.5)$$

while in a second order phase transition, as for superconductivity or ferromagnetism, F and G can be written as

$$F(\chi) = \frac{\chi^4}{4} - \frac{\chi^2}{2}, \quad G(\chi) = \frac{\chi^2}{2}. \quad (1.6)$$

Let us remark from the beginning that both the cases (1.5) and (1.6) comply with our general assumptions (2.3)–(2.4) provided you take $\chi_* \leq 0$ and $\chi^* \geq 1$.

1.3. Admissible values for the phase variable

The physical constraint on χ , that is $\chi_* \leq \chi \leq \chi^*$, is not a priori guaranteed by the choice of the free energy (1.4): in particular, the functions F and G prescribed in (1.5) and (1.6) are smooth on the whole of \mathbb{R} (as in [3]). A different possible choice for Ψ introduced in the literature (cf., e.g., [5,22]) and accounting for this constraint is

$$\Psi(\vartheta, \chi, \nabla\chi) = -\frac{c_0}{2}\vartheta^2 + I_{[\chi_*, \chi^*]}(\chi) + \mathcal{F}(\chi) + \vartheta\mathcal{G}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

where \mathcal{F} , \mathcal{G} are sufficiently smooth functions characterizing the phase transition. In this case the free energy is defined for any value of χ but it is $+\infty$ if $\chi \notin [\chi_*, \chi^*]$ (while $I_{[\chi_*, \chi^*]}(\chi) = 0$ if $\chi \in [\chi_*, \chi^*]$). In our approach, instead, we will show that the constraint on χ is ensured by the evolution of the system, i.e., it will be proved that the equations of the system are somehow *consistent* as they yield $\chi_* \leq \chi \leq \chi^*$ whenever the initial value is in the same range. The proof of this property of our model will be detailed in the sequel and relies on a maximum principle argument.

1.4. Pseudo-potential of dissipation

Secondly, we introduce the pseudo-potential of dissipation Φ (see [28]) that depends on χ_t and $\nabla\vartheta$. Let us just comment on the choice of these dissipative variables: χ_t is related to microscopic transformations which are responsible for the phase transition, i.e. for the evolution of the order structure of the system, while $\nabla\vartheta$ is concerned with the heat flux. For the sake of completeness, let us recall that the pseudo-potential of dissipation Φ is nonnegative, convex with respect to the dissipative variables, and it attains its minimum 0 for a null dissipation, that is when $(\chi_t, \nabla\vartheta) = (0, \mathbf{0})$. We prescribe

$$\Phi(\chi_t, \nabla\vartheta) = \frac{\mu}{2}|\chi_t|^2 + \frac{\lambda}{2\vartheta}|\nabla\vartheta|^2 \quad (1.7)$$

with μ and λ denoting positive coefficients.

1.5. Constitutive relations

Hence, constitutive relations can be written for B , \mathbf{H} , s , \mathbf{Q} . They are recovered from the free energy (for non-dissipative contributions) and the pseudo-potential of dissipation (for dissipative parts). We have

$$B = \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t} = \vartheta_c F'(\chi) + \vartheta G'(\chi) + \mu \chi_t \quad (1.8)$$

and

$$\mathbf{H} = \frac{\partial \Psi}{\partial (\nabla \chi)} = \nu \nabla \chi \quad (1.9)$$

as well as

$$s = -\frac{\partial \Psi}{\partial \vartheta} = c_0 \vartheta - G(\chi) \quad (1.10)$$

and

$$\mathbf{Q} = -\frac{\partial \Phi}{\partial (\nabla \vartheta)} = -\frac{\lambda}{\vartheta} \nabla \vartheta = -\lambda \nabla \log \vartheta. \quad (1.11)$$

Let us point out that the choice of the free energy (1.4) leads to a linear contribution for the temperature in (1.10). This will yield sufficient regularity on the solution, from which we will be able to prove uniqueness. We also point out that the term $-(c_0/2)\vartheta^2$ could be seen as a first order approximation of the following, well-known, form of the energy potential

$$\Psi(\vartheta, \dots) = -c_0 \vartheta \log \vartheta + \dots$$

used, e.g., in [5]. In this case, the entropy s would be related to the temperature ϑ through a logarithmic nonlinearity. Note that, on the contrary, a logarithmic nonlinearity forcing ϑ to be strictly positive is present in our expression (1.11) for \mathbf{Q} , analogously to the choice made, e.g., in [5].

1.6. From energy balance to our equation

The reader may be curious about the derivation of (1.2) from (1.3). Let us recall constitutive relations (1.8)–(1.11) and the well-known Helmholtz relation

$$e = \Psi + \vartheta s. \quad (1.12)$$

By applying the chain rule in (1.3), some terms cancel and one can rewrite the energy balance as

$$\vartheta (s_t + \operatorname{div} \mathbf{Q} - R) = \left(\frac{\partial \Phi}{\partial \chi_t}, \frac{\partial \Phi}{\partial (\nabla \vartheta)} \right) \cdot (\chi_t, \nabla \vartheta) \quad (1.13)$$

letting $\mathbf{Q} = \mathbf{q}/\vartheta$ and $R = r/\vartheta$. We point out that the fact that Φ is convex, l.s.c., proper, nonnegative and it attains its minimum 0 when $(\chi_t, \nabla \vartheta) = (0, \mathbf{0})$ ensure that the right-hand side of (1.13) is nonnegative. As $\vartheta > 0$, the Clausius–Duhem inequality

$$s_t + \operatorname{div} \mathbf{Q} - R \geq 0 \quad (1.14)$$

complies with (1.13). Hence, dividing (1.13) by ϑ and neglecting the nonnegative and higher order dissipative resulting contribution on the right-hand side, Eq. (1.2) follows.

1.7. Nonlinearity in the source term

We stress that the entropy source R in (1.2) is related to the source r appearing in (1.3) by $R = r/\vartheta$. Thus, it seems reasonable to include in our analysis the possibility for R to depend on the temperature (and possibly to present some singularities). This is one of the features of our paper: we actually deal with entropy sources (positive or negative, as sinks) depending also on the temperature ϑ . Indeed, possible choices satisfying our assumptions are (cf. the later Remark 2.2)

$$R(x, t, \vartheta) = \frac{R_1(x, t)}{\vartheta^2} - R_2(x, t)$$

which would correspond to $r(x, t, \vartheta) = (R_1(x, t)/\vartheta) - R_2(x, t)\vartheta$, or

$$R(x, t, \vartheta) = R_3(x, t)\vartheta - R_4(x, t)$$

which can be viewed as a linearization of R around some equilibrium value of ϑ . In such cases, the possible data R_1, R_2 or R_3, R_4 are smooth enough and at least R_1 should be nonnegative throughout $\Omega \times (0, T)$.

1.8. System of PDEs and initial–boundary value problem

Now, combining constitutive relations (1.8)–(1.11) with (1.1) and (1.2) leads to the following PDE system

$$c_0\vartheta_t - G'(\chi)\chi_t - \lambda\Delta \log \vartheta = R(x, t, \vartheta), \tag{1.15}$$

$$\mu\chi_t - \nu\Delta\chi + F'(\chi)\vartheta_c + G'(\chi)\vartheta = 0 \tag{1.16}$$

which is addressed in $Q := \Omega \times (0, T)$. Then, concerning boundary conditions, prescribed in $\Gamma \times (0, T)$, we fix a Dirichlet condition for the temperature and a Neumann homogeneous condition for the phase parameter

$$\log \vartheta = \log \vartheta_\Gamma, \quad \partial_n \chi = 0. \tag{1.17}$$

Finally, initial conditions are set in Ω

$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0. \tag{1.18}$$

For the sake of simplicity, in the mathematical analysis performed in subsequent sections we will take the physical constants $c_0, \lambda, \mu, \nu, \vartheta_c$ all equal to 1. Let us just point out that Dirichlet conditions for $\log \vartheta$ are very important in our approach since they allow us to perform the basic a priori estimates and use the Poincaré inequality.

Let us point out in our model is that we are dealing with a non-smooth entropy source $R(\vartheta)$ in (1.15) that is assumed to be increasing with respect to $\vartheta \in (0, +\infty)$ up to Lipschitz perturbations. This fact turns out to be interesting both for analytical and modelling aspects. Indeed, we are able to treat a diffusive equation for the temperature with nonlinear and singular diffusion along with a non-smooth contribution in source term. Then, for modelling aspects, we can figure the entropy source R directly as $r(\vartheta)/\vartheta$ (compare (1.13) with (1.3)). The second aspect we notice is that the physical property $\chi \in [\chi_*, \chi^*]$ comes as a consequence of our analysis and then it is ensured by the evolution of the system itself and not required a priori.

Under suitable assumptions on the data and on the regularity of the involved nonlinearities, we can prove existence of a solution in any time interval $(0, T)$ for a weak formulation of our system

(1.15)–(1.18). Moreover, in a wide setting we are able to show that also the solution component ϑ is bounded by adapting a Moser type argument. Finally, if $R(x, t, \vartheta)$ is uniformly Lipschitz continuous with respect to ϑ , we prove uniqueness of the solution.

1.9. Phase field model with special heat flux law

What is also interesting in our contribution is that system (1.15)–(1.16) may be interpreted as a nonlinear Caginalp phase-field model (see [13]) with special heat flux law. Indeed, take the following energy functional

$$\mathcal{Y}(\vartheta, \chi) = -\frac{1}{2\vartheta} + \vartheta F(\chi) + G(\chi) + \frac{\nu\vartheta}{2} |\nabla\chi|^2. \quad (1.19)$$

Hence, arguing as in the derivation of the Penrose–Fife model (cf. [12,21,29]), we recover the entropy s , the internal energy e , and the phase field equation (for χ) as

$$s = -\frac{\partial\mathcal{Y}}{\partial\vartheta} = -\frac{1}{2\vartheta^2} - F(\chi) - \frac{\nu}{2} |\nabla\chi|^2, \quad (1.20)$$

$$e = \mathcal{Y} + \vartheta s = -\frac{1}{\vartheta} + G(\chi), \quad (1.21)$$

$$\mu\chi_t + \frac{\delta H}{\delta\chi} = 0, \quad H = \int_{\Omega} \frac{1}{\vartheta} \mathcal{Y} \quad (1.22)$$

where $\delta H/\delta\chi$ denotes the variational (Fréchet) derivative of the functional H . Then, we can write the first principle neglecting microscopic motions and forces (here \tilde{r} represents a mere source term)

$$e_t + \operatorname{div} \mathbf{q} = \tilde{r}. \quad (1.23)$$

We now let

$$\mathbf{q} = -\lambda \nabla \log \vartheta \quad (1.24)$$

and this special choice of the heat flux can be compared with the ones studied in the papers [14,15], where Penrose–Fife models with special heat flux laws have been investigated. Then, combine (1.22) and (1.23) with (1.19)–(1.21) and (1.24), so to obtain the following system

$$u_t - G'(\chi)\chi_t - \lambda \Delta \log u = -\tilde{r}, \quad (1.25)$$

$$\mu\chi_t - \nu \Delta \chi + F'(\chi) + uG'(\chi) = 0 \quad (1.26)$$

in which u has now the physical meaning of $1/\vartheta$. Note that system (1.25)–(1.26) is formally equivalent to (1.15)–(1.16) (with $c_0 = \vartheta_c = 1$ and $-\tilde{r}$ in place of R). Thus, it turns out that our analysis applies to the Caginalp system (1.25)–(1.26) in which non-smooth heat sources depending on the temperature are admitted.

1.10. Plan of the paper

We conclude the Introduction by giving an outline of the paper. In Section 2 we set our problem in a Sobolev spaces framework and make precise assumptions on the data and the involved functions, also stating main results. In Section 3 we introduce an approximating problem and prove the existence of solutions for it. Next, Section 4 is devoted to derive some a priori estimates and to pass to the limit as the approximating parameter goes to 0. A maximum principle is also checked to show that the constraint on χ is always satisfied. Boundedness of ϑ and uniqueness proofs are the subjects of the last two Sections 5 and 6, respectively.

2. Main results

In this section, we carefully describe the problem we are going to deal with and state our results. First of all, we introduce the notation regarding the domain in which the evolution is considered. In the sequel, Ω is a bounded open set in \mathbb{R}^3 , whose boundary Γ is assumed to be of class C^2 . Moreover, ∂_n is the (say, outward) normal derivative on Γ . Given a finite final time T , we set for convenience

$$Q_t := \Omega \times (0, t) \quad \text{for every } t \in (0, T] \quad \text{and} \quad Q := Q_T. \tag{2.1}$$

Next, we describe the structure of our system. We are given four constants $\vartheta_*, \vartheta^*, \chi_*, \chi^* \in \mathbb{R}$ such that

$$0 < \vartheta_* \leq 1 \leq \vartheta^* \quad \text{and} \quad \chi_* < \chi^* \tag{2.2}$$

and four functions F, G, β , and π

$$F, G : \mathbb{R} \rightarrow \mathbb{R}, \quad \beta : Q \times (0, +\infty) \rightarrow \mathbb{R}, \quad \text{and} \quad \pi : Q \times \mathbb{R} \rightarrow \mathbb{R}$$

satisfying

$$F, G \in C^2(\mathbb{R}), \quad F \text{ is bounded from below and } G \text{ is nonnegative,} \tag{2.3}$$

$$F', G' \leq 0 \quad \text{in } (-\infty, \chi_*), \quad \text{and} \quad F', G' \geq 0 \quad \text{in } (\chi^*, +\infty), \tag{2.4}$$

$$\beta \text{ is Lipschitz continuous in } Q \times [\delta, 1/\delta] \text{ for every } \delta \in (0, 1), \tag{2.5}$$

$$\beta_{,x}, \beta_{,t}, \beta', \text{ and } \pi \text{ are Carathéodory functions, with the notation,} \tag{2.6}$$

$$\beta_{,x}(x, t, r) := \nabla \beta(x, t, r), \quad \beta_{,t}(x, t, r) := \partial_t \beta(x, t, r), \quad \beta'(x, t, r) := \partial_r \beta(x, t, r), \tag{2.7}$$

$$0 \leq \beta'(x, t, r) \leq \beta_1(r) \tag{2.8}$$

for a.a. $(x, t) \in Q$, every $r \in (0, +\infty)$, and some $\beta_1 \in C^0(0, +\infty)$,

$$|\beta_{,x}(x, t, r)| + |\beta_{,t}(x, t, r)| \leq M_\beta(1 + |\beta(x, t, r)|) \tag{2.9}$$

for a.a. $(x, t) \in Q$, every $r \in (0, +\infty)$, and some $M_\beta \in [0, +\infty)$,

$$\beta(x, t, 1) = 0 \quad \text{for every } (x, t) \in Q, \tag{2.10}$$

$$|\pi(x, t, r)| \leq \lambda|r| + \pi_0(x, t) \tag{2.11}$$

for a.a. $(x, t) \in Q$, every $r \in \mathbb{R}$, some $\lambda \in [0, +\infty)$, and some $\pi_0 \in L^2(Q)$.

Furthermore, we set for convenience

$$\widehat{\beta}(x, t, r) := \int_1^r \beta(x, t, s) ds \quad \text{for every } (x, t, r) \in Q \times (0, +\infty). \tag{2.12}$$

Then, observing that (by (2.8)) $\beta(x, t, \cdot)$ is nondecreasing, it turns out that $\widehat{\beta}$ is nonnegative and convex with respect to the third variable.

