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Differential Operators on Rational Projective Curves*

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Fix a singular, projective, rational curve $\mathscr{X}$ over an algebraically closed field k of characteristic zero with normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$, and write $\mathscr{D}(\mathscr{X}) = \Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{X}})$ for its ring of global differential operators. We prove

THEOREM 1. *The ring $\mathscr{D}(\mathscr{X})$ is a Noetherian domain, finitely generated as a k-algebra, with a unique minimal non-zero ideal $J(\mathscr{X})$. Moreover, $F(\mathscr{X}) = \mathscr{D}(\mathscr{X})/J(\mathscr{X})$ is finite-dimensional over k.*

In the case when π is injective we give complete description of the structure of $\mathscr{D}(\mathscr{X})$.

THEOREM 2. *If π is injective then $\mathscr{D}(\mathscr{X})$ is contained in a unique equivalent maximal order, S, and $J(\mathscr{X})$ is the unique non-zero ideal of S. Moreover,*

$$F(\mathscr{X}) \cong \begin{pmatrix} M_t(k) & k^{(t)} \\ 0 & k \end{pmatrix} \subset M_{t+1}(k) \cong S/J(\mathscr{X}).$$

Here, t is the arithmetic genus of $\mathscr{X}$. Finally, $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is a simple $\mathscr{D}(\mathscr{X})$-module.

It follows from Theorem 2 that, unlike the analogous result for affine curves, $\mathscr{D}(\mathscr{X})$ is not Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$. However, $\mathscr{D}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathscr{Y})$ for any singular, projective curve $\mathscr{Y}$ with injective normalisation map $\rho: \mathbb{P}^1 \to \mathscr{Y}$. We also show that the map $\Gamma(\mathscr{X}, \_): \mathscr{D}_{\mathscr{X}}$-**mod** $\to \mathscr{D}(\mathscr{X})$-**mod** is not exact. However, every other aspect of the Beilinson–Bernstein equivalence of categories between $\mathscr{D}_{\mathbb{P}^1}$-**mod** and $\mathscr{D}(\mathbb{P}^1)$-**mod** does have an analogue for $\mathscr{X}$. For example, $\mathscr{D}_{\mathscr{X}} \otimes \_$ is exact, quasi-coherent $\mathscr{D}_{\mathscr{X}}$-modules are generated by global sections, and $\mathscr{D}_{\mathscr{X}}$-**mod** is a quotient category of $\mathscr{D}(\mathscr{X})$-**mod**.

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Contents.

0. Introduction

0.1. Fix once and for all an algebraically closed field k of characteristic zero. All varieties will be irreducible, algebraic varieties over k. The ring of differential operators $\mathscr{D}(\mathscr{X})$ defined over an affine curve $\mathscr{X}$ was studied in [SS] and was shown to have many pleasant properties. In particular, it is a Noetherian domain, finitely generated as a k-algebra, and has a unique minimal non-zero ideal, $J(\mathscr{X})$. Moreover, the factor $\mathscr{D}(\mathscr{X})/J(\mathscr{X})$ is a finite-dimensional k-algebra. The situation is particularly pleasant when the normalisation map $\pi: \tilde{\mathscr{X}} \to \mathscr{X}$ is injective, as in this case $\mathscr{D}(\mathscr{X})$ is a simple ring and is even Morita equivalent to $\mathscr{D}(\tilde{\mathscr{X}})$. (Note that $\tilde{\mathscr{X}}$ is smooth and so, as is well known, $\mathscr{D}(\tilde{\mathscr{X}})$ is a simple Noetherian ring.) The basic technique in [SS] is to transfer ring-theoretic properties from $\mathscr{D}(\tilde{\mathscr{X}})$ to $\mathscr{D}(\mathscr{X})$ via the $\mathscr{D}(\mathscr{X})$-$\mathscr{D}(\tilde{\mathscr{X}})$-bimodule $\mathscr{D}(\tilde{\mathscr{X}}, \mathscr{X}) = \{\theta \in \mathscr{D}(\tilde{\mathscr{X}}): \theta \circ \mathcal{O}(\tilde{\mathscr{X}}) \subseteq \mathcal{O}(\mathscr{X})\}$. Here, $\circ$ denotes the action of the differential operator θ on $\mathcal{O}(\tilde{\mathscr{X}})$. (We use $\circ$ rather than $*$, as was used in [SS], since $*$ will be used frequently to denote a dual object.)

0.2. The basic aim of this paper is to study the analogous questions when $\mathscr{X}$ is a projective curve and $\mathscr{D}(\mathscr{X})$ is the ring of globally defined differential operators over $\mathscr{X}$.

Let us begin by formally defining the relevant objects for projective curves. Thus, let $\mathscr{X}$ be a projective curve with sheaf of regular functions $\mathcal{O}_{\mathscr{X}}$. Then, following Grothendieck [EGA], we first define $\mathscr{D}_{\mathscr{X}}$, the sheaf of rings of differential operators on $\mathscr{X}$. If U is an open affine subset of $\mathscr{X}$ set $\mathscr{D}^0_{\mathscr{X}}(U) = \mathcal{O}(U)$ and, inductively define

$$\mathscr{D}^n_{\mathscr{X}}(U) = \{\theta \in \operatorname{End}_k(\mathcal{O}_{\mathscr{X}}(U)): \theta a - a\theta \in \mathscr{D}^{n-1}_{\mathscr{X}}(U) \text{ for all } a \in \mathcal{O}_{\mathscr{X}}(U)\}.$$

Then $\mathscr{D}_{\mathscr{X}}(U) = \bigcup_n \mathscr{D}^n_{\mathscr{X}}(U)$ with multiplication given by composition of operators. Of course $\mathcal{O}_{\mathscr{X}}(U) \subseteq \mathscr{D}_{\mathscr{X}}(U)$ and $\mathscr{D}_{\mathscr{X}}$ is defined to be the unique quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module with sections $\mathscr{D}_{\mathscr{X}}(U)$ on an open affine subset U.

We denote by $\mathscr{D}(\mathscr{X})$ the global sections, $\Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{X}})$, of this sheaf, and call it simply the ring of differential operators on $\mathscr{X}$. The category of sheaves of (left) $\mathscr{D}_{\mathscr{X}}$-modules which are quasi-coherent as $\mathscr{O}_{\mathscr{X}}$-modules will be denoted by $\mathscr{D}_{\mathscr{X}}$-**mod**, while the category of left $\mathscr{D}(\mathscr{X})$-modules is denoted by $\mathscr{D}(\mathscr{X})$-**mod**.

If $\mathscr{X}$ has normalisation $\tilde{\mathscr{X}}$, with normalisation map $\pi: \tilde{\mathscr{X}} \to \mathscr{X}$, then, as in [SS], a basic technique in studying $\mathscr{D}(\mathscr{X})$ will be to relate it to $\mathscr{D}(\tilde{\mathscr{X}})$ via the $\mathscr{D}(\mathscr{X})$-$\mathscr{D}(\tilde{\mathscr{X}})$-bimodule $\mathscr{D}(\tilde{\mathscr{X}}, \mathscr{X})$. Roughly speaking, this is defined to be the global sections of the sheaf of differential operators from $\tilde{\mathscr{X}}$ to $\mathscr{X}$. More precisely, take an open affine cover $\{U_i\}$ of $\mathscr{X}$ and let $\tilde{U}_i = \pi^{-1}(U_i)$. We write

$$\mathscr{D}(\tilde{U}_i, U_i) = \{\theta \in \mathscr{D}_{\tilde{\mathscr{X}}}(\tilde{U}_i) : \theta \circ \mathscr{O}_{\tilde{\mathscr{X}}}(\tilde{U}_i) \subseteq \mathscr{O}_{\mathscr{X}}(U_i)\}$$

and

$$\mathscr{D}(\tilde{\mathscr{X}}, \mathscr{X}) = \bigcap_i \mathscr{D}(\tilde{U}_i, U_i).$$

The sheaf-theoretic definition of $\mathscr{D}(\tilde{\mathscr{X}}, \mathscr{X})$ is independent of the choice of open affine cover. The $\mathscr{D}(\tilde{\mathscr{X}})$-$\mathscr{D}(\mathscr{X})$-bimodule $\mathscr{D}(\mathscr{X}, \tilde{\mathscr{X}})$ is defined similarly and the details may be found in Section 2.

0.3. Any attempt to generalise the methods of [SS] to an arbitrary projective curve $\mathscr{X}$ soon runs into difficulties. The problem is that $\mathscr{D}_{\tilde{\mathscr{X}}}$ may not have many global sections; indeed if $\tilde{\mathscr{X}}$ is a smooth projective curve of genus at least two then it is an easy consequence of the Riemann–Roch Theorem that $\mathscr{D}(\tilde{\mathscr{X}})$ consists merely of the constant functions k, while if $\tilde{\mathscr{X}}$ has genus one then $\mathscr{D}(\tilde{\mathscr{X}}) \cong k[\theta]$. It follows from [Mn]—see in particular [Mn, Theorem D and Corollary 1.11]—that $\mathscr{D}(\mathscr{X})$ is similarly small when $\mathscr{X}$ has genus at least one.

0.4. Thus in this paper we shall restrict attention to a singular, rational, projective curve $\mathscr{X}$; so $\mathscr{X}$ now has normalisation the projective line $\mathbb{P}^1$ with normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$. In this case the problems mentioned above do not arise since $\mathscr{D}(\mathbb{P}^1)$ is an infinite-dimensional, primitive k-algebra. Indeed, it is an easy exercise to show that $\mathscr{D}(\mathbb{P}^1) \cong U(\mathfrak{sl}_2(k))/(\Omega)$, where Ω is the Casimir element. Furthermore, it is not difficult to show that $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ is a non-zero one-sided ideal of both $\mathscr{D}(\mathbb{P}^1)$ and $\mathscr{D}(\mathscr{X})$, and this allows one to transfer basic properties of $\mathscr{D}(\mathbb{P}^1)$ to $\mathscr{D}(\mathscr{X})$. In this way, one proves the following result (see Theorem 2.4).

Theorem A. *Let $\mathscr{X}$ be a rational projective curve. Then*

(a) *$\mathscr{D}(\mathscr{X})$ is a Noetherian domain of left and right (Gabriel–Rentschler) Krull dimension one.*

(b) *$\mathscr{D}(\mathscr{X})$ is a finitely generated k-algebra.*

(c) $\operatorname{End}_{\mathscr{D}(\mathscr{X})} M$ *is a finite-dimensional k-vector space for every $\mathscr{D}(\mathscr{X})$-module M of finite length.*

(d) *$\mathscr{D}(\mathscr{X})$ has a unique, minimal non-zero ideal $J(\mathscr{X})$. Moreover $F(\mathscr{X}) = \mathscr{D}(\mathscr{X})/J(\mathscr{X})$ is a finite-dimensional k-vector space.*

0.5. If $\mathscr{Y}$ is an affine curve then one of the significant results from [SS] is that $\mathscr{D}(\mathscr{Y})$ is simple and Morita equivalent to $\mathscr{D}(\tilde{\mathscr{Y}})$ if (and only if) the normalisation map $\pi: \tilde{\mathscr{Y}} \to \mathscr{Y}$ is injective. For a rational projective curve no such result is possible as one has the following result (see Proposition 2.10 and Theorem 3.19(a)).

Proposition B. *Let $\mathscr{X}$ be a singular, rational projective curve. Then $\mathscr{D}(\mathscr{X})$ is not Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$.*

If $\pi: \tilde{\mathscr{X}} \to \mathscr{X}$ is not injective then this proposition follows fairly easily from the analogous result for affine curves (see Section 2). However, if π is injective the reason is much more subtle and is intimately related to the next result (see Proposition 3.4).

Proposition B′. *If $\mathscr{X}$ is a singular, rational projective curve, then the global sections functor $\Gamma(\mathscr{X}, -)$: $\mathscr{D}_{\mathscr{X}}$-**mod** $\to \mathscr{D}(\mathscr{X})$-**mod** is not exact.*

0.6. For the rest of this introduction (and for the bulk of the paper) assume that the normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$ is injective. In order to explain the connection between Propositions B and B′, we need to recall the famous equivalence of categories of Beilinson and Bernstein [BB]. In the very special case of $\mathbb{P}^1$, this theorem states that every sheaf in $\mathscr{D}_{\mathbb{P}^1}$-**mod** is generated by its global sections and that the global sections functor $\Gamma(\mathbb{P}^1, \_): \mathscr{D}_{\mathbb{P}^1}$-**mod** $\to \mathscr{D}(\mathbb{P}^1)$-**mod** is exact. In other words the functor $\Gamma(\mathbb{P}^1, \_)$ makes the categories $\mathscr{D}_{\mathbb{P}^1}$-**mod** and $\mathscr{D}(\mathbb{P}^1)$-**mod** equivalent. The inverse functor is sheafification: $\mathscr{D}_{\mathbb{P}^1} \otimes \_$.

In order to relate this result to the structure of $\mathscr{D}(\mathscr{X})$-**mod**, we use the fact that the categories $\mathscr{D}_{\mathbb{P}^1}$-**mod** and $\mathscr{D}_{\mathscr{X}}$-**mod** are equivalent via the natural functors (see Section 3 or [SS, Sect. 6]). These two results imply that the following diagram is commutative.

$$\begin{array}{ccc} \mathscr{D}_{\mathscr{X}}\text{-}\mathbf{mod} & \longleftarrow & \mathscr{D}_{\mathbb{P}^1}\text{-}\mathbf{mod} \\ {\scriptstyle \mathscr{D}_{\mathscr{X}} \otimes \_} \uparrow & & \uparrow {\scriptstyle \mathscr{D}_{\mathbb{P}^1} \otimes \_} \\ \mathscr{D}(\mathscr{X})\text{-}\mathbf{mod} & \xrightarrow{\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \otimes \_} & \mathscr{D}(\mathbb{P}^1)\text{-}\mathbf{mod} \end{array} \qquad (\dagger)$$

Heuristically, if $\mathscr{D}(\mathscr{X})$ were Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$, then that equivalence would have to be via the module $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$. But, by (†), this would imply an equivalence of categories between $\mathscr{D}(\mathscr{X})$-**mod** and $\mathscr{D}_{\mathscr{X}}$-**mod**, contradicting Proposition B′.

In order to justify this heuristic argument, one needs a detailed understanding of the bimodules $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ and $Q = \mathscr{D}(\mathbb{P}^1, \mathscr{X})$ and this is given in Sections 3 and 4 of this paper. It is shown there that Q is a progenerator as a right $\mathscr{D}(\mathbb{P}^1)$-module and that $Q \cong P^* = \operatorname{Hom}_{\mathscr{D}(\mathbb{P}^1)}(P, \mathscr{D}(\mathbb{P}^1))$. On the other hand,

$$\mathscr{D}(\mathscr{X}) = \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} P \subseteq S(\mathscr{X}) = \operatorname{End} Q.$$

Thus, if $\mathscr{D}(\mathscr{X})$ were Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$, then it would be a maximal order and so equal to $S(\mathscr{X})$. Thus, any Morita equivalence would indeed have to be via Q and P.

0.7. These detailed results about P and Q allow one to turn (†) into an effective dictionary between modules and sheaves. This dictionary easily implies the first four parts of the following result proved in 3.15 (the final part of the theorem is rather more subtle and we will discuss its proof later in the Introduction).

THEOREM C. (a) *The localisation functor* $\mathscr{D}_{\mathscr{X}} \otimes \_$ *is exact (equivalently, P is a projective right $\mathscr{D}(\mathscr{X})$-module).*

(b) *Every sheaf in* $\mathscr{D}_{\mathscr{X}}$-**mod** *is generated by its global sections (equivalently, ${}_{\mathscr{D}(\mathbb{P}^1)}P$ is a generator).*

(c) *The functor* $\mathscr{D}_{\mathscr{X}} \otimes \_$ *makes* $\mathscr{D}_{\mathscr{X}}$-**mod** *into a quotient category of* $\mathscr{D}(\mathscr{X})$-**mod**.

(d) *If U is an open affine subset of $\mathscr{X}$, then $\mathscr{D}_{\mathscr{X}}(U)$ is a flat right $\mathscr{D}(\mathscr{X})$-module.*

(e) *$H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is a simple left $\mathscr{D}(\mathscr{X})$-module. Moreover, the only $\mathscr{D}(\mathscr{X})$-modules killed by $\mathscr{D}_{\mathscr{X}} \otimes \_$ are direct sums of copies of $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$.*

This theorem should be regarded as saying that the only part of the Beilinson–Bernstein equivalence of categories that fails to hold for $\mathscr{X}$ is the exactness of $\Gamma(\mathscr{X}, \_)$.

