

Reflexivity of the Automorphism and Isometry Groups of the Suspension of $\mathcal{B}(\mathcal{H})$

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The aim of this paper is to show that the automorphism and isometry groups of the suspension of $\mathcal{B}(\mathcal{H})$, \mathcal{H} being a separable infinite-dimensional Hilbert space,

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

The study of reflexive linear subspaces of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on the Hilbert space \mathcal{H} represents one of the most active research areas in operator theory (see [Had] for a beautiful general view of reflexivity of this kind). In the past decade, similar questions concerning certain important sets of transformations acting on Banach algebras rather than Hilbert spaces have also attracted attention. The originators of the research in this direction are Kadison and Larson. In [Kad], Kadison studied local derivations from a von Neumann algebra \mathcal{R} into a dual \mathcal{R} -bimodule \mathcal{M} . A continuous linear map from \mathcal{R} into \mathcal{M} is called a local derivation if it agrees with some derivation at each point (the derivations possibly differing from point to point) in the algebra. This investigation was motivated by study of the Hochschild cohomology of operator algebras. The main result in [Kad], Theorem A, states that in the above setting every local derivation is a derivation. Independently, Larson and Sourour proved in [LaSo] that the same conclusion holds for local derivations of $\mathcal{B}(\mathcal{X})$, where \mathcal{X} is a Banach space. Since then, a considerable amount of work has been done concerning local derivations of various algebras. See, for example, [Bre, BrSe1, Cri, Shu, ZhXi]. Besides derivations, there are at least two other very important classes of transformations on operator algebras which certainly deserve attention, namely, the

group of automorphisms and the group of surjective isometries. In [Lar, Some concluding remarks (5), p. 298], Larson initiated the study of local automorphisms (the definition should be self-explanatory) of Banach algebras. In his joint paper with Sourour [LaSo], it was proved that if \mathcal{X} is an infinite-dimensional Banach space, then every surjective local automorphism of $\mathcal{B}(\mathcal{X})$ is an automorphism (see also [BrSe1]). For a separable infinite-dimensional Hilbert space \mathcal{H} , it was shown in [BrSe2] that the above conclusion holds true without the assumption on surjectivity, i.e., every local automorphism of $\mathcal{B}(\mathcal{H})$ is an automorphism.

Let us now define our concept of reflexivity. Let \mathcal{X} be a Banach space (in fact, in the cases we are interested in this is a C^* -algebra) and for any subset $\mathcal{E} \subset \mathcal{B}(\mathcal{X})$ let

$$\text{ref}_{\text{al}} \mathcal{E} = \{ T \in \mathcal{B}(\mathcal{X}) : Tx \in \mathcal{E}x \text{ for all } x \in \mathcal{X} \}$$

and

$$\text{ref}_{\text{to}} \mathcal{E} = \{ T \in \mathcal{B}(\mathcal{X}) : Tx \in \overline{\mathcal{E}x} \text{ for all } x \in \mathcal{X} \},$$

where bar denotes norm-closure. The collection \mathcal{E} of transformations is called algebraically reflexive if $\text{ref}_{\text{al}} \mathcal{E} = \mathcal{E}$. Similarly, \mathcal{E} is said to be topologically reflexive if $\text{ref}_{\text{to}} \mathcal{E} = \mathcal{E}$. In this terminology, the main result in [BrSe2] can be reformulated by saying that the automorphism group of $\mathcal{B}(\mathcal{H})$ is algebraically reflexive. Similarly, Theorem 1.2 in [LaSo] states that the Lie algebra of all generalized derivations on $\mathcal{B}(\mathcal{X})$ is algebraically reflexive. Obviously, the topological reflexivity is a stronger property than the algebraic reflexivity. Among the previously mentioned papers, there is only one which concerns topological reflexivity. Namely, Corollary 2 in [Shu] asserts that the derivation algebra of any C^* -algebra is topologically reflexive. Hence, not only are the local derivations derivations in this case, but every bounded linear map which agrees with the limit of some sequence of derivations at each point is a derivation.

As for the automorphism groups of C^* -algebras, such a general result as in [Shu] does not hold true. If \mathcal{A} is a Banach algebra, then denote by $\text{Aut}(\mathcal{A})$ and $\text{Iso}(\mathcal{A})$ the group of automorphisms (i.e. multiplicative linear bijections) and the group of surjective linear isometries of \mathcal{A} , respectively. Now if X is an uncountable discrete topological space, then it is not difficult to verify that the groups $\text{Aut}(C_0(X))$ and $\text{Iso}(C_0(X))$ of the C^* -algebra $C_0(X)$ of all continuous complex valued functions on X vanishing at infinity are not algebraically reflexive. Concerning topological reflexivity, there are even von Neumann algebras whose automorphism and isometry groups are not topologically reflexive. For example, the infinite dimensional commutative von Neumann algebras acting on a separable Hilbert space have this nonreflexivity property as it was shown in [BaMo].

Let us now mention some positive results. In [Mol1] we proved that if \mathcal{H} is a separable infinite-dimensional Hilbert space, then $\text{Aut}(\mathcal{B}(\mathcal{H}))$ and $\text{Iso}(\mathcal{B}(\mathcal{H}))$ are topologically reflexive. In [Mol2] we studied the reflexivity of the automorphism and isometry groups of C^* -algebras in the famous Brown–Douglas–Fillmore theory, i.e. the extensions of the C^* -algebra of all compact operators on \mathcal{H} by commutative separable unital C^* -algebras. We proved there that the groups Aut and Iso are algebraically reflexive in the case of every such extension, but, for example, in the probably most important case of extensions by $C(\mathbb{T})$ (\mathbb{T} is the perimeter of the unit disc), our groups are not topologically reflexive. This result seems to be surprising even in the case of the Toeplitz extension.

In this present paper we study our reflexivity problem for the suspension of $\mathcal{B}(\mathcal{H})$. The suspension $S\mathcal{A}$ of a C^* -algebra \mathcal{A} is the tensor product $C_0(\mathbb{R}) \otimes \mathcal{A}$ which is well-known to be isomorphic to the C^* -algebra $C_0(\mathbb{R}, \mathcal{A})$ of all continuous functions from \mathbb{R} to \mathcal{A} which vanish at infinity. The suspension plays very important role in K-theory since the K_1 -group of \mathcal{A} is the K_0 -group of $S\mathcal{A}$. In Corollary 5 below we obtain that the automorphism and isometry groups of the suspension of $\mathcal{B}(\mathcal{H})$ are algebraically reflexive. In fact, in what follows we consider more general C^* -algebras of the form $C_0(X) \otimes \mathcal{B}(\mathcal{H}) \cong C_0(X, \mathcal{B}(\mathcal{H}))$, where X is a locally compact Hausdorff space.

Turning to the results of this paper, in our first theorem we describe the general form of the elements of $\text{Aut}(C_0(X, \mathcal{B}(\mathcal{H})))$ and $\text{Iso}(C_0(X, \mathcal{B}(\mathcal{H})))$. From now on, let \mathcal{H} stand for an infinite-dimensional separable Hilbert space.