Remark 2.1. We note that the second inequality of (2.8) and (2.10) imply that

$$|\beta(x, t, r)| \leq \beta_0(r) := \left| \int_1^r \beta_1(s) ds \right| \quad \forall (x, t, r) \in Q \times (0, +\infty). \tag{2.13}$$

Therefore, we see by (2.9) that even β_x and β_t satisfy an analogous inequality and infer that (2.5) follows from the other assumptions. We have written (2.5) in advance to give a meaning to the pointwise values of β .

Remark 2.2. In the applications just the difference $R = \pi - \beta$ is interesting and we observe that the choice

$$R(x, t, r) = \frac{R_1(x, t)}{r^2} + R_2(x, t) \quad \text{with } R_1 \geq 0 \tag{2.14}$$

is included in our assumption. Indeed, we can take $\beta(x, t, r) = R_1(x, t) - R_1(x, t)/r^2$ for $r > 0$ and $\pi(x, t, r) = R_1(x, t) + R_2(x, t)$, so that $R = \pi - \beta$ and (2.10) hold at the same time. However, we have to assume that R_1 is Lipschitz continuous and that $|\nabla R_1| + |\partial_t R_1| \leq cR_1$ in Q for some constant c in order to fulfill (2.9). As R_1 is bounded and nonnegative, (2.8) holds as well. Moreover, we ask that $R_2 \in L^2(Q)$ in view of (2.11). A different choice for R is the following

$$R(x, t, r) = R_3(x, t)r - R_4(x, t) \tag{2.15}$$

and can be obtained in our setting by simply taking $\beta = 0$ and $\pi = R$. Here, we can assume $R_3 \in L^\infty(Q)$ and $R_4 \in L^2(Q)$.

Notation 2.3. Let I be a real interval and $\psi : Q \times I \rightarrow \mathbb{R}$ be a Carathéodory function. We use the same symbol ψ to denote the operator acting on measurable functions on Q as follows. If $v : Q \rightarrow \mathbb{R}$ is measurable

$$\psi(v) \text{ denotes the function } (x, t) \mapsto \psi(x, t, v(x, t)), \quad (x, t) \in Q. \tag{2.16}$$

Note that $\psi(v)$ actually is measurable due to the Carathéodory assumption on ψ . Similar definitions and symbols are used for functions depending on the space variable. Namely, if $v : \Omega \rightarrow \mathbb{R}$ is measurable

$$\psi(t, v) \text{ denotes the function } x \mapsto \psi(x, t, v(x)), \quad x \in \Omega, \tag{2.17}$$

for a.a. $t \in (0, T)$. We obtain a time dependent operator. As a consequence, if $v \in L^2(Q)$, the symbol $\psi(t, v(t))$ denotes the measurable function $x \mapsto \psi(x, t, v(x, t))$, $x \in \Omega$. Notation (2.16)–(2.17) will be used, in particular, with (some of the functions listed below will be introduced later on)

$$\psi = \beta, \widehat{\beta}, \beta_\varepsilon, \widehat{\beta}_\varepsilon, \beta_{,x}, \beta_{,t}, \beta', \beta_{\varepsilon,x}, \beta_{\varepsilon,t}, \beta'_\varepsilon, \pi. \tag{2.18}$$

Furthermore, we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad V_0 := H_0^1(\Omega), \quad W := \{v \in H^2(\Omega): \partial_n v = 0\}. \quad (2.19)$$

We endow H , V , and W with their usual scalar products and norms and use a self-explaining notation, like $\|\cdot\|_V$. For the sake of simplicity, the same symbol will be used both for a space and for any power of it. It is understood that H is embedded in V_0^* in the usual way, i.e., $\langle u, v \rangle = (u, v)$ for every $u \in H$ and $v \in V_0$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between V_0^* and V_0 and (\cdot, \cdot) is the inner product of H .

Now, we list our assumptions on the boundary and initial data. We are given three functions ϑ_Γ , ϑ_0 , and χ_0 such that

$$\vartheta_\Gamma \in L^2(0, T; H^{1/2}(\Gamma)) \cap H^1(0, T; H^{-1/2}(\Gamma)), \quad \vartheta_* \leq \vartheta_\Gamma \leq \vartheta^* \quad \text{a.e. on } \Gamma \times (0, T), \quad (2.20)$$

$$\vartheta_0 \in H, \quad \vartheta_* \leq \vartheta_0 \leq \vartheta^* \quad \text{a.e. in } \Omega, \quad (2.21)$$

$$\chi_0 \in V, \quad \chi_* \leq \chi_0 \leq \chi^* \quad \text{a.e. in } \Omega, \quad (2.22)$$

where ϑ_* , ϑ^* , χ_* , and χ^* are introduced in (2.2).

At this point, we are ready to state our problem. We look for a triplet (ϑ, χ, ξ) satisfying the regularity conditions and the equations listed below:

$$\vartheta \in L^\infty(0, T; H), \quad \vartheta > 0 \quad \text{a.e. in } Q, \quad \text{and} \quad \ln \vartheta \in L^2(0, T; V), \quad (2.23)$$

$$\chi \in L^2(0, T; W) \cap H^1(0, T; H), \quad (2.24)$$

$$G(\chi), F'(\chi), G'(\chi) \in L^2(Q), \quad (2.25)$$

$$\xi \in L^2(Q), \quad (2.26)$$

$$\partial_t(\vartheta - G(\chi)) \in L^2(0, T; V_0)^*, \quad (2.27)$$

$$\partial_t(\vartheta - G(\chi)) - \Delta \ln \vartheta + \xi = \pi(\vartheta) \quad \text{in } L^2(0, T; V_0)^* \quad \text{and} \quad \xi = \beta(\vartheta), \quad (2.28)$$

$$\partial_t \chi - \Delta \chi + F'(\chi) + G'(\chi)\vartheta = 0 \quad \text{a.e. in } Q, \quad (2.29)$$

$$\ln \vartheta = \ln \vartheta_\Gamma \quad \text{a.e. on } \Gamma \times (0, T), \quad (2.30)$$

$$(\vartheta - G(\chi))(0) = \vartheta_0 - G(\chi_0) \quad \text{and} \quad \chi(0) = \chi_0. \quad (2.31)$$

Even though ξ is a known function of ϑ , we refer to the triplet (ϑ, χ, ξ) instead of the pair (ϑ, χ) when we speak of a solution, just for convenience. Moreover, we note that (2.23) and (2.25) yield $G'(\chi)\vartheta \in L^2(0, T; L^1(\Omega))$. However, we see by comparison in (2.29) that $G'(\chi)\vartheta \in L^2(Q)$. Next, we observe that $G(\chi_0)$ makes sense in $L^\infty(\Omega)$. Furthermore, we point out that the first condition in (2.31) reduces to $\vartheta(0) = \vartheta_0$ whenever one knows that $G(\chi) \in C^0([0, T]; V_0)^*$. Actually, some additional smoothness for $G(\chi)$ as well as for $F'(\chi)$ and $G'(\chi)$ surely holds if the nonlinearities satisfy some growth conditions, thanks to (2.24). The same is trivially true whenever χ is bounded, and our existence result stated below ensures such a property. Finally, (2.24) itself entails the homogeneous Neumann boundary condition for χ (see (2.19)). Here is our first result.

Theorem 2.4. *Let (2.2)–(2.11) and (2.20)–(2.22) be fulfilled. Then, there exists a triplet (ϑ, χ, ξ) satisfying the regularity requirements (2.23)–(2.27) and solving problem (2.28)–(2.31). Moreover, every solution (ϑ, χ, ξ) fulfills the inequalities*

$$\chi_* \leq \chi \leq \chi^* \quad \text{a.e. in } Q. \quad (2.32)$$

In particular, χ is bounded.

One can wonder whether the component ϑ of the solution is bounded as well. Actually, such a property holds whenever we reinforce the assumption on the structure of our system a little, namely

$$\pi_0 \in L^q(Q) \quad \text{for some } q > 5/2. \tag{2.33}$$

We can state the following result.

Theorem 2.5. *Assume (2.33) in addition to the hypotheses of Theorem 2.4. Then, the component ϑ of any solution (ϑ, χ, ξ) to problem (2.28)–(2.31) is bounded.*

Remark 2.6. From Theorem 2.4 it follows that $G(\chi)$, $F'(\chi)$, and $G'(\chi)$ are smoother, namely

$$G(\chi), F'(\chi), G'(\chi) \in L^\infty(Q) \cap L^2(0, T; V) \cap H^1(0, T; H) \tag{2.34}$$

due to (2.3) and (2.24). In particular, we deduce that

$$\partial_t \vartheta \in L^2(0, T; V)_0^* \tag{2.35}$$

by comparison in (2.27). On the other hand, using the regularity conditions $\ln \vartheta \in L^2(0, T; V)$ and $\vartheta \in L^\infty(Q)$ (see (2.23) and Theorem 2.5), we see that $\nabla \vartheta^m = m \vartheta^{m-1} \nabla \ln \vartheta$ for every $m > 0$, whence

$$\vartheta^m \in L^2(0, T; V) \quad \text{for every } m \in (0, +\infty). \tag{2.36}$$

Moreover, as $F'(\chi)$ and $G'(\chi)$ are bounded, from (2.24) and (2.29) we see that $\partial_t \chi - \Delta \chi$ is bounded too. Hence, we have

$$\chi \in L^p(0, T; W^{2,p}(\Omega)) \cap W^{1,p}(0, T; L^p(\Omega)) \quad \text{for every } p \geq 1 \tag{2.37}$$

thanks to the general theory of linear parabolic equations.

Finally, we state our uniqueness result. As it often happens for doubly nonlinear problems, uniqueness for solutions cannot be proved unless more restrictive assumption on the structure are made. In particular, we cannot allow a singular β like (2.14).

Theorem 2.7. *Assume that the hypotheses of Theorem 2.4 are fulfilled. Moreover, assume that $R := \pi - \beta$ satisfies the uniform Lipschitz condition*

$$|R(x, t, r) - R(x, t, s)| \leq \lambda_R |r - s| \quad \text{for every } (x, t) \in Q \text{ and } r, s \in (0, +\infty) \tag{2.38}$$

for some $\lambda_R \geq 0$. Then, problem (2.28)–(2.31) has at most one solution (ϑ, χ, ξ) satisfying $\vartheta \in L^\infty(Q)$.

Corollary 2.8. *If (2.33) and (2.38) are fulfilled at the same time in addition to the hypotheses of Theorem 2.4, then problem (2.28)–(2.31) has exactly one solution (ϑ, χ, ξ) such that ϑ is bounded. Moreover, (2.32), (2.36), and (2.37) are satisfied.*

The paper is organized as follows. The next section deals with approximating problems. Theorem 2.4 is proved in Section 4 and our argument relies on some a priori estimates on the approximate solutions and on monotonicity and compactness methods. Section 5 is devoted to prove Theorem 2.5 and uses a Moser type procedure. Finally, the proof of Theorem 2.7 is given in the last section.

In our proofs, we use the well-known inequalities we are going to recall. As $\Omega \subset \mathbb{R}^3$ is bounded and smooth, the Poincaré inequality

$$\|v\|_V \leq M_\Omega \|\nabla v\|_H \quad \text{for every } v \in V_0 \tag{2.39}$$

holds, and the space V is continuously embedded in $L^6(\Omega)$, i.e.,

$$\|v\|_{L^6(\Omega)} \leq M_\Omega \|v\|_V \quad \text{for every } v \in V. \tag{2.40}$$

For the sake of completeness, we recall a related embedding result for parabolic spaces (see, e.g., [18, formula (3.2), p. 8]). For $m \geq 1$, we have

$$L^\infty(0, T; L^m(\Omega)) \cap L^2(0, T; V)_0 \subset L^{q(m)}(Q) \quad \text{where } q(m) := \frac{2}{3}(m + 3)$$

the embedding being continuous, i.e.,

$$\begin{aligned} \|v\|_{L^{q(m)}(Q)} &\leq M_{\Omega, T, m} (\|v\|_{L^\infty(0, T; L^m(\Omega))} + \|\nabla v\|_{L^2(Q)}) \\ &\text{for every } v \in L^\infty(0, T; L^m(\Omega)) \cap L^2(0, T; V)_0. \end{aligned} \tag{2.41}$$

In particular, we observe that

$$L^\infty(0, T; H) \cap L^2(0, T; V)_0 \subset L^{10/3}(Q), \tag{2.42}$$

$$L^\infty(0, T; L^3(\Omega)) \cap L^2(0, T; V)_0 \subset L^4(Q) \tag{2.43}$$

and that the corresponding estimates (2.41) hold.

The above inequalities are widely used in the sequel, as well as the elementary Young inequality

$$ab \leq \delta a^p + c_{\delta, p} b^{p'} \quad \forall a, b \geq 0, \quad \forall \delta > 0 \tag{2.44}$$

where $p, p' > 1$ satisfy $(1/p) + (1/p') = 1$ and $c_{\delta, p} := (p')^{-1}(\delta p)^{-p'/p}$.

We conclude this section by stating a general rule we use for constants, in order to avoid a boring notation. Throughout the paper, the symbol c stands for different constants which depend only on Ω , on the final time T , and on the constants and the norms of the functions involved in the assumptions of either our statements or our approximation. In particular, c is independent of the approximation parameter ε we introduce in the next section. A notation like c_δ allows the constant to depend on the positive parameter δ , in addition. Hence, the meaning of c and c_δ might change from line to line and even in the same chain of inequalities. On the contrary, we use different symbols (see, e.g., (2.39)) to denote precise constants which we could refer to. By the way, several of such constants could be the same (like in (2.39) and in (2.40)), since sharpness is not needed.