0.8. The second consequence of the detailed description of P and Q is that it allows one to give a complete description of the structure of $\mathscr{D}(\mathscr{X})$. For, it is easy to see that the minimal non-zero ideal $J(\mathscr{X})$ of $\mathscr{D}(\mathscr{X})$ is nothing more than the unique non-zero ideal of $S(\mathscr{X}) = \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)}(Q)$. Thus, if $\mathbf{m}$ is the non-zero ideal of $\mathscr{D}(\mathbb{P}^1)$, then

$$\mathscr{D}(\mathscr{X})/J(\mathscr{X}) \cong F = \{\theta \in \operatorname{End} Q^*/\mathbf{m}Q^* : \theta(P/\mathbf{m}Q^*) \subseteq P/\mathbf{m}Q^*\}$$

and this provides $\mathscr{D}(\mathscr{X})$ with the structure of a pull-back:

$$\begin{array}{ccc} \mathscr{D}(\mathscr{X}) & \longrightarrow & S(\mathscr{X}) \\ \downarrow & & \downarrow \\ F & \hookrightarrow & S(\mathscr{X})/J(\mathscr{X}) \end{array} \tag{‡}$$

This allows us to almost completely characterise the properties of $\mathscr{D}(\mathscr{X})$ (see Subsections 4.4–4.10 and Corollary 7.13).

THEOREM D. (a) *Let t be the aithmetic genus of $\mathscr{X}$. Then*

$$\mathscr{D}(\mathscr{X})/J(\mathscr{X}) \cong \begin{pmatrix} M_t(k) & k^{(t)} \\ 0 & k \end{pmatrix} \subset M_{t+1}(k) = S(\mathscr{X})/J(\mathscr{X}).$$

(b) *If I and L are the ideals of $\mathscr{D}(\mathscr{X})$ defined by*

$$I/J(\mathscr{X}) = \begin{pmatrix} 0 & k^{(t)} \\ 0 & k \end{pmatrix} \qquad and \qquad L/J(\mathscr{X}) = \begin{pmatrix} M_t(k) & k^{(t)} \\ 0 & 0 \end{pmatrix},$$

then $H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ is a simple left $\mathscr{D}(\mathscr{X})/I$-module, while $H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ is a simple left $\mathscr{D}(\mathscr{X})/L$-module.

(c) *$\mathscr{D}(\mathscr{X})$ has global dimension two.*

(d) *$K_i(\mathscr{D}(\mathscr{X}))) \cong K_i(k) \oplus K_i(k) \oplus K_i(k)$ for all $i \geqslant 0$.*

In Theorem 7.10 we prove that, ironically, one cannot replace $\mathbb{P}^1$, in Proposition B, by any other curve with injective normalisation $\mathbb{P}^1$.

THEOREM E. *If $\mathscr{X}$ and $\mathscr{Y}$ are singular, projective, rational curves with injective normalisation maps then $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$ are Morita equivalent. This equivalence is via the natural functor $\mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes \_$ and sends $H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ to $H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$.*

0.9. The proof that $H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ is simple is rather roundabout but in outline is as follows. First, we show in Section 6 that any two pull-backs of the form described in (‡) are Morita equivalent. In Section 7, we use this to prove Theorem E. Finally, as is observed in Section 5, $H^1(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$ is one-dimensional for the plane cuspidal cubic curve, $\mathscr{X}_1$. Thus Theorem E implies that $H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ is simple for any curve $\mathscr{X}$. This then allows one to determine the kernel of $\mathscr{D}_{\mathscr{X}} \otimes -$ in Theorem C and to show that the value of the integer t in Theorem D (which is the only part of that result that does not follow easily from (‡)) is actually the arithmetic genus.

It is perhaps worth remarking that Section 5 also gives a direct, elementary description of the various objects $\mathscr{D}(\mathscr{X})$, $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$, $S(\mathscr{X})$, etc., in the case when $\mathscr{X} = \mathscr{X}_1$. Thus the reader may prefer to read this section before studying the general results of Sections 2–4.

0.10. Section 8 is devoted to the study of the associated graded ring of $\mathscr{D}(\mathscr{X})$. The main result is:

THEOREM F. (a) $\operatorname{gr} \mathscr{D}(\mathscr{X}) \subset \operatorname{gr} \mathscr{D}(\mathbb{P}^1)$ *when* $\mathscr{D}(\mathscr{X})$ *and* $\mathscr{D}(\mathbb{P}^1)$ *are given the natural filtration by order of differential operator.*

(b) $\operatorname{gr} \mathscr{D}(\mathbb{P}^1)/\operatorname{gr} \mathscr{D}(\mathscr{X})$ *is a finite-dimensional k-vector space.*

(c) $\operatorname{gr} \mathscr{D}(\mathscr{X})$ *is an affine, Noetherian, commutative domain.*

In Section 9 some twisted rings of differential operators on $\mathscr{X}$ are studied. If $\mathscr{L}$ is an invertible sheaf on $\mathscr{X}$ then one defines the sheaf of differential operators with coefficients in $\mathscr{L}$ to be $\mathscr{D}_{\mathscr{L}} = \mathscr{L} \otimes \mathscr{D}_{\mathscr{X}} \otimes \mathscr{L}^{-1}$. Denote by $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ the global sections of this sheaf. It is shown in Section 9 that Theorem A also holds for $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$. However, the main result is:

THEOREM G. (a) $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ *is Morita equivalent to one of the rings* $\mathscr{D}(\mathbb{P}^1)$, $\mathscr{D}(\mathscr{X})$, *or* $U(\mathfrak{sl}_2)/(\Omega + 1)$. *Moreover, all these possibilities can occur.*

(b) *If* $\mathscr{L}$ *is generated by global sections and* $H^1(\mathscr{X}, \mathscr{L}) = 0$ *then* $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ *is Morita equivalent to* $\mathscr{D}(\mathbb{P}^1)$. *Further, the global sections functor provides an equivalence of categories between* $\mathscr{D}_{\mathscr{L}}$-**mod** *and* $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-**mod**, *and* $\Gamma(\mathscr{X}, \mathscr{L})$ *is a simple* $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$*-module.*

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1. ORDERS EQUIVALENT TO $\mathscr{D}(\mathbb{P}^1)$

1.1. In [RS] the authors show that orders equivalent to the first Weyl algebra A_1 inherit many of the nice properties of A_1, in particular they are finitely generated Noetherian domains. This formed one of the main steps of the proof given in [SS, Mu] that $\mathscr{D}(\mathscr{X})$, for $\mathscr{X}$ an affine curve, is a finitely generated Noetherian domain. When $\mathscr{X}$ is a rational projective curve the role of A_1 is taken by the ring $\mathscr{D}(\mathbb{P}^1)$ and so one needs to understand the structure of orders equivalent to this ring. That is the aim of this section. In effect the results and proofs of this section are very similar to those of [RS] except that one continually has to cope with the extra complications arising from the fact that $\mathscr{D}(\mathbb{P}^1)$ has a finite-dimensional module and that this module has homological dimension greater than 1.

1.2. Given an Ore domain R, write Quot(R) for its quotient division ring. A second subring S of Quot(R) is an *order equivalent to* R if there exist non-zero elements $a, b, c, d \in \mathrm{Quot}(R)$ such that $aRb \subseteq S$ and $cSd \subseteq R$. The ring R is a *maximal order* in Quot(R) if it is equivalent to no order S with $R \subsetneq S$. A finitely generated (right) R-submodule of Quot(R) is called a fractional (right) R-ideal. Given a fractional right (left) R-ideal I we will identify $\mathrm{Hom}_R(I, R)$ with

$$I^* = (I_R)^* = \{\theta \in \mathrm{Quot}(R) : \theta I \subseteq R\}$$

respectively

$$({}_RI)^* = \{\theta \in \mathrm{Quot}(R) : I\theta \subseteq R\}.$$

The module I is called *reflexive* if $I = I^{**}$. Similarly, $\mathrm{End}(I_R)$ will be identified with $\mathrm{O}(I_R) = \{\theta \in \mathrm{Quot}(R) : \theta I \subseteq I\}$. As usual, the subscript will be dropped if it is clear from the context. Note that $\mathrm{O}(I)$ is a natural example of an order equivalent to R.

1.3. There are a number of elementary properties of orders that will be used frequently, usually without comment, in the sequel. For the reader's convenience we now state the ones that will be used most frequently. Throughout R will denote an Ore domain with quotient division ring Quot(R) and I will be a fractional right R-ideal.

(a) $(I_R)^* \subseteq ({}_{\mathrm{End}(I)}I)^*$.

(b) If I_R is projective then I is certainly reflexive. The converse holds if R has global dimension $\leqslant 2$, written gldim $R \leqslant 2$ (see [Ba, Proposition 5.2]).

(c) If I_R is projective then, by the dual basis lemma, $1 \in II^*$ and hence ${}_{\mathrm{End}(I)}I$ is a generator; that is, $\mathrm{End}(I) = I({}_{\mathrm{End}(I)}I)^*$.

(d) If R is a maximal order and I is reflexive, then $\mathrm{End}(I)$ is also a maximal order (see [MR, Proposition 5.1.11, p. 136]).

(e) Suppose that R is a maximal order and that S is an order equivalent to R. If I is a fractional left S-ideal such that $R \subseteq \mathrm{O}({}_SI)$ then $R = \mathrm{O}(I)$. In particular, if J is a fractional right R-ideal, then $R = \mathrm{End}({}_{\mathrm{O}(J)}J)$ and $(J_R)^* = ({}_{\mathrm{End}(J)}J)^*$.

(f) If R is a maximal order and ${}_{\mathrm{O}(I)}I$ is a generator then I_R is projective. (Use (e).)

(g) If R has a unique minimal non-zero ideal $\mathbf{m}$ and I is projective then $\mathrm{O}(I)$ has a unique minimal non-zero ideal $I\mathbf{m}I^*$. If R has only finitely many non-zero ideals then $\mathrm{O}(I)$ has no more ideals than R and one has equality if and only if I_R is a progenerator. (Use the fact that $II^* = O(I)$.

Thus $X \mapsto I^*XI$ gives an injective map from the lattice of ideals of $O(I)$ to the lattice of ideals of R contained in $\text{Trace}(I) = I^*I$.)

We leave the (routine) verification of the results stated above as an exercise to the reader.

1.4. The properties of a ring that interest us are the following:

1.4(a) R is a Noetherian domain of left and right (Gabriel–Rentschler) Krull dimension one.

1.4(b) R contains a central subfield k and is a finitely generated k-algebra.

1.4(c) $\text{End}_R(M)$ is a finite-dimensional k-vector space for every R-module M of finite length.

1.4(d) For every non-zero (left or right) ideal J of R, J^{**}/J is finite-dimensional over k.

1.4(e) R has a unique, minimal non-zero ideal K. Moreover R/K is finite-dimensional.

1.5. The point behind the properties given in (1.4) is, of course, that we intend to show that they are satisfied whenever $R = \mathscr{D}(\mathscr{X})$ for a rational, projective curve $\mathscr{X}$. We begin by showing that these properties are satisfied by $\mathscr{D}(\mathbb{P}^1)$. As remarked in the Introduction, $\mathscr{D}(\mathbb{P}^1)$ is isomorphic to an appropriate factor ring of the enveloping algebra $U(\mathfrak{sl}_2(k))$ and so the desired result follows from the next lemma.

LEMMA. *Let R be an infinite-dimensional, primitive factor ring of $U(\mathfrak{sl}_2(k))$. Then R satisfies the properties* (1.4) *and, moreover, is a maximal order.*

Proof. That R has Krull dimension one is proved in [Sm], while the rest of 1.4(a), (b), (c), and (e) are standard (see, for example, [Di] or [St, Sect. 1]). That R is a maximal order is proved in [St, Lemma 3.1]. Thus it remains to prove 1.4(d). Since this fact is presumably well known (and the proof closely resembles that of [St, Theorem 2.6]) some of the details will be left to the reader.

Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis elements for $\mathfrak{sl}_2$ and write $\Omega = h(h-2) + 4ef$ for the Casimir element. We will identify e, f, and h with their images in R; thus $\Omega - \lambda = 0$ in R, for some $\lambda \in k$. Let J be a right ideal of R such that J^{**}/J

is infinite-dimensional and set $K=J^{**}$. Set $\mathscr{C}=\{e^n\}$ and $\mathscr{S}=\{f^n\}\subset R$. Since e and f act ad-nilpotently on $U(\mathfrak{sl}_2)$ it follows easily that $\mathscr{C}$ and $\mathscr{S}$ are Ore sets in R. Moreover, $R_{\mathscr{C}}$ and $R_{\mathscr{S}}$ are both isomorphic to localisations of A_1 and hence are hereditary. Now an elementary exercise shows that $(J^*)_{\mathscr{C}}=(J_{\mathscr{C}})^*$ and hence that $K_{\mathscr{C}}=(J_{\mathscr{C}})^{**}$. Since $R_{\mathscr{C}}$ is hereditary this forces $K_{\mathscr{C}}=J_{\mathscr{C}}$ and, similarly, $K_{\mathscr{S}}=J_{\mathscr{S}}$. Now, by 1.4(a), there exist right ideals $J\subseteq J_1\subsetneq J_2\subseteq K$ such that J_2/J_1 is an infinite-dimensional simple module. Thus $(J_2)_{\mathscr{C}}=(J_1)_{\mathscr{C}}$. Thus we may pick $x\in J_2\backslash J_1$ such that $xe\in J_1$. However, one also has $xf^n\in J_1$ for some integer n and, as $\Omega-\lambda=0$, $x(h(h-2)-\lambda)\in J_1$. It follows that $J_2/J_1=xR+J_1/J_1$ is finite-dimensional, a contradiction.

Remark. Combined with [St, Proposition 3.5], the above lemma shows that $U=U(\mathfrak{sl}_2)/(\Omega+1)$ is a simple ring of infinite global homological dimension in which every right ideal is reflexive. It is amusing to note that every right ideal of U is either projective or has infinite homological dimension.

1.6. The following observation will be used repeatedly and without particular comment. Let A be a k-algebra, $I\subset J$ right ideals of A such that J/I is finite-dimensional and K a finitely generated left A-module. Then JK/IK is a homomorphic image of $J/I\otimes_A A^{(n)}$, for some integer n, and hence is finite-dimensional.

LEMMA. *Let R be a ring satisfying properties* (1.4) *and assume that R is a maximal order in* Quot(R). *Let P be a projective, fractional right ideal of R. Then $S=\operatorname{End}(P)$ is also a maximal order satisfying* (1.4).

Proof. The fact that S is a maximal order follows from 1.3(d) while S is finitely generated over k by [MS, Corollary 1]. By the dual basis lemma, $S=PP^*$ and so $\_\otimes_S P$ provides an injective map from the lattice of right ideals of S to the lattice of R-submodules of P. This immediately shows that property 1.4(a) holds for S while 1.4(e) follows from 1.3(g). If M is a right S-module of finite length then $M\otimes P$ is torsion and hence of finite length as a right R-module. If $\theta\in\operatorname{End}_S(M)$, with $\theta\neq 0$, then θ induces an endomorphism $\theta\otimes 1\in\operatorname{End}_R(M\otimes P)$. Tensoring again with P^* shows that $\theta\otimes 1$ is non-zero. Thus

$$\dim_k \operatorname{End}_S(M)\leqslant \dim_k \operatorname{End}_R(M\otimes P)<\infty.$$

Finally, suppose that J is a right ideal of S for which J^{**}/J is infinite-dimensional. Since $J^{**}/J=J^{**}PP^*/JPP^*$, the R-module $J^{**}P/JP$ must also be infinite-dimensional. Therefore, by 1.4(d), there exists $\theta\in\operatorname{Quot}(R)$ such that $\theta JP\subseteq R$ but $\theta J^{**}P\nsubseteq R$. Equivalently, $\theta J\subseteq(P_R)^*$ but

$\theta J^{**} \nsubseteq P^*$. Since $(P_R)^* = ({}_S P)^*$ by 1.3(e), this in turn forces $P\theta J \subseteq S$ but $P\theta J^{**} \nsubseteq S$. Since $J^* = J^{***}$ this is absurd. Thus (1.4) holds for right S-modules. Interchanging the roles of P and P^* shows that it also holds for left S-modules.