THEOREM 1. *Let X be a locally compact Hausdorff space. A linear map $\Phi: C_0(X, \mathcal{B}(\mathcal{H})) \rightarrow C_0(X, \mathcal{B}(\mathcal{H}))$ is an automorphism if and only if there exist a function $\tau: X \rightarrow \text{Aut}(\mathcal{B}(\mathcal{H}))$ and a bijection $\varphi: X \rightarrow X$ so that*

$$\Phi(f)(x) = [\tau(x)](f(\varphi(x))) \quad (f \in C_0(X, \mathcal{B}(\mathcal{H})), x \in X). \quad (1)$$

Similarly, a linear map $\Phi: C_0(X, \mathcal{B}(\mathcal{H})) \rightarrow C_0(X, \mathcal{B}(\mathcal{H}))$ is a surjective isometry if and only if there exist a function $\tau: X \rightarrow \text{Iso}(\mathcal{B}(\mathcal{H}))$ and a bijection $\varphi: X \rightarrow X$ so that Φ is of the form (1).

Moreover, if the linear map $\Phi: C_0(X, \mathcal{B}(\mathcal{H})) \rightarrow C_0(X, \mathcal{B}(\mathcal{H}))$ is an automorphism, respectively a surjective isometry, then for the maps τ, φ appearing in (1) we obtain that $x \mapsto \tau(x), x \mapsto \tau(x)^{-1}$ are strongly continuous and that $\varphi: X \rightarrow X$ is a homeomorphism.

In the following two results we show that the algebraic reflexivity of our groups in the case of $C_0(X)$ implies the algebraic reflexivity of $\text{Aut}(C_0(X) \otimes \mathcal{B}(\mathcal{H}))$ and $\text{Iso}(C_0(X) \otimes \mathcal{B}(\mathcal{H}))$.

THEOREM 2. *Let X be a locally compact Hausdorff space. If the automorphism group of $C_0(X)$ is algebraically reflexive, then so is the automorphism group of $C_0(X, \mathcal{B}(\mathcal{H}))$.*

THEOREM 3. *Let X be a σ -compact locally compact Hausdorff space. If the isometry group of $C_0(X)$ is algebraically reflexive, then so is the isometry group of $C_0(X, \mathcal{B}(\mathcal{H}))$.*

To obtain the algebraic reflexivity of the automorphism and isometry groups of the suspension of $\mathcal{B}(\mathcal{H})$ we prove the following assertion.

THEOREM 4. *Let $\Omega \subset \mathbb{R}^n$ be an open convex set. The automorphism and isometry groups of $C_0(\Omega)$ are algebraically reflexive.*

The proof of this result will show how difficult it might be to treat our reflexivity problem for tensor product of general C^* -algebras or even for the suspension of any C^* -algebra with algebraically reflexive automorphism and isometry groups.

Finally, we arrive at the statement announced in the abstract.

COROLLARY 5. *The automorphism and isometry groups of the suspension of $\mathcal{B}(\mathcal{H})$ are algebraically reflexive.*

As for the natural question of whether the groups above are topologically reflexive, we have the immediate negative answer as follows.

EXAMPLE. Let (φ_n) be a sequence of homeomorphisms of \mathbb{R} which converges uniformly to a noninjective function φ . Define linear maps Φ_n, Φ on $C_0(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ by

$$\Phi_n(f) = f \circ \varphi_n \quad \text{and} \quad \Phi(f) = f \circ \varphi \quad (f \in C_0(\mathbb{R}, \mathcal{B}(\mathcal{H})), n \in \mathbb{N}).$$

Then Φ_n is an isometric automorphism of $C_0(\mathbb{R}, \mathcal{B}(\mathcal{H}))$, the sequence $(\Phi_n(f))$ converges to $\Phi(f)$ for every $f \in C_0(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ but Φ is not surjective.

2. PROOFS

We begin with the following lemma on a characterization of certain closed ideals in $C_0(X, \mathcal{B}(\mathcal{H}))$.

LEMMA 2.1. *Let X be a locally compact Hausdorff space. A closed ideal \mathcal{I} in $C_0(X, \mathcal{B}(\mathcal{H}))$ is of the form*

$$\mathcal{I} = \mathcal{I}_{x_0} = \{f \in C_0(X, \mathcal{B}(\mathcal{H})) : f(x_0) = 0\}$$

for some point $x_0 \in X$ if and only if \mathcal{I} is proper subset of a maximal ideal \mathcal{I}_m in $C_0(X, \mathcal{B}(\mathcal{H}))$, there is no closed ideal properly in between \mathcal{I} and \mathcal{I}_m , and \mathcal{I} is not the intersection of two different maximal ideals in $C_0(X, \mathcal{B}(\mathcal{H}))$.

Proof. The structure of closed ideals in Banach algebras of vector valued functions is well-known. See, for example, [Nai, Remark on p. 342]. Using this result, \mathcal{I} is a closed ideal in $C_0(X, \mathcal{B}(\mathcal{H}))$ if and only if it is of the form

$$\mathcal{I} = \{f \in C_0(X, \mathcal{B}(\mathcal{H})) : f(x) \in \mathcal{I}_x\},$$

where every \mathcal{I}_x is a closed ideal of $\mathcal{B}(\mathcal{H})$, i.e. by the separability of \mathcal{H} , every \mathcal{I}_x is either $\{0\}$ or $\mathcal{C}(\mathcal{H})$ or $\mathcal{B}(\mathcal{H})$. By the help of Uryson's lemma on the construction of continuous functions on X with compact support, one can readily verify that the maximal ideals in $C_0(X, \mathcal{B}(\mathcal{H}))$ are exactly those ideals which are of the form

$$\mathcal{I}\{f \in C_0(X, \mathcal{B}(\mathcal{H})) : f(x_0) \in \mathcal{C}(\mathcal{H})\}$$

for some point $x_0 \in X$. Now, the statement of the lemma follows quite easily. ■

Proof of Theorem 1. We begin with the proof of the statement on isometries. Let Φ be a surjective linear isometry of $C_0(X, \mathcal{B}(\mathcal{H}))$. As a consequence of a deep result due to Kaup (see, for example, [DFR]) we obtain that every surjective linear isometry ϕ between C^* -algebras \mathcal{A} and \mathcal{B} has a certain algebraic property, namely ϕ is a triple isomorphism, i.e., it satisfies the equality

$$\phi(ab^*c) + \phi(cb^*a) = \phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a)$$

for every $a, b, c \in \mathcal{A}$. This implies that ϕ preserves the closed ideals in both directions. Indeed, if $\mathcal{I} \subset \mathcal{A}$ is a closed ideal, then, since $\mathcal{I} = \mathcal{I}^*$, we have

$$\phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a) \in \phi(\mathcal{I}) \quad (a, c \in \mathcal{A}, b \in \mathcal{I}).$$

Let $\mathcal{I}' = \phi(\mathcal{I})$. We obtain that $a'\mathcal{I}'^*c' + c'\mathcal{I}'^*a' \in \mathcal{I}'$ ($a', c' \in \mathcal{B}$). Since \mathcal{I}' is a closed linear subspace of \mathcal{B} , if c' runs through an approximate identity, we deduce

$$a'\mathcal{I}'^* + \mathcal{I}'^*a' \in \mathcal{I}' \quad (a' \in \mathcal{B}). \quad (2)$$

If now a' runs through an approximate identity, then we have

$$\mathcal{I}'^* \subset \mathcal{I}'. \quad (3)$$

We infer from (2) and (3) that $a'\mathcal{I}' + \mathcal{I}'a' \subset \mathcal{I}'$ ($a' \in \mathcal{B}$), i.e., \mathcal{I}' is a closed Jordan ideal of \mathcal{B} . It is well-known that in the case of C^* -algebras, every closed Jordan ideal is an (associative) ideal (see, for example, [CiYo, 5.3. Theorem]) and hence the same is true for \mathcal{I}' .