3. Approximating problems

This section contains a preliminary work in the direction of proving Theorem 2.4 and deals with a suitable approximation of problem (2.28)–(2.31). Namely, we replace the strong nonlinearities that appear in Eq. (2.28) with smooth functions depending on the parameter $\varepsilon \in (0, 1)$. First of all, we see the logarithm \ln as a maximal monotone operator in \mathbb{R} with domain $(0, +\infty)$. Precisely, for $r \in \mathbb{R}$, the set $\ln r$ is the singleton $\{\ln r\}$ if $r > 0$, while it is empty if $r \leq 0$. Then, we can consider the function $\text{Ln}_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ defined as follows

$$\text{Ln}_\varepsilon r := \varepsilon r + \text{In}_\varepsilon r \quad \text{where } \text{In}_\varepsilon \text{ is the Yosida regularization of } \ln. \tag{3.1}$$

We note that In_ε is monotone and Lipschitz continuous with constant $1/\varepsilon$ (see, e.g., [11, Proposition 2.6, p. 28]). Thus, $\varepsilon \leq \text{Ln}'_\varepsilon(r) \leq \varepsilon + (1/\varepsilon)$ for every $r \in \mathbb{R}$. The functions In_ε and Ln_ε act on

L^2 -spaces as well and Notation 2.3 is extended to them. Moreover, we replace F and G by new functions, still termed F and G , satisfying

$$F' \text{ and } G' \text{ are bounded} \tag{3.2}$$

in addition to (2.3)–(2.4). Indeed, we can arbitrarily modify F and G outside $[\chi_*, \chi^*]$ due to the last part of Theorem 2.4. Finally, we replace the possibly singular function β by a C^∞ function β_ε , in order to justify the chain rules we have to use. We proceed by extension, truncation, and regularization. For $\varepsilon > 0$, we set

$$\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \Gamma) < \varepsilon\} \quad \text{and} \quad \Omega'_\varepsilon := \{x \in \mathbb{R}^3 \setminus \bar{\Omega} : \text{dist}(x, \Gamma) < \varepsilon\}.$$

As Ω is of class C^2 , there exists $\varepsilon_0 \in (0, 1)$ such that, for every $x \in \Omega'_\varepsilon$, there exists a unique point $\tilde{x} \in \Omega_{\varepsilon_0}$ satisfying

$$x' := \frac{x + \tilde{x}}{2} \in \Gamma \quad \text{and} \quad x - \tilde{x} \text{ is orthogonal to } \Gamma \text{ at } x' \tag{3.3}$$

the correspondence $x \mapsto \tilde{x}$ being a bi-Lipschitz diffeomorphism of class C^1 from Ω'_ε onto Ω_{ε_0} . Then, we define $\tilde{\Omega}$ and the extension-by-reflection operator $\tilde{\cdot} : L^\infty(\Omega) \rightarrow L^\infty(\tilde{\Omega})$ as follows

$$\begin{aligned} \tilde{\Omega} &:= \bar{\Omega} \cup \Omega'_\varepsilon \quad \text{and} \quad \text{for } v \in L^\infty(\Omega) \text{ and a.a. } x \in \tilde{\Omega} \text{ we set} \\ \tilde{v}(x) &= v(x) \quad \text{if } x \in \Omega \quad \text{and} \quad \tilde{v}(x) = v(\tilde{x}) \quad \text{if } x \in \Omega'_\varepsilon. \end{aligned} \tag{3.4}$$

Next, we extend further by reflection as well. We define \tilde{Q} and $\mathcal{E} : L^\infty(Q) \rightarrow L^\infty(\tilde{Q})$ as follows

$$\begin{aligned} \tilde{Q} &:= \tilde{\Omega} \times (-T, 2T) \quad \text{and} \quad \text{for } v \in L^\infty(Q) \text{ and a.a. } (x, t) \in \tilde{Q} \text{ we set} \\ (\mathcal{E}v)(x, t) &= \tilde{v}(x, t) \quad \text{if } t \in (0, T), \quad (\mathcal{E}v)(x, t) = \tilde{v}(x, -t) \quad \text{if } t \in (-T, 0) \quad \text{and} \\ (\mathcal{E}v)(x, t) &= \tilde{v}(x, 2T - t) \quad \text{if } t \in (T, 2T). \end{aligned} \tag{3.5}$$

It is clear that the extension operator \mathcal{E} is linear and continuous. More precisely, we have

$$\sup_{\tilde{Q}} \mathcal{E}v = \sup_Q v \quad \text{and} \quad \inf_{\tilde{Q}} \mathcal{E}v = \inf_Q v \quad \text{for every } v \in L^\infty(Q). \tag{3.6}$$

Moreover, one can check that for every $v \in L^\infty(Q)$ we have

$$\mathcal{E}v \geq 0 \quad \text{a.e. in } \tilde{Q} \quad \text{whenever} \quad v \geq 0 \quad \text{a.e. in } Q, \tag{3.7}$$

$$\|\nabla \mathcal{E}v\|_{L^\infty(\tilde{Q})} \leq M \|\nabla v\|_{L^\infty(Q)} \quad \text{if } \nabla v \in L^\infty(Q), \tag{3.8}$$

$$\|\partial_t \mathcal{E}v\|_{L^\infty(\tilde{Q})} \leq M \|\partial_t v\|_{L^\infty(Q)} \quad \text{if } \partial_t v \in L^\infty(Q), \tag{3.9}$$

$$\text{lip}(\mathcal{E}v) \leq M \text{lip } v \quad \text{if } v \text{ is Lipschitz continuous} \tag{3.10}$$

for some constant M , where $\text{lip } v$ is the Lipschitz constant of v . At this point, for $\varepsilon \in (0, 1)$ we define the operators $\tilde{\beta}, \tilde{\beta}_\varepsilon : \tilde{Q} \times \mathbb{R} \rightarrow \mathbb{R}$ by using the extension operator \mathcal{E} and a truncation procedure as follows

$$\tilde{\beta}(x, t, r) := (\mathcal{E}\beta(\cdot, \cdot, r))(x, t), \tag{3.11}$$

$$\tilde{\beta}_\varepsilon(x, t, r) := \tilde{\beta}(x, t, r_\varepsilon) \quad \text{where } r_\varepsilon := \max\{\varepsilon, \min\{r, 1/\varepsilon\}\} \tag{3.12}$$

and observe that $\tilde{\beta}_\varepsilon$ is globally Lipschitz continuous. Indeed, by recalling (2.5) and setting for convenience

$$L_\delta := \text{lip } \beta|_{Q \times [\delta, 1/\delta]} \quad \text{for } \delta \in (0, 1) \tag{3.13}$$

we clearly see that ML_ε is a Lipschitz constant for $\tilde{\beta}_\varepsilon$. Moreover, as \mathcal{E} is linear and (3.7) holds, we infer that both $\tilde{\beta}(x, t, \cdot)$ and $\tilde{\beta}_\varepsilon(x, t, \cdot)$ are nondecreasing on \mathbb{R} for every $(x, t) \in \tilde{Q}$. Furthermore, both $\tilde{\beta}$ and $\tilde{\beta}_\varepsilon$ vanish at $r = 1$. In particular, their values at every $r \in \mathbb{R}$ have the sign of $r - 1$.

Finally, we are ready to define a C^∞ approximation $\beta_\varepsilon : Q \times \mathbb{R} \rightarrow \mathbb{R}$ of β . We regularize $\tilde{\beta}_\varepsilon$ by convolution and restrict the regularization we obtain to $Q \times \mathbb{R}$. Namely, we fix a nonnegative $\zeta \in C^\infty(\mathbb{R}^5)$ supported in the unit ball B of \mathbb{R}^5 and normalized in $L^1(\mathbb{R}^5)$. Then, by assuming $\varepsilon_0 \leq T$ and $\varepsilon \in (0, \varepsilon_0)$ (such restrictions are not stressed in the sequel, but it is understood that they are satisfied), we recall (3.13) and set

$$\delta_\varepsilon := \frac{\varepsilon}{1 + L_\varepsilon} \quad \text{and} \quad \zeta_\varepsilon(x, t, r) := \delta_\varepsilon^{-5} \zeta((x, t, r)/\delta_\varepsilon) \quad \text{for } (x, t, r) \in \mathbb{R}^5, \tag{3.14}$$

$$\begin{aligned} \beta_\varepsilon(x, t, r) &:= (\tilde{\beta}_\varepsilon * \zeta_\varepsilon)(x, t, r) \\ &= \int_{\delta_\varepsilon B} \tilde{\beta}_\varepsilon(x - y, t - \tau, r - s) \zeta_\varepsilon(y, \tau, s) dy d\tau ds \\ &= \int_B \tilde{\beta}_\varepsilon(x - \delta_\varepsilon y, t - \delta_\varepsilon \tau, r - \delta_\varepsilon s) \zeta(y, \tau, s) dy d\tau ds \quad \text{for } (x, t, r) \in Q \times \mathbb{R}. \end{aligned} \tag{3.15}$$

The reason of the above choice of δ_ε is that we would like to have

$$|\beta_\varepsilon(x, t, r) - \tilde{\beta}_\varepsilon(x, t, r)| \leq M\varepsilon \quad \text{for every } (x, t, r) \in Q \times \mathbb{R} \tag{3.16}$$

and some constant M . Actually, (3.16) holds with the constant M that makes (3.10) true, as we show at once. We have indeed

$$\begin{aligned} |\beta_\varepsilon(x, t, r) - \tilde{\beta}_\varepsilon(x, t, r)| &= \left| \int_B (\tilde{\beta}_\varepsilon(x - \delta_\varepsilon y, t - \delta_\varepsilon \tau, r - \delta_\varepsilon s) - \tilde{\beta}_\varepsilon(x, t, r)) \zeta(y, \tau, s) dy d\tau ds \right| \\ &\leq \int_B ML_\varepsilon |(\delta_\varepsilon y, \delta_\varepsilon \tau, \delta_\varepsilon s)| \zeta(y, \tau, s) dy d\tau ds \leq ML_\varepsilon \delta_\varepsilon \leq M\varepsilon \end{aligned}$$

since ML_ε is a Lipschitz constant for $\tilde{\beta}_\varepsilon$, as just observed. With a similar argument, we see that

$$\beta_\varepsilon \text{ is Lipschitz continuous with } \text{lip } \beta_\varepsilon \leq ML_\varepsilon \tag{3.17}$$

since such a property holds for $\tilde{\beta}_\varepsilon$. In the sequel we use the following more precise facts

$$\begin{aligned} \sup_{Q \times [\delta, 1/\delta]} |\beta_\varepsilon| &\leq \sup_{\tilde{Q} \times [\delta/2, 1/\delta + \delta/2]} |\tilde{\beta}| \quad \text{and} \quad \text{lip } \beta_\varepsilon|_{Q \times [\delta, 1/\delta]} \leq \text{lip } \tilde{\beta}|_{\tilde{Q} \times [\delta/2, 1/\delta + \delta/2]} \\ &\text{for } \delta \in (0, 1) \text{ and } \varepsilon < \delta/2. \end{aligned} \tag{3.18}$$

Indeed, we have $\delta_\varepsilon \leq \varepsilon \leq \delta/2$. Hence, if $(x, t) \in Q$ and $\delta \leq r \leq 1/\delta$, the values of $\tilde{\beta}_\varepsilon$ in (3.15) actually are values of $\tilde{\beta}$ at points of the set $\tilde{Q} \times [\delta/2, 1/\delta + \delta/2]$, where $\tilde{\beta}$ is bounded and Lipschitz continuous. Therefore, both the supremum and the Lipschitz constant are preserved by the convolution, since ζ is normalized, and (3.18) follow. Finally, we point out that

$$\beta_\varepsilon(x, t, \cdot) \text{ is nondecreasing on } \mathbb{R} \text{ for every } (x, t) \in Q \tag{3.19}$$

since such a property holds for $\tilde{\beta}_\varepsilon$ and ζ is nonnegative. Moreover, we set for convenience

$$\widehat{\beta}_\varepsilon(x, t, r) := \int_1^r \beta_\varepsilon(x, t, s) \, ds \quad \text{for a.a. } (x, t) \in Q \text{ and every } r \in \mathbb{R}. \tag{3.20}$$

Clearly, $\widehat{\beta}_\varepsilon$ is convex with respect to the third variable. Furthermore, as $\tilde{\beta}_\varepsilon(x, t, r)$ and $r - 1$ have the same sign as just observed, we see that (3.16) implies

$$\widehat{\beta}_\varepsilon(x, t, r) \geq \int_1^r (\beta_\varepsilon(x, t, s) - \tilde{\beta}_\varepsilon(x, t, s)) \, ds \geq -M\varepsilon|r - 1| \quad \text{for every } (x, t, r) \in Q \times \mathbb{R}. \tag{3.21}$$

Finally, it is clear that the first of (3.18) implies the analogue for $\widehat{\beta}_\varepsilon$, namely

$$\sup_{Q \times [\delta, 1/\delta]} |\widehat{\beta}_\varepsilon| \leq c_\delta \quad \text{for every } \delta \in (0, 1) \tag{3.22}$$

for some constant c_δ .

At this point, we are ready to state the approximating problem, which consists in finding a triplet $(\vartheta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ having the proper regularity and satisfying

$$\partial_t(\vartheta_\varepsilon - G(\chi_\varepsilon)) - \Delta \text{Ln}_\varepsilon \vartheta_\varepsilon + \xi_\varepsilon = \pi(\vartheta_\varepsilon) \quad \text{in } L^2(0, T; V)_0^* \quad \text{and} \quad \xi_\varepsilon = \beta_\varepsilon(\vartheta_\varepsilon), \tag{3.23}$$

$$\partial_t \chi_\varepsilon - \Delta \chi_\varepsilon + F'(\chi_\varepsilon) + G'(\chi_\varepsilon)\vartheta_\varepsilon = 0 \quad \text{a.e. in } Q, \tag{3.24}$$

$$\vartheta_\varepsilon = \vartheta_\Gamma \quad \text{a.e. on } \Gamma \times (0, T), \tag{3.25}$$

$$\vartheta_\varepsilon(0) = \vartheta_0 \quad \text{and} \quad \chi_\varepsilon(0) = \chi_0. \tag{3.26}$$

The following result holds.

Theorem 3.1. *Let the assumptions of Theorem 2.4 be fulfilled. Moreover, assume (3.1), (3.2), and (3.15). Then, problem (3.23)–(3.26) has a solution satisfying*

$$\vartheta_\varepsilon \in L^2(0, T; V) \cap H^1(0, T; V)_0^*, \tag{3.27}$$

$$\chi_\varepsilon \in L^2(0, T; W) \cap H^1(0, T; H). \tag{3.28}$$

We avoid proving Theorem 3.1 in order not to make the paper too long. Indeed, the a priori estimates in Section 4 suggest how to proceed. Anyway, a rigorous proof could be done by regularizing π and using, e.g., a Galerkin procedure.

On the contrary, we prove some auxiliary results regarding the approximating nonlinearities. The first formula we state is showed in [6, Lemma 6.1]. We repeat the short proof here, for convenience, and include it in the following proposition.

Proposition 3.2. *We have*

$$\ln_\varepsilon^{-1}(s) = e^s + \varepsilon s \quad \text{for every } s \in \mathbb{R}, \tag{3.29}$$

$$\frac{\ln r}{1 + \varepsilon} \leq \ln_\varepsilon r \leq \ln r \quad \text{for every } r \geq 1, \tag{3.30}$$

$$\text{Ln}'_\varepsilon(r) \geq 1 \quad \text{for every } r \leq 1 \quad \text{and} \quad \text{Ln}'_\varepsilon(r) \geq \frac{1}{2r} \quad \text{for every } r \geq 1, \tag{3.31}$$

$$\ell_* \leq \ln_\varepsilon r \leq \ell^* \quad \text{for every } r \in [\vartheta_*, \vartheta^*] \tag{3.32}$$

where we have set $\ell_* := \min\{0, \ln \vartheta_*\}$ and $\ell^* := \max\{0, \ln \vartheta^*\}$. Moreover, we have

$$\frac{1}{2\vartheta^*} \leq \text{Ln}'_\varepsilon(r) \leq \frac{2}{\vartheta_*} \quad \text{for every } r \in [\vartheta_*, \vartheta^*] \tag{3.33}$$

for ε small enough.