1.7. We now combine the earlier results and prove

THEOREM. *Let R be a primitive, infinite-dimensional factor ring of $U(\mathfrak{sl}_2(k))$, Q a non-zero right ideal of R and T a ring for which $Q \subseteq T \subseteq \operatorname{End}(Q_R)$. Then T satisfies properties* (1.4).

Proof. We begin with two simplifications. There is one primitive factor ring, call it R_1, of $U(\mathfrak{sl}_2)$ that has infinite homological dimension (see [St, Proposition 3.5]). In this case $R_1 = \operatorname{End}(P_S)$ where S is a second primitive factor ring of $U(\mathfrak{sl}_2)$ and P is a projective right ideal of S. If $R = R_1$ then $P \subseteq R$ and so

$$QP \subseteq Q \subseteq T \subseteq \operatorname{End} Q_R \subseteq \{\theta \in \operatorname{Quot}(R) : \theta QP \subseteq QP\} = \operatorname{End}(QP_S).$$

Thus replacing Q by QP and R by S, if necessary, we may assume that R has finite homological dimension, gldim $R = d < \infty$. By [St, Theorem 2.6], again, this implies that $d \leqslant 2$.

Secondly, let $\mathbf{m}$ be the minimal non-zero ideal of R. By Lemma 1.5, Q^{**}/Q is finite-dimensional and so $Q^{**}\mathbf{m} = Q\mathbf{m} \subseteq Q$. Also, $\operatorname{End} Q^{**} \subseteq \operatorname{End} Q^{**}\mathbf{m}$. But, by 1.3(d), $\operatorname{End} Q^{**}$ is a maximal order and so $\operatorname{End} Q^{**} = \operatorname{End} Q^{**}\mathbf{m}$. Thus, replacing Q by $Q\mathbf{m}$, we may assume that $\operatorname{End} Q = \operatorname{End} Q^{**}$. Since gldim $R \leqslant 2$, the significance of these reductions is that Q^{**} is projective (see 1.3(b)) and so Lemmas 1.5 and 1.6 imply that $E = \operatorname{End} Q$ does satisfy (1.4).

We may now apply [RS, Proposition 1]. This implies that T is a left Noetherian ring of left Krull dimension one and that 1.4(c) holds for left T-modules. Let $M = QT$. Then

$$T \subseteq \mathbb{I}_E(M) = \{\theta \in E : M\theta \subseteq M\}.$$

Since $\mathbb{I}(M)/M \cong \operatorname{End}_E(E/M)$, this implies that T/M is finite-dimensional. Thus, by [RS, Proposition 2], T is a finitely generated k-algebra. Since Q_R^{**} is projective it is clear that the minimal non-zero ideal of E is $V = Q^{**}\mathbf{m}Q^* = Q\mathbf{m}Q^*$. Thus, if I is an ideal of T, then

$$I \supseteq MIM = (EQTIE)QT \supseteq VQT = VM.$$

Thus VM is the unique minimal non-zero ideal of T. Moreover since E/V and T/M are finite-dimensional, so is T/VM. Since we will not need 1.4(d) in the future, we leave it as an easy exercise to the reader.

Thus the properties of (1.4) do hold for left T-modules and in order to complete the proof, we need to show that 1.4(a) and 1.4(c) hold on the right. To do this we use an idea from [RS]. Let $\tilde{M} = ({}_E M)^{**}$. By 1.3(d), $F = \mathrm{End}({}_E\tilde{M})$ is a maximal order and so, by 1.3(e), $F = \mathrm{End}\,{}_R X$ where $X = {}_R[(Q_R)^* \tilde{M}]^{**}$. Thus by Lemma 1.6, F satisfies (1.4). Moreover, $VM = V\tilde{M} \subseteq T \subseteq \mathrm{End}\, VM = F$. Since VM is therefore a right ideal of F, the earlier part of the proof may be used to show that T satisfies (1.4) on the right.

2. General Structure of Differential Operators

2.1. Let $\mathscr{Y}$ be an affine algebraic curve with normalisation $\tilde{\mathscr{Y}}$. The basic technique used in [SS] was to study the structure of $\mathscr{D}(\mathscr{Y})$ via the object

$$\mathscr{D}(\tilde{\mathscr{Y}}, \mathscr{Y}) = \{\theta \in \mathscr{D}(\tilde{\mathscr{Y}}) : \theta \circ \mathcal{O}(\tilde{\mathscr{Y}}) \subseteq \mathcal{O}(\mathscr{Y})\}.$$

Now suppose that $\mathscr{X}$ is a singular rational projective curve, so $\mathscr{X}$ has normalisation $\mathbb{P}^1$. In this section we introduce and give the elementary properties of the analogous objects $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ and $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$. These will, however, be sufficient to prove a number of results about the general structure of $\mathscr{D}(\mathscr{X})$; in particular to show that it is a finitely generated Noetherian k-algebra.

2.2. We begin by fixing some notation. Unless otherwise stated $\mathscr{X}$ will denote a rational, projective curve (and, as mentioned in the Introduction, all varieties are assumed to be algebraic and irreducible). Since $\mathscr{X}$ is rational it has normalisation $\mathbb{P}^1$ and there is a canonical morphism $\pi : \mathbb{P}^1 \to \mathscr{X}$. If U is a non-empty, open, affine subset of $\mathscr{X}$ then $\tilde{U} = \pi^{-1}(U)$ is defined to be the open affine subset of $\mathbb{P}^1$ with the property that $\mathcal{O}(\tilde{U})$ is the integral closure of $\mathcal{O}(U)$ in its field of fractions $K(\mathbb{P}^1) = K(\mathscr{X})$. In this paper $\tilde{U}$ will always denote the set defined in this way. A useful fact, and one that we will use without further reference, is that any (non-empty) open subset U of $\mathscr{X}$, with $U \neq \mathscr{X}$, is affine (see [Ha, Exercise IV.1.4, p. 298]).

Given an open affine subset U of $\mathscr{X}$, define

$$\mathscr{D}(\tilde{U}, U) = \{\theta \in \mathscr{D}(\tilde{U}) : \theta \circ \mathcal{O}(\tilde{U}) \subseteq \mathcal{O}(U)\}$$

and

$$\mathscr{D}(U, \tilde{U}) = \{\theta \in \mathscr{D}(K(\mathbb{P}^1)) : \theta \circ \mathcal{O}(U) \subseteq \mathcal{O}(\tilde{U})\}.$$

Observe that $\mathscr{D}(\tilde{U}, U)$ is naturally a $\mathscr{D}(U)$-$\mathscr{D}(\tilde{U})$-bimodule while $\mathscr{D}(U, \tilde{U})$ is a $\mathscr{D}(\tilde{U})$-$\mathscr{D}(U)$-bimodule. The direct image functor π_*, defined by

$(\pi_* \mathscr{D}_{\mathbb{P}^1})(U) = \mathscr{D}_{\mathbb{P}^1}(\tilde{U})$, makes $\pi_* \mathscr{D}_{\mathbb{P}^1}$ into a sheaf of rings, quasi-coherent as an $\mathcal{O}_{\mathscr{X}}$-module, the $\mathcal{O}_{\mathscr{X}}$-module action coming from the inclusion $\mathcal{O}_{\mathscr{X}}(U) \subseteq (\pi_* \mathcal{O}_{\mathbb{P}^1})(U) = \mathcal{O}_{\mathbb{P}^1}(\tilde{U})$. Denote by $\pi_* \mathscr{D}_{\mathbb{P}^1}$-**mod** the category of sheaves of left $\pi_* \mathscr{D}_{\mathbb{P}^1}$-modules which are quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules. The argument of [Ha, Exercise II.5.17] shows that $\pi_*: \mathscr{D}_{\mathbb{P}^1}$-**mod** $\to \pi_* \mathscr{D}_{\mathbb{P}^1}$-**mod** is an equivalence of categories. Denote by $\mathscr{D}(\pi_* \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathscr{X}})$ the sheaf of $\mathscr{D}_{\mathscr{X}}$-$\pi_* \mathscr{D}_{\mathbb{P}^1}$-bimodules with sections $\mathscr{D}(\tilde{U}, U)$ over U. Define the sheaf of right $\mathscr{D}_{\mathbb{P}^1}$-modules $\mathscr{D}(\mathcal{O}_{\mathbb{P}^1}, \pi^{-1}\mathcal{O}_{\mathscr{X}})$ by $\pi_* \mathscr{D}(\mathcal{O}_{\mathbb{P}^1}, \pi^{-1}\mathcal{O}_{\mathscr{X}}) = \mathscr{D}(\pi_* \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathscr{X}})$. Similarly $\mathscr{D}(\mathcal{O}_{\mathscr{X}}, \pi_* \mathcal{O}_{\mathbb{P}^1})$ is the sheaf of $\pi_* \mathscr{D}_{\mathbb{P}^1}$-$\mathscr{D}_{\mathscr{X}}$-bimodules with sections $\mathscr{D}(U, \tilde{U})$ over U and $\mathscr{D}(\pi^{-1}\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathbb{P}^1})$ is the sheaf of left $\mathscr{D}_{\mathbb{P}^1}$-modules defined by $\pi_* \mathscr{D}(\pi^{-1}\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathbb{P}^1}) = \mathscr{D}(\mathcal{O}_{\mathscr{X}}, \pi_* \mathcal{O}_{\mathbb{P}^1})$. We write

$$\mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \Gamma(\mathscr{X}, \mathscr{D}(\pi_* \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathscr{X}}))$$

and

$$\mathscr{D}(\mathscr{X}, \mathbb{P}^1) = \Gamma(\mathscr{X}, \mathscr{D}(\mathcal{O}_{\mathscr{X}}, \pi_* \mathcal{O}_{\mathbb{P}^1})).$$

Notice that $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ is a $\mathscr{D}(\mathscr{X})$-$\mathscr{D}(\mathbb{P}^1)$-bimodule and that $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is a $\mathscr{D}(\mathbb{P}^1)$-$\mathscr{D}(\mathscr{X})$-bimodule.

Given any open affine cover $\{U_i\}$ of $\mathscr{X}$, then it is easy to see that

$$Q = \mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \bigcap_i \mathscr{D}(\tilde{U}_i, U_i)$$

and

$$P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1) = \bigcap_i \mathscr{D}(U_i, \tilde{U}_i).$$

In this paper, P and Q will only be used to denote these objects.

2.3. The structure of $\mathscr{D}(\mathbb{P}^1)$ is easy to determine. For, pick two points on $\mathbb{P}^1$, which we may as well assume to be at 0 and ∞, and take the affine cover $U_1 = \mathbb{P}^1 \setminus \{\infty\}$ and $U_2 = \mathbb{P}^1 \setminus \{0\}$. Thus each U_i is isomorphic to the affine line $\mathbb{A}^1$. Taking t to be a coordinate function on $\mathbb{A}^1 \cong U_1$ we have

$$\mathscr{D}(U_1) = k[t, \partial] \qquad \text{for} \quad \partial = \partial/\partial t$$

while

$$\mathscr{D}(U_2) = k[t^{-1}, \partial/\partial(t^{-1})] = k[t^{-1}, t^2\partial].$$

An easy computation shows that

$$\mathscr{D}(\mathbb{P}^1) = \mathscr{D}(U_1) \cap \mathscr{D}(U_2) = k[\partial, t\partial, t^2\partial].$$

However, it tends to be rather difficult to describe $\mathscr{D}(\mathscr{X})$ explicitly in this manner and so one of the basic techniques of this paper will be to use P and Q to transfer properties of $\mathscr{D}(\mathbb{P}^1)$ to $\mathscr{D}(\mathscr{X})$. Their basic structure is described by the next few results.

PROPOSITION. *$\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ is a non-zero right ideal of $\mathscr{D}(\mathbb{P}^1)$ and a left ideal of $\mathscr{D}(\mathscr{X})$.*

Proof. By its construction, $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ is a right ideal of $\mathscr{D}(\mathbb{P}^1)$ and a left ideal of $\mathscr{D}(\mathscr{X})$ and so it only remains to prove that $\mathscr{D}(\mathbb{P}^1, \mathscr{X}) \neq 0$. Let $\{U_1, ..., U_m\}$ be an open affine cover of $\mathscr{X}$ and, as usual, write $\{\tilde{U}_1, ..., \tilde{U}_m\}$ for the corresponding cover of $\mathbb{P}^1$, where $\tilde{U}_i = \pi^{-1}(U_i)$ for each i. Now, $\mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \bigcap_i \mathscr{D}(\tilde{U}_i, U_i)$. Since $\mathscr{D}(\mathbb{P}^1)$ is an Ore domain, in order to prove that $\mathscr{D}(\mathbb{P}^1, \mathscr{X}) \neq 0$, it suffices to prove $\mathscr{D}(\tilde{U}_i, U_i) \cap \mathscr{D}(\mathbb{P}^1) \neq 0$, for each i. Fix $U = U_i$ for some i. Observe that $\tilde{U} = \mathbb{A}^1 \backslash H$, for some finite set of points H, in affine 1-space $\mathbb{A}^1$. Thus, if t is a coordinate on $\mathbb{A}^1$, then $\mathcal{O}(\tilde{U}) = k[t]_f$, where $0 \neq f \in k[t]$ defines H. As indicated earlier, $\mathscr{D}(\mathbb{P}^1)$ is just the subring $k[\partial, t\partial, t^2\partial]$ of $\mathscr{D}(\mathbb{A}^1) = k[t, \partial]$. Now $\mathscr{D}(\tilde{U}, U)$ contains the conductor $\operatorname{ann}_{\mathcal{O}(U)}(\mathcal{O}(\tilde{U})/\mathcal{O}(U))$ and so there exists $g \in k[t] \cap \mathscr{D}(\tilde{U}, U)$ with $g \neq 0$. The identity

$$t^p \partial^{p+r} = (t^p \partial^p) \partial^r = \left(\prod_{i=0}^{p-1} (t\partial - i) \right) \partial^r \in \mathscr{D}(\mathbb{P}^1),$$

for any integers p and r, ensures that $g\partial^r \in \mathscr{D}(\mathbb{P}^1)$ for $r = \deg g$. Thus $\mathscr{D}(\tilde{U}, U) \cap \mathscr{D}(\mathbb{P}^1) \neq 0$; as required.

2.4. The first main result is an easy consequence of Proposition 2.3.

THEOREM A. *Let X be a rational, projective curve. Then*

(1) *$\mathscr{D}(\mathscr{X})$ is a Noetherian domain of left and right Krull dimension one.*

(b) *$\mathscr{D}(\mathscr{X})$ is a finitely generated k-algebra.*

(c) *If M is a $\mathscr{D}(\mathscr{X})$-module of finite length, then* $\operatorname{End}_{\mathscr{D}(\mathscr{X})} M$ *is a finite-dimensional k-vector space.*

(d) *$\mathscr{D}(\mathscr{X})$ has a unique, minimal non-zero ideal, $J = J(\mathscr{X})$. Moreover $F(\mathscr{X}) = \mathscr{D}(\mathscr{X})/J(\mathscr{X})$ is finite-dimensional over k.*

Proof. By Proposition 2.3, $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ is a non-zero right ideal of $\mathscr{D}(\mathbb{P}^1)$ for which

$$\mathscr{D}(\mathbb{P}^1, \mathscr{X}) \subseteq \mathscr{D}(\mathscr{X}) \subseteq \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathbb{P}^1, \mathscr{X}).$$

Now apply Theorem 1.7.

One question raised by Theorem A is: What is the structure of $\mathscr{D}(\mathscr{X})/J(\mathscr{X})$? In Sections 4 and 7 we will completely answer this question in the case when the normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$ is injective.