By Lemma 2.1 we infer that our map Φ preserves the ideals

$$\mathcal{I}_x = \{f \in C_0(X, \mathcal{B}(\mathcal{H})) : f(x) = 0\} \quad (x \in X)$$

in both directions. This gives us that there exists a bijection $\varphi: X \rightarrow X$ for which

$$\Phi(f)(x) = 0 \Leftrightarrow f(\varphi(x)) = 0 \quad (4)$$

holds true for every $f \in C_0(X, \mathcal{B}(\mathcal{H}))$ and $x \in X$. For any $x \in X$, let us define $\tau(x)$ by the formula

$$[\tau(x)](f(\varphi(x))) = \Phi(f)(x) \quad (f \in C_0(X, \mathcal{B}(\mathcal{H}))). \quad (5)$$

Because of (4) we obtain that $\tau(x)$ is a well-defined injective linear map on $\mathcal{B}(\mathcal{H})$. Since Φ is surjective, we have the surjectivity of $\tau(x)$. Now, we compute

$$\begin{aligned} & [\tau(x)](f(\varphi(x)) g(\varphi(x))^* f(\varphi(x))) \\ &= \Phi(fg^*f)(x) \\ &= \Phi(f)(x) \Phi(g)(x)^* \Phi(f)(x) \\ &= [\tau(x)](f(\varphi(x))) ([\tau(x)](g(\varphi(x))))^* [\tau(x)](f(\varphi(x))) \end{aligned}$$

for every $f, g \in C_0(X, \mathcal{B}(\mathcal{H}))$. This implies that $\tau(x)$ is a triple automorphism of $\mathcal{B}(\mathcal{H})$. Since the triple homomorphisms preserve the partial isometries and every operator with norm less than 1 is the average of unitaries, it follows that $\tau(x)$ is a contraction. Applying the same argument to the inverse of $\tau(x)$, we obtain that $\tau(x) \in \text{Iso}(\mathcal{B}(\mathcal{H}))$. This proves that Φ is of the form (1) given in the statement of our theorem.

Let now $\Phi: C_0(X, \mathcal{B}(\mathcal{H})) \rightarrow C_0(X, \mathcal{B}(\mathcal{H}))$ be a linear map of the form

$$\Phi(f)(x) = [\tau(x)](f(\varphi(x))) \quad (f \in C_0(X, \mathcal{B}(\mathcal{H})), x \in X), \quad (6)$$

where $\tau: X \rightarrow \text{Iso}(\mathcal{B}(\mathcal{H}))$ and $\varphi: X \rightarrow X$ is a bijection. The function φ is continuous. Indeed, this follows easily from the equality $\|f(\varphi(x))\| = \|\Phi(f)(x)\|$ and from Uryson's lemma. To see the strong continuity of $\tau: X \rightarrow \text{Iso}(\mathcal{B}(\mathcal{H}))$, let (x_α) be a net in X converging to $x \in X$. Let $y_\alpha = \varphi(x_\alpha)$, $y = \varphi(x)$. We may suppose that every y_α belongs to a fixed

compact neighbourhood of y . If $f \in C_0(X)$ is identically 1 on this neighbourhood, then for every operator $A \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned} [\tau(x_\alpha)](A) &= [\tau(x_\alpha)](f(\varphi(x_\alpha)) A) = \Phi(fA)(x_\alpha) \rightarrow \Phi(fA)(x) \\ &= [\tau(x)](f(\varphi(x)) A) = [\tau(x)](A). \end{aligned}$$

Next, from the equality

$$\begin{aligned} &\|[\tau(x_\alpha)^{-1}](A) - [\tau(x)^{-1}](A)\| \\ &= \|[\tau(x_\alpha)^{-1} \tau(x) \tau(x)^{-1}](A) - [\tau(x)^{-1}](A)\| \\ &= \|[\tau(x)]([\tau(x)^{-1}](A)) - [\tau(x_\alpha)]([\tau(x)^{-1}](A))\| \end{aligned}$$

we get the strong continuity of the map $x \mapsto \tau(x)^{-1}$. We prove that φ^{-1} is also continuous. Since Φ maps into $C_0(X, \mathcal{B}(\mathcal{H}))$, it is quite easy to see from (6) that $f \circ \varphi \in C_0(X)$ holds true for every $f \in C_0(X)$. If $K \subset X$ is an arbitrary compact set and $f \in C_0(X)$ is a function which is identically 1 on K , then it follows from $f \circ \varphi \in C_0(X)$ that there exists a compact set $K' \subset X$ for which $\varphi(x) \in K^c$ holds true for all $x \in K'^c$. Thus, we have $K \subset \varphi(K')$. Let (x_α) be a net in X such that $(\varphi(x_\alpha))$ converges to some $\varphi(x)$. Obviously, we may suppose that every $\varphi(x_\alpha)$ belongs to a compact neighbourhood K of $\varphi(x)$. By what we have just seen, there exists a compact set $K' \subset X$ which contains the net (x_α) and the point x as well. Since K' is compact, the net (x_α) has a convergent subnet. Because of the continuity of the bijection φ , it is easy to see that the limit of this subnet is x . The continuity of φ^{-1} is now apparent. Finally, one can verify quite readily that Φ is a surjective linear isometry of $C_0(X, \mathcal{B}(\mathcal{H}))$.