Proof. For $r \in \mathbb{R}$, let $\rho_\varepsilon(r) > 0$ be defined by the equation

$$\rho_\varepsilon(r) + \varepsilon \ln \rho_\varepsilon(r) = r. \tag{3.34}$$

Take now any $s \in \mathbb{R}$. Then, $r := e^s + \varepsilon s$ satisfies $\rho_\varepsilon(r) = e^s$. On the other hand, we have

$$\ln_\varepsilon r = \frac{r - \rho_\varepsilon(r)}{\varepsilon} \tag{3.35}$$

by definition of Yosida regularization. We deduce that $\ln_\varepsilon r = s$ and (3.29) follows. To prove (3.30), we observe that for $s \geq 0$ we have $e^{\varepsilon s} \geq 1 + \varepsilon s$ and $e^s \geq 1$. We infer that

$$e^{(1+\varepsilon)s} \geq e^s + \varepsilon s \geq e^s.$$

If $r \geq 1$, applying this to $s := \ln_\varepsilon r$ (which is nonnegative since $\ln_\varepsilon 1 = 0$), we obtain

$$e^{(1+\varepsilon)\ln_\varepsilon r} \geq e^{\ln_\varepsilon r} + \varepsilon \ln_\varepsilon r \geq e^{\ln_\varepsilon r}$$

and (3.30) follows from (3.29) by applying the logarithm. Next, we prove (3.31). To this aim, we compute $\text{Ln}'_\varepsilon(r)$ from (3.35) and (3.34). We have

$$\text{Ln}'_\varepsilon(r) = \frac{1 - \rho'_\varepsilon(r)}{\varepsilon} = \frac{1}{\varepsilon} \left(1 - \frac{\rho_\varepsilon(r)}{\rho_\varepsilon(r) + \varepsilon} \right) = \frac{1}{\rho_\varepsilon(r) + \varepsilon}. \tag{3.36}$$

On the other hand, we observe that (3.34) and $\rho_\varepsilon(r) > 1$ imply $r > \rho_\varepsilon(r) \geq 1$. Hence, $\rho_\varepsilon(r) \leq 1$ for $r \leq 1$. We conclude that

$$\text{Ln}'_\varepsilon(r) = \varepsilon + \text{Ln}'_\varepsilon(r) \geq \varepsilon + \frac{1}{1 + \varepsilon} = \frac{1 + \varepsilon + \varepsilon^2}{1 + \varepsilon} \geq 1$$

for every $r \leq 1$. Assume now $r \geq 1$. Then, $r + \varepsilon \ln r \geq r$, whence $\rho_\varepsilon(r) \leq r$. Accounting for (3.36), we infer that

$$\text{Ln}'_\varepsilon(r) \geq \frac{1}{r + \varepsilon} \geq \frac{1}{2r}$$

and the second of (3.31) follows. To prove (3.32), we observe that $\exp \ell_* + \varepsilon \ell_* \leq \vartheta_*$ and $\exp \ell^* + \varepsilon \ell^* \geq \vartheta^*$. We deduce that $\exp \ell_* + \varepsilon \ell_* \leq r \leq \exp \ell^* + \varepsilon \ell^*$ for every $r \in [\vartheta_*, \vartheta^*]$, and (3.32) follows from (3.29). Finally, we prove (3.33). Owing to (3.36) and to the monotonicity of ρ_ε , we see that $r \in [\vartheta_*, \vartheta^*]$ implies that

$$\frac{1}{\rho_\varepsilon(\vartheta^*) + \varepsilon} \leq \ln'_\varepsilon(r) \leq \frac{1}{\rho_\varepsilon(\vartheta_*) + \varepsilon}.$$

On the other hand, it is clear that $\rho_\varepsilon(r')$ tends to r' as $\varepsilon \rightarrow 0$ for every $r' > 0$ (see also, e.g., [11, Proposition 2.6, p. 28]). Then, (3.33) immediately follows if ε is small enough. \square

Next, we deal with the analogue of (2.9) for β_ε . At the same time, we prove an inequality involving $\widehat{\beta}_\varepsilon$. A notation like (2.7) is extended to such functions.

Proposition 3.3. *We have*

$$|\beta_{\varepsilon,x}(x, t, r)| + |\beta_{\varepsilon,t}(x, t, r)| \leq c(1 + |\beta_\varepsilon(x, t, r)|) \quad \text{and} \quad |\widehat{\beta}_{\varepsilon,t}(x, t, s)| \leq c(\widehat{\beta}_\varepsilon(r) + |r| + 1) \quad (3.37)$$

for every $(x, t, r) \in Q \times \mathbb{R}$, some constant c , and ε small enough.

Proof. As far as the first inequality is concerned, we deal, e.g., with $\beta_{\varepsilon,t}$, since the argument for the space derivatives is quite similar. We first prove that

$$|\partial_t \widetilde{\beta}_\varepsilon(x, t, r)| \leq MM_\beta(1 + |\widetilde{\beta}_\varepsilon(x, t, r)|) \quad (3.38)$$

for every $(x, t) \in \widetilde{Q}$ and $r \in \mathbb{R}$, where M and M_β are the constants satisfying (3.8)–(3.10) and (2.9), respectively. Assume first $r \in [\varepsilon, 1/\varepsilon]$. Then, (3.38) coincides with the analogue for $\widetilde{\beta}$ due to (3.12), and this easily follows from (3.9), (3.11), and (2.9). Assume now $r < \varepsilon$. Then, we have

$$|\partial_t \widetilde{\beta}_\varepsilon(x, t, r)| = |\partial_t \widetilde{\beta}(x, t, \varepsilon)| \leq MM_\beta(1 + |\widetilde{\beta}(x, t, \varepsilon)|) = MM_\beta(1 + |\widetilde{\beta}_\varepsilon(x, t, r)|)$$

just using the above with $r = \varepsilon$. As the argument is similar if $r > 1/\varepsilon$, (3.38) is established. In order to prove the first of (3.37), we notice that

$$\pm \beta_{\varepsilon,t} = \pm \partial_t (\widetilde{\beta}_\varepsilon * \zeta_\varepsilon) = \pm (\partial_t \widetilde{\beta}_\varepsilon) * \zeta_\varepsilon \leq c(1 + |\widetilde{\beta}_\varepsilon|) * \zeta_\varepsilon = c + c|\widetilde{\beta}_\varepsilon| * \zeta_\varepsilon$$

with $c := MM_\beta$, since (3.38) holds and the convolution with the nonnegative normalized kernel ζ_ε preserves order and constants. Therefore, we have to bound the last convolution with the right-hand side of the inequality we want to prove. Assume first $(x, t) \in Q$ and $r \geq 1 + \varepsilon$. Then, $\widetilde{\beta}_\varepsilon(y, \tau, s) \geq 0$ for $(y, \tau) \in \widetilde{Q}$ and $|s - r| < \delta_\varepsilon$ (since $\delta_\varepsilon \leq \varepsilon$), and we have

$$(|\widetilde{\beta}_\varepsilon| * \zeta_\varepsilon)(x, t, r) = (\widetilde{\beta}_\varepsilon * \zeta_\varepsilon)(x, t, r) = \beta_\varepsilon(x, t, r) = |\beta_\varepsilon(x, t, r)|.$$

If $r < 1 - \varepsilon$ the argument is similar. Finally, if $|r - 1| \leq \varepsilon$, by assuming $\varepsilon \leq 1/4$, we have $\delta_\varepsilon \leq 1/4$, $r - \delta_\varepsilon \geq \max\{\varepsilon, 1/2\}$, and $r + \delta_\varepsilon \leq \min\{1/\varepsilon, 3/2\}$, whence

$$|(|\widetilde{\beta}_\varepsilon| * \zeta_\varepsilon)(x, t, r)| = |(\widetilde{\beta}_\varepsilon * \zeta_\varepsilon)(x, t, r)| \leq \sup_{\widetilde{Q} \times [r - \delta_\varepsilon, r + \delta_\varepsilon]} |\widetilde{\beta}_\varepsilon| \leq \sup_{\widetilde{Q} \times [1/2, 3/2]} |\widetilde{\beta}_\varepsilon|$$

since ζ_ε is normalized in $L^1(\mathbb{R}^5)$, and the first of (3.37) clearly follows. Then, we easily derive the second inequality. As β_ε is monotone with respect to the third variable, we have

$$\begin{aligned}
 |\widehat{\beta}_{\varepsilon,t}(x, t, r)| &= \left| \int_1^r \beta_{\varepsilon,t}(x, t, s) ds \right| \leq c \left| \int_1^r (1 + |\beta_{\varepsilon}(x, t, s)|) ds \right| \\
 &\leq c|r - 1| + c \int_1^r (\beta_{\varepsilon}(x, t, s) - \beta_{\varepsilon}(x, t, 1)) ds + c \left| \int_1^r |\beta_{\varepsilon}(x, t, 1)| ds \right| \\
 &\leq c(\widehat{\beta}_{\varepsilon}(x, t, r) + |r| + 1)
 \end{aligned}$$

since $\beta_{\varepsilon}(x, t, 1)$ is bounded due to (3.18) with $\delta = 1/2$. \square

The last preliminary remarks we make regard relationships between the approximating nonlinearities and the boundary datum ϑ_{Γ} . In the next section, we need to consider a known smooth function that coincides with ϑ_{Γ} on the boundary. Thus, a natural choice is the harmonic extension $\vartheta_{\mathcal{H}}$ of ϑ_{Γ} . Precisely, we define $\vartheta_{\mathcal{H}} : Q \rightarrow \mathbb{R}$ by the conditions

$$\vartheta_{\mathcal{H}}(t) \in V, \quad \vartheta_{\mathcal{H}}(t)|_{\Gamma} = \vartheta_{\Gamma}(t), \quad \text{and} \quad \Delta \vartheta_{\mathcal{H}}(t) = 0 \quad \text{in } \Omega, \quad \text{for a.a. } t \in (0, T). \tag{3.39}$$

Then, assumption (2.20) and the general theory of harmonic functions (in particular, the maximum principle) ensure that $\vartheta_{\mathcal{H}} \in L^2(0, T; V) \cap H^1(0, T; H)$ and that the following estimates hold

$$\begin{aligned}
 \|\vartheta_{\mathcal{H}}\|_{L^2(0,T;V) \cap H^1(0,T;H)} &\leq c \|\vartheta_{\Gamma}\|_{L^2(0,T;H^{1/2}(\Gamma)) \cap H^1(0,T;H^{-1/2}(\Gamma))} \quad \text{and} \\
 \vartheta_* &\leq \vartheta_{\mathcal{H}} \leq \vartheta^* \quad \text{a.e. in } Q.
 \end{aligned} \tag{3.40}$$

Proposition 3.4. *We have*

$$\|\text{Ln}_{\varepsilon} \vartheta_{\mathcal{H}}\|_{L^{\infty}(Q) \cap L^2(0,T;V) \cap H^1(0,T;H)} \leq c, \tag{3.41}$$

$$\|\beta_{\varepsilon}(\vartheta_{\mathcal{H}})\|_{L^{\infty}(Q) \cap L^2(0,T;V) \cap H^1(0,T;H)} \leq c \tag{3.42}$$

for some constant c and ε small enough.

Proof. Estimate (3.40) and (3.32) imply that $\ell_* \leq \text{Ln}_{\varepsilon} \vartheta_{\mathcal{H}} \leq \ell^*$ a.e. in Q , where the notation of Proposition 3.2 has been used. Owing to (3.33) as well, (3.41) immediately follows. To prove the L^{∞} -bound of (3.42), it suffices to recall (2.2), (2.13), and the first of (3.18). Finally, we prove the estimate regarding the time derivative (the argument for space derivatives is similar). We have

$$\begin{aligned}
 \|\partial_t \beta_{\varepsilon}(\vartheta_{\mathcal{H}})\|_{L^2(Q)} &\leq \|\beta_{\varepsilon,t}(\vartheta_{\mathcal{H}})\|_{L^2(Q)} + \|\beta'_{\varepsilon}(\vartheta_{\mathcal{H}}) \partial_t \vartheta_{\mathcal{H}}\|_{L^2(Q)} \\
 &\leq c \|\beta_{\varepsilon,t}(\vartheta_{\mathcal{H}})\|_{L^{\infty}(Q)} + \sup_{Q \times [\vartheta_*, \vartheta^*]} |\beta'_{\varepsilon}| \|\partial_t \vartheta_{\mathcal{H}}\|_{L^2(Q)} \leq c
 \end{aligned}$$

due to (3.37), the L^{∞} -estimate just proved, the second of (3.18), and (3.40), provided that ε is small enough. \square

4. Existence

In this section, we prove Theorem 2.4. More precisely, we first check the last part of the statement, i.e., we show that every solution to problem (2.28)–(2.31) satisfies the bounds (2.32). Then, we consider the approximating problem (3.23)–(3.26) taking the further assumption (3.2) into account, and perform a number of a priori estimates on its solution $(\vartheta_{\varepsilon}, \chi_{\varepsilon}, \xi_{\varepsilon})$. Finally, we let ε tend to zero by using monotonicity and compactness methods. This leads to the proof of Theorem 2.4 under the

additional condition (3.2). However, it is clear that such a procedure proves Theorem 2.4 in the general case. Indeed, the triplet (ϑ, χ, ξ) we find solves anyone of the problems obtained by replacing the original functions F and G by other ones that coincides with the given F and G on $[\chi_*, \chi^*]$, due to (2.32).

Hence, we assume (3.2) throughout the present section. As such a condition is added to our assumptions, we allow the values of the different constants c to depend even on the Lipschitz constants of F and G , following the general rule explained at the end of Section 2. Moreover, it is understood that ε belongs to $(0, 1)$ and is even smaller as in the previous section (see, e.g., Propositions 3.3 and 3.4). Finally, δ is a positive parameter, say $\delta \in (0, 1)$, whose value is chosen according to our convenience.

4.1. The maximum principle argument

We prove that every solution (ϑ, χ, ξ) to problem (2.28)–(2.31) satisfies (2.32). Fix a Lipschitz continuous function $H : \mathbb{R} \rightarrow \mathbb{R}$ of class C^1 such that

$$H(r) = 0 \text{ if } r \in [\chi_*, \chi^*] \text{ and } H'(r) > 0 \text{ if } r \notin [\chi_*, \chi^*]. \tag{4.1}$$

As $\chi \in L^2(0, T; V)$, it turns out that $v := H(\chi)$ is an admissible test function for (2.29). Then, we multiply (2.29) by $H(\chi)$ and integrate over Q_t , where $t \in (0, T]$ is arbitrary. After integrating by parts, we obtain

$$\int_{Q_t} \partial_t \chi H(\chi) + \int_{Q_t} \nabla \chi \cdot \nabla H(\chi) + \int_{Q_t} (F'(\chi) + G'(\chi)\vartheta)H(\chi) = 0. \tag{4.2}$$

If \widehat{H} denotes the primitive of H vanishing at χ_* , we have

$$\int_{Q_t} \partial_t \chi H(\chi) = \int_{\Omega} \widehat{H}(\chi(t)) - \int_{\Omega} \widehat{H}(\chi_0) = \int_{\Omega} \widehat{H}(\chi(t)) \geq 0$$

due to (2.22) and (4.1). The next integral of (4.2) is nonnegative, obviously, and the last one has the same property. Indeed, its integrand is nonnegative since $\vartheta > 0$ and the three functions F' , G' , and H have the same sign, due to (2.4) and (4.1). Therefore, all the integrals of (4.2) vanish identically. In particular, we deduce that $\widehat{H}(\chi(t)) = 0$ a.e. in Ω , for every $t \in [0, T]$, and this clearly implies (2.32).

Now, we start proving estimates on the solution $(\vartheta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ to problem (3.23)–(3.26), as mentioned at the beginning of the present section. When we shortly say that we test an equation by some function v , we mean that we test such an equation at time s by $v(s)$, we integrate first over Ω and then over $(0, t)$ with respect to s , where $t \in (0, T)$ is arbitrary, and we integrate by parts, if necessary.