2.5. COROLLARY. *Let $\mathscr{X}$ be a rational, projective curve. Then*

(i) *$\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ is a finitely generated, non-zero right ideal of $\mathscr{D}(\mathbb{P}^1)$ and a finitely generated left ideal of $\mathscr{D}(\mathscr{X})$.*

(ii) *$\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is a finitely generated, non-zero, fractional right $\mathscr{D}(\mathscr{X})$-ideal and a finitely generated, fractional left $\mathscr{D}(\mathbb{P}^1)$-ideal.*

(iii) $\mathscr{D}(\mathbb{P}^1, \mathscr{X}) \subseteq [{}_{\mathscr{D}(\mathbb{P}^1)}\mathscr{D}(\mathscr{X}, \mathbb{P}^1)]^* \cap [\mathscr{D}(\mathscr{X}, \mathbb{P}^1)_{\mathscr{D}(\mathscr{X})}]^*$ *and*

$$\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \subseteq [{}_{\mathscr{D}(\mathscr{X})}\mathscr{D}(\mathbb{P}^1, \mathscr{X})]^* \cap [\mathscr{D}(\mathbb{P}^1, \mathscr{X})_{\mathscr{D}(\mathbb{P}^1)}]^*.$$

Proof. Since $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathbb{P}^1)$ are Noetherian, part (i) is immediate from Proposition 2.3. Pick an open affine cover $\{U_1, ..., U_m\}$ of $\mathscr{X}$ with corresponding cover $\{\tilde{U}_1, ..., \tilde{U}_m\}$ for $\mathbb{P}^1$. Then

$$\mathscr{D}(\mathscr{X}, \mathbb{P}^1)\, \mathscr{D}(\mathbb{P}^1, \mathscr{X}) \subseteq \bigcap \mathscr{D}(U_i, \tilde{U}_i)\, \mathscr{D}(\tilde{U}_i, U_i) \subseteq \bigcap \mathscr{D}(\tilde{U}_i) = \mathscr{D}(\mathbb{P}^1). \quad (2.5.1)$$

Similarly, $\mathscr{D}(\mathbb{P}^1, \mathscr{X})\, \mathscr{D}(\mathscr{X}, \mathbb{P}^1) \subseteq \mathscr{D}(\mathscr{X})$. Thus, if q is any non-zero element of $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$, then as left modules

$$\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \cong \mathscr{D}(\mathscr{X}, \mathbb{P}^1) q \subseteq \mathscr{D}(\mathbb{P}^1).$$

Thus $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is a finitely generated left $\mathscr{D}(\mathbb{P}^1)$-module. Similarly,

$$\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \cong q\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \subseteq \mathscr{D}(\mathscr{X})$$

and so, by Theorem 2.4, $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is a finitely generated right $\mathscr{D}(\mathscr{X})$-module. Finally, as $\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \supseteq \mathscr{D}(\mathscr{X}) + \mathscr{D}(\mathbb{P}^1)$, it is certainly non-zero. This proves part (ii). Part (iii) is an immediate consequence of (2.5.1) and the sentence thereafter.

2.6. In [SS] the module $\mathscr{D}(\tilde{\mathscr{Y}}, \mathscr{Y})$ for $\mathscr{Y}$ an affine curve with normalisation $\tilde{\mathscr{Y}}$, was used extensively to understand the structure of $\mathscr{D}(\mathscr{Y})$. For a rational, projective curve $\mathscr{X}$, the corresponding module $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ will again play a significant role, but its properties are much more subtle and so we will also need to use $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ far more than $\mathscr{D}(\mathscr{Y}, \tilde{\mathscr{Y}})$ was used in [SS]. The reason for this is that $\mathscr{D}(\mathbb{P}^1)$ is no longer hereditary and so the question of when $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ and $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ are projective becomes subtle and important. For example, the main theorem of the next section shows that when π is injective (but $\mathscr{X}$ is singular) $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is not projective yet

$$\mathscr{D}(\mathscr{X}) = \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathscr{X}, \mathbb{P}^1) \subsetneq \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathbb{P}^1, \mathscr{X}).$$

In contrast, if $\mathscr{Y}$ is an affine curve for which $\pi: \tilde{\mathscr{Y}} \to \mathscr{Y}$ is injective then $\mathscr{D}(\mathscr{Y}) = \text{End}_{\mathscr{D}(\tilde{\mathscr{Y}})} \mathscr{D}(\tilde{\mathscr{Y}}, \mathscr{Y}) = \text{End}_{\mathscr{D}(\tilde{\mathscr{Y}})} \mathscr{D}(\mathscr{Y}, \tilde{\mathscr{Y}})$.

2.7. Consider $\mathscr{D}(\mathbb{P}^1)$. As has been seen in this, and the last, section there are two convenient representations of $\mathscr{D}(\mathbb{P}^1)$, either as

$$\mathscr{D}(\mathbb{P}^1) = k[\partial, t\partial, t^2\partial] \subset A_1 = k[t, \partial] \tag{2.7.1}$$

with maximal ideal $\mathbf{m} = \partial\mathscr{D}(\mathbb{P}^1) + t\partial\mathscr{D}(\mathbb{P}^1) + t^2\partial\mathscr{D}(\mathbb{P}^1)$, or as the factor $U(\mathfrak{sl}_2(k))/(\Omega)$ of the enveloping algebra of $\mathfrak{sl}_2$. If e, f, h are the generators of $\mathfrak{sl}_2$ described in (1.5) then the equivalence of the two representations is obtained by setting

$$e = \partial, \qquad h = -2t\partial, \qquad \text{and} \qquad f = -t^2\partial.$$

In this paper we will have occasion to consider various specific right ideals of $\mathscr{D}(\mathbb{P}^1)$. The basic facts we require are contained in the next lemma.

LEMMA. *Write $\mathscr{D}(\mathbb{P}^1)$ as in (2.7.1), and set*

$$W = (t\partial - 1)\,\mathscr{D}(\mathbb{P}^1) + t^2\partial\mathscr{D}(\mathbb{P}^1) \qquad \text{and} \qquad V = t\partial\mathscr{D}(\mathbb{P}^1) + t^2\partial\mathscr{D}(\mathbb{P}^1).$$

Then

(i) *$M = \mathscr{D}(\mathbb{P}^1)/W$ is a simple right $\mathscr{D}(\mathbb{P}^1)$-module.*

(ii) *As a right $\mathscr{D}(\mathbb{P}^1)$-module, $N = A_1/tA_1 \cong \mathscr{D}(\mathbb{P}^1)/V$ is a non-split module of length two. It has socle $N_1 = (\partial\mathscr{D}(\mathbb{P}^1) + tA_1)/tA_1 \cong M$ while $N/N_1 \cong \mathscr{D}(\mathbb{P}^1)/\mathbf{m}$.*

(iii) *$(t^2A_1 + (t\partial - 1)A_1) \cap \mathscr{D}(\mathbb{P}^1) = W$.*

(iv) *$tA_1 \cap \mathscr{D}(\mathbb{P}^1) = V$.*

Proof. We begin with (iv). Certainly $tA_1 \cap \mathscr{D}(\mathbb{P}^1) \supseteq V$. Now, by (2.7.1)

$$\mathscr{D}(\mathbb{P}^1) = k[\partial] \oplus (t\partial\mathscr{D}(\mathbb{P}^1) + t^2\partial\mathscr{D}(\mathbb{P}^1)) \tag{2.7.2}$$

and so, if the above containment is strict, $tA_1 \cap \mathscr{D}(\mathbb{P}) \cap k[\partial] \neq 0$. This is clearly absurd and so (iv) is proven. Now consider part (ii). Since $A_1/tA_1 = tA_1 + k[\partial]/tA_1 = \mathscr{D}(\mathbb{P}^1) + tA_1/tA$, certainly as right $\mathscr{D}(\mathbb{P}^1)$-modules $A_1/tA_1 \cong \mathscr{D}(\mathbb{P}^1)/V$. By (2.7.2), $\mathscr{D}(\mathbb{P}^1)/V \cong k[\partial]$, as a right $k[\partial]$-module. Thus any proper factor module of $\mathscr{D}(\mathbb{P}^1)/V$ (over either $\mathscr{D}(\mathbb{P}^1)$ or $k[\partial]$) must be finite-dimensional. But the only finite-dimensional factor of $\mathscr{D}(\mathbb{P}^1)$ is $\mathscr{D}(\mathbb{P}^1)/\mathbf{m} \cong k$. Since $\mathbf{m} \supsetneq V$, this forces $\mathscr{D}(\mathbb{P}^1)/V$ to have length two, corresponding to the chain

$$V \subset \mathbf{m} = \partial\mathscr{D}(\mathbb{P}^1) + V \subset \mathscr{D}(\mathbb{P}^1).$$

Clearly $\mathbf{m}+tA_1/tA_1=N_1$. Moreover, if N were a split module, then it would have to have a finite-dimensional submodule; a contradiction. Thus $N_1=\mathrm{Soc}(N)$.

Next, consider the simple module $\mathbf{m}/V\cong \partial\mathscr{D}(\mathbb{P}^1)/\partial\mathscr{D}(\mathbb{P}^1)\cap V$. Now $\partial(t\partial-1)=t\partial^2\in V$ and $\partial(t^2\partial)=t\partial(t\partial+1)\in V$. Consequently, the right annihilator, $Z=\text{r-ann}(\partial+V/V)$ contains $W=(t\partial-1)\,\mathscr{D}(\mathbb{P}^1)+(t^2\partial)\,\mathscr{D}(\mathbb{P}^1)$. The argument of the last paragraph shows that every proper factor module of $\mathscr{D}(\mathbb{P}^1)/W$ is finite-dimensional. Since $\mathbf{m}/V\cong\mathscr{D}(\mathbb{P}^1)/Z$ is an infinite-dimensional, simple module, this therefore implies that $W=Z$. Thus W is a maximal right ideal of $\mathscr{D}(\mathbb{P}^1)$. This completes the proof of (ii) and (i).

Finally, $X=t^2A_1+(t\partial-1)A_1$ is a proper right ideal of A_1 and certainly $X\cap\mathscr{D}(\mathbb{P}^1)\supseteq W$. The maximality of W therefore forces $X\cap\mathscr{D}(\mathbb{P}^1)=W$.

An alternative proof to part (ii) of the above lemma is to note that $\mathscr{D}(\mathbb{P}^1)/V$ is a Verma module for $U(\mathfrak{sl}_2(k))$. Since this module is transparently not simple, standard theory says that it has length two with socle $\mathbf{m}/V$. Similarly, M is a Verma module which, since $W\not\subseteq\mathbf{m}$, must be the simple Verma module. However, the details of this approach take almost as long as the elementary proof given above.

2.8. For the rest of this section we will consider the case when the normalisation map $\pi:\mathbb{P}^1\to\mathscr{X}$ is not injective and show that, in this case, $\mathscr{D}(\mathscr{X})$ is not a maximal order. Unfortunately this does not seem to follow easily from the corresponding result proved in [SS, Theorem 3.7] for affine curves; a point we will return to in Remark 2.11.

LEMMA. *Let $\mathscr{X}$ be a rational, projective curve such that $\pi:\mathbb{P}^1\to\mathscr{X}$ is not injective. Then, for an appropriate choice of coordinate function t, one has*

$$\mathscr{D}(\mathbb{P}^1,\mathscr{X})\subseteq(t^2-t)\,\partial\mathscr{D}(\mathbb{P}^1)\subset\mathscr{D}(\mathbb{P}^1)=k[\partial,t\partial,t^2\partial].$$

Proof. By deleting one smooth point from $\mathscr{X}$ and its (single) preimage in $\mathbb{P}^1$, one obtains an open affine subset, U of $\mathscr{X}$, such that $\tilde{U}=\pi^{-1}(U)\cong\mathbb{A}^1$. By an appropriate choice of the coordinate function t, we may assume that $\pi(0)=\pi(1)$. Thus,

$$\mathscr{O}(U)\subseteq k+t(t-1)k[t]\subset\mathscr{O}(\tilde{U})=k[t].$$

By [SS, Proposition 4.4(a)], $\mathscr{D}(\tilde{U},U)\subseteq(t^2-t)k[t,\partial]$. Thus, $\mathscr{D}(\mathbb{P}^1,\mathscr{X})\subseteq\mathscr{D}(\mathbb{P}^1)\cap(t^2-t)k[t,\partial]$ and it suffices to prove

2.9. SUBLEMMA. $\mathscr{D}(\mathbb{P}^1)\cap(t^2-t)k[t,\partial]=(t^2-t)\,\partial\mathscr{D}(\mathbb{P}^1)$.

Proof. Clearly $Z=(t^2-t)\,\partial\mathscr{D}(\mathbb{P}^1)\subseteq X=\mathscr{D}(\mathbb{P}^1)\cap(t^2-t)k[t,\partial]$. Now certainly

$$X\subseteq\mathscr{D}(\mathbb{P}^1)\cap(t-1)k[t,\partial]=Y.$$

Moreover (after the change of variable $t \mapsto t+1$), Lemma 2.7(iv) implies that $Y=(t-1)\,\partial\mathscr{D}(\mathbb{P}^1)+(t-1)^2\,\partial\mathscr{D}(\mathbb{P}^1)$. Since $(t-1)\,\partial \in Y\backslash X$, certainly $Y \neq X$. Now consider Z. Then

$$\begin{aligned} Y/Z &\cong (\partial\mathscr{D}(\mathbb{P}^1)+(t-1)\,\partial\mathscr{D}(\mathbb{P}^1))/t\partial\mathscr{D}(\mathbb{P}^1) \\ &= (\partial\mathscr{D}(\mathbb{P}^1)+t\partial\mathscr{D}(\mathbb{P}^1))/t\partial\mathscr{D}(\mathbb{P}^1) \end{aligned}$$

and hence Y/Z is a homomorphic image of $\mathscr{D}(\mathbb{P}^1)/(t\partial-1)\,\mathscr{D}(\mathbb{P}^1)+t^2\partial\mathscr{D}(\mathbb{P}^1)=M$. But, by Lemma 2.7(i), M is simple. Since $Y\neq X$, this forces $X=Z$, as required.

2.10. Proposition. *Let $\mathscr{X}$ be a rational, projective curve such that $\pi\colon \mathbb{P}^1 \to \mathscr{X}$ is not injective. Then $\mathscr{D}(\mathscr{X})$ is not a maximal order. In particular, $\mathscr{D}(\mathscr{X})$ is not a simple ring and is not Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$.*

Proof. Suppose that $\mathscr{D}(\mathscr{X})$ is a maximal order and set $Q=\mathscr{D}(\mathbb{P}^1,\mathscr{X})$ and $P=\mathscr{D}(\mathscr{X},\mathbb{P}^1)$. Throughout the proof L^* will stand for the dual of L, as a $\mathscr{D}(\mathbb{P}^1)$-module. From the chain

$$\begin{aligned} Q^{**}\mathbf{m} \subseteq Q \subset \mathscr{D}(\mathscr{X}) &\subseteq \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} Q \\ &\subseteq \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)}(Q^{**}) = \operatorname{End}(Q^{**}\mathbf{m}) \end{aligned}$$

one obtains that $\mathscr{D}(\mathscr{X})=\operatorname{End} Q=\operatorname{End}(Q^{**})$. Since $\operatorname{gldim}\mathscr{D}(\mathbb{P}^1)=2$ (see [St, Theorem 2.6]), Q^{**} is a projective right ideal of $\mathscr{D}(\mathbb{P}^1)$. Thus $\mathscr{D}(\mathscr{X})=\operatorname{End} Q^{**}=Q^{**}Q^*$, by the dual basis lemma. Moreover, by (1.3)(g), the unique minimal, non-zero ideal of $\mathscr{D}(\mathscr{X})$ is just $Q^{**}\mathbf{m}Q^*$. Now consider $Q^{**}P$. Since this is an ideal of $\mathscr{D}(\mathscr{X})$, certainly $Q^{**}P \supseteq Q^{**}\mathbf{m}Q^*$. Thus $Q^*Q^{**}P \supseteq Q^*Q^{**}\mathbf{m}Q^*$. Now Q^*Q^{**} is an ideal of $\mathscr{D}(\mathbb{P}^1)$, and so either $Q^*Q^{**}=\mathscr{D}(\mathbb{P}^1)$ or $Q^*Q^{**}=\mathbf{m}$. Since $\mathbf{m}^2=\mathbf{m}$, both possibilities force $P\supseteq \mathbf{m}P=\mathbf{m}Q^*$. By Lemma 2.8, $Q\subseteq (t^2-t)\,\partial\mathscr{D}(\mathbb{P}^1)$. Thus

$$P \supseteq \mathbf{m}Q^* \supseteq \mathbf{m}\mathscr{D}(\mathbb{P}^1)\,\partial^{-1}(t^2-t)^{-1} \ni \partial\partial^{-1}(t^2-t)^{-1}=(t^2-t)^{-1}.$$

But now let U be the open affine subset of $\mathscr{X}$ defined in Lemma 2.8. Then $P=\mathscr{D}(\mathscr{X},\mathbb{P}^1)\subseteq \mathscr{D}(U,\tilde{U})$ and so

$$k[t]=\mathscr{O}(\tilde{U}) \supseteq P\circ\mathscr{O}(U) \ni (t^2-t)^{-1}(1)=(t^2-t)^{-1};$$

a contradiction.

