Oscillatory Property for Second Order Linear Delay Differential Equations

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The purpose of this paper is to establish some new oscillation criteria for the second order linear delay differential equation

\[ x''(t) + a(t) x(g(t)) = 0, \]  

where \( a(t) \in C[0, \infty) \rightarrow [0, \infty), \) \( a(t) \not= 0 \) on \([t_0, \infty), \) \( g(t) \in C[0, \infty) \rightarrow [0, \infty), \)

\[ 0 \leq g(t) \leq t, \ t \geq 0, \ \lim_{t \to \infty} g(t) = \infty. \]  

A nontrivial solution to (1) is called oscillatory if it exists on a half-line and has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all solutions of (1) is oscillatory.

The following lemmas are basic for all later discussions and the first lemma was due to Erbe, the proof may be found in [1].

**Lemma 1.** Let \( g(t) \) satisfy (2) and assume \( x(t) \in C^{(2)} \in [t_0, \infty) \) satisfies

\[ x(t) > 0, \quad x'(t) > 0, \quad x''(t) \leq 0, \quad t \in [t_0, \infty). \]  

Then for each \( 0 < \varepsilon < 1, \) there is a \( T > t_0 \) such that

\[ x(g(t)) > \varepsilon x(t) \frac{g(t)}{t}, \quad t \geq T. \]  

**Lemma 2.** If

\[ \int_{t_0}^{\infty} \mu(s) a(s) \, ds < \infty, \]  

\[ \int_{t_0}^{\infty} a(s) \, ds < \infty, \]

then

\[ a(t) x(g(t)) \leq \mu(t) x(t), \quad t \geq t_0. \]
and $x(t)$ is a nonoscillatory solution of (1), then for all large $t$,

$$
\int_{t}^{\infty} \omega^2(s) \, ds < \infty,
$$

(6)

$$
w(t) \geq \int_{t}^{\infty} w^2(s) \, ds + \varepsilon \int_{t}^{\infty} \mu(s) a(s) \, ds,
$$

(7)

where $\mu(t) = \frac{g(t)}{t}$, $w(t) = \frac{x'(t)}{x(t)}$, and $0 < \varepsilon < 1$ is a constant which is independent of $x(t)$.

Proof. Without loss of generality, we can assume $x(t) > 0$ for $t \geq t_1 \geq t_0$. From (2) there is $t_2 \geq t_1$ such that $x(g(t)) > 0$ for $t \geq t_2$. Since $a(t) > 0$, $x''(t) \leq 0$ for $t \geq t_2$. Hence $x'(t)$ is nonincreasing, and $x'(t) > 0$, otherwise, it leads to a contradiction. By Lemma 2, for each constant $0 < \varepsilon < 1$ there is $t_3 \geq t_2$ such that

$$
x''(t) + \varepsilon \mu(t) a(t) x(t) \leq 0 \quad \text{for} \quad t \geq t_3.
$$

(8)

Let $w(t) = x'(t)/x(t)$, from (8) we have

$$
w'(t) + w^2(t) + \varepsilon \mu(t) a(t) \leq 0, \quad t \leq t_3.
$$

(9)

Integrating (9) from $t_3$ to $t_1$, we obtain

$$
w(t) - w(t_3) + \int_{t_3}^{t} w^2(s) \, ds + \varepsilon \int_{t_3}^{t} \mu(s) a(s) \, ds \leq 0.
$$

(10)

On the other hand, (9) implies $w'(t) + w^2(t) \leq 0$ and hence $(d/dt)(-1/w(t)) + t \leq 0$, namely, $0 < w(t) < 1/(t + c)$ where $c$ is a constant. Thus $\lim_{t \to \infty} w(t) = 0$. Using (5) and taking limit in (10), we have (6). Thus (7) holds for all large $t$. This completes the proof.

We introduce the function sequence

$$
\{\alpha_n(t)\}, \quad n = 0, 1, 2, ..., \quad t \in [t_0, \infty),
$$

(11)

where $\alpha_0(t) = \varepsilon \int_{t}^{\infty} \mu(s) a(s) \, ds$, $\alpha_n(t) = \int_{t}^{\infty} \alpha_{n-1}^2(s) \, ds + \alpha_0(t)$, $n = 1, 2, ...$, and $\varepsilon$ is a constant with $0 < \varepsilon < 1$.

**Theorem 3.** Assume that (5) holds, and that there exists a constant $0 < \varepsilon < 1$ such that one of the following conditions is satisfied:

1. there exists a positive integer $m$ such that $\alpha_n(t)$ is defined for $n = 1, 2, ..., m - 1$, but

$$
\lim_{t \to \infty} \int_{t}^{t_0} \alpha_{m-1}^2(s) \, ds = \infty,
$$

(12)
(ii) \( \alpha_n(t) \) is defined for \( n = 1, 2, \ldots \), but there is \( t^* \geq t_0 \) such that

\[
\lim_{n \to \infty} \alpha_n(t^*) = \infty.
\]  

(13)

Then the equation (1) is oscillatory.

Proof. Assume the contrary. Then (1) has a nonoscillatory solution \( x(t) \). Assume that \( x(t) > 0 \) for \( t \geq t_1 \geq t_0 \). Since \( -x(t) \) is a solution also. Let \( w(t) = x'(t)/x(t) \). By Lemma 2, we have \( w(t) > \alpha_0(t) \) for \( t \geq t_1 \). Hence

\[
w^2(t) > \alpha_0^2(t).
\]  

(14)

If (i) is satisfied. Suppose that \( m = 1 \). From (14) we get

\[
\int_{t_1}^t w^2(s) \, ds < \int_{t_1}^t \alpha_0^2(s) \, ds < \infty.
\]

Thus \( \alpha_1(t) \) is defined, this contradicts the condition (i). Suppose that \( m > 1 \), then by (11) and (7), we have that

\[
\alpha_{m-1}(t) < w(t).
\]  

(15)

This implies

\[
\int_{t_1}^t \alpha_{m-1}^2(s) \, ds < \int_{t_1}^t w^2(s) \, ds < \infty.
\]

This contradicts the condition (12).