4.2. First a priori estimate

We test (3.23) by $v := \vartheta_\varepsilon - \vartheta_{\mathcal{H}} + \delta(\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})$. At the same time, we test (3.24) by $\partial_t \chi_\varepsilon$. Then, we sum the equalities we get to each other and note that two terms involving G cancel. After rearranging a little and adding the same integral to both sides for convenience, we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\vartheta_\varepsilon(t) - \vartheta_{\mathcal{H}}(t)|^2 + \int_{Q_t} \partial_t G(\chi_\varepsilon) \vartheta_{\mathcal{H}} + \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_\varepsilon \cdot \nabla \vartheta_\varepsilon \\ & + \int_{Q_t} (\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}}))(\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) \end{aligned}$$

$$\begin{aligned}
 & + \delta \int_{Q_t} \partial_t \vartheta_\varepsilon \operatorname{Ln}_\varepsilon \vartheta_\varepsilon + \delta \int_{Q_t} |\nabla(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon - \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 \\
 & + \delta \int_{Q_t} (\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}}))(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon - \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}}) \\
 & + \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_\varepsilon(t)|^2 + \frac{1}{2} \int_\Omega |\chi_\varepsilon(t)|^2 \\
 = & \frac{1}{2} \int_\Omega |\vartheta_0 - \vartheta_{\mathcal{H}}(0)|^2 - \int_{Q_t} \partial_t \vartheta_{\mathcal{H}}(\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) + \int_{Q_t} \nabla \operatorname{Ln}_\varepsilon \vartheta_\varepsilon \cdot \nabla \vartheta_{\mathcal{H}} \\
 & - \int_{Q_t} \beta_\varepsilon(\vartheta_{\mathcal{H}})(\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) + \int_{Q_t} \pi(\vartheta_\varepsilon)(\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) \\
 & + \delta \int_{Q_t} \partial_t \vartheta_\varepsilon \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}} + \delta \int_{Q_t} G'(\chi_\varepsilon) \partial_t \chi_\varepsilon (\operatorname{Ln}_\varepsilon \vartheta_\varepsilon - \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}}) \\
 & - \delta \int_{Q_t} \nabla \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}} \cdot \nabla (\operatorname{Ln}_\varepsilon \vartheta_\varepsilon - \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}}) - \delta \int_{Q_t} \beta_\varepsilon(\vartheta_{\mathcal{H}})(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon - \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}}) \\
 & + \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 + \delta \int_{Q_t} \pi(\vartheta_\varepsilon)(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon - \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}}) - \int_{Q_t} F'(\chi_\varepsilon) \partial_t \chi_\varepsilon + \frac{1}{2} \int_\Omega |\chi_\varepsilon(t)|^2. \tag{4.3}
 \end{aligned}$$

Now, we observe with the help of monotonicity that all terms on the left-hand side are nonnegative but two of them. We deal with the first one that needs some treatment. Using (2.3), (3.40), and (2.20)–(2.22), we have

$$\begin{aligned}
 \int_{Q_t} \partial_t G(\chi_\varepsilon) \vartheta_{\mathcal{H}} & = \int_\Omega G(\chi_\varepsilon(t)) \vartheta_{\mathcal{H}}(t) - \int_\Omega G(\chi_0) \vartheta_{\mathcal{H}}(0) - \int_{Q_t} G(\chi_\varepsilon) \partial_t \vartheta_{\mathcal{H}} \\
 & \geq \vartheta_* \int_\Omega G(\chi_\varepsilon(t)) - c - \frac{1}{2} \int_{Q_t} |G(\chi_\varepsilon)|^2 - \frac{1}{2} \int_{Q_t} |\partial_t \vartheta_{\mathcal{H}}|^2 \geq \vartheta_* \int_\Omega G(\chi_\varepsilon(t)) - c \int_{Q_t} |\chi_\varepsilon|^2 - c.
 \end{aligned}$$

The second one is the following

$$\delta \int_{Q_t} \partial_t \vartheta_\varepsilon \operatorname{Ln}_\varepsilon \vartheta_\varepsilon = \delta \int_\Omega \mathcal{L}_\varepsilon(\vartheta_\varepsilon(t)) - \delta \int_\Omega \mathcal{L}_\varepsilon(\vartheta_0) \geq \delta \int_\Omega \mathcal{L}_\varepsilon(\vartheta_\varepsilon(t)) - c$$

where we have set

$$\mathcal{L}_\varepsilon(r) := \int_1^r \operatorname{Ln}_\varepsilon s \, ds = \frac{\varepsilon}{2}(r^2 - 1) + \int_1^r \ln_\varepsilon s \, ds \quad \text{for } r \in \mathbb{R}. \tag{4.4}$$

Note that \mathcal{L}_ε is convex and bounded from below uniformly with respect to ε since $\ln_\varepsilon 1 = 0$. Now, we consider the right-hand side and deal with the non-trivial terms of it. We have (cf. (2.20) and (3.40))

$$- \int_{Q_t} \partial_t \vartheta_{\mathcal{H}}(\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) \leq \frac{1}{4} \int_{Q_t} |\partial_t \vartheta_{\mathcal{H}}|^2 + \int_{Q_t} |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}|^2 \leq \int_{Q_t} |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}|^2 + c.$$

Next, we consider (cf. (3.40)–(3.41))

$$\begin{aligned} \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_\varepsilon \cdot \nabla \vartheta_{\mathcal{H}} &= \int_{Q_t} \nabla (\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}) \cdot \nabla \vartheta_{\mathcal{H}} + \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_{\mathcal{H}} \cdot \nabla \vartheta_{\mathcal{H}} \\ &\leq \frac{\delta}{8} \int_{Q_t} |\nabla (\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 + \frac{2}{\delta} \int_{Q_t} |\nabla \vartheta_{\mathcal{H}}|^2 + c \\ &\leq \frac{\delta}{8} \int_{Q_t} |\nabla (\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 + c_\delta. \end{aligned}$$

Moreover, as $|\vartheta_\varepsilon| \leq |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}| + |\vartheta_{\mathcal{H}}|$, by (3.42), (2.11), and (3.40) it is easily seen that

$$\begin{aligned} - \int_{Q_t} \beta_\varepsilon(\vartheta_{\mathcal{H}})(\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) + \int_{Q_t} \pi(\vartheta_\varepsilon)(\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) &\leq c \int_{Q_t} (1 + |\vartheta_\varepsilon| + |\pi_0|) |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}| \\ &\leq c \int_{Q_t} |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}|^2 + c. \end{aligned}$$

We deal with the next integral of (4.3) as follows (cf. (3.41))

$$\begin{aligned} \delta \int_{Q_t} \partial_t \vartheta_\varepsilon \text{Ln}_\varepsilon \vartheta_{\mathcal{H}} &= \delta \int_{Q_t} \partial_t (\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) \text{Ln}_\varepsilon \vartheta_{\mathcal{H}} + \delta \int_{Q_t} \partial_t \vartheta_{\mathcal{H}} \text{Ln}_\varepsilon \vartheta_{\mathcal{H}} \\ &= \delta \int_{\Omega} (\vartheta_\varepsilon(t) - \vartheta_{\mathcal{H}}(t)) \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}(t) - \delta \int_{\Omega} (\vartheta_0 - \vartheta_{\mathcal{H}}(0)) \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}(0) \\ &\quad - \delta \int_{Q_t} (\vartheta_\varepsilon - \vartheta_{\mathcal{H}}) \partial_t \text{Ln}_\varepsilon \vartheta_{\mathcal{H}} + \delta \int_{Q_t} \partial_t \vartheta_{\mathcal{H}} \text{Ln}_\varepsilon \vartheta_{\mathcal{H}} \\ &\leq \frac{\delta}{2} \int_{\Omega} |\vartheta_\varepsilon(t) - \vartheta_{\mathcal{H}}(t)|^2 + \int_{Q_t} |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}|^2 + c_\delta. \end{aligned}$$

Subsequently, by the Poincaré inequality (2.39) we have

$$\begin{aligned} \delta \int_{Q_t} G'(\chi_\varepsilon) \partial_t \chi_\varepsilon (\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}) &\leq \frac{\delta}{8M_\Omega^2} \int_{Q_t} |\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}|^2 + 2\delta M_\Omega^2 \int_{Q_t} |G'(\chi_\varepsilon) \partial_t \chi_\varepsilon|^2 \\ &\leq \frac{\delta}{8} \int_{Q_t} |\nabla (\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 + 2\delta M_\Omega^2 \sup |G'|^2 \int_{Q_t} |\partial_t \chi_\varepsilon|^2. \end{aligned}$$

The next integral is easy. We have indeed (by (3.41))

$$-\delta \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_{\mathcal{H}} \cdot \nabla (\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}) \leq \frac{\delta}{8} \int_{Q_t} |\nabla (\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 + c_\delta$$

while the other two terms involving Ln_ε are estimated owing to the Poincaré inequality once more this way (cf. (3.42) and (2.11))

$$\begin{aligned}
 & -\delta \int_{Q_t} \beta_\varepsilon(\vartheta_{\mathcal{H}})(\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}) + \delta \int_{Q_t} \pi(\vartheta_\varepsilon)(\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}) \\
 & \leq \frac{\delta}{8M_\Omega^2} \int_{Q_t} |\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}|^2 + 2\delta M_\Omega^2 \int_{Q_t} |\pi(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}})|^2 \\
 & \leq \frac{\delta}{8} \int_{Q_t} |\nabla(\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 + c \int_{Q_t} |\pi(\vartheta_\varepsilon)|^2 + c \int_{Q_t} |\beta_\varepsilon(\vartheta_{\mathcal{H}})|^2 \\
 & \leq \frac{\delta}{8} \int_{Q_t} |\nabla(\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 + c \int_{Q_t} |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}|^2 + c.
 \end{aligned}$$

Next, we treat the second last integral of (4.3). We have

$$- \int_{Q_t} F'(\chi_\varepsilon) \partial_t \chi_\varepsilon \leq \frac{1}{4} \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + \int_{Q_t} |F'(\chi_\varepsilon)|^2 \leq \frac{1}{4} \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + c \int_{Q_t} |\chi_\varepsilon|^2 + c.$$

Finally, we deal with the last term as follows

$$\frac{1}{2} \int_\Omega |\chi_\varepsilon(t)|^2 = \frac{1}{2} \int_\Omega |\chi_0|^2 + \int_{Q_t} \chi_\varepsilon \partial_t \chi_\varepsilon \leq \frac{1}{4} \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + \int_{Q_t} |\chi_\varepsilon|^2 + |\Omega|t(\sup |F'|)^2.$$

Hence, we can recall (4.3) and all the estimates we have derived. Forgetting some nonnegative terms on the left-hand side we have

$$\begin{aligned}
 & \frac{1-\delta}{2} \int_\Omega |\vartheta_\varepsilon(t) - \vartheta_{\mathcal{H}}(t)|^2 + \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_\varepsilon \cdot \nabla \vartheta_\varepsilon + \frac{\delta}{2} \int_{Q_t} |\nabla(\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}})|^2 \\
 & + \left(\frac{1}{2} - 2\delta M_\Omega^2 \sup |G'|^2 \right) \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_\varepsilon(t)|^2 + \frac{1}{2} \int_\Omega |\chi_\varepsilon(t)|^2 \\
 & \leq c \int_{Q_t} |\vartheta_\varepsilon - \vartheta_{\mathcal{H}}|^2 + \int_{Q_t} |\chi_\varepsilon|^2 + c_\delta.
 \end{aligned}$$

Therefore, we can choose δ small enough and apply the Gronwall lemma. By the Poincaré inequality, we conclude that

$$\begin{aligned}
 & \|\vartheta_\varepsilon - \vartheta_{\mathcal{H}}\|_{L^\infty(0,T;H)} + \|(\text{Ln}'_\varepsilon(\vartheta_\varepsilon))^{1/2} \nabla \vartheta_\varepsilon\|_{L^2(0,T;H)} + \|\text{Ln}_\varepsilon \vartheta_\varepsilon - \text{Ln}_\varepsilon \vartheta_{\mathcal{H}}\|_{L^2(0,T;V)} \\
 & + \|\partial_t \chi_\varepsilon\|_{L^2(0,T;H)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V)} \leq c.
 \end{aligned}$$

Then, thanks to (3.40) and (3.41) we obtain the basic a priori estimate

$$\begin{aligned}
 & \|\vartheta_\varepsilon\|_{L^\infty(0,T;H)} + \|(\text{Ln}'_\varepsilon(\vartheta_\varepsilon))^{1/2} \nabla \vartheta_\varepsilon\|_{L^2(0,T;H)} \\
 & + \|\text{Ln}_\varepsilon \vartheta_\varepsilon\|_{L^2(0,T;V)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V) \cap H^1(0,T;H)} \leq c.
 \end{aligned} \tag{4.5}$$

4.3. First consequence

As all the terms of (3.24) but the Laplacian are bounded in $L^2(0, T; H)$ due to (4.5) and (3.2), we immediately derive that the Laplacian is bounded as well, whence

$$\|\chi_\varepsilon\|_{L^2(0, T; W)} \leq c \tag{4.6}$$

by the homogeneous Neumann boundary condition satisfied by χ_ε and the general theory of elliptic equations.

4.4. Second consequence

We set $Q_\varepsilon^+ := \{(x, t) \in Q : \vartheta_\varepsilon(x, t) \geq 1\}$ and $Q_\varepsilon^- := \{(x, t) \in Q : \vartheta_\varepsilon(x, t) \leq 1\}$. Then, inequalities (3.31) yield

$$\begin{aligned} \int_Q \text{Ln}'_\varepsilon(\vartheta_\varepsilon) |\nabla \vartheta_\varepsilon|^2 &\geq \int_{Q_\varepsilon^-} \text{Ln}'_\varepsilon(\vartheta_\varepsilon) |\nabla \vartheta_\varepsilon|^2 \geq \int_{Q_\varepsilon^-} |\nabla \vartheta_\varepsilon|^2, \\ \int_Q \text{Ln}'_\varepsilon(\vartheta_\varepsilon) |\nabla \vartheta_\varepsilon|^2 &\geq \int_{Q_\varepsilon^+} \text{Ln}'_\varepsilon(\vartheta_\varepsilon) |\nabla \vartheta_\varepsilon|^2 \geq \frac{1}{2} \int_{Q_\varepsilon^+} \frac{|\nabla \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \end{aligned}$$

whence (4.5) implies that

$$\int_{Q_\varepsilon^-} |\nabla \vartheta_\varepsilon|^2 \leq c \quad \text{and} \quad \int_{Q_\varepsilon^+} |\nabla \vartheta_\varepsilon^{1/2}|^2 \leq c. \tag{4.7}$$

Now, we set $\Omega_\varepsilon^-(t) := \{x \in \Omega : \vartheta_\varepsilon(t) \leq 1\}$ and $\Omega_\varepsilon^+(t) := \{x \in \Omega : \vartheta_\varepsilon(t) \geq 1\}$ for a.a. $t \in (0, T)$ and use (4.7) in order to estimate $\nabla \vartheta_\varepsilon$ in a suitable norm. Accounting for the first (4.7), we see that

$$\int_0^T \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega_\varepsilon^-(t))}^2 dt \leq c \int_0^T \|\nabla \vartheta_\varepsilon(t)\|_{L^2(\Omega_\varepsilon^-(t))}^2 dt = c \int_{Q_\varepsilon^-} |\nabla \vartheta_\varepsilon|^2 \leq c. \tag{4.8}$$