2.11. *Remark.* Let $\mathscr{X}$ be as in Proposition 2.10 and U be the open affine subset defined by Lemma 2.8. Then [SS, Theorem 3.7] proves that $\mathscr{D}(U)$ is not a maximal order and it is tempting to suppose that this fact should quickly imply that $\mathscr{D}(\mathscr{X})$ is not a maximal order. This seems not to be the case. For example, consider the idealizer

$$R = \mathbb{I}(k[t, \partial]\partial) = k + k[t, \partial]\partial,$$

which is certainly not a maximal order. Then, clearly R contains the maximal order $\mathscr{D}(\mathbb{P}^1)$. Note that this example is very similar to the situation that occurs for the simple node. Indeed, if $\mathscr{X}$ is the nodal curve in $\mathbb{P}^2$, then (with U as above) $\mathcal{O}(U) = k + (t^2 - t)k[t] \subset k[t] = \mathcal{O}(\tilde{U})$. Thus

$$\mathscr{D}(U) = k + (t^2 - t)k[t, \partial] \subset \mathbb{I}((t^2 - t)k[t, \partial])$$

(see [SS, Proposition 4.4(b)]). It follows easily from Sublemma 2.9 that

$$\mathscr{D}(\mathscr{X}) = k + \mathscr{D}(\mathbb{P}^1, \mathscr{X}) = k + (t^2 - t)\,\partial\mathscr{D}(\mathbb{P}^1) = \mathscr{D}(U) \cap \mathscr{D}(\mathbb{P}^1).$$

The description of $\mathscr{D}(\mathscr{X})$, for the plane nodal curve, was first obtained by I. Musson. We would like to thank him for communicating his computations to us as they suggested the approach of Lemma 2.8.

3. Curves with Injective Normalisation Map

3.1. In this section $\mathscr{X}$ will always stand for a projective curve with normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$. The aims of this section are to complete the proof of Proposition B of the Introduction, by showing that $\mathscr{D}(\mathscr{X})$ is not Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$, and to investigate the relationship between the categories $\mathscr{D}_{\mathscr{X}}$-**mod** and $\mathscr{D}(\mathscr{X})$-**mod**. To do this will require a detailed analysis of the modules $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ and $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$. As in Section 2, we will usually write $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ and $Q = \mathscr{D}(\mathbb{P}^1, \mathscr{X})$.

3.2. If U is an open affine subset of $\mathscr{X}$ and $\tilde{U} = \pi^{-1}(U)$ then, thanks to [SS], the structure of $\mathscr{D}(U)$ and $\mathscr{D}(\tilde{U}, U)$, etc., is well understood. The following special case of Beilinson and Bernstein's famous theorem allows us to pull some of this information down to $\mathscr{D}(\mathscr{X})$.

Theorem. (i) *There is an equivalence of categories between* $\mathscr{D}_{\mathbb{P}^1}$-**mod** *and* $\mathscr{D}(\mathbb{P}^1)$-**mod**, *the category of left* $\mathscr{D}(\mathbb{P}^1)$*-modules. This is provided by the mutually inverse functors*

$$\mathscr{M} \mapsto \Gamma(\mathbb{P}^1, \mathscr{M}) \qquad \text{and} \qquad M \mapsto \mathscr{D}_{\mathbb{P}^1} \otimes M$$

for $\mathscr{M} \in \mathscr{D}_{\mathbb{P}^1}$-**mod** *and* $M \in \mathscr{D}(\mathbb{P}^1)$-**mod**. *Moreover, these functors restrict to give an equivalence between the category of coherent* $\mathscr{D}_{\mathbb{P}^1}$*-modules and that of finitely generated* $\mathscr{D}(\mathbb{P}^1)$*-modules.*

(ii) *Let* $\{\tilde{U}_1, ..., \tilde{U}_m\}$ *be an open affine cover of* $\mathbb{P}^1$. *Then* $\bigoplus \mathscr{D}(\tilde{U}_i)$ *is a faithfully flat right* $\mathscr{D}(\mathbb{P}^1)$*-module.*

Proof. Part (i) is a special case of [BB]. That part (ii) is a consequence of (i) is observed in [HS].

3.3. By construction, $\mathscr{D}(\mathscr{X}, \mathbb{P}^1) = \Gamma(\mathbb{P}^1, \mathscr{D}(\pi^{-1}\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathbb{P}^1}))$. If U is an open affine subset of $\mathscr{X}$ and $\tilde{U} = \pi^{-1}(U)$ the corresponding subset of $\mathbb{P}^1$, then

$$\Gamma(\tilde{U}, \mathscr{D}(\pi^{-1}\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathbb{P}^1})) = \Gamma(U, \mathscr{D}(\mathcal{O}_{\mathscr{X}}, \pi_*\mathcal{O}_{\mathbb{P}^1})) = \mathscr{D}(U, \tilde{U}).$$

Thus our next result is an immediate consequence of Theorem 3.2.

COROLLARY. *Let* $\mathscr{X}$ *be a rational, projective curve with normalisation map* $\pi: \mathbb{P}^1 \to \mathscr{X}$. *If* U *is an open affine subset of* $\mathscr{X}$ *with* $\pi^{-1}(U) = \tilde{U}$ *then* $\mathscr{D}(\tilde{U})\, \mathscr{D}(\mathscr{X}, \mathbb{P}^1) = \mathscr{D}(U, \tilde{U})$.

3.4. The structure of $\mathscr{D}(\mathscr{X})$ would be easy to determine if Theorem 3.2 also held with $\mathbb{P}^1$ replaced by $\mathscr{X}$. Unfortunately it is easy to see that this cannot be the case.

PROPOSITION. *Let* $\mathscr{Y}$ *be any projective curve (not necessarily rational) such that* $\mathscr{Y}$ *is not isomorphic to* $\mathbb{P}^1$. *Then* $\Gamma(\mathscr{Y}, \_)$ *is not an exact functor from* $\mathscr{D}_{\mathscr{Y}}$**-mod** *to* $\mathscr{D}(\mathscr{Y}$**-mod**.

Proof. The key point is that, by [Ha, Exercise IV.1.8 and Exercise III.5.3], one has $H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) \neq 0$. Now, $\mathcal{O}_{\mathscr{Y}}$ is certainly a coherent $\mathscr{D}_{\mathscr{Y}}$-module, while the constant sheaf $K(\mathscr{Y})$, of rational functions on $\mathscr{Y}$, is a $\mathscr{D}_{\mathscr{Y}}$-module which is quasi-coherent as an $\mathcal{O}_{\mathscr{Y}}$-module. Thus, as $K(\mathscr{Y})$ is flasque, $H^1(\mathscr{Y}, K(\mathscr{Y})) = 0$. Therefore, on applying $\Gamma(\mathscr{Y}, \_)$ to the exact sequence in $\mathscr{D}_{\mathscr{Y}}$**-mod**

$$0 \to \mathcal{O}_{\mathscr{Y}} \to K(\mathscr{Y}) \to K(\mathscr{Y})/\mathcal{O}_{\mathscr{Y}} \to 0,$$

one obtains the exact sequence

$$0 \to k \to K(\mathscr{Y}) \to \Gamma(\mathscr{Y}, K(Y)/\mathcal{O}_{\mathscr{Y}}) \to H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) \to 0. \tag{3.4.1}$$

Since $H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) \neq 0$, this implies that $\Gamma(\mathscr{Y}, \_)$ is not exact.

Remark. Observe that the sequence (3.4.1) gives $H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$ the structure of a $\mathscr{D}(\mathscr{Y})$-module. This module will play an important role in understanding the structure of $\mathscr{D}(\mathscr{Y})$.

The following alternative description of the $\mathscr{D}(\mathscr{X})$-module structure on $H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ will prove useful. Let $\mathscr{U} = \{U_1, U_2\}$ be an open affine cover of

$\mathscr{X}$. Then it follows from the construction of [Ha, Lemma 4.4, p. 221] that the natural isomorphism

$$\check{H}^1(\mathscr{U}, \mathscr{O}_{\mathscr{X}}) = \mathscr{O}(U_1 \cap U_2)/\mathscr{O}(U_1) + \mathscr{O}(U_2) \cong H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$$

is a $\mathscr{D}(\mathscr{X})$-module map.

3.5. *For the rest of this section, $\mathscr{X}$ will denote a rational, projective curve for which the normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$ is injective.*

Ironically, given this assumption, Proposition 3.4 gives the only significant way in which Theorem 3.2 fails to hold for $\mathscr{D}_{\mathscr{X}}$-**mod** (see, in particular Theorem 3.15). One reason for this is that, by [SS, Sect. 6], there is an equivalence of categories between $\mathscr{D}_{\mathscr{X}}$-**mod** and $\mathscr{D}_{\mathbb{P}^1}$-**mod**. As a consequence, one has the following method for factoring the functor $\mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} -$.

PROPOSITION. *Let $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ and write $\mathscr{Q} = \mathscr{D}(\pi_* \mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathscr{X}})$ for the sheaf defined in (2.2). Define a map $\eta: \pi_* \mathscr{D}_{\mathbb{P}^1}$-**mod** $\to \mathscr{D}_{\mathscr{X}}$**mod** by $\eta = \mathscr{Q} \otimes_{\pi_* \mathscr{D}(\mathbb{P}^1)} -$. Then the following diagram commutes*

$$\begin{array}{ccccc}
\mathscr{D}(\mathscr{X})\text{-}\mathbf{mod} & & \xrightarrow{P \otimes_{\mathscr{D}(\mathscr{X})} -} & & \mathscr{D}(\mathbb{P}^1)\text{-}\mathbf{mod} \\
\downarrow{\scriptstyle \mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} -} & & & & \downarrow{\scriptstyle \mathscr{D}_{\mathbb{P}^1} \otimes_{\mathscr{D}(\mathbb{P}^1)} -} \\
\mathscr{D}_{\mathscr{X}}\text{-}\mathbf{mod} & \xleftarrow[\eta]{} & \pi_* \mathscr{D}_{\mathbb{P}^1}\text{-}\mathbf{mod} & \xleftarrow[\pi_*]{} & \mathscr{D}_{\mathbb{P}^1}\text{-}\mathbf{mod}
\end{array} \qquad (3.5.1)$$

Moreover, the three maps $\mathscr{D}_{\mathbb{P}^1} \otimes_{\mathscr{D}(\mathbb{P}^1)} -$, π_*, and η are all equivalences of categories.

Proof. As usual, let U be an open affine subset of $\mathscr{X}$ and set $\tilde{U} = \pi^{-1}(U)$. Then, as in (3.3),

$$\pi_*(\mathscr{D}_{\mathbb{P}^1} \otimes_{\mathscr{D}(\mathbb{P}^1)} P) = \pi_* \mathscr{D}(\pi^{-1} \mathscr{O}_{\mathscr{X}}, \mathscr{O}_{\mathbb{P}^1}) = \mathscr{D}(\mathscr{O}_{\mathscr{X}}, \pi_* \mathscr{O}_{\mathbb{P}^1})$$

is the sheaf of left $\pi_* \mathscr{D}_{\mathbb{P}^1}$-modules with sections $\mathscr{D}(U, \tilde{U})$ over U. Thus, $\mathscr{Q} \otimes \pi_*(\mathscr{D}_{\mathbb{P}^1} \otimes P)$ is the sheaf of left $\mathscr{D}_{\mathscr{X}}$-modules with sections

$$\Gamma(U, \mathscr{Q} \otimes \pi_*(\mathscr{D}_{\mathbb{P}^1} \otimes P)) = \mathscr{D}(\tilde{U}, U) \otimes_{\mathscr{D}(\tilde{U})} \mathscr{D}(U, \tilde{U}) = \mathscr{D}(U),$$

where the final equality follows from [SS, Proposition 3.14]. Thus, as U was an arbitrary, open affine subset of $\mathscr{X}$, we have proved that

$$\mathscr{Q} \otimes_{\pi_* \mathscr{D}_{\mathbb{P}^1}} \pi_*(\mathscr{D}_{\mathbb{P}^1} \otimes_{\mathscr{D}(\mathbb{P}^1)} P) \otimes_{\mathscr{D}(\mathscr{X})} - = \mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} -,$$

as is required to show that (3.5.1) commutes.

That the maps $\mathcal{D}_{\mathbb{P}^1} \otimes_{\mathcal{D}(\mathbb{P}^1)} -$ and π_* are equivalences of categories, follows from Theorem 3.2 and (2.2), respectively. Finally, [SS, Sect. 6.1] shows that the mutually inverse functors $\mathcal{D}(\pi_* \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \otimes -$ and $\mathcal{D}(\mathcal{O}_{\mathcal{X}}, \pi_* \mathcal{O}_{\mathcal{X}}) \otimes -$ give an equivalence between $\pi_* \mathcal{D}_{\mathbb{P}^1}$-**mod** and $\mathcal{D}_{\mathcal{X}}$-**mod**.

3.6. It is clear from Proposition 3.5 that one needs to understand $\mathcal{D}(\mathcal{X}, \mathbb{P}^1)$ in terms of the sheaf-theoretic data, and much of this section will be devoted to obtaining such an understanding. The following abstract result will prove useful.

LEMMA. *Let $\{R_j : j \in J\}$ be a finite set of Noetherian domains. Suppose further that $R = \bigcap_{j \in J} R_j$ is Noetherian with* $\operatorname{Quot}(R) = \operatorname{Quot}(R_j)$, *for each j.*

(a) *If M is a fractional left R-ideal with $M = \bigcap_{j \in J} R_j M$ then*

$$\operatorname{End}_R(M) = \bigcap_{j \in J} \operatorname{End}_{R_j}(R_j M).$$

(b) *Suppose that each R_j is flat as a right R-module. If N is a fractional right R-ideal then $R_j N^* = (NR_{j R_j})^*$, for each j.*

Proof. (a) As usual, identify $\operatorname{End}_R M$ with $O({}_R M) = \{q \in \operatorname{Quot}(R) : Mq \subseteq M\}$ and $\operatorname{End}_{R_j}(R_j M)$ with $O({}_{R_j} R_j M)$. Then, clearly $\operatorname{End}_R M \subseteq \bigcap \operatorname{End}(R_j M)$. Conversely, if $q \in \operatorname{Quot}(R)$ is such that $R_j Mq \subseteq R_j M$, for each j, then $Mq \subseteq \bigcap R_j M = M$.

(b) If A and B are fractional left R-ideals, then the obvious homomorphisms make

$$0 \to A \cap B \to A \oplus B \to A + B \to 0$$

into an exact sequence. Since $R_j \otimes -$ is exact, $R_j(A \cap B) = R_j A \cap R_j B$. In particular, the obvious induction implies that $R_j(\bigcap_{i=1}^t Ra_i) = \bigcap_{i=1}^t R_j a_i$ for any $a_i \in \operatorname{Quot}(R)$ and $j \in J$. Now if $N = \sum_{i=1}^t n_i R$ for some $n_i \in \operatorname{Quot}(R)$ then $N^* = \bigcap_{i=1}^t Rn_i^{-1}$. Combined with the above observation, this implies that

$$R_j N^* = \bigcap_i R_j n_i^{-1} = \left\{ \sum_i n_i R_j \right\}^*_{R_j} = \{NR_{j R_j}\}^*.$$

3.7. The following results from [SS] will be used frequently.

PROPOSITION. *Let U be an affine curve such that the normalisation map $\pi : \tilde{U} \to U$ is injective. Then*

(i) *$\mathcal{D}(U, \tilde{U}) = \mathcal{D}(\tilde{U}, U)^*$, as a module over both $\mathcal{D}(U)$ and $\mathcal{D}(\tilde{U})$.*

(ii) *Similarly, $\mathcal{D}(\tilde{U}, U) = \mathcal{D}(U, \tilde{U})^*$ as a module over either ring.*

(iii) $\mathscr{D}(\tilde{U}) = \mathrm{End}_{\mathscr{D}(U)} \mathscr{D}(\tilde{U}, U)$ *and* $\mathscr{D}(U) = \mathrm{End}\, \mathscr{D}(\tilde{U}, U)_{\mathscr{D}(\tilde{U})}$. *Moreover* $\mathscr{D}(\tilde{U})$ *and* $\mathscr{D}(U)$ *are Morita equivalent.*

(iv) *Let* V *be a second affine curve (possibly* $V = \tilde{U}$*), such that* V *has injective normalisation map* $\rho: \tilde{U} \to V$, *and set*

$$\mathscr{D}(U, V) = \{\theta \in \mathscr{D}(K(U)) : \theta \circ \mathcal{O}(U) \subseteq \mathcal{O}(V)\}.$$

Then $\mathscr{D}(U, V) = \mathscr{D}(\tilde{U}, V)\, \mathscr{D}(U, \tilde{U})$. *Thus* $[_{\mathscr{D}(V)} \mathscr{D}(U, V)]^* = \mathscr{D}(V, U)$.