Let us turn to the proof of our statement concerning automorphisms. So, let Φ be an automorphism of $C_0(X, \mathcal{B}(\mathcal{H}))$. Every automorphism Ψ of a C^* -algebra \mathcal{A} is continuous and its norm equals the norm of its inverse. This follows, for example, from [Sak, 4.1.12. Lemma] and from the proof of [Sak, 4.1.13. Proposition] where it is proved that $\|a\|/\|\Psi\| \leq \|\Psi(a)\|$ ($a \in \mathcal{A}$) which implies that $\|\Psi^{-1}\| \leq \|\Psi\|$. Using these facts, one can get the form (1) in a way very similar to that was followed in the case of isometries. Let now $\Phi: C_0(X, \mathcal{B}(\mathcal{H})) \rightarrow C_0(X, \mathcal{B}(\mathcal{H}))$ be a linear map of the form

$$\Phi(f)(x) = [\tau(x)](f(\varphi(x))) \quad (f \in C_0(X, \mathcal{B}(\mathcal{H})), x \in X), \quad (7)$$

where $\tau: X \rightarrow \text{Aut}(\mathcal{B}(\mathcal{H}))$ and $\varphi: X \rightarrow X$ is a bijection. We show that φ is continuous. Let (x_α) be a net in X converging to $x \in X$. By (7) we have

$$f(\varphi(x_\alpha)) I = [\tau(x_\alpha)](f(\varphi(x_\alpha)) I) \rightarrow [\tau(x)](f(\varphi(x)) I) = f(\varphi(x)) I$$

for every $f \in C_0(X)$. Referring to Uryson's lemma again, we infer that $\varphi(x_\alpha) \rightarrow \varphi(x)$. This verifies the continuity of φ . We claim that the function τ is bounded. In fact, by the principle of uniform boundedness, in the opposite case we would obtain that there exists an operator $A \in \mathcal{B}(\mathcal{H})$ for which $[\tau(\cdot)](A)$ is not bounded. Then there is a sequence (x_n) in X with the property that $\|[\tau(x_n)](A)\| > n^3$ ($n \in \mathbb{N}$). Using Uryson's lemma, it is an easy task to construct a nonnegative function $f \in C_0(X)$ for which $f(\varphi(x_n)) \geq 1/n^2$. Indeed, for every $n \in \mathbb{N}$ let $f_n: X \rightarrow [0, 1]$ be a continuous function with compact support such that $f_n(\varphi(x_n)) = 1$ and define $f = \sum_n (1/n^2) f_n$. We have $\|\Phi(fA)(s_n)\| = \|f(\varphi(x_n))[\tau(x_n)](A)\| > n$ ($n \in \mathbb{N}$) which contradicts the boundedness of the function $\Phi(fA)$. The strong continuity of τ can be proved as it was done in the case of isometries. Using the inequality

$$\begin{aligned} & \|[\tau(x_\alpha)^{-1}](A) - [\tau(x)^{-1}](A)\| \\ &= \|[\tau(x_\alpha)^{-1} \tau(x) \tau(x)^{-1}](A) - [\tau(x)^{-1}](A)\| \\ &\leq \|\tau(x_\alpha)^{-1}\| \|[\tau(x)]([\tau(x)^{-1}](A)) - [\tau(x_\alpha)]([\tau(x)^{-1}](A))\| \\ &= \|\tau(x_\alpha)\| \|[\tau(x)]([\tau(x)^{-1}](A)) - [\tau(x_\alpha)]([\tau(x)^{-1}](A))\| \end{aligned}$$

and the boundedness of τ , we get the strong continuity of the map $x \mapsto \tau(x)^{-1}$. The proof can be completed as in the case of isometries. ■

The following two lemmas are needed in the proof of Theorem 2.

LEMMA 2.2. *Let τ, τ_1, τ_2 be automorphisms of $\mathcal{B}(\mathcal{H})$ and let $\lambda \in \mathbb{C}, 0 \neq \lambda_1, \lambda_2 \in \mathbb{C}$ be scalars so that*

$$\lambda\tau(A) = \lambda_1\tau_1(A) + \lambda_2\tau_2(A) \quad (A \in \mathcal{B}(\mathcal{H})).$$

Then we have $\tau_1 = \tau_2$.

Proof. Since the automorphisms of $\mathcal{B}(\mathcal{H})$ are all spatial (see, for example, [Che, 3.2. Corollary]), hence there exist invertible operators $T, T_1, T_2 \in \mathcal{B}(\mathcal{H})$ such that

$$\lambda TAT^{-1} = \lambda_1 T_1AT_1^{-1} + \lambda_2 T_2AT_2^{-1} \quad (A \in \mathcal{B}(\mathcal{H})). \tag{8}$$

It is apparent that if $a, b, x, y, u, v \in X$ and

$$a \otimes b = x \otimes y + u \otimes v,$$

then either $\{x, u\}$ or $\{y, v\}$ is linearly dependent. Using this elementary observation and putting $A = x \otimes y$ into (8), we infer that either $\{T_1x, T_2x\}$ is linearly dependent for all $x \in \mathcal{H}$ or $\{T_1^{-1*}y, T_2^{-1*}y\}$ is linearly dependent

for all $y \in \mathcal{H}$. In both cases we have the linear dependence of $\{T_1, T_1\}$ which results in $\tau_1 = \tau_2$. ■

In the proof of the next lemma we need the concept of Jordan homomorphisms. A linear map ϕ between algebras \mathcal{A} and \mathcal{B} is called a Jordan homomorphism if it satisfies

$$\phi(A)^2 = \phi(A^2) \quad (A \in \mathcal{A}).$$

If, in addition, \mathcal{A} and \mathcal{B} have involutions and

$$\phi(A)^* = \phi(A^*) \quad (A \in \mathcal{A}),$$

then we say that ϕ is a Jordan *-homomorphism.

LEMMA 2.3. *Let $\Phi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a bounded linear map with the property that for every $A \in \mathcal{B}(\mathcal{H})$ there exist a number $\lambda_A \in \mathbb{C}$ and an automorphism $\tau_A \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ so that $\Phi(A) = \lambda_A \tau_A(A)$. Then there exist a number $\lambda \in \mathbb{C}$ and an automorphism $\tau \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ such that $\Phi(A) = \lambda \tau(A)$ ($A \in \mathcal{B}(\mathcal{H})$).*

Proof. First suppose that $\Phi(I) = 0$. Assume that there exists a projection $0, I \neq P \in \mathcal{B}(\mathcal{H})$ for which $\Phi(P) \neq 0$. Applying an appropriate transformation, we may suppose that $\Phi(P) = P$. Then we have $\Phi(I - P) = -P$. If ε, δ are different nonzero numbers, then by our assumption we infer that $\Phi(\varepsilon P + \delta(I - P))$ is a scalar multiple of an invertible operator which, on the other hand, equals $(\varepsilon - \delta)P$. This clearly implies that $\varepsilon = \delta$, which is a contradiction. Hence, we obtain that $\Phi(P) = 0$ holds true for every projection $P \in \mathcal{B}(\mathcal{H})$. Using the spectral theorem and the continuity of Φ , we conclude that $\Phi = 0$.