On the other hand, we have $\nabla \vartheta_\varepsilon = 2\vartheta_\varepsilon^{1/2} \nabla \vartheta_\varepsilon^{1/2}$ a.e. in Q_ε^+ . Therefore, using the Hölder inequality, we obtain for a.a. $t \in (0, T)$

$$\begin{aligned} \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega_\varepsilon^+(t))} &\leq 2\|\vartheta_\varepsilon^{1/2}(t)\|_{L^4(\Omega_\varepsilon^+(t))} \|\nabla \vartheta_\varepsilon^{1/2}(t)\|_{L^2(\Omega_\varepsilon^+(t))} \\ &\leq 2\|\vartheta_\varepsilon(t)\|_{L^2(\Omega)}^{1/2} \|\nabla \vartheta_\varepsilon^{1/2}(t)\|_{L^2(\Omega_\varepsilon^+(t))} \end{aligned}$$

and squaring, integrating over $(0, T)$, and owing to (4.5) and to the second (4.7), we derive that

$$\int_0^T \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega_\varepsilon^+(t))}^2 dt \leq 4\|\vartheta_\varepsilon\|_{L^\infty(0, T; H)} \int_{Q_\varepsilon^+} |\nabla \vartheta_\varepsilon^{1/2}|^2 \leq c. \tag{4.9}$$

Finally, we observe that for a.a. $t \in (0, T)$ we have

$$\begin{aligned} \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega)}^{4/3} &= \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega_\varepsilon^-(t))}^{4/3} + \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega_\varepsilon^+(t))}^{4/3} \quad \text{whence} \\ \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega)}^2 &\leq c(\|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega_\varepsilon^-(t))}^2 + \|\nabla \vartheta_\varepsilon(t)\|_{L^{4/3}(\Omega_\varepsilon^+(t))}^2) \end{aligned}$$

so that, using (4.8) and (4.9), we conclude that

$$\|\nabla \vartheta_\varepsilon\|_{L^2(0,T;L^{4/3}(\Omega))} \leq c. \tag{4.10}$$

4.5. Second a priori estimate

We observe that

$$(\partial_t \vartheta_\varepsilon)\beta_\varepsilon(\vartheta_\varepsilon) = \partial_t \widehat{\beta}_\varepsilon(\vartheta_\varepsilon) - \widehat{\beta}_{\varepsilon,t}(\vartheta_\varepsilon) \quad \text{and} \quad \nabla \beta_\varepsilon(\vartheta_\varepsilon) = \beta_{\varepsilon,x}(\vartheta_\varepsilon) + \beta'_\varepsilon(\vartheta_\varepsilon)\nabla \vartheta_\varepsilon.$$

Therefore, if we test (3.23) by $\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}})$, rearrange, and add the same quantity to both sides in order to take advantage of (3.21), we obtain

$$\begin{aligned} &\int_{\Omega} (\widehat{\beta}_\varepsilon(t, \vartheta_\varepsilon(t)) + M\varepsilon|\vartheta_\varepsilon(t) - 1|) + \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_\varepsilon \cdot \beta'_\varepsilon(\vartheta_\varepsilon)\nabla \vartheta_\varepsilon + \int_{Q_t} (\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}}))^2 \\ &= \int_{\Omega} \widehat{\beta}_\varepsilon(0, \vartheta_0) + \int_{Q_t} \widehat{\beta}_{\varepsilon,t}(\vartheta_\varepsilon) + \int_{Q_t} \partial_t \vartheta_\varepsilon \beta_\varepsilon(\vartheta_{\mathcal{H}}) + M\varepsilon \int_{\Omega} |\vartheta_\varepsilon(t) - 1| \\ &\quad - \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_\varepsilon \cdot \beta_{\varepsilon,x}(\vartheta_\varepsilon) + \int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_\varepsilon \cdot \nabla \beta_\varepsilon(\vartheta_{\mathcal{H}}) \\ &\quad + \int_{Q_t} (\partial_t G(\chi_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}}) + \pi(\vartheta_\varepsilon))(\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}})). \end{aligned} \tag{4.11}$$

All the terms on the left-hand side are nonnegative, and we estimate each integral on the right-hand side separately. The first term is bounded due to (2.21) and (3.22). The next one is treated by accounting for the second of (3.37) this way

$$\begin{aligned} \int_{Q_t} \widehat{\beta}_{\varepsilon,t}(\vartheta_\varepsilon) &\leq c \int_{Q_t} (\widehat{\beta}_\varepsilon(\vartheta_\varepsilon) + |\vartheta_\varepsilon| + 1) \leq c \int_{Q_t} \widehat{\beta}_\varepsilon(\vartheta_\varepsilon) + c \\ &\leq c \int_{Q_t} (\widehat{\beta}_\varepsilon(\vartheta_\varepsilon) + M\varepsilon|\vartheta_\varepsilon - 1|) + c \end{aligned}$$

since ϑ_ε has already been estimated in $L^\infty(0, T; H)$ by (4.5). For that reason, the fourth integral of (4.11) is bounded as well and we deal with the third one. We integrate it by parts and have

$$\int_{Q_t} \partial_t \vartheta_\varepsilon \beta_\varepsilon(\vartheta_{\mathcal{H}}) = \int_{\Omega} \vartheta_\varepsilon(t) \beta_\varepsilon(t, \vartheta_{\mathcal{H}}(t)) - \int_{\Omega} \vartheta_0 \beta_\varepsilon(0, \vartheta_{\mathcal{H}}(0)) - \int_{Q_t} \vartheta_\varepsilon \partial_t \beta_\varepsilon(\vartheta_{\mathcal{H}})$$

and one immediately sees that the last right-hand side is bounded, due to (4.5) and (3.42). For the same reason, the next integral of (4.11) that involves $\vartheta_{\mathcal{H}}$ is bounded as well. Now, we treat the

term containing $\beta_{\varepsilon,x}$. By using the first of (3.37) and accounting for (4.5) and (3.42) once more, we easily have

$$-\int_{Q_t} \nabla \text{Ln}_\varepsilon \vartheta_\varepsilon \cdot \beta_{\varepsilon,x}(\vartheta_\varepsilon) \leq \delta \int_{Q_t} (|\beta_\varepsilon(\vartheta_\varepsilon)| + 1)^2 + c_\delta \leq \delta \int_{Q_t} (\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}}))^2 + c_\delta.$$

Finally, owing to (3.2), (3.42), and (2.11), we obtain

$$\begin{aligned} & \int_{Q_t} (\partial_t G(\chi_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}}) + \pi(\vartheta_\varepsilon))(\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}})) \\ & \leq \delta \int_{Q_t} |\beta_\varepsilon(\vartheta_\varepsilon) - \beta_\varepsilon(\vartheta_{\mathcal{H}})|^2 + c_\delta \int_{Q_t} (|\partial_t \chi_\varepsilon|^2 + |\vartheta_\varepsilon|^2 + 1) \end{aligned}$$

and the last integral is bounded due to (4.5). Therefore, if we collect (4.11) and the inequalities we have proved, choose δ small enough, and apply the Gronwall lemma, we derive that

$$\|\beta_\varepsilon(\vartheta_\varepsilon)\|_{L^2(0,T;H)} \leq c. \tag{4.12}$$

4.6. Consequence

Estimates (4.5) and (4.12) and our assumptions (3.2) and (2.11) on G and π ensure that

$$\|\partial_t \vartheta_\varepsilon\|_{L^2(0,T;V)} \leq c \tag{4.13}$$

just by comparison in (3.23).

Lemma 4.1. Assume $z, z_n \in L^2(Q)$, $z > 0$ a.e. in Q , and $z_n \rightarrow z$ a.e. in Q . Moreover, let $\{\varepsilon_n\}$ be a positive real sequence converging to 0. Then, $\{\beta_{\varepsilon_n}(z_n)\}$ converges to $\beta(z)$ a.e. in Q .

Proof. It suffices to show that, for every $\delta \in (0, 1)$, we have

$$\beta_{\varepsilon_n}(z_n) \rightarrow \beta(z) \text{ almost uniformly in } Q^\delta := \{(x, t) \in Q : \delta \leq z(x, t) \leq 1/\delta\}.$$

Thus, fix $\delta \in (0, 1)$ and $\eta > 0$. We have to show that a subset $Q_\eta^\delta \subset Q^\delta$ exists such that $|Q \setminus Q_\eta^\delta| \leq \eta$ and $\beta_{\varepsilon_n}(z_n) \rightarrow \beta(z)$ uniformly in Q_η^δ , where $|\cdot|$ stands for the Lebesgue measure in \mathbb{R}^4 . By the Severini–Egorov theorem, we find $Q_\eta^\delta \subset Q^\delta$ such that $|Q \setminus Q_\eta^\delta| \leq \eta$ and $z_n \rightarrow z$ uniformly in Q_η^δ , and we can prove that $\beta_{\varepsilon_n}(z_n) \rightarrow \beta(z)$ uniformly in Q_η^δ . Fix \bar{n} such that

$$\varepsilon_n \leq \frac{\delta}{2} \text{ and } \frac{\delta}{2} \leq z_n \leq \frac{2}{\delta} \text{ in } Q_\eta^\delta \text{ for every } n \geq \bar{n}.$$

On the other hand, we have

$$\|\beta_{\varepsilon_n}(z_n) - \beta(z)\|_{L^\infty(Q_\eta^\delta)} \leq \|\beta_{\varepsilon_n}(z_n) - \beta_{\varepsilon_n}(z)\|_{L^\infty(Q_\eta^\delta)} + \|\beta_{\varepsilon_n}(z) - \beta(z)\|_{L^\infty(Q_\eta^\delta)}. \tag{4.14}$$

Assume now $n \geq \bar{n}$. Then, $\varepsilon_n \leq \delta$, whence $\varepsilon_n \leq z \leq 1/\varepsilon_n$. Thus, $\beta(z) = \tilde{\beta}_{\varepsilon_n}(z)$ (by (3.11)). We infer that the last term of (4.14) is $\leq M\varepsilon_n$ by (3.16). On the other hand, as $\varepsilon_n \leq \delta/2$, we can use the second of (3.18). Therefore, we conclude that

$$\|\beta_{\varepsilon_n}(z_n) - \beta(z)\|_{L^\infty(Q_\eta^\delta)} \leq c_\delta \|z_n - z\|_{L^\infty(Q_\eta^\delta)} + M\varepsilon_n$$

and deduce that $\beta_{\varepsilon_n}(z_n)$ converges to $\beta(z)$ uniformly in Q_η^δ . \square

4.7. Conclusion of the proof

The estimates (4.5), (4.6), (4.10), (4.12), and (4.13) proved in the previous steps and classical weak and weak star compactness results ensure that suitable limit functions exist in order that the following convergences hold (at least for a subsequence)

$$\vartheta_\varepsilon \rightharpoonup \vartheta \quad \text{weakly star in } L^\infty(0, T; H) \cap H^1(0, T; V)_0^*, \tag{4.15}$$

$$\ln_\varepsilon \vartheta_\varepsilon \rightharpoonup \ell \quad \text{weakly in } L^2(0, T; V), \tag{4.16}$$

$$\chi_\varepsilon \rightharpoonup \chi \quad \text{weakly star in } L^\infty(0, T; V) \cap L^2(0, T; W) \cap H^1(0, T; H), \tag{4.17}$$

$$\nabla \vartheta_\varepsilon \rightharpoonup \nabla \vartheta \quad \text{weakly in } L^2(0, T; L^{4/3}(\Omega)), \tag{4.18}$$

$$\beta_\varepsilon(\vartheta_\varepsilon) \rightharpoonup \xi \quad \text{weakly in } L^2(0, T; H). \tag{4.19}$$

Note that (4.15) and (4.18) imply that ϑ_ε converges to ϑ weakly in $L^2(0, T; W^{1,4/3}(\Omega))$. Now, we observe that the Sobolev exponent $(4/3)^*$ of $W^{1,4/3}(\Omega)$ is $12/5 > 2$. Hence, $W^{1,4/3}(\Omega)$ is compactly embedded in H . On the other hand, even W is compactly embedded in H . Therefore, by applying [27, Theorem 5.1, p. 58] and possibly taking another subsequence, we derive that

$$\vartheta_\varepsilon \rightarrow \vartheta \quad \text{and} \quad \chi_\varepsilon \rightarrow \chi \quad \text{strongly in } L^2(0, T; H) \text{ and a.e. in } Q. \tag{4.20}$$

This allows us to identify all the limits of the nonlinear terms. As far as the logarithm is concerned, we note that (4.15) and (4.16) imply that $\ln_\varepsilon \vartheta_\varepsilon$ converges to ℓ weakly in $L^2(0, T; H)$ (cf. (3.1)). Hence, we can conclude that $\vartheta > 0$ and $\ell = \ln \vartheta$ a.e. in Q (see, e.g., [11, Proposition 2.5, p. 27] for a similar result). From $\vartheta > 0$ a.e. in Q and (4.20) for ϑ_ε , we see that we can apply Lemma 4.1 and infer that $\xi = \beta(\vartheta)$ a.e. in Q . Finally, the limits of the remaining nonlinear terms (i.e., those related to $G, F', G',$ and π) can be identified by using the convergences a.e. given by (4.20) and accounting for our assumptions (2.3) and (2.11). This concludes the proof of Theorem 2.4.

5. Boundedness

In this section, we prove Theorem 2.5 by estimating the L^p -norm of ϑ (or of a suitable function of it) with a constant independent of p by using a Moser type technique. As usual, if z is either a function or a real number, the symbol z^+ denotes its positive part. Moreover, it is understood that n is a positive integer and δ is a positive parameter, say $\delta \in (0, 1)$. Furthermore, we set for convenience

$$u^* := \ln \vartheta^*. \tag{5.1}$$

Finally, we assume $q \leq 4$ (see (2.33)) without loss of generality.

In the sequel, we perform two a priori estimates. In each of them, the use of the chain rule for time derivatives has to be justified, and the trouble is a lack of regularity for $\partial_t \vartheta$, which is not known to belong to $L^2(0, T; H)$. The first lemma we prove overcomes such a difficulty and uses just the regularity we actually know for ϑ , namely

$$\vartheta \in L^\infty(0, T; H) \cap H^1(0, T; V)_0^*, \quad \vartheta > 0 \quad \text{a.e. in } Q, \quad \text{and} \quad \ln \vartheta \in L^2(0, T; V) \tag{5.2}$$

the regularity for the time derivative being a consequence of Theorem 2.4 (see Remark 2.6). Actually, the last of (5.2) does not play a special role in the lemma, in which a general continuous increasing function $\phi' : (0, +\infty) \rightarrow \mathbb{R}$ is considered.