(v) $\mathscr{D}(U, V) \otimes \mathcal{O}(U) \cong \mathscr{D}(U, V) \circ \mathcal{O}(U) = \mathcal{O}(V)$.

Proof. Only parts (iv) and (v) are not explicitly in [SS]. Certainly, $\mathscr{D}(U, V) \supseteq \mathscr{D}(\tilde{U}, V)\, \mathscr{D}(U, \tilde{U})$. Conversely, $\mathscr{D}(V, \tilde{U})\, \mathscr{D}(U, V) \subseteq \mathscr{D}(U, \tilde{U})$ and so, by pre-multiplying by $\mathscr{D}(\tilde{U}, V)$, one obtains $\mathscr{D}(U, V) = \mathscr{D}(\tilde{U}, V)\, \mathscr{D}(V, \tilde{U})\, \mathscr{D}(U, V) \subseteq \mathscr{D}(\tilde{U}, V)\, \mathscr{D}(U, \tilde{U})$. In order to prove (v), recall from [SS, Remark 4.2 and Proposition 3.3] that $\mathcal{O}(U)$ is a simple left $\mathscr{D}(U)$-module and $\mathscr{D}(U, V) \circ \mathcal{O}(U) = \mathcal{O}(V)$. Since $\mathscr{D}(U, V)$ is a projective right $\mathscr{D}(U)$-module, $\mathscr{D}(U, V) \otimes \mathcal{O}(U)$ is also simple. Since the multiplication map $\mu: \mathscr{D}(U, V) \otimes \mathcal{O}(U) \to \mathscr{D}(U, V) \circ \mathcal{O}(U)$ is certainly onto, this implies that μ must be an isomorphism.

3.8. Corollary. *Let* $\mathscr{X}$ *be a rational, projective curve such that the normalisation map* $\pi: \mathbb{P}^1 \to \mathscr{X}$ *is injective. Then*

(i) $\mathscr{D}(\mathscr{X}) = \mathrm{End}_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$.

(ii) $\mathscr{D}(\mathbb{P}^1, \mathscr{X}) = [_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathscr{X}, \mathbb{P}^1)]^*$ *and is a projective right ideal of* $\mathscr{D}(\mathbb{P}^1)$.

Proof. As usual, write $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ and $Q = \mathscr{D}(\mathbb{P}^1, \mathscr{X})$. Pick an open affine cover $\{U_i\}$ for $\mathscr{X}$ and set $\tilde{U}_i = \pi^{-1}(U_i)$, for each i. Then Lemma 3.6 and Corollary 3.3 combine to prove that

$$\mathrm{End}_{\mathscr{D}(\mathbb{P}^1)} P = \bigcap \mathrm{End}_{\mathscr{D}(\tilde{U}_i)} \mathscr{D}(\tilde{U}_i) P = \bigcap \mathrm{End}_{\mathscr{D}(\tilde{U}_i)} \mathscr{D}(U_i, \tilde{U}_i).$$

But for each $i, \pi|_{\tilde{U}_i}$ is injective, and so, by Proposition 3.7, $\mathrm{End}_{\mathscr{D}(\tilde{U}_i)} \mathscr{D}(U_i, \tilde{U}_i) = \mathscr{D}(U_i)$. Thus $\mathrm{End}\, P = \mathscr{D}(\mathscr{X})$ and part (i) is proven.

By Corollary 2.5(iii), $Q \subseteq (_{\mathscr{D}(\mathbb{P}^1)}P)^*$. Conversely, if $q \in P^*$ then $Pq \subseteq \mathscr{D}(\mathbb{P}^1)$ and so $\mathscr{D}(\tilde{U}_i)Pq \subseteq \mathscr{D}(\tilde{U}_i)$, for each i. Thus, Corollary 3.3 implies that $\mathscr{D}(U_i, \tilde{U}_i)q \subseteq \mathscr{D}(\tilde{U}_i)$ and so $q \in \mathscr{D}(\tilde{U}_i, U_i)$, by Proposition 3.7 again. Thus, $q \in \bigcap \mathscr{D}(\tilde{U}_i, U_i) = Q$; thereby proving that $Q = P^*$. Thus, Q is reflexive as a right $\mathscr{D}(\mathbb{P}^1)$-module which, since $\mathrm{gldim}\, \mathscr{D}(\mathbb{P}^1) = 2$, is equivalent to $Q_{\mathscr{D}(\mathbb{P}^1)}$ being projective.

Remark. Variants of this result will be used elsewhere in the paper. Unfortunately, any result, sufficiently general that it covers all the applications, is too cumbersome to be useful. Thus we will content ourselves with the (somewhat vague) comment that, whenever P and Q are the global sections of sheaves, for which the local results of Proposition 3.7 and Corollary 3.3 hold, then the conclusion of Corollary 3.8 will also hold.

3.9. The aim of the next few results is to continue the study of $Q = \mathscr{D}(\mathbb{P}^1, \mathscr{X})$ and $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ by showing that ${}_{\mathscr{D}(\mathbb{P}^1)}P$ is a generator and that $Q_{\mathscr{D}(\mathbb{P}^1)}$ is actually a progenerator—and hence that End Q is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$. To prove these statements it will certainly suffice to prove that $Q \not\subseteq \mathbf{m}$, the unique, non-zero ideal of $\mathscr{D}(\mathbb{P}^1)$, and this in turn is proved through an understanding of the $\mathscr{D}(\mathbb{P}^1)$-modules $\mathscr{D}(\tilde{U}_i, U_i) \cap \mathscr{D}(\mathbb{P}^1)$. We begin with some preparatory lemmas.

LEMMA. *Write $\mathscr{D}(\mathbb{P}^1) = k[\partial, t\partial, t^2\partial] \subset A_1 = k[t, \partial]$ for some choice of coordinate function t. Let I be a maximal right ideal of A_1 such that $t^n \in I$, for some integer n. Then, as $\mathscr{D}(\mathbb{P}^1)$-modules, either $\mathscr{D}(\mathbb{P}^1)/I \cap \mathscr{D}(\mathbb{P}^1) \cong A_1/tA_1$ or*

$$\mathscr{D}(\mathbb{P}^1)/I \cap \mathscr{D}(\mathbb{P}^1) \cong M = \mathscr{D}(\mathbb{P}^1)/(t\partial - 1)\,\mathscr{D}(\mathbb{P}^1) + t^2\partial\mathscr{D}(\mathbb{P}^1).$$

Moreover, M is a simple, infinite-dimensional $\mathscr{D}(\mathbb{P}^1)$-module.

Proof. Since $A_1/I \cong A_1/tA_1$, one sees that

$$\mathscr{D}(\mathbb{P}^1)/I \cap \mathscr{D}(\mathbb{P}^1) \cong \mathscr{D}(\mathbb{P}^1) + I/I \cong x\mathscr{D}(\mathbb{P}^1) + tA_1/tA_1,$$

for some $x \in A_1$. Thus $\mathscr{D}(\mathbb{P}^1)/I \cap \mathscr{D}(\mathbb{P}^1)$ is isomorphic to a submodule of A_1/tA_1 and the result follows from Lemma 2.7(ii).

3.10. LEMMA. *Let $\mathscr{D}(\mathbb{P}^1) \subset A_1$ be as in Lemma 3.9. Suppose that J is a right ideal of A_1 such that:*

(a) *$t^n \in J$, for some $n \geqslant 1$, and*

(b) *$(t\partial - 1) \cdots (t\partial - r) \in J$, for some $r \geqslant 1$.*

Then $\mathscr{D}(\mathbb{P}^1)/J \cap \mathscr{D}(\mathbb{P}^1)$ has no subfactor isomorphic to $\mathscr{D}(\mathbb{P}^1)/\mathbf{m}$.

Proof. Recall that, by Lemma 1.5, $\mathscr{D}(\mathscr{P}^1)/K$ has finite length for every non-zero right ideal K, of $\mathscr{D}(\mathscr{P}^1)$. Thus if K_1 and K_2 are right ideals of $\mathscr{D}(\mathscr{P}^1)$ such that neither $\mathscr{D}(\mathscr{P}^1)/K_1$ nor $\mathscr{D}(\mathscr{P}^1)/K_2$ has $\mathscr{D}(\mathscr{P}^1)/\mathbf{m}$ as a subfactor, then nor does $\mathscr{D}(\mathscr{P}^1)/K_1 \cap K_2$. But, for example, by [SS, Proposition 4.15], $A_1/t^nA_1 \cong (A_1/tA_1)^{(n)}$ and A_1/tA_1 is a simple A_1-module. Thus, as $t^n \in J$, one has $J = \bigcap_{j=1}^{m} J_i$, for some maximal right ideals J_i of A_1 such that $A_1/J_i \cong A_1/tA_1$, for each i. Combining these two observations means that we may, in proving the lemma, assume that $J=J_j$ is a maximal right ideal of A_1. But now $\prod_1^r(t\partial - i)\in J\cap\mathscr{D}(\mathscr{P}^1)$ and so $J\cap\mathscr{D}(\mathscr{P}^1)\not\subseteq \mathbf{m}$. Thus, Lemma 3.9 implies that $\mathscr{D}(\mathscr{P}^1)/J\cap\mathscr{D}(\mathscr{P}^1)\cong M$, an infinite-dimensional, simple $\mathscr{D}(\mathscr{P}^1)$-module.

3.11. *Remark.* Let $\mathscr{D}(\mathscr{P}^1)\subset A_1$ be as in Lemma 3.9. Then Theorem 3.2 implies that A_1 is flat as a right $\mathscr{D}(\mathbb{P}^1)$-module. However, flatness of ${}_{\mathscr{D}(\mathbb{P}^1)}A_1$ fails in a rather dramatic way. *Indeed let M be as in Lemma 3.9 and $N=A_1/tA_1$. Then $0\to M\xrightarrow{i} N$ is exact, $M_{\mathscr{D}(\mathbb{P}^1)}$ has length one, and $N_{\mathscr{D}(\mathbb{P}^1)}$ has length two. In contrast, $M\otimes A_1\xrightarrow{i\otimes 1} N\otimes A_1\to 0$ is exact, $M\otimes A_1$ has length two, yet $N\otimes A_1$ is simple.*

The first sentence of the above claim is just Lemma 2.7(ii). An easy exercise shows that $M\otimes A_1\cong A_1/(t\partial-1)A_1+t^2\partial A_1$ has length two, the proper factor module being $A_1/(t\partial-1)A_1+t^2A_1$. Similarly,

$$N\otimes A_1\cong A_1/t\partial A_1+t^2\partial A_1=A_1/tA_1$$

is simple.

3.12. Let $\mathscr{X}$ be a singular, projective curve with injective normalisation map $\pi\colon\mathbb{P}^1\to\mathscr{X}$. In order to apply Lemma 3.10 to $\mathscr{D}(\mathbb{P}^1,\mathscr{X})$ we need to be careful about the open affine cover that we choose for $\mathscr{X}$. We do it as follows. Fix an open affine cover $\{U_1,...,U_m\}$ of $\mathscr{X}$ such that each U_i is either smooth or has exactly one singular point, say x_i. (If U_i is smooth, set x_i to be any point of U_i.) As usual, set $\tilde{U}_i=\pi^{-1}(U_i)$, for $1\leqslant i\leqslant m$, and write $y_i=\pi^{-1}(x_i)$. Now let $\sigma_i\colon\tilde{U}_i\cong\mathbb{A}^1\backslash H_i$, for some finite set of points H_i. We choose this isomorphism so that $y_i\mapsto 0$. Thus, we can take a coordinate t_i on $\mathbb{A}^1$ such that, with $\partial_i=\partial/\partial t_i$, one has

(i) $\mathscr{O}(\tilde{U}_i)=k[t_i]_{f_i}$, where $0\neq f_i\in k[t_i]$, defines H_i.

(ii) $\pi\sigma_i^{-1}(0)=x_i$, the only possible singular point of U_i.

(iii) As $\pi|_{\tilde{U}_i}$ is injective, $\mathscr{O}(U_i)\supset t_i^n\mathscr{O}(\tilde{U}_i)$, for some n. It follows that

$$\mathscr{D}(\tilde{U}_i,U_i)\supseteq t_i^n\,\mathscr{D}(\tilde{U}_i)+\prod_{j=1}^{n-1}(t_i\partial_i-j)\,\mathscr{D}(\tilde{U}_i)$$

(see the proof of [SS, Theorem 3.4]). Observe that if U_i is not singular then $\mathscr{O}(U_i)=\mathscr{O}(\tilde{U}_i)$ and so $\mathscr{D}(\tilde{U}_i,U_i)=\mathscr{D}(\tilde{U}_i)$.

3.13. LEMMA. *Keep the notation of* (3.12). *Then* $\mathscr{D}(\mathbb{P}^1)/\mathscr{D}(\mathbb{P}^1)\cap\mathscr{D}(\tilde{U}_i,U_i)$ *has no subfactor isomorphic to* $\mathscr{D}(\mathbb{P}^1)/\mathbf{m}$.

Proof. Since $\mathscr{D}(\mathbb{P}^1)\cap\mathscr{D}(\tilde{U}_i,U_i)=\mathscr{D}(\mathbb{P}^1)\cap(A_1\cap\mathscr{D}(\tilde{U}_i,U_i))$, the proposition follows immediately from Lemma 3.10 and 3.12(iii).

3.14. We can now complete the description of the properties of P and Q that we began in 3.8.

PROPOSITION. *Let $\mathscr{X}$ be a singular, projective curve with injective normalisation $\pi: \mathbb{P}^1 \to \mathscr{X}$. Then*

(a) *${}_{\mathscr{D}(\mathbb{P}^1)}\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is a generator and $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)_{\mathscr{D}(\mathscr{X})}$ is projective. Indeed, $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)\,\mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \mathscr{D}(\mathbb{P}^1)$.*

(b) *$\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ is a progenerative right ideal of $\mathscr{D}(\mathbb{P}^1)$.*

(c) *$\mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \{\mathscr{D}(\mathscr{X}, \mathbb{P}^1)_{\mathscr{D}(\mathscr{X})}\}^*$.*

Proof. By Corollary 3.8, Q is a projective right ideal of $\mathscr{D}(\mathbb{P}^1)$ while by Corollary 2.5(iii), $Q \subseteq P^*$ and $P \subseteq Q^*$, as modules over both $\mathscr{D}(\mathbb{P}^1)$ and $\mathscr{D}(\mathscr{X})$. Thus, to prove parts (a) and (b) of the Proposition, it suffices to show that $PQ = \mathscr{D}(\mathbb{P}^1)$.

Retain the notation of (3.12). Now,

$$Q = \bigcap_i \mathscr{D}(\tilde{U}_i, U_i) = \bigcap_i \mathscr{D}(\tilde{U}_i, U_i) \cap \mathscr{D}(\mathbb{P}^1).$$

Thus Lemma 3.13 and the first two sentences of the proof of Lemma 3.10 combine to show that $\mathscr{D}(\mathbb{P}^1)/Q$ has no subfactor isomorphic to $\mathscr{D}(\mathbb{P}^1)/\mathbf{m}$. Therefore $Q \not\subseteq \mathbf{m}$ and $\mathscr{D}(\mathbb{P}^1)Q = \mathscr{D}(\mathbb{P}^1)$. Since $\mathscr{D}(\mathbb{P}^1) \subseteq P$, this suffices to show that $PQ = \mathscr{D}(\mathbb{P}^1)$.