Next suppose that $\Phi(I) \neq 0$. Apparently, we may assume that $\Phi(I) = I$. By the linearity of Φ , for an arbitrary projection $0, I \neq P \in \mathcal{B}(\mathcal{H})$ we obtain

$$I = \Phi(I) = \Phi(P) + \Phi(I - P) = \lambda_P Q + \lambda_{I-P} R,$$

where Q, R are idempotents different from $0, I$. Taking squares on both sides in the equality

$$I - \lambda_{I-P} R = \lambda_P Q,$$

we have

$$I + \lambda_{I-P}^2 R - 2\lambda_{I-P} R = \lambda_P^2 Q.$$

But we also have

$$\lambda_P(I - \lambda_{I-P} R) = \lambda_P^2 Q.$$

Comparing these equalities and using $R \neq 0, I$, we deduce that $\lambda_P = 1$. This means that $\Phi(P)$ is an idempotent. Therefore, Φ sends projections to idempotents. Now, a standard argument shows that Φ is a Jordan endomorphism of $\mathcal{B}(\mathcal{H})$ (see, for example, the proof of [Moll, Theorem 2]). Clearly, the range of Φ contains a rank-one operator (e.g. a rank-one idempotent) and an operator with dense range (e.g. the identity). Using our former result [Moll, Theorem 1], we infer that Φ is either an automorphism or an antiautomorphism. This latter concept means that Φ is a bijective linear map with the property that $\Phi(AB) = \Phi(B)\Phi(A)$ ($A, B \in \mathcal{B}(\mathcal{H})$). But Φ cannot be an antiautomorphism. In fact, in this case we would obtain that the image $\Phi(S)$ of a unilateral shift S has a right inverse. But, on the other hand, since Φ is locally a scalar multiple of an automorphism of $\mathcal{B}(\mathcal{H})$, it follows that $\Phi(S)$ is not right invertible. This contradiction justifies our assertion. ■

Before proving Theorem 2 we recall that the automorphisms of the function algebra $C_0(X)$ are of the form $f \mapsto f \circ \varphi$, where $\varphi: X \rightarrow X$ is a homeomorphism.

Proof of Theorem 2. Let $\Phi: C_0(X, \mathcal{B}(\mathcal{H})) \rightarrow C_0(X, \mathcal{B}(\mathcal{H}))$ be a local automorphism of $C_0(X, \mathcal{B}(\mathcal{H}))$, i.e. Φ is a bounded linear map which agrees with some automorphism at each point in $C_0(X, \mathcal{B}(\mathcal{H}))$. By Theorem 1, for every $f \in C_0(X, \mathcal{B}(\mathcal{H}))$ there exist a homeomorphism $\varphi_f: X \rightarrow X$ and a function $\tau_f: X \rightarrow \text{Aut}(\mathcal{B}(\mathcal{H}))$ such that

$$\Phi(f)(x) = [\tau_f(x)](f(\varphi_f(x))) \quad (x \in X).$$

It follows that for every $f \in C_0(X)$ there exists a homeomorphism $\psi_f: X \rightarrow X$ for which $\Phi(fI) = (f \circ \psi_f)I$. Since, by assumption, the automorphism group of $C_0(X)$ is reflexive, we obtain that there is a homeomorphism $\varphi: X \rightarrow X$ for which

$$\Phi(fI) = (f \circ \varphi)I \quad (f \in C_0(X)). \quad (9)$$

Let $f \in C_0(X)$ and $x \in X$. Consider the linear map $\Psi: A \mapsto \Phi(fA)(x)$ on $\mathcal{B}(\mathcal{H})$. From the form (1) of the automorphisms of $C_0(X, \mathcal{B}(\mathcal{H}))$ it follows that Ψ has the property that for every $A \in \mathcal{B}(\mathcal{H})$ there exist a number λ_A and an automorphism $\tau_A \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ such that

$$\Psi(A) = \lambda_A \tau_A(A).$$

Now, Lemma 2.3 tells us that there exist functions $\tau_f: X \rightarrow \text{Aut}(\mathcal{B}(\mathcal{H}))$ and $\lambda_f: X \rightarrow \mathbb{C}$ such that

$$\Phi(fA)(x) = [\tau_f(x)](\lambda_f(x) A) \quad (f \in C_0(X), A \in \mathcal{B}(\mathcal{H}), x \in X).$$

From (9) we obtain that $\lambda_f = f \circ \varphi$ and hence we have

$$\Phi(fA)(x) = [\tau_f(x)](f(\varphi(x)) A) \quad (f \in C_0(X), A \in \mathcal{B}(\mathcal{H}), x \in X). \quad (10)$$

Let $x \in X$ be fixed for a moment. Pick functions $f, g \in C_0(X)$ with the property that $f(\varphi(x)), g(\varphi(x)) \neq 0$. Because of linearity we get

$$\begin{aligned} & [\tau_f(x)](f(\varphi(x)) A) + [\tau_g(x)](g(\varphi(x)) A) \\ &= \Phi(fA)(x) + \Phi(gA)(x) = \Phi((f+g)A)(x) \\ &= [\tau_{f+g}(x)](f(\varphi(x)) A + g(\varphi(x)) A) \quad (A \in \mathcal{B}(\mathcal{H})). \end{aligned}$$

Using Lemma 2.2 we infer that $\tau_f(x) = \tau_g(x)$. By the formula (10) it follows readily that there is a function $\tau: X \rightarrow \text{Aut}(\mathcal{B}(\mathcal{H}))$ for which

$$\Phi(fA)(x) = [\tau(x)](f(\varphi(x)) A) \quad (f \in C_0(X), A \in \mathcal{B}(\mathcal{H}), x \in X). \quad (11)$$

Since the linear span of the set of functions fA ($f \in C_0(X), A \in \mathcal{B}(\mathcal{H})$) is dense in $C_0(X, \mathcal{B}(\mathcal{H}))$ (see, for example, [Mur, 6.4.16. Lemma]), the equality in (11) gives us that

$$\Phi(f)(x) = [\tau(x)](f(\varphi(x))) \quad (x \in X)$$

holds true for every $f \in C_0(X, \mathcal{B}(\mathcal{H}))$. By Theorem 1, the proof is complete. \blacksquare

The next lemma that we shall make use in the proof of Theorem 3 states that every bounded linear map on $\mathcal{B}(\mathcal{H})$ which is locally a scalar multiple of a surjective isometry, equals globally a scalar multiple of a surjective isometry. For the proof we recall the folk result (in fact this is a consequence of a theorem of Kadison) that every surjective linear isometry of $\mathcal{B}(\mathcal{H})$ is either of the form

$$A \mapsto UAV$$

or of the form

$$A \mapsto UA^{tr}V,$$

where U, V are unitary operators and tr denotes the transpose with respect to an arbitrary but fixed complete orthonormal system in \mathcal{H} . In what follows $\mathcal{P}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ denote the set of all projections and all unitaries on \mathcal{H} , respectively.

LEMMA 2.4. *Let $\Phi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a bounded linear map with the property that for every $A \in \mathcal{B}(\mathcal{H})$ there exist a number $\lambda_A \in \mathbb{C}$ and a*

surjective linear isometry $\tau_A \in \text{Iso}(\mathcal{B}(\mathcal{H}))$ so that $\Phi(A) = \lambda_A \tau_A(A)$. Then there exist a number $\lambda \in \mathbb{C}$ and a surjective linear isometry $\tau \in \text{Iso}(\mathcal{B}(\mathcal{H}))$ for which $\Phi(A) = \lambda \tau(A)$ ($A \in \mathcal{B}(\mathcal{H})$).