Lemma 5.1. Assume $\vartheta \in L^\infty(0, T; H) \cap H^1(0, T; V)_0^*$ and $\vartheta > 0$ a.e. in Q . Moreover, let $\phi : (0, +\infty) \rightarrow \mathbb{R}$ be a convex function of class C^1 and assume that $\phi'(\vartheta) \in L^2(0, T; V)_0$. Then, if $\Phi : \mathbb{R} \rightarrow (-\infty, +\infty]$ denotes the extension

$$\Phi(r) := \phi(r) \quad \text{if } r > 0, \quad \Phi(0) := \lim_{r \rightarrow 0^+} \phi(r), \quad \text{and} \quad \Phi(r) := +\infty \quad \text{if } r < 0 \tag{5.3}$$

then the function $t \mapsto \int_\Omega \Phi(\vartheta(t))$ is absolutely continuous on $[0, T]$ and we have

$$\int_0^t \langle \partial_t \vartheta(s), \phi'(\vartheta(s)) \rangle ds = \int_\Omega \Phi(\vartheta(t)) - \int_\Omega \Phi(\vartheta(0)) \quad \text{for every } t \in [0, T]. \tag{5.4}$$

Proof. We first observe that Φ is convex, proper, and lower semicontinuous in \mathbb{R} . In addition, we notice that

$$\phi'(u) \in \partial\Phi(u) \quad \text{a.e. in } \Omega \quad \text{if } u \in H, \quad u > 0 \quad \text{a.e. in } \Omega, \quad \text{and} \quad \phi'(u) \in H \tag{5.5}$$

and the conjugate function Φ^* of Φ satisfies $\partial\Phi^* = (\partial\Phi)^{-1}$. For a.a. $t \in (0, T)$ we set for convenience $v(t) := \phi'(\vartheta(t))$ and observe that both $\vartheta(t)$ and $v(t)$ lie in H , and $\vartheta(t) > 0$ a.e. in Ω . Thus, it turns out that $v(t) \in \partial\Phi(\vartheta(t))$ a.e. in Ω by (5.5), and consequently

$$\vartheta(t) \in \partial\Phi^*(v(t)) \quad \text{a.e. in } \Omega. \tag{5.6}$$

Moreover, defining the functionals $J : H \rightarrow (-\infty, +\infty]$ and $J_0 : V_0 \rightarrow (-\infty, +\infty]$ as follows

$$J(v) := \int_\Omega \Phi^*(v) \quad \text{if } \Phi^*(v) \in L^1(\Omega), \quad J(v) := +\infty \quad \text{otherwise,} \quad \text{and} \quad J_0 := J|_{V_0} \tag{5.7}$$

we note that (cf., e.g., [1, Proposition 2.8, p. 61]) J is convex, proper, and l.s.c., and its subdifferential operator $\partial J : H \rightarrow 2^H$ is exactly induced by $\partial\Phi^*$ via the almost everywhere in Ω inclusion. Then, for a.a. $t \in (0, T)$ (5.6) entails $\vartheta(t) \in \partial J(v(t))$, that is,

$$J(v(t)) \leq J(w) + \langle \vartheta(t), v(t) - w \rangle \quad \forall w \in H.$$

Therefore, as $v(t) \in V_0$ (and $v(t) \in D(\partial J) \subseteq D(J)$), the functional J_0 is proper and

$$J_0(v(t)) \leq J_0(w) + \langle \vartheta(t), v(t) - w \rangle \quad \forall w \in V_0$$

whence the inclusion $\vartheta(t) \in \partial J_0(v(t))$ holds for the subdifferential $\partial J_0 : V_0 \rightarrow 2^{V_0^*}$ as well. Then, introducing the conjugate functionals and the subdifferentials

$$J^* : H \rightarrow (-\infty, +\infty], \quad J_0^* : V_0^* \rightarrow (-\infty, +\infty] \quad \text{and} \quad \partial J^* : H \rightarrow 2^H, \quad \partial J_0^* : V_0^* \rightarrow 2^{V_0}$$

and recalling that $\partial J^* = (\partial J)^{-1}$ and $\partial J_0^* = (\partial J_0)^{-1}$, we observe that $v(t) \in \partial J^*(\vartheta(t))$ and $v(t) \in \partial J_0^*(\vartheta(t))$ for a.a. $t \in (0, T)$. Therefore, being understood that ϑ denotes the V_0^* -valued continuous representative, from the latter we conclude that (see, e.g., [11, Lemma 3.3, p. 73] for a similar result) $J_0^*(\vartheta)$ is absolutely continuous in $[0, T]$ and

$$\int_0^t \langle \partial_t \vartheta(s), \phi'(\vartheta(s)) \rangle ds = \int_0^t \langle \partial_t \vartheta(s), v(s) \rangle ds = J_0^*(\vartheta(t)) - J_0^*(\vartheta(0)) \quad \forall t \in [0, T]. \tag{5.8}$$

Moreover, the same representative ϑ is H -valued and weakly continuous from $[0, T]$ to H . Now, as $\Phi^{**} \equiv \Phi$ we remind that for $u, w \in H$ one has

$$J^*(u) := \int_{\Omega} \Phi(u) \quad \text{if } \Phi(u) \in L^1(\Omega), \quad J^*(u) := +\infty \quad \text{otherwise,} \tag{5.9}$$

$$w \in \partial J^*(u) \quad \text{if and only if } w \in \partial \Phi(u) \quad \text{a.e. in } \Omega. \tag{5.10}$$

We also claim that

$$J_0^*(u) = J^*(u) \quad \text{if } u \in H. \tag{5.11}$$

For $u \in H$ we have indeed

$$J_0^*(u) = \sup_{w \in V_0} \{ \langle u, w \rangle - J_0(w) \} \leq \sup_{w \in H} \{ \langle u, w \rangle - J(w) \} = J^*(u).$$

On the other hand, in view of [17, Lemma 2.3] (or, also for related results, [2, Lemma 2.4 and Section 2]), it turns out that for all $w \in D(J)$ there exists a sequence $\{w_n\} \subset V_0$ such that $w_n \rightarrow w$ in H and $J_0(w_n) = J(w_n) \rightarrow J(w)$ as $n \rightarrow \infty$, whence

$$\langle u, w \rangle - J(w) = \lim_{n \rightarrow \infty} \{ \langle u, w_n \rangle - J(w_n) \} = \lim_{n \rightarrow \infty} \{ \langle u, w_n \rangle - J_0(w_n) \} \leq J_0^*(u)$$

and consequently $J^*(u) \leq J_0^*(u)$ as the inequality $\langle u, w \rangle - J(w) \leq J_0^*(u)$ holds for all $w \in H$. Then, (5.11) is proved. By combining (5.9) and (5.11) we conclude that

$$J_0^*(\vartheta(t)) = J^*(\vartheta(t)) = \int_{\Omega} \Phi(\vartheta(t)) \quad \forall t \in [0, T].$$

This yields the assertion of the lemma. \square

Remark 5.2. We can replace Φ by ϕ in the right-hand side of (5.4) in a number of cases. For instance, if we know that (the continuous representative of) ϑ satisfies $\vartheta(t) > 0$ a.e. in Ω for every $t \in [0, T]$, then $\Phi(\vartheta(t)) = \phi(\vartheta(t))$ a.e. in Ω for every t as well. Similarly, we have the same conclusion, independently of the strict positivity of ϑ , whenever $\Phi(0) = +\infty$.

Lemma 5.3. Set

$$\phi_n(r) := \int_{\vartheta^*}^r (e^{2 \min\{n, (\ln s - u^*)^+\}} - 1) ds \quad \text{for } r \in (0, +\infty). \tag{5.12}$$

Then, positive constants α_* and C^* exist such that

$$\phi_n(r) \geq \alpha_* e^{3 \min\{n, (\ln r - u^*)^+\}} - C^* \tag{5.13}$$

for every $r \in (0, +\infty)$ and any positive integer n .

Proof. Assume first $\vartheta^* \leq r \leq \vartheta^* e^n$. Then, we have

$$\phi_n(r) = \int_{\vartheta^*}^r ((s/\vartheta^*)^2 - 1) ds = \frac{\vartheta^*}{3} \left(\frac{r}{\vartheta^*} \right)^3 - r + \frac{2\vartheta^*}{3} \geq \frac{\vartheta^*}{6} \left(\frac{r}{\vartheta^*} \right)^3 - C^*$$

for some $C^* > 0$, whence $\alpha_* := \vartheta^*/6$ works in (5.13) for this case. Moreover, we can assume $C^* \geq \alpha_*$, so that (5.13) holds even for $r \in (0, \vartheta^*)$, since $\phi_n(r) = 0$ for such values of r . Finally, if $r \geq \vartheta^*e^n$, we have $r \geq r' := \vartheta^*e^n$ and we already know that (5.13) holds with $r = r'$. We deduce that

$$\phi_n(r) \geq \phi_n(r') \geq \alpha_* e^{3 \min\{n, (\ln r' - u^*)^+\}} - C^* = \alpha_* e^{3n} - C^* = \alpha_* e^{3 \min\{n, (\ln r - u^*)^+\}} - C^*.$$

This concludes the proof. \square

Lemma 5.4. Assume $p \in [1, +\infty)$ and set

$$\psi_n(r) := \int_{\vartheta^*}^r (\min\{n, (\ln s - u^*)^+\})^{2p-1} ds \quad \text{for } r \in (0, +\infty). \tag{5.14}$$

Then, we have

$$\psi_n(r) \geq \frac{1}{2p} \min\{n, (\ln r - u^*)^+\}^{2p} \quad \text{for every } r \in (0, +\infty). \tag{5.15}$$

Proof. If $\vartheta^* \leq r \leq \vartheta^*e^n$, we have

$$\psi_n(r) = \int_{u^*}^{\ln r} e^y (y - u^*)^{2p-1} dy \geq \int_{u^*}^{\ln r} (y - u^*)^{2p-1} dy = \frac{1}{2p} (\ln r - u^*)^{2p}$$

and (5.15) follow. If instead, $r \geq \vartheta^*e^n$, we observe that $\psi_n(r) \geq \psi_n(r')$, where $r' := \vartheta^*e^n$. On the other hand, we already know that (5.15) holds with $r = r'$. Hence, we easily conclude that the desired inequality is true even in this case. Finally, if $r < \vartheta^*$, we have $\psi_n(r) = 0$ and (5.15) trivially holds. \square

Now, we start estimating.

5.1. First a priori estimate

We set

$$u := \ln \vartheta, \quad w_n := \min\{n, (u - u^*)^+\}, \quad \text{and} \quad v_n := e^{2w_n} - 1. \tag{5.16}$$

We want to use v_n as a test function in (2.28) and apply Lemma 5.1 with $\phi = \phi_n$ given by (5.12). To this aim, we note that $u \in L^2(0, T; V)$ and that $v_n = \phi_n(u)$, where ϕ_n is a Lipschitz continuous function. Hence, $v_n \in L^2(0, T; V)$. Moreover, v_n vanishes on the boundary since $\ln \vartheta r \leq u^*$ by (2.20). Therefore, $v_n \in L^2(0, T; V)_0$. Furthermore, ϕ_n is a C^1 convex function on $(0, +\infty)$ and $v_n = \phi'_n(\vartheta)$. Hence, we are allowed both to test (2.28) by v_n and to apply Lemma 5.1. Note that $\phi_n(\vartheta_0) = 0$ since $\vartheta_0 \leq \vartheta^*$ by (2.21). Then, by extending $\phi_n(r)$ with 0 value for $r = 0$ (cf. (5.3)) and setting

$$f := -\beta(\vartheta^*) + \pi(\vartheta) + \partial_t G(\chi) \tag{5.17}$$

from (5.4) we obtain

$$\int_{\Omega} \phi_n(\vartheta(t)) + \int_{Q_t} \nabla u \cdot \nabla v_n + \int_{Q_t} (\beta(\vartheta) - \beta(\vartheta^*))v_n = \int_{Q_t} f v_n. \tag{5.18}$$

We treat each integral, separately. For the first one, we have

$$\int_{\Omega} \phi_n(\vartheta(t)) \geq \alpha_* \int_{\Omega} e^{3w_n(t)} - C^* = \alpha_* \|e^{w_n(t)}\|_{L^3(\Omega)}^3 - c$$

thanks to Lemma 5.3. The next term of (5.18) is treated this way

$$\int_{Q_t} \nabla u \cdot \nabla v_n = \int_{Q_t} \nabla w_n \cdot \nabla v_n = 2 \int_{Q_t} e^{2w_n} |\nabla w_n|^2 = 2 \int_{Q_t} |\nabla e^{w_n}|^2$$

and the last term on the left-hand side is nonnegative too. Indeed, v_n is nonnegative and $\beta(\vartheta) \geq \beta(\vartheta^*)$ where $v_n > 0$ since β is monotone. Thus, let us consider the right-hand side. We first notice that $f \in L^2(Q)$. Therefore, using the Hölder, Sobolev, and Poincaré inequalities (see (2.40) and (2.39)), we have

$$\begin{aligned} \int_{Q_t} f v_n &\leq \int_{Q_t} |f| (|e^{w_n(s)}|^2 + 1) \\ &\leq \int_0^t \|f(s)\|_{L^2(\Omega)} \|e^{w_n(s)}\|_{L^3(\Omega)} \|e^{w_n(s)}\|_{L^6(\Omega)} ds + \|f\|_{L^1(Q)} \\ &\leq \delta \int_0^t \|e^{w_n(s)}\|_{L^6(\Omega)}^2 ds + c_\delta \int_0^t \|f(s)\|_H^2 \|e^{w_n(s)}\|_{L^3(\Omega)}^2 ds + c \\ &\leq 2\delta \int_0^t \|e^{w_n(s)} - 1\|_{L^6(\Omega)}^2 ds + c_\delta \int_0^t \|f(s)\|_H^2 \|e^{w_n(s)}\|_{L^3(\Omega)}^2 ds + c_\delta \\ &\leq 2\delta M_\Omega^4 \int_{Q_t} |\nabla e^{w_n}|^2 + c_\delta \int_0^t \|f(s)\|_H^2 \|e^{w_n(s)}\|_{L^3(\Omega)}^2 ds + c_\delta \\ &\leq 2\delta M_\Omega^4 \int_{Q_t} |\nabla e^{w_n}|^2 + c_\delta \int_0^t \|f(s)\|_H^2 \|e^{w_n(s)}\|_{L^3(\Omega)}^3 ds + c_\delta. \end{aligned} \tag{5.19}$$

At this point, we collect all the above estimates, choose δ small enough, and apply the Gronwall lemma (see, e.g., [11, Lemma A.4, p. 156]) noting that $\|f(\cdot)\|_H^2 \in L^1(0, T)$ since $f \in L^2(Q)$. We obtain

$$\|e^{w_n}\|_{L^\infty(0, T; L^3(\Omega))}^3 + \int_Q |\nabla e^{w_n}|^2 \leq c. \tag{5.20}$$

5.2. Consequence

From (5.20) and (2.43), we deduce that $\exp(w_n)$ is bounded in $L^4(Q)$. Hence, we can let n tend to infinity and infer that $\exp((u - u^*)^+) \in L^4(Q)$. Now, we observe that

$$e^{(u-u^*)^+} = e^{-u^*} e^u = \vartheta / \vartheta^* \quad \text{where } \vartheta > \vartheta^*.$$

As ϑ is positive, we conclude that

$$\vartheta \in L^4(Q). \tag{5.21}$$

Now, we rewrite Eq. (2.29) in the form

$$\partial_t \chi - \Delta \chi = -F'(\chi) - G'(\chi)\vartheta$$

and observe that the right-hand side belongs to $L^4(Q)$ due to (5.21) and the Lipschitz continuity of F and G . By the general theory for parabolic equations, we infer that $\partial_t \chi \in L^4(Q)$, whence also

$$\|\partial_t G(\chi)\|_{L^q(Q)} \leq c \|\partial_t \chi\|_{L^4(Q)} < +\infty \tag{5.22}$$

since $q \leq 4$, as we have assumed at the beginning of this section. On the other hand, we can estimate the right-hand side of (2.28) in a better way owing to (4.12). Indeed, recalling (2.33), we conclude that

$$\|\pi(\vartheta)\|_{L^q(Q)} \leq c \|\vartheta\|_{L^4(Q)} + \|\pi_0\|_{L^q(Q)} < +\infty. \tag{5.23}$$

5.3. The Moser type procedure

Our aim is to prove an iterative estimate for

$$w := (\ln \vartheta - u^*)^+ \tag{5.24}$$

depending on the parameter $p \in [1, +\infty)$. It is understood that the values of the constant c do not depend on p . We define

$$u := \ln \vartheta, \quad w_n := \min\{n, (u - u^*)^+\}, \quad \text{and} \quad v_n := w_n^{2p-1}. \tag{5.25}$$