In order to prove part (c) of the proposition, it remains to show that $Q \supseteq \{P_{\mathscr{D}(\mathscr{X})}\}^*$. Let $q \in \operatorname{Quot} \mathscr{D}(\mathscr{X})$ be such that $qP \subseteq \mathscr{D}(\mathscr{X})$. Then, by the last paragraph, $q\mathscr{D}(\mathbb{P}^1) = qPQ \subseteq Q$, and so $q \in Q$; as required.

3.15. The significance of Proposition 3.14 is that, when combined with Proposition 3.5, it shows that "half" of the Beilinson–Bernstein equivalence of categories (Theorem 3.2) still holds for $\mathscr{D}(\mathscr{X})$.

THEOREM. *Let $\mathscr{X}$ be a projective curve such that the normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$ is injective. Then:*

(a) *Every sheaf $\mathscr{M} \in \mathscr{D}_{\mathscr{X}}$-**mod** is generated over $\mathscr{D}_{\mathscr{X}}$ by global sections.*

(b) *The functor $\mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} \_ : \mathscr{D}(\mathscr{X})$-**mod** $\to \mathscr{D}_{\mathscr{X}}$-**mod** is exact.*

(c) *$\mathscr{D}_{\mathscr{X}}$-**mod** is a quotient category of $\mathscr{D}(\mathscr{X})$-**mod**.*

(d) *The modules $A \in \mathscr{D}(\mathscr{X})$-**mod** with $\mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} A = 0$ are precisely those $\mathscr{D}(\mathscr{X})$-modules annihilated by $\mathscr{D}(\mathbb{P}^1, \mathscr{X})\,\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$.*

(e) *If U is an open affine subset of $\mathscr{X}$, then $\mathscr{D}(U)$ is flat as a right $\mathscr{D}(\mathscr{X})$-module.*

Remark. This proves the first four parts of Theorem C of the Introduction.

Proof. Consider $P_{\mathscr{D}(\mathscr{X})}$. By Proposition 3.14(a), it is projective and hence $P \otimes_{\mathscr{D}(\mathscr{X})} -$ is exact. By Proposition 3.14(a), (c), and the dual basis lemma, $\mathrm{End}_{\mathscr{D}(\mathscr{X})} P = P\{P_{\mathscr{D}(\mathscr{X})}\}^* = PQ = \mathscr{D}(\mathbb{P}^1)$. Now if R is a ring and T is a finitely generated, projective right R-module then the exact functor $T \otimes_R$ makes $\mathrm{End}_R\, T$-**mod** into a quotient category of R-**mod**. The associated torsion category consists of those R-modules annihilated by $\mathrm{Trace}(T) = T^*T$ (see [S, Proposition XI.8.6]). Thus, in our situation, $P \otimes -$ makes $\mathscr{D}(\mathbb{P}^1)$-**mod** a quotient category of $\mathscr{D}(\mathscr{X})$-**mod**. Now consider the commutative diagram (3.5.1). Since the three maps at the bottom and right of that diagram are equivalences, we conclude that $\mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} -$ is exact and that $\mathscr{D}_{\mathscr{X}}$-**mod** is a quotient category of $\mathscr{D}(\mathscr{X})$-**mod**. This proves (b) and (c) while (e) is an immediate consequence of (b). Part (a) is equivalent to proving that $\mathscr{D}_{\mathscr{X}}$ is a generator in $\mathscr{D}_{\mathscr{X}}$-**mod**. (To see this, use the fact that $\mathscr{M}$ is generated by sections if and only if $\mathscr{M}$ is a homomorphic image of a free sheaf $\mathscr{D}_{\mathscr{X}}^{(I)}$.) But, by Proposition 3.14(a), ${}_{\mathscr{D}(\mathbb{P}^1)}P$ is a generator, and hence so is $\mathscr{D}_{\mathscr{X}} \cong \mathscr{Q} \otimes \pi_*(\mathscr{D}_{\mathbb{P}^1} \otimes P)$. Thus part (a) is proved. Finally, (d) follows from observing that, by (3.5.1), $\mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} A = 0$ if and only if $P \otimes_{\mathscr{D}(\mathscr{X})} A = 0$. But, as we observed above, this last condition is equivalent to $\mathrm{Trace}(P) A = 0$, and $\mathrm{Trace}(P) = [P_{\mathscr{D}(\mathscr{X})}]^* P = QP$.

3.16. *Remark.* There are now two methods of proving that $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathbb{P}^1)$ are not Morita equivalent. The first is by an analysis of the $\mathscr{D}(\mathscr{X})$-module $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ and will be given shortly as it contains other useful information. The second is more direct and is, in outline, as follows.

Since $\mathscr{D}(\mathscr{X}) = \mathrm{End}_{\mathscr{D}(\mathbb{P}^1)} P$ and ${}_{\mathscr{D}(\mathbb{P}^1)}P$ is a generator, any Morita equivalence forces ${}_{\mathscr{D}(\mathbb{P}^1)}P$ to be projective (count ideals!). Then, by going round the diagram (3.5.1), one finds that $\mathscr{D}_{\mathscr{X}} \cong \mathscr{Q} \otimes \pi_*(\mathscr{D}_{\mathbb{P}^1} \otimes P)$ is a projective object of $\mathscr{D}_{\mathscr{X}}$-**mod** and hence $\mathrm{Hom}_{\mathscr{D}_{\mathscr{X}}}(\mathscr{D}_{\mathscr{X}}, -) = \Gamma(\mathscr{X}, -)$ is exact. But this contradicts Proposition 3.4.

The above argument also shows that $\mathscr{D}(\mathscr{X})$ is not a maximal order (use 1.3(e) and the fact that $\mathrm{End}_{\mathscr{D}(\mathbb{P}^1)}(Q)$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$).

3.17. Corollary. *Let $\mathscr{X}$ be a projective curve with injective normalisation $\pi: \mathbb{P}^1 \to \mathscr{X}$. Suppose that U is an open affine subset of $\mathscr{X}$ and set $\tilde{U} = \pi^{-1}(U)$. Then $\mathscr{D}(\mathscr{X}, \mathbb{P}^1)\, \mathscr{D}(U) = \mathscr{D}(U, \tilde{U})$.*

Remark. This result is quite surprising in view of the fact that $\mathscr{D}(\mathbb{P}^1, \mathscr{X})\, \mathscr{D}(\tilde{U}) \neq \mathscr{D}(\tilde{U}, U)$, when U is singular. The proof of this statement is not required later (and, in any case, is proved similarly to the corollary) and so we leave it to the reader.

Proof. Let $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ and suppose that $P\mathscr{D}(U) \subsetneq \mathscr{D}(U, \tilde{U})$. Now, $\mathscr{D}(U)$ is hereditary by [SS, Corollary 3.5] and so $\{P\mathscr{D}(U)_{\mathscr{D}(U)}\}^* \supsetneq \mathscr{D}(U, \tilde{U})^*$. Thus

$$\begin{aligned}
\mathscr{D}(U)Q &= \mathscr{D}(U)(P_{\mathscr{D}(\mathscr{X})})^* && \text{by Proposition 3.14(c)}\\
&= \{P\mathscr{D}(U)\}^* && \text{by Theorem 3.15(e) and Lemma 3.6(b)}\\
&\supsetneq \mathscr{D}(U, \tilde{U})^* &&\\
&= \mathscr{D}(\tilde{U}, U) && \text{by Lemma 3.7(ii)}\\
&= \mathscr{D}(U)Q && \text{by Theorem 3.15(a);}
\end{aligned}$$

a contradiction.

3.18. COROLLARY. *Let $\mathscr{X}$ be a projective curve with injective normalisation $\pi: \mathbb{P}^1 \to \mathscr{X}$. Then $\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \otimes_{\mathscr{D}(\mathscr{X})} H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) = 0$.*

Proof. By deleting two points on $\mathscr{X}$, we may take an open affine cover $\{U_1, U_2\}$ for $\mathscr{X}$. As usual, set $\tilde{U}_i = \pi^{-1}(U_i)$; thus $\{\tilde{U}_1, \tilde{U}_2\}$ is an open affine cover of $\mathbb{P}^1$. Corollary 3.17 implies that $P\mathscr{D}(U_i) = \mathscr{D}(U_i, \tilde{U}_i)$ for each i. Moreover, $P_{\mathscr{D}(\mathscr{X})}$ is projective, by Proposition 3.14(a). Thus

$$\begin{aligned}
P \otimes_{\mathscr{D}(\mathscr{X})} \mathcal{O}(U_i) &\cong P \otimes_{\mathscr{D}(\mathscr{X})} \mathscr{D}(U_i) \otimes_{\mathscr{D}(U_i)} \mathcal{O}(U_i)\\
&\cong \mathscr{D}(U_i, \tilde{U}_i) \otimes_{\mathscr{D}(U_i)} \mathcal{O}(U_i) \cong \mathcal{O}(\tilde{U}_i),
\end{aligned}$$

where the final isomorphism comes from 3.7(v). A Čech cohomology computation (see [Ha, pp. 218–222]) implies that

$$H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \cong \mathcal{O}(U_1 \cap U_2)/(\mathcal{O}(U_1) + \mathcal{O}(U_2)).$$

Finally, by combining the last two displayed equations, one finds that

$$\begin{aligned}
P \otimes_{\mathscr{D}(\mathscr{X})} H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) &\cong \mathcal{O}(\tilde{U}_1 \cap \tilde{U}_2)/(\mathcal{O}(\tilde{U}_1) + \mathcal{O}(\tilde{U}_2))\\
&\cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0;
\end{aligned}$$

as required.

3.19. THEOREM. *Let $\mathscr{X}$ be a singular, rational curve with injective normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$. Then*

(a) *$\mathscr{D}(\mathscr{X})$ is not a maximal order and hence is not Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$.*

(b) *$\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is not projective as a left $\mathscr{D}(\mathbb{P}^1)$-module and not a generator as a right $\mathscr{D}(\mathscr{X})$-module. Thus $\mathscr{D}(\mathscr{X}, \mathbb{P}^1) \neq \{\mathscr{D}(\mathbb{P}^1, \mathscr{X})_{\mathscr{D}(\mathbb{P}^1)}\}^*$.*

(c) *The left annihilator* $\operatorname{ann}_{\mathscr{D}(\mathscr{X})} H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) \supseteq \mathscr{D}(\mathbb{P}^1, \mathscr{X})\, \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ *and hence* $\mathscr{D}(\mathbb{P}^1, \mathscr{X})\, \mathscr{D}(\mathscr{X}, \mathbb{P}^1) \neq \mathscr{D}(\mathscr{X})$.

(d) $\mathscr{D}(\mathscr{X}) = \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathscr{X}, \mathbb{P}^1) \subsetneq \operatorname{End}(\mathscr{D}(\mathbb{P}^1, \mathscr{X})_{\mathscr{D}(\mathbb{P}^1)})$.

Remark. Part (a) of the above theorem, combined with Proposition 2.10, proves Proposition B of the Introduction.

Proof. (a) Suppose that $\mathscr{D}(\mathscr{X})$ is a maximal order. Since $\mathscr{D}(\mathscr{X}) \subseteq \operatorname{End} Q_{\mathscr{D}(\mathbb{P}^1)}$, this implies that $\mathscr{D}(\mathscr{X}) = \operatorname{End} Q_{\mathscr{D}(\mathbb{P}^1)}$. Since $Q_{\mathscr{D}(\mathbb{P}^1)}$ is a progenerator, by Proposition 3.14, $\mathscr{D}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$. Now consider P. Since $P_{\mathscr{D}(\mathscr{X})}$ is projective with $\operatorname{End} P_{\mathscr{D}(\mathscr{X})} = \mathscr{D}(\mathbb{P}^1)$ (see Proposition 3.14, again), the Morita equivalence forces $P_{\mathscr{D}(\mathscr{X})}$ to be a progenerator. But, as $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) \neq 0$, this contradicts Corollary 3.18.

(b) Recall, from Proposition 3.14 and Corollary 3.8, that $P_{\mathscr{D}(\mathscr{X})}$ is projective, ${}_{\mathscr{D}(\mathbb{P}^1)}P$ is a generator, $\operatorname{End} P_{\mathscr{D}(\mathscr{X})} = \mathscr{D}(\mathbb{P}^1)$, and $\operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} P = \mathscr{D}(\mathscr{X})$. Thus, if either $P_{\mathscr{D}(\mathscr{X})}$ is a generator or ${}_{\mathscr{D}(\mathbb{P}^1)}P$ is projective, then this would force $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathbb{P}^1)$ to be Morita equivalent, a contradiction. Finally, by Corollary 3.8, Q is a projective $\mathscr{D}(\mathbb{P}^1)$-module and hence so is its dual. Thus $P \neq Q^*$.

(c) This is immediate from Corollary 3.18.

(d) By Corollary 3.8, $\mathscr{D}(\mathscr{X}) = \operatorname{End} P \subseteq \operatorname{End} Q$. But $Q_{\mathscr{D}(\mathbb{P}^1)}$ is a progenerator and so $\operatorname{End} Q$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$. Thus part (d) follows from part (a).

Remark. Note the curious lack of symmetry between P and Q, as evidenced by Corollary 3.8(ii), Remark 3.17, and part (b) of the theorem. Of course, it is this lack of symmetry that has caused most of the complications in the proofs in this paper.

4. The Ideal Structure of $\mathscr{D}(\mathscr{X})$

4.1. Throughout this section we fix a projective, singular curve $\mathscr{X}$ with injective normalisation $\pi: \mathbb{P}^1 \to \mathscr{X}$ and, as before, write $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ and $Q = \mathscr{D}(\mathbb{P}^1, \mathscr{X})$. The aim of this section is to give an almost complete description of the ideal structure of $\mathscr{D}(\mathscr{X})$ (this description will be completed in Section 7). Throughout the section W^* will denote the dual of W considered as a $\mathscr{D}(\mathbb{P}^1)$-module.

4.2. Recall that, from Proposition 3.14 and Theorem 3.19, $Q_{\mathscr{D}(\mathbb{P}^1)}$ is a progenerator and that $\mathscr{D}(\mathscr{X}) = \operatorname{End}({}_{\mathscr{D}(\mathbb{P}^1)}P) \subsetneq \operatorname{End}(Q_{\mathscr{D}(\mathbb{P}^1)})$. Moreover, $P_{\mathscr{D}(\mathscr{X})}$ is projective while ${}_{\mathscr{D}(\mathbb{P}^1)}P$ is a generator. Since $\mathscr{D}(\mathbb{P}^1)$ has a unique non-zero ideal $\mathbf{m}$, the ring $S = \operatorname{End} Q_{\mathscr{D}(\mathbb{P}^1)}$ has a unique non-zero ideal

$Q\mathbf{m}Q^* = Q\mathbf{m}P^{**}$. Similarly, by Theorem 2.4, $\mathscr{D}(\mathscr{X}) = \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} P$ has a unique minimal, non-zero ideal $J(\mathscr{X})$. In fact slightly more is true:

LEMMA. *One has* $\mathbf{m}P^{**} = \mathbf{m}P$ *and hence* S *and* $\mathscr{D}(\mathscr{X})$ *share the common minimal non-zero ideal* $J(\mathscr{X}) = Q\mathbf{m}Q^* = Q\mathbf{m}P$. *Moreover,* $J(\mathscr{X}) = \operatorname{Hom}_{\mathscr{D}(\mathbb{P}^1)}(P^{**}, \mathbf{m}P^{**})$.

Proof. By Lemma 1.5, P^{**}/P is a finite-dimensional $\mathscr{D}(\mathbb{P}^1)$-module and so $\mathbf{m}P^{**} \subseteq P$. Thus $\mathbf{m}P^{**} = \mathbf{m}^2P^{**} \subseteq \mathbf{m}P$ and $\mathbf{m}P^{**} = \mathbf{m}P$. Next consider $J(\mathscr{X})$. Clearly, $Q\mathbf{m}Q^* = Q\mathbf{m}P^{**} = Q\mathbf{m}P$ is an ideal of $\mathscr{D}(\mathscr{X})$. Conversely, if W is any non-zero ideal of $\mathscr{D}(\mathscr{X})$, then $W \supseteq Q(\mathbf{m}PWQ\mathbf{m})P = Q\mathbf{m}P$. Thus $J(\mathscr{X}) = Q\mathbf{m}P = Q\mathbf{m}P^{**}$.