Proof. Just as in the proof of Lemma 2.3, first suppose that $\Phi(I) = 0$. Assume that there exists a projection $0, I \neq P \in \mathcal{B}(\mathcal{H})$ for which $\Phi(P) \neq 0$. Apparently, we may suppose that $\Phi(P) = P$. Then we have $\Phi(I - P) = -P$. Since for any different nonzero numbers $\varepsilon, \delta \in \mathbb{C}$, the operator $\varepsilon P + \delta(I - P)$ is invertible, we obtain that $(\varepsilon - \delta)P = \Phi(\varepsilon P + \delta(I - P))$ is a scalar multiple of an invertible operator. But this is a contradiction and hence we have $\Phi(P) = 0$ for every projection P . This gives us that $\Phi = 0$.

So, let us suppose that $\Phi(I) \neq 0$. Clearly, we may assume that $\Phi(I) = I$ and that the constants λ_A are all nonnegative. Let $P \neq 0, I$ be a projection. Let λ, μ be nonnegative numbers and let U, V be partial isometries for which $\Phi(P) = \lambda U, \Phi(I - P) = \mu V$. We have

$$\lambda U + \mu V = I \quad \text{and} \quad \varepsilon \lambda U + \delta \mu V \in \mathbb{C} \mathcal{U}(\mathcal{H}) \quad (|\varepsilon| = |\delta| = 1). \quad (12)$$

Since $P \neq 0, I$, it follows that $\lambda, \mu > 0$. Choose different ε and δ with $|\varepsilon| = |\delta| = 1$. Since by (12) it follows that the operator

$$\delta I + (\varepsilon - \delta) \lambda U = \varepsilon \lambda U + \delta(I - \lambda U) = \varepsilon \lambda U + \delta \mu V$$

is normal, we obtain that U and then that V are both normal partial isometries. Therefore, U has a matrix representation

$$U = \begin{bmatrix} U_0 & 0 \\ 0 & 0 \end{bmatrix}$$

where U_0 is unitary on a proper closed linear subspace \mathcal{H}_0 of \mathcal{H} . In accordance with (12), we have the following matrix representation of V

$$V = \begin{bmatrix} (I - \lambda U_0)/\mu & 0 \\ 0 & I/\mu \end{bmatrix}.$$

Using the characteristic property $VV^*V = V$ of partial isometries, we get that $\mu = 1$ and, by symmetry, that $\lambda = 1$. Taking the matrix representations above into account, it is easy to see that $I - U_0$ is a normal partial isometry and that $\varepsilon U_0 + \delta(I - U_0)$ is a scalar multiple of a unitary operator for every $\varepsilon, \delta \in \mathbb{C}$ with $|\varepsilon| = |\delta| = 1$. Since $I - U_0$ is a normal partial isometry, the spectrum of U_0 must consist of such numbers c of modulus one, for which either $1 - c$ has modulus one or $1 - c = 0$. This gives us that $\sigma(U_0) \subset \{1, e^{i\pi/3}, e^{-i\pi/3}\}$. Let P_1, P_2, P_3 denote the projections onto the subspaces

$\ker(U_0 - I)$, $\ker(U_0 - e^{i\pi/3}I)$, $\ker(U_0 - e^{-i\pi/3}I)$ of \mathcal{H}_0 , respectively. We assert that two of the operators P_1, P_2, P_3 are necessarily zero. In fact, if for example, $P_2, P_3 \neq 0$, then it follows from the second property in (12) that

$$|\varepsilon e^{i\pi/3} + \delta e^{-i\pi/3}| = |\varepsilon e^{-i\pi/3} + \delta e^{i\pi/3}|$$

for every, ε, δ of modulus one. But this is an obvious contradiction. The other cases can be treated in a similar way. Therefore, we have $\Phi(P) = U \in \{1, e^{i\pi/3}, e^{-i\pi/3}\} \mathcal{P}(\mathcal{H})$ for every projection P on \mathcal{H} . Now, let P be a projection having infinite rank and infinite corank. Since in this case P is unitarily equivalent to $I - P$, it follows that P and $I - P$ can be connected by a continuous curve within the set of projections. Consequently, we obtain that $\Phi(P)$ and $\Phi(I - P)$ have the same nonzero eigenvalue. Since $\Phi(I - P) = I - \Phi(P)$, it follows that this eigenvalue is 1. Thus we obtain that $\Phi(P)$ is a projection. If P is a finite rank projection, then P is the difference of two projections having infinite rank and corank. Then we obtain that $\Phi(P)$ is difference of two projections and consequently $\Phi(P)$ is self-adjoint. On the other hand, we have $\Phi(P) \in \{1, e^{i\pi/3}, e^{-i\pi/3}\} \mathcal{P}(\mathcal{H})$. These result in $\Phi(P) \in \mathcal{P}(\mathcal{H})$ and we deduce that Φ sends every projection to a projection. It now follows that Φ is a Jordan *-endomorphism of $\mathcal{B}(\mathcal{H})$. Since, by our condition, the range of Φ contains a rank-one operator and an operator with dense range, using [Mol1, Theorem 1] again, we infer that Φ is either a *-automorphism or a *-antiautomorphism of $\mathcal{B}(\mathcal{H})$. In both cases we obtain that Φ is a surjective isometry of $\mathcal{B}(\mathcal{H})$ and this completes the proof. ■

LEMMA 2.5. *If $\mathcal{M} \subset \mathbb{C}\mathcal{U}(\mathcal{H})$ is a linear subspace, then \mathcal{M} is either 1-dimensional or 0-dimensional.*

Proof. In the first version of the paper we gave a direct proof of this lemma. We are grateful to the referee who kindly pointed out that this is just a simple consequence of [RaRo, Remark iii, p. 691] that asserts that every linear space of normal operators is commutative. Nevertheless, to get a completely elementary and trivial proof one can argue as follows. Let $A, B \in \mathcal{M}$. For every $\lambda \in \mathbb{C}$ we have $(A + \lambda B)^*(A + \lambda B) \in \mathbb{C}I$. Since A^*A, B^*B are scalar, choosing $\lambda = 1$ and then $\lambda = i$, it follows easily that A^*B is also scalar. This clearly gives us that A, B are linearly dependent. ■

LEMMA 2.6. *Let X be a locally compact Hausdorff space. Let $\mathcal{M} \subset \mathbb{C}^X$ be a linear subspace containing a nowhere vanishing function $f_0 \in \mathcal{M}$ and having the property that $|f| \in C_0(X)$ for every $f \in \mathcal{M}$. Then there is a function $t: X \rightarrow \mathbb{C}$ of modulus one such that $t\mathcal{M} \subset C_0(X)$.*

Proof. We know that the function $|f + f_0|^2 - |f|^2 - |f_0|^2$ is continuous for every $f \in \mathcal{M}$. This gives us that $\overline{ff_0}$ is continuous for every $f \in \mathcal{M}$. Let $t = |f_0|/\overline{f_0}$. Then we have $|t| = 1$ and the function $(tf)|f_0| = (tf)(\overline{tf_0}) = \overline{ff_0}$ is continuous. Consequently, we obtain $tf \in C_0(X)$. ■

For the proof of Theorem 3 we recall the well-known Banach–Stone theorem stating that the surjective isometries of the function algebra $C_0(X)$ are all of the form $f \mapsto \tau \cdot f \circ \varphi$, where $\tau: X \rightarrow \mathbb{C}$ is a continuous function of modulus one and $\varphi: X \rightarrow X$ is a homeomorphism.