By arguing as done for the first estimate, we see that $v_n \in L^2(0, T; V)_0$, thus an admissible test function for (2.28), and that Lemma 5.1 can be applied with $\Phi = \psi_n$ given by (5.14), by noting that $v_n = \psi'_n(\vartheta)$ and letting $\psi_n(0) = 0$. Therefore, as $\psi_n(\vartheta_0) = 0$ by (2.21), we obtain

$$\int_{\Omega} \psi_n(\vartheta(t)) + \int_{Q_t} \nabla u \cdot \nabla v_n + \int_{Q_t} (\beta(\vartheta) - \beta(\vartheta^*))v_n = \int_{Q_t} f v_n \tag{5.26}$$

where f is still given by (5.17). Thanks to Lemma 5.4, we immediately derive that

$$\int_{\Omega} \psi_n(\vartheta(t)) \geq \frac{1}{2p} \int_{\Omega} (w_n(t))^{2p}.$$

The next term on the left-hand side of (5.26) is easily treated as follows

$$\int_{Q_t} \nabla u \cdot \nabla v_n = (2p - 1) \int_{Q_t} w_n^{2p-2} |\nabla w_n|^2 = \frac{2p - 1}{p^2} \int_{Q_t} |\nabla w_n^p|^2 \geq \frac{1}{p} \int_{Q_t} |\nabla w_n^p|^2$$

and the last one is nonnegative, since $v_n \geq 0$ and $\beta(\vartheta) \geq \beta(\vartheta^*)$ where $v_n > 0$. In order to deal with the right-hand side, let us observe that f belongs to $L^q(Q)$ thanks to (5.22)–(5.23) and (2.13). Therefore, if q' denotes the conjugate exponent of q , we have

$$\int_{Q_t} f v_n \leq \|f\|_{L^q(Q)} \|v_n\|_{L^{q'}(Q)} \leq c \|w_n^{2p-1}\|_{L^{q'}(Q)} = c \|w_n^p\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/p}.$$

Collecting the above estimates, we obtain

$$\| (w_n(t))^p \|_H^2 + \int_{Q_t} |\nabla w_n^p|^2 \leq cp \|w_n^p\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/p} \quad \text{for every } t \in [0, T].$$

As both terms on the left-hand side are nonnegative, each of them satisfies the same bound. Therefore, owing to (2.42), we derive that

$$\|w_n^p\|_{L^{10/3}(Q)}^2 \leq c(\|w_n^p\|_{L^\infty(0,T;H)}^2 + \|\nabla w_n^p\|_{L^2(0,T;H)}^2) \leq cp \|w_n^p\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/p}.$$

At this point, we note that (5.24) and $\vartheta \in L^2(Q)$ trivially imply that $w \in L^r(Q)$ for every $r \in [1, +\infty)$. So we let n tend to infinity and conclude that

$$\|w^p\|_{L^{10/3}(Q)}^2 \leq cp \|w^p\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/p}$$

where w is given by (5.24). In other words, we have

$$\|w\|_{L^{10p/3}(Q)} \leq (cp)^{1/(2p)} \|w\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/(2p)}.$$

Finally, using the Hölder inequality and terming $|Q|$ the Lebesgue measure of Q , we infer that

$$\begin{aligned} \|w\|_{L^{10p/3}(Q)} &\leq (cp)^{1/(2p)} |Q|^{1/(4p^2q')} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} \\ &\leq (c|Q|^{1/(2pq')})^{1/(2p)} p^{1/(2p)} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} \leq (cp)^{1/(2p)} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} \end{aligned}$$

since $|Q|^{1/(2pq')}$ is bounded with respect to $p \geq 1$. As we can assume the last constant c to be ≥ 1 , we conclude that

$$\|w\|_{L^{10p/3}(Q)} \leq (cp)^{1/(2p)} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} \tag{5.27}$$

with $c \geq 1$.

5.4. Conclusion of the proof

We rewrite (5.27) in the form

$$\|w\|_{L^{\sigma \cdot 2pq'}(Q)} \leq (cp)^{1/(2p)} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} \quad \text{where } \sigma := \frac{5}{3q'} \tag{5.28}$$

and observe that $\sigma > 1$ since $q' < 5/3$ by (2.33). Now, we apply (5.28) to the divergent sequence $\{p_k\}$ defined by $p_k := \sigma^k$ and obtain

$$\|w\|_{L^{2p_{k+1}q'}(Q)} \leq (cp_k)^{1/(2p_k)} \|w\|_{L^{2p_kq'}(Q)}^{(2p_k-1)/(2p_k)}. \tag{5.29}$$

Setting for convenience

$$\ell_k := \ln^+ \|w\|_{L^{2p_kq'}(Q)}$$

and taking the positive part of the logarithm of both sides of (5.29), we derive that

$$\ell_{k+1} \leq \frac{1}{2p_k} \ln(cp_k) + \frac{2p_k - 1}{2p_k} \ell_k \leq \frac{1}{2p_k} \ln(cp_k) + \ell_k$$

the logarithms being nonnegative since $c \geq 1$. As this holds for every $k \geq 0$, we have that

$$\ln^+ \|w\|_{L^{2p_k q'}(Q)} = \ell_k \leq C := \ell_0 + \sum_{i=0}^{\infty} \frac{1}{2p_i} \ln(cp_i), \quad \text{whence } \|w\|_{L^{2p_k q'}(Q)} \leq e^C \tag{5.30}$$

by noting that the series actually converges since $p_i = \sigma^i$ with $\sigma > 1$.

At this point, we can easily conclude the proof. Indeed, from (5.30), we immediately deduce that $w \in L^\infty(Q)$. Hence, coming back to (5.24), we derive that ϑ is bounded from above.

6. Uniqueness

In this section, we prove Theorem 2.7. The tool we use is the operator $\mathcal{R} : V_0^* \rightarrow V_0$ given by the Riesz representation theorem, namely

$$\text{for } v^* \in V_0^* \text{ and } v \in V_0, \quad v = \mathcal{R}v^* \quad \text{means} \quad v^* = -\Delta v. \tag{6.1}$$

We note that

$$\langle -\Delta u, \mathcal{R}v \rangle = \int_{\Omega} uv \quad \text{for every } u \in V_0 \text{ and } v \in H, \tag{6.2}$$

$$(u^*, \mathcal{R}v^*) = (u^*, v^*)_* \quad \text{for every } u^*, v^* \in V_0^*, \tag{6.3}$$

$$\int_0^t \langle \partial_t u(s), \mathcal{R}u(s) \rangle ds = \frac{1}{2} \|u(t)\|_*^2 - \frac{1}{2} \|u(0)\|_*^2$$

for every $u \in H^1(0, T; V_0^*)$ and for a.a. $t \in (0, T)$. (6.4)

In (6.4), $\|\cdot\|_*$ is the norm in V_0^* dual to the norm $v \mapsto \|\nabla v\|_H$ in V_0 , and the symbol on the right-hand side of (6.3) is the corresponding inner product. By the Poincaré inequality, such norm and product in V_0^* are equivalent to the standard ones and we mainly use them for convenience. Moreover, we recall that $R = \pi - \beta$ satisfies assumption (2.38). Finally, despite of the general rule explained at the end of Section 2, we decide to compute all the constants we use in our estimates with care. In particular, we denote by M_Ω a constant that makes the following relations true

$$\|v\|_{H^{-1}(\Omega)} \leq M_\Omega \|v\|_* \quad \text{for every } v \in V_0^* \quad \text{and} \quad \|v\|_* \leq M_\Omega \|v\|_H \quad \text{for every } v \in H \tag{6.5}$$

as well as the analogous of (2.40) for $L^4(\Omega)$. In (6.5), the first norm is the standard one in $V_0^* = H^{-1}(\Omega)$.

To prove our uniqueness result, we have to show that any pair of solutions $(\vartheta_1, \chi_1, \xi_1)$ and $(\vartheta_2, \chi_2, \xi_2)$ to problem (2.28)–(2.31) satisfying the regularity requirements (2.23)–(2.27) and such that

$$\vartheta_1, \vartheta_2, \chi_1, \chi_2 \in L^\infty(Q)$$

coincide, i.e. $\vartheta_1 = \vartheta_2$ and $\chi_1 = \chi_2$. So, pick such solutions and observe that both of them satisfy the boundary condition (2.30) and the initial conditions (2.31) with the same boundary and initial data.

In particular, we can define M to be the maximum of the L^∞ -norms of the four functions ϑ_1 , ϑ_2 , χ_1 , and χ_2 . We note that G , F' , and G' are Lipschitz continuous on $[-M, M]$, since they are smooth by (2.3). In the sequel, L is the maximum of their Lipschitz constants on such an interval. Moreover, we can apply Remark 2.6 to $(\chi_1$ and χ_2 since ϑ_2 is bounded, and deduce that (2.37) holds for χ_2 .

Now, we write Eqs. (2.28) for both solutions and test their difference by $\mathcal{R}(\vartheta_1 - \vartheta_2)$. At the same time, we write Eqs. (2.29) for both solutions and test their difference by $\mu \partial_t(\chi_1 - \chi_2)$, where $\mu \in (0, 1)$ is a parameter whose value will be chosen later on. Finally, we sum the equalities we get to each other, rearrange, and add the same integral to both sides, for convenience. If we use the notation $\vartheta := \vartheta_1 - \vartheta_2$ and $\chi := \chi_1 - \chi_2$, owing to (6.2)–(6.4) we obtain

$$\begin{aligned} & \frac{1}{2} \|\vartheta(t)\|_*^2 + \int_{Q_t} (\ln \vartheta_1 - \ln \vartheta_2) \vartheta + \mu \int_{Q_t} |\partial_t \chi|^2 + \frac{\mu}{2} \int_{\Omega} |\nabla \chi(t)|^2 + \frac{\mu}{2} \int_{\Omega} |\chi(t)|^2 \\ &= \int_0^t (\partial_t G(\chi_1(s)) - \partial_t G(\chi_2(s)), \vartheta(s))_* ds + \int_0^t ((R(\vartheta_1) - R(\vartheta_2))(s), \vartheta(s))_* ds \\ & \quad + \mu \int_{Q_t} (F'(\chi_2) - F'(\chi_1)) \partial_t \chi + \mu \int_{Q_t} (G'(\chi_2) \vartheta_2 - G'(\chi_1) \vartheta_1) \partial_t \chi + \frac{\mu}{2} \int_{\Omega} |\chi(t)|^2. \end{aligned} \tag{6.6}$$

The only term on the left-hand side that needs some treatment is the second one. We have

$$\int_{Q_t} (\ln \vartheta_1 - \ln \vartheta_2) \vartheta \geq \frac{1}{M} \int_{Q_t} |\vartheta|^2$$

since $0 < \vartheta_i \leq M$ for $i = 1, 2$. We deal with the first term on the right-hand side and use (2.37) for χ_2 as mentioned above. We get

$$\begin{aligned} & \int_0^t (\partial_t G(\chi_1(s)) - \partial_t G(\chi_2(s)), \vartheta(s))_* ds \\ &= \int_0^t (G'(\chi_1(s)) \partial_t \chi(s) + (G'(\chi_1(s)) - G'(\chi_2(s))) \partial_t \chi_2(s), \vartheta(s))_* ds \\ &\leq M_{\Omega} \int_0^t \|G'(\chi_1(s)) \partial_t \chi(s)\|_H \|\vartheta(s)\|_* ds \\ & \quad + M_{\Omega} \int_0^t \|G'(\chi_1(s)) - G'(\chi_2(s))\|_{L^4(\Omega)} \|\partial_t \chi_2(s)\|_{L^4(\Omega)} \|\vartheta(s)\|_* ds \\ &\leq M_{\Omega} L \int_0^t \|\partial_t \chi(s)\|_H \|\vartheta(s)\|_* ds + M_{\Omega} L \int_0^t \|\chi(s)\|_{L^4(\Omega)} \|\partial_t \chi_2(s)\|_{L^4(\Omega)} \|\vartheta(s)\|_* ds \\ &\leq \frac{\mu}{8} \int_{Q_t} |\partial_t \chi|^2 + \frac{2M_{\Omega}^2 L^2}{\mu} \int_0^t \|\vartheta(s)\|_*^2 ds \\ & \quad + \int_0^t \|\chi(s)\|_{L^4(\Omega)}^2 ds + \frac{M_{\Omega}^2 L^2}{4} \int_0^t \|\partial_t \chi_2(s)\|_{L^4(\Omega)}^2 \|\vartheta(s)\|_*^2 ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{\mu}{8} \int_{Q_t} |\partial_t \chi|^2 + \frac{2M_\Omega^2 L^2}{\mu} \int_0^t \|\vartheta(s)\|_*^2 ds \\ &+ M_\Omega^2 \int_{Q_t} |\nabla \chi|^2 + M_\Omega^2 \int_{Q_t} |\chi|^2 + \frac{M_\Omega^2 L^2}{4} \int_0^t \|\partial_t \chi_2(s)\|_{L^4(\Omega)}^2 \|\vartheta(s)\|_*^2 ds. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int_0^t ((R(\vartheta_1) - R(\vartheta_2))(s), \vartheta(s))_* ds &\leq M_\Omega \lambda_R \int_0^t \|\vartheta(s)\|_H \|\vartheta(s)\|_* ds \\ &\leq \frac{1}{2M} \int_{Q_t} |\vartheta|^2 + MM_\Omega^2 \lambda_R^2 \int_0^t \|\vartheta(s)\|_*^2 ds, \\ \mu \int_{Q_t} (F'(\chi_2) - F'(\chi_1)) \partial_t \chi &\leq \frac{\mu}{8} \int_{Q_t} |\partial_t \chi|^2 + 2\mu L^2 \int_{Q_t} |\chi|^2, \\ \mu \int_{Q_t} (G'(\chi_2) \vartheta_2 - G'(\chi_1) \vartheta_1) \partial_t \chi &= \mu \int_{Q_t} (G'(\chi_2) - G'(\chi_1)) \vartheta_1 \partial_t \chi - \mu \int_{Q_t} G'(\chi_2) \vartheta \partial_t \chi \\ &\leq \mu LM \int_{Q_t} |\chi| |\partial_t \chi| + \mu L \int_{Q_t} |\vartheta| |\partial_t \chi| \\ &\leq \frac{\mu}{4} \int_{Q_t} |\partial_t \chi|^2 + 2\mu L^2 M^2 \int_{Q_t} |\chi|^2 + 2\mu L^2 \int_{Q_t} |\vartheta|^2. \end{aligned}$$

Finally, we treat the last term of (6.6) this way

$$\frac{\mu}{2} \int_\Omega |\chi(t)|^2 = \mu \int_{Q_t} \chi \partial_t \chi \leq \frac{\mu}{8} \int_{Q_t} |\partial_t \chi|^2 + \mu \int_{Q_t} |\chi|^2.$$

At this point, we observe that $s \mapsto \|\partial_t \chi_2(s)\|_{L^4(\Omega)}^2$ belongs to $L^1(0, T)$ by (2.37). Hence, if we choose μ in order that $2\mu L^2 = 1/(4M)$, we see that (6.6) and all the estimates we have performed allow us to apply the Gronwall lemma. This yields $\vartheta = 0$ and $\chi = 0$ a.e. in Q , whence the solutions coincide and the proof is complete.

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