If (ii) is satisfied, then we can obtain as in the above case that

\[
\alpha_n(t) < w(t), \quad n = 1, 2, \ldots.
\]  

(16)

From (16), \( \lim_{n \to \infty} \alpha_n(t) \leq w(t) < \infty \). This contradicts the condition (13).

Applying Theorem 3, we obtain the following result. It was first proved by Wong [5] under stronger condition. Our result extends the well known oscillation criteria of Hille [2], Kneser [3], and Opial [4] in the ordinary differential equation case and Erbe [1] in the delay differential equation case.

**Corollary 4.** Assume that there exist \( t_0 \) and constant, such that one of the following conditions for \( t \geq t_0 \) is satisfied:

\begin{align*}
(\text{i}) & \quad \epsilon \mu(t) a(t) \geq C_0/t^2, \\
(\text{ii}) & \quad \alpha_0(t) \geq C_0/t, \\
(\text{iii}) & \quad \int_{t_0}^t \alpha_0^2(s) \, ds \geq C_0 \alpha_0(t),
\end{align*}

(17) (18) (19)

where \( C_0 > \frac{1}{4} \) is a constant. Then (1) is oscillatory.

Proof. If (18) holds, then \( \alpha_1(t) = \int_{t_0}^t \alpha_0^2(s) \, ds + \alpha_0(t) \geq C_1/t \), where \( C_1 = C_0^2 + C_0 \). Hence \( \alpha_2(t) = \int_{t_0}^t \alpha_1^2(s) \, ds + \alpha_1(t) \geq C_2/t \). Generally, we have that \( \alpha_n(t) \geq C_n/t \), where \( C_n = C_{n-1}^2 + C_0 \). It is easy to see that \( C_n > C_{n-1}, \quad n = 1, 2, \ldots \). Since \( C_0 > \frac{1}{4} \), \( \lim_{t \to t_0} C_n = \infty \). Thus \( \lim_{t \to t_0} \alpha_n(t) = \infty, \quad t \in [t_0, \infty) \). By Theorem 3, (1) is oscillatory.
If (17) holds, suppose \( \int_0^\infty \mu(s) a(s) \, ds = \infty \), then by (10) we easily obtain that \( \lim_{t \to \infty} w(t) = \lim_{t \to \infty} \frac{x'(t)}{x(t)} = -\infty \), which is a contradiction.

If \( \int_0^\infty \mu(s) a(s) \, ds < \infty \), it is obvious that (18) is satisfied, hence (1) is oscillatory.

If (19) holds, it is easy to obtain

\[
\alpha_n(t) \geq \alpha_0(t)(1 + C_{n-1}), \quad C_n = C_0(1 + C_{n-1})^2, \quad n = 1, 2, \ldots, \tag{20}
\]

Since \( \alpha_0(t) \) is not identical with zero, there is \( t^* \geq t_0 \) such that \( \alpha(t^*) > 0 \).

From (20) we get \( C_n > C_{n-1}, \quad n = 1, 2, \ldots \). Since \( C_0 > \frac{1}{4} \) from (20) \( \lim_{n \to \infty} C_n = \infty \), that is \( \lim_{n \to \infty} \alpha_n(t^*) = \infty \). By Theorem 3, (1) is oscillatory.

**Theorem 5.** Let (5) hold. If there exist a constant \( \varepsilon, \quad 0 < \varepsilon < 1 \), and non-negative integer \( m \) such that \( \alpha_m(t) \) is defined, and

\[
\lim_{t \to \infty} t\alpha_m(t) > 1, \tag{21}
\]

then is oscillatory.

**Proof.** Let \( x(t) \) be nonoscillatory solution of (1), say \( x(t) > 0 \) for \( t \geq t_1 \geq t_0 \). Setting \( w(t) = \frac{x'(t)}{x(t)} \) and proceeding as in the proof of Lemma 2, we get \( 0 < w(t) < 1/(t + c) \) where \( c \) is a constant. On the other hand, since \( \alpha_m(t) \leq w(t) \) for \( t \geq t_1 \), it follows that \( \alpha_m(t) < 1/(t + c) \), and so \( (t + c) \alpha_m(t) \leq 1 \) which contradicts (21).

**Corollary 2.6.** Assume that condition (5) holds. If there is a constant \( \varepsilon, \quad 0 < \varepsilon < 1 \), such that \( \alpha_n(t) \) is defined for \( n = 1, 2, \ldots \), and

\[
\lim_{n \to \infty} \alpha_n(t) = \alpha(t) < \infty, \quad t \in [t_0, \infty). \tag{22}
\]

Moreover, if

\[
\lim_{t \to \infty} t\alpha(t) > 1, \tag{23}
\]

then (1) is oscillatory.

**Proof.** Suppose that the condition (21) is not satisfied, then for constant \( \varepsilon \) and each fixed nonnegative integer \( n \), there is \( t_1 \geq t_0 \) such that

\[
t\alpha_n(t) \leq 1, \quad t \geq t_1. \tag{24}
\]

This implies \( t\alpha(t) \leq 1, \quad t \geq t_1 \), which contradicts (23). Thus there is a constant \( \varepsilon \) and nonnegative integer \( m \) such that (21) holds, and so (1) is oscillatory.
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