Proof of Theorem 3. Let $\Phi: C_0(X, \mathcal{B}(\mathcal{H})) \rightarrow C_0(X, \mathcal{B}(\mathcal{H}))$ be a local surjective isometry. Pick a function $f \in C_0(X)$ and a point $x \in X$, and consider the linear map $\Psi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$

$$\Psi: A \mapsto \Phi(fA)(x).$$

It follows from Theorem 1 that for every $A \in \mathcal{B}(\mathcal{H})$ there exist a number λ_A and a surjective isometry $\tau_A \in \text{Iso}(\mathcal{B}(\mathcal{H}))$ such that $\Psi(A) = \lambda_A \tau_A(A)$. By Lemma 2.4 we infer that there exist a nonnegative number $\lambda_{f,x}$ and a surjective linear isometry $\tau_{f,x} \in \text{Iso}(\mathcal{B}(\mathcal{H}))$ for which

$$\Phi(fA)(x) = \lambda_{f,x} \tau_{f,x}(A) \quad (13)$$

holds true for every $f \in C_0(X)$, $A \in \mathcal{B}(\mathcal{H})$ and $x \in X$. Now, let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator and $x \in X$. The linear map

$$f \mapsto \Phi(fU)(x)$$

maps $C_0(X)$ into $\mathbb{C}\mathcal{U}(\mathcal{H})$. Since the range of this map is a linear subspace, by Lemma 2.5 we infer that it is either 1-dimensional or 0-dimensional. Thus there is a linear functional $F_{U,x}: C_0(X) \rightarrow \mathbb{C}$ and a unitary operator $[\tau(x)](U)$ such that

$$\Phi(fU)(x) = F_{U,x}(f)[\tau(x)](U) \quad (f \in C_0(X), U \in \mathcal{U}(\mathcal{H}), x \in X).$$

Clearly, the map $F_U: C_0(X) \rightarrow \mathbb{C}^X$ defined by $F_U(f)(x) = F_{U,x}(f)$ is linear and we have

$$\Phi(fU)(x) = F_U(f)(x)[\tau(x)](U) \quad (f \in C_0(X), U \in \mathcal{U}(\mathcal{H}), x \in X). \quad (14)$$

Since Φ is a local surjective isometry of $C_0(X, \mathcal{B}(\mathcal{H}))$, it follows from Theorem 1 that for every $f \in C_0(X)$ there exist a strongly continuous function $\tau_{f,U}: X \rightarrow \text{Iso}(\mathcal{B}(\mathcal{H}))$ and a homeomorphism $\varphi_{f,U}: X \rightarrow X$ such that

$$\Phi(fU)(x) = f(\varphi_{f,U}(x))[\tau_{f,U}(x)](U) \quad (x \in X). \quad (15)$$

Apparently, we have $|F_U(f)| = |f| \circ \varphi_{f,U}$. Because of the σ -compactness of X , it is a quite easy consequence of Uryson's lemma that there exists a strictly positive function in $C_0(X)$. Therefore, the range of F_U contains a nowhere vanishing function and has the property that the absolute value of every function belonging to this range is continuous. By Lemma 2.6, there exists a function $t: X \rightarrow \mathbb{C}$ of modulus one such that the functions $tF_U(f)$ are all continuous ($f \in C_0(X)$). Consequently, we may suppose that the map F_U in (14) maps $C_0(X)$ into itself. Comparing (14) and (15) we have

$$F_U(f)(x)[\tau(x)](U) = f(\varphi_{f,U}(x))[\tau_{f,U}(x)](U) \quad (x \in X). \quad (16)$$

If $f \in C_0(X)$ is a nowhere vanishing function, then by the continuity of the functions $F_U(f)$, $f \circ \varphi_{f,U}$ and $[\tau_{f,U}(\cdot)](U)$, it follows that $[\tau(\cdot)](U)$ is also continuous. From (16) we have

$$F_U(f) = f(\varphi_{f,U}(x))[\tau_{f,U}(x)](U)[\tau(x)](U)^* \quad (x \in X).$$

In particular, this implies that the function

$$x \mapsto [\tau_{f,U}(x)](U)[\tau(x)](U)^*$$

can be considered as a continuous scalar valued function of modulus one. Hence, F_U is a local surjective isometry of $C_0(X)$. By our assumption this means that F_U is a surjective isometry, i.e. there exist a continuous function $t_U: X \rightarrow \mathbb{C}$ of modulus one and a homeomorphism $\varphi_U: X \rightarrow X$ such that $F_U(f) = t_U \cdot f \circ \varphi_U$ ($f \in C_0(X)$, $U \in \mathcal{U}(\mathcal{H})$). Having a look at (14), it is obvious that we may suppose that Φ satisfies

$$\Phi(fU)(x) = f(\varphi_U(x))[\tau(x)](U) \quad (f \in C_0(X), U \in \mathcal{U}(\mathcal{H}), x \in X),$$

where $[\tau(x)](U)$ is unitary. If $f \in C_0(X)$ is nonnegative, we see from (13) that

$$f(\varphi_U(x)) = \lambda_{f,x} = f(\varphi_I(x))$$

and

$$[\tau(x)](U) = \tau_{f,x}(U) \quad (U \in \mathcal{U}(\mathcal{H}), x \in X).$$

This verifies the existence of a homeomorphism φ of X and, due to the fact that every operator in $\mathcal{B}(\mathcal{H})$ is a linear combination of unitaries, the existence of a function $\tau: X \rightarrow \text{Iso}(\mathcal{B}(\mathcal{H}))$ for which

$$\Phi(fU)(x) = f(\varphi(x))[\tau(x)](U) \quad (U \in \mathcal{U}(\mathcal{H}), x \in X)$$

holds true for every nonnegative function $f \in C_0(X)$. Since every function in $C_0(X)$ is the linear combination of nonnegative functions in $C_0(X)$, we finally obtain that

$$\Phi(fA)(x) = f(\varphi(x))[\tau(x)](A) \quad (f \in C_0(X), A \in \mathcal{B}(\mathcal{H}), x \in X).$$

Referring to the fact once again that the linear span of the elementary tensors fA ($f \in C_0(X), A \in \mathcal{B}(\mathcal{H})$) is dense in $C_0(X, \mathcal{B}(\mathcal{H}))$, we arrive at the form

$$\Phi(f)(x) = [\tau(x)](f(\varphi(x))) \quad (f \in C_0(X, \mathcal{B}(\mathcal{H})), x \in X).$$

By Theorem 1, the proof is complete. \blacksquare

We now turn to the proof of Theorem 4. The next result describes the form of local surjective isometries of the function algebra $C_0(X)$.

LEMMA 2.7. *Let X be a first countable locally compact Hausdorff space. Let $F: C_0(X) \rightarrow C_0(X)$ be a local surjective isometry. Then there exist a continuous function $t: X \rightarrow \mathbb{C}$ of modulus one and a homeomorphism g of X onto a subspace of X so that*

$$F(f) \circ g = t \cdot f \quad (f \in C_0(X)). \quad (17)$$

Proof. By Banach–Stone theorem on the form of surjective linear isometries of $C_0(X)$ it follows that for every $f \in C_0(X)$ there exist a homeomorphism $\varphi_f: X \rightarrow X$ and a continuous function $\tau_f: X \rightarrow \mathbb{C}$ of modulus one such that

$$F(f) = \tau_f \cdot f \circ \varphi_f. \quad (18)$$

For any $x \in X$ let \mathcal{L}_x denote the set of all functions $p \in C_0(X)$ which map into the interval $[0, 1]$, $p(x) = 1$ and $p(y) < 1$ for every $x \neq y \in X$. By Uryson's lemma and the first countability of X , it is easy to verify that \mathcal{L}_x is nonempty. Let $p, p' \in \mathcal{L}_x$. By (18) there exist $y, y' \in X$ for which $|F(p)| \in \mathcal{L}_y$, $|F(p')| \in \mathcal{L}_{y'}$. Similarly, since $(p + p')/2 \in \mathcal{L}_x$, there is a point $y'' \in X$ for

which $|F((p + p')/2)| \in \mathcal{S}_{y''}$. Apparently, we have $y = y'$ and $F(p)(y) = F(p')(y')$. This shows that there are functions $t: X \rightarrow \mathbb{C}$ and $g: X \rightarrow X$ such that

$$t(x) = F(p)(g(x)) \quad (19)$$

holds true for every $x \in X$ and $p \in \mathcal{S}_x$. Clearly, $|t(x)| = 1$. Pick $x \in X$. It is easy to see that for any strictly positive function $f \in C_0(X)$ with $f(x) = 1$ we have a function $p \in \mathcal{S}_x$ such that $p(y) < f(y)$ ($x \neq y \in X$). Now, let $f \in C_0(X)$ be an arbitrary nonnegative function. Then there is a positive constant c for which the function $y \mapsto c + f(x) - f(y)$ is positive. Hence, we can choose a function $p \in \mathcal{S}_x$ such that $cp(y) < cp(x) + f(x) - f(y)$ ($x \neq y \in X$). This means that the nonnegative function $cp + f$ takes its maximum only at x . By (19) we infer

$$t(x)(cp(x) + f(x)) = F(cp + f)(g(x)).$$

Clearly, we have

$$t(x)(cp(x)) = F(cp)(g(x)),$$

too. Therefore, we obtain

$$t \cdot f = F(f) \circ g \quad (20)$$

for every nonnegative f and then for every function in $C_0(X)$. We prove that g is a homeomorphism of X onto the range of g . To see this, first observe that for every function $p \in \mathcal{S}_y$ and net (y_α) in X , the condition that $p(y_\alpha) \rightarrow 1$ implies that $y_\alpha \rightarrow y$. Let (x_α) be a net in X converging to $x \in X$. Pick $p \in \mathcal{S}_x$. Since F is a local surjective isometry, we have a homeomorphism φ of X for which

$$p = |t \cdot p| = |F(p) \circ g| = p \circ \varphi \circ g.$$

Since this implies that $p(\varphi(g(x_\alpha))) \rightarrow 1$, we obtain $\varphi(g(x_\alpha)) \rightarrow x = \varphi(g(x))$ and hence we have $g(x_\alpha) \rightarrow g(x)$. So, g is continuous. The injectivity of g follows from (20) immediately using the fact that the nonnegative elements of $C_0(X)$ separate the points of X . As for the continuity of g^{-1} and t , these follow from (20) again and from Uryson's lemma. ■

Now, we are in a position to prove our last theorem.

Proof of Theorem 4. It is well-known that every open convex subset of \mathbb{R}^n is homeomorphic to the open unit ball B of \mathbb{R}^n . Hence, it is sufficient

to show that the automorphism and isometry groups of $C_0(B)$ are algebraically reflexive. Furthermore, by the form of the automorphisms and surjective isometries of the function algebra $C_0(X)$ we are certainly done if we prove the statement only for the isometry group. So, let $F: C_0(B) \rightarrow C_0(B)$ be a local surjective isometry. Then F is of the form (17). The only thing that we have to verify is that the function g appearing in this form is surjective. Consider the function $f \in C_0(B)$ defined by $f(x) = 1/(1 + \|x\|)$. Clearly, we may assume that $F(f) = f$. From (17) we infer that

$$\frac{1}{1 + \|x\|} = \frac{1}{1 + \|g(x)\|} \quad (x \in S).$$

Therefore, the continuous function g maps the surface S_r of the closed ball $r\bar{B}$ ($0 \leq r < 1$) into itself. It is obvious that every proper closed subset of S_r is homeomorphic to a subset of \mathbb{R}^{n-1} . By Borsuk–Ulam theorem we get that g takes the same value at some antipodal points of S_r . But this contradicts the injectivity of g . Consequently, the range of g contains every set S_r ($0 \leq r < 1$) which means that g is bijective. This completes the proof. ■

The proof of Theorem 4 shows how difficult it might be to treat our reflexivity problem for the suspension of arbitrary C^* -algebras. We mean the role of the use of Borsuk–Ulam theorem in the above argument. To reinforce this opinion, let us consider only the particular case of commutative C^* -algebras. Let X be a locally compact Hausdorff space and suppose that the automorphism and isometry groups of $C_0(X)$ are algebraically reflexive. If $F: C_0(\mathbb{R} \times X) \rightarrow C_0(\mathbb{R} \times X)$ is a local surjective isometry, then Lemma 2.7 gives the form of F . The problem is to verify that the function g appearing in (17) is surjective. This would be easy if there were an injective nonnegative function in $C_0(\mathbb{R} \times X)$. Unfortunately, this is not the case even when X is a singleton. Anyway, if $n \geq 3$, there is no injective function in $C_0(\mathbb{R}^n)$ at all. Therefore, to attack the problem of the surjectivity of g , we had to invent a different approach which was the use of Borsuk–Ulam theorem. To mention another point, it is easy to see that in general the automorphisms as well as the surjective isometries of the tensor product $C_0(X_1) \otimes C_0(X_2) \cong C_0(X_1 \times X_2)$ have nothing to do with the automorphisms and surjective isometries of $C_0(X_1)$ and $C_0(X_2)$, respectively. However, according to Theorem 1, in the case of the tensor product $C_0(X) \otimes \mathcal{B}(\mathcal{H})$ every automorphism as well as surjective isometry is an easily identifiable mixture of a “functional algebraic” and an “operator algebraic” part. This observation was of fundamental importance when verifying the result in Corollary 5. These might justify the suspicion why we feel our reflexivity problem really difficult even for the suspension of general commutative C^* -algebras.

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