Now, consider $V = \operatorname{Hom}(P^{**}, \mathbf{m}P^{**})$ which we identify with

$$\{\theta \in \operatorname{Quot}(R) \colon P^{**}\theta \subseteq \mathbf{m}P^{**}\}.$$

Now $P^{**}J(\mathscr{X}) = P^{**}P^*\mathbf{m}P^{**} = \mathscr{D}(\mathbb{P}^1)\mathbf{m}P^{**} = \mathbf{m}P^{**}$ and so $J(\mathscr{X}) \subseteq V$. Conversely, if $\theta \in V$, then Proposition 3.14 implies that $S\theta = P^*P^{**}\theta \subseteq P^*\mathbf{m}P^{**} = J(\mathscr{X})$. Thus $J(\mathscr{X}) = V$.

4.3. Since $J(\mathscr{X})$ is the unique non-zero ideal of S, clearly $S/J(\mathscr{X})$ is isomorphic to a full matrix ring, $M_n(k)$, for some integer n. Thus, in order to determine the ideal structure of $\mathscr{D}(\mathscr{X})$, we need only consider the structure of $\mathscr{D}(\mathscr{X})/J(\mathscr{X})$ as a subring of $M_n(k)$.

THEOREM. *Let* $\mathscr{X}$ *be a projective, singular curve with injective normalisation* $\pi \colon \mathbb{P}^1 \to \mathscr{X}$. *Then, in the above notation,*

$$F(\mathscr{X}) = \mathscr{D}(\mathscr{X})/J(\mathscr{X}) \cong \begin{pmatrix} M_t(k) & M_{t,s}(k) \\ 0 & M_s(k) \end{pmatrix} \subset M_{t+s}(k) = S/J(\mathscr{X})$$

for some positive integers t *and* s.

Proof. First note that, as ${}_{\mathscr{D}(\mathbb{P}^1)}P^{**}$ is projective, Lemma 4.2 implies that

$$\operatorname{End}(P^{**}/\mathbf{m}P^{**}) \cong \operatorname{End}(P^{**})/P^*\mathbf{m}P^{**} \cong M_n(k)$$

(and hence that $n = \dim_k P^{**}/\mathbf{m}P^{**}$). Consider

$$H = \{\bar{\theta} \in \operatorname{End}(P^{**}/\mathbf{m}\,P^{**}) \colon \bar{\theta}(P/\mathbf{m}P^{**}) \subseteq P/\mathbf{m}P^{**}\} \subset S/J(\mathscr{X}).$$

We claim that $H = \operatorname{End}(P)/J(\mathscr{X})$. For, suppose that $\bar{\theta} \in H$. Since ${}_{\mathscr{D}(\mathbb{P}^1)}P^{**}$ is projective, we may pick a preimage $\theta \in \operatorname{End} P^{**}$ that induces $\bar{\theta}$. Observe that $\mathbf{m}P^{**}\phi \subseteq \mathbf{m}P^{**}$ for any $\phi \in \operatorname{End} P^{**}$. Thus $\theta \in \operatorname{End}(P)$, and $H \subseteq \mathrm{End}(P)/J(\mathcal{X})$. Conversely, if $\psi \in \mathrm{End}(P)$ then $\mathbf{m}P^{**}\psi = \mathbf{m}P\psi \subseteq \mathbf{m}P = \mathbf{m}P^{**}$. Thus, modulo $J(\mathcal{X})$, ψ does induce an element of H, as is required to prove the claim.

Finally, write $P^{**}/\mathbf{m}P^{**} = K \oplus P/\mathbf{m}P^{**}$ as left modules over $k \cong \mathcal{D}(\mathbb{P}^1)/\mathbf{m}$. Then

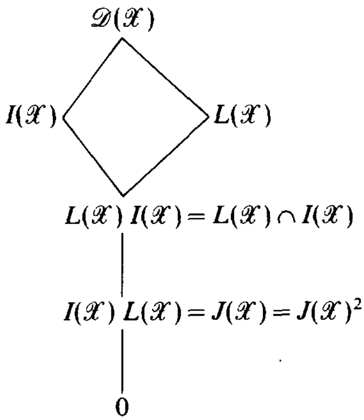
$$S/J(\mathcal{X}) = \mathrm{End}(P^{**}/\mathbf{m}P^{**}) \cong \begin{pmatrix} \mathrm{End}(K) & \mathrm{Hom}(K, P/\mathbf{m}P^{**}) \\ \mathrm{Hom}(P/\mathbf{m}P^{**}, K) & \mathrm{End}(P/\mathbf{m}P^{**}) \end{pmatrix}.$$

Under this identification

$$\mathrm{End}(P)/J(\mathcal{X}) = H \cong \begin{pmatrix} \mathrm{End}(K) & \mathrm{Hom}(K, P/\mathbf{m}P^{**}) \\ 0 & \mathrm{End}(P/\mathbf{m}P^{**}) \end{pmatrix}$$
$$= \begin{pmatrix} M_t(k) & M_{t,s}(k) \\ 0 & M_s(k) \end{pmatrix},$$

where $t = \dim_k K$ and $s = \dim_k P/\mathbf{m}P^{**}$.

4.4. COROLLARY. *Keep the notation of* (4.3) *and* (4.4). *Then* $\mathcal{D}(\mathcal{X})$ *has exactly four non-zero ideals*



Here

$$I(\mathcal{X}) = \mathcal{D}(\mathbb{P}^1, \mathcal{X})\, \mathcal{D}(\mathcal{X}, \mathbb{P}^1) = \text{r-ann}(S/\mathcal{D}(\mathcal{X})) = \text{l-ann}(H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$$

while

$$L\mathcal{X}) = \text{l-ann}(\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})) = \text{l-ann}(S/\mathcal{D}(\mathcal{X})).$$

In each case, annihilators are taken as those of $\mathcal{D}(\mathcal{X})$*-modules.*

Proof. Let $\bar{y}$ denote the image in $S/J(\mathscr{X})$ of $y \in S$ and consider the structure of $\overline{\mathscr{D}(\mathscr{X})}$ given by Theorem 4.3. This shows that $\mathscr{D}(\mathscr{X})$ has exactly four non-zero ideals, say $I(\mathscr{X})$, $L(\mathscr{X})$, $T(\mathscr{X})$, and $J(\mathscr{X})$ defined by

$$\overline{I(\mathscr{X})} = \begin{pmatrix} 0 & M_{t,s}(k) \\ 0 & M_s(k) \end{pmatrix} \qquad \overline{L(\mathscr{X})} = \begin{pmatrix} M_t(k) & M_{t,s}(k) \\ 0 & 0 \end{pmatrix}$$

and

$$\overline{T(\mathscr{X})} = \begin{pmatrix} 0 & M_{t,s}(k) \\ 0 & 0 \end{pmatrix},$$

as subrings of

$$\overline{\mathscr{D}(\mathscr{X})} = \begin{pmatrix} M_t(k) & M_{t,s}(k) \\ 0 & M_s(k) \end{pmatrix}.$$

Certainly, this implies that $I(\mathscr{X}) = \text{r-ann}(S/\mathscr{D}(\mathscr{X}))$ and $L(\mathscr{X}) = \text{l-ann}(S/\mathscr{D}(\mathscr{X}))$. Next consider QP. Since $S = \operatorname{End}(Q_{\mathscr{D}(\mathbb{P}^1)})$, certainly $SQP \subseteq QP$, and so $QP \subseteq I(\mathscr{X})$. Conversely, using Proposition 3.14(a),

$$QPS = QPQQ^* = Q\mathscr{D}(\mathbb{P}^1)Q^* = QQ^* = S.$$

By inspection, these two properties of QP force $QP = I(\mathscr{X})$. Since $I(\mathscr{X})$ is maximal, and $QP \subseteq \text{l-ann}_{\mathscr{D}(\mathscr{X})} H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$, by Corollary 3.19(c), this also implies that $QP = \text{l-ann}\, H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$.

Finally, consider $k = \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$. This is evidently a simple left $\mathscr{D}(\mathscr{X})$-module and its annihilator must be a maximal ideal of $\mathscr{D}(\mathscr{X})$. Now, by Theorem 3.15(a), $\Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ generates the sheaf $\mathscr{O}_{\mathscr{X}}$ and so, by Theorem 3.15(d), $\text{l-ann}\, \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) \neq QP$. Thus, the only possibility is for $L(\mathscr{X})$ to equal $\text{l-ann}\, \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$.

4.5. COROLLARY. *In the notation of* (4.3), $s = 1$, *that is,*

$$\mathscr{D}(\mathscr{X})/J(\mathscr{X}) \cong \begin{pmatrix} M_t(k) & k^{(t)} \\ 0 & k \end{pmatrix} \subset M_{t+1}(k) = S/J(\mathscr{X})$$

for some integer t.

Proof. By Corollary 4.4, $M_s(k) = \mathscr{D}(\mathscr{X})/L(\mathscr{X})$ has a 1-dimensional representation $k \cong \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$. Thus $s = 1$.

Remark. We will show later that $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is also a simple $\mathscr{D}(\mathscr{X})$-module, and hence that $t = \dim_k H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is the arithmetic genus of $\mathscr{X}$.

4.6. For the rest of this section we examine the ring-theoretic consequences of Theorem 4.4. In particular, with the exception of determining the value of t, this is enough to prove Theorem D of the Introduction.

COROLLARY. *$\mathscr{D}(\mathscr{X})$ has global homological dimension two.*

Proof. Let $\mathrm{pd}(W)$ stand for the projective dimension of a module W. We apply [RS2, Theorem 5(ii)], with $R=\mathscr{D}(\mathscr{X})$ and $A=L(\mathscr{X})$. Since $SA=S$, this implies that

$$\sup\{\operatorname{gldim} S, \operatorname{gldim} \mathscr{D}(\mathscr{X})/L(\mathscr{X})\}$$
$$\leqslant \operatorname{gldim} \mathscr{D}(\mathscr{X}) \leqslant \sup\{\operatorname{gldim} S+\mathrm{pd}_{\mathscr{D}(\mathscr{X})} S, 1+\operatorname{gldim} \mathscr{D}(\mathscr{X})/L(\mathscr{X})\}. \quad (4.6.1)$$

Now, as S is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$, [St, Theorem 2.6] implies that $\operatorname{gldim} S=\operatorname{gldim} \mathscr{D}(\mathbb{P}^1)=2$. Clearly $\operatorname{gldim} \mathscr{D}(\mathscr{X})/L(\mathscr{X})=0$. Thus it remains to consider ${}_{\mathscr{D}(\mathscr{X})}S$. Now, by Corollary 4.4, $({}_{\mathscr{D}(\mathscr{X})}S)^*=\text{r-ann}(S/\mathscr{D}(\mathscr{X})_{\mathscr{D}(\mathscr{X})})=I(\mathscr{X})$. But, $I(\mathscr{X})S=S\ni 1$ and so the dual basis lemma implies that $\mathrm{pd}({}_{\mathscr{D}(\mathscr{X})}S)=0$. Substituting these numbers back into (4.6.1) yields $\operatorname{gldim} \mathscr{D}(\mathscr{X})=2$.

4.7. Consider rings and ring homomorphisms

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & \bar{U} \\ & & \uparrow{\scriptstyle \pi} \\ & & U \end{array}$$

(where π is surjective). Then the *pull-back* $R=(U, H, \bar{U})$ is defined to be

$$R=\{(u, h)\in U\oplus H : \pi(u)=\sigma(h)\}$$

and will often be denoted by the diagram

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & \bar{U} \\ \uparrow & & \uparrow{\scriptstyle \pi} \\ R & \dashrightarrow & U \end{array}$$

Another immediate consequence of Corollary 4.5 is

COROLLARY. *In the notation of* 4.3, *$\mathscr{D}(\mathscr{X})$ is the pull-back*

$$\begin{pmatrix} M_t(k) & M_{t,1}(k) \\ 0 & k \end{pmatrix} = \begin{array}{ccc} H & \overset{\sigma}{\hookrightarrow} & S/J(\mathscr{X})=M_{t+1}(k) \\ \uparrow & & \uparrow{\scriptstyle \pi} \\ \mathscr{D}(\mathscr{X}) & \dashrightarrow & S \end{array}$$

4.8. COROLLARY. *Keep the notation of Theorem* 4.3. *Then S is the unique maximal order containing and equivalent to $\mathscr{D}(\mathscr{X})$.*

Proof. By Corollary 3.8, $Q=\mathscr{D}(\mathbb{P}^1,\mathscr{X})$ is a projective $\mathscr{D}(\mathbb{P}^1)$-module and hence $S=\operatorname{End} Q_{\mathscr{D}(\mathbb{P}^1)}$ is a maximal order. Since S and $\mathscr{D}(\mathscr{X})$ have the same minimal, non-zero ideal $J=J(\mathscr{X})$, they are certainly equivalent orders. Moreover, as $J^2=J$, certainly $\operatorname{End}(J_S)=(J_S)^*\supseteq S$. Since S is a maximal order, $S=(J_S)^*=({}_SJ)^*$.

Now suppose that T is a second maximal order containing and equivalent to $\mathscr{D}(\mathscr{X})$. Then $aTb\subset\mathscr{D}(\mathscr{X})$ for some $a,b\in\mathscr{D}(\mathscr{X})\backslash\{0\}$. Thus

$$JTJ=(JaJ)\,T(JbJ)\subseteq JaTbJ\subseteq\mathscr{D}(\mathscr{X})\subset S.$$

In particular, $JT\subset(J_S)^*=S$. Thus $T\subseteq({}_SJ)^*=S$; and the maximality of T forces $T=S$.

4.9. The description of $\mathscr{D}(\mathscr{X})$ in Theorem 4.3 is sufficient to give a complete description of the K-groups of $\mathscr{D}(\mathscr{X})$.

THEOREM. *Let $\mathscr{X}$ be a singular curve with injective normalisation $\pi\colon\mathbb{P}^1\to\mathscr{X}$. Then for any $i\geqslant 0$,*

$$K_i(\mathscr{D}(\mathscr{X}))\cong K_i(k)\oplus K_i(k)\oplus K_i(k).$$

Proof. Keep the notation of Corollary 4.4. Then $I(\mathscr{X})S=S$ but $SI(\mathscr{X})=I(\mathscr{X})$. Thus S is a finitely generated projective left $\mathscr{D}(\mathscr{X})$-module. Similarly, as $L(\mathscr{X})S=L(\mathscr{X})$, $S_{\mathscr{D}(\mathscr{X})}$ is finitely generated projective. Therefore, if $\mathbf{B}=\{M\in\mathscr{D}(\mathscr{X})\text{-}\mathbf{mod}\colon S\otimes_{\mathscr{D}(\mathscr{X})}M=0\}$, then [Ho, Proposition 3.1] implies that $K_i(\mathscr{D}(\mathscr{X}))\cong K_i(S)\oplus K_i(\mathbf{B})$, for $i\geqslant 0$. Now, $\mathbf{B}$ is equivalent to $\mathscr{D}(\mathscr{X})/L(\mathscr{X})\text{-}\mathbf{mod}\cong k\text{-}\mathbf{mod}$, as in (4.5). Thus $K_i(\mathbf{B})\cong K_i(k)$, for each i. Finally

$$\begin{aligned} K_i(S)&\cong K_i(\mathscr{D}(\mathbb{P}^1))\\ &\cong K_i(\mathbb{P}^1) &&\text{by [Ho, Corollary 4.3]}\\ &\cong K_i(k)\oplus K_i(k) &&\text{by [Q, Theorem 8.3.1].}\end{aligned}$$

4.10. Recall from [S] some facts about torsion classes. Let R be a ring. An hereditary torsion class is a class of (left) R-modules closed under quotients, submodules, extensions, and direct sums. The associated filter is $\mathscr{F}=\{I \text{ a left ideal of } R\colon R/I\in\mathscr{T}\}$. If V is an R-module then the intersection of torsion classes containing V is a torsion class, denoted by $\langle V\rangle$. A simple module is in $\langle V\rangle$ if and only if it is a subfactor of V.

Now let R be a prime Noetherian ring. Then we define the localisation of R at the class $\mathscr{T}$ to be

$$R_{\mathscr{T}}=\{x\in Q(R)\colon Ix\subseteq R,\text{ for some } I\in\mathscr{F}\}.$$

The next result is proved in [S, Proposition XI.2.4, p. 229].

PROPOSITION. *If $Q(R) \supset S \supset R$ and S_R is flat then*

$$\mathscr{T} = \{M \in R\text{-}\mathbf{mod} : S \otimes_R M = 0\}$$

is an hereditary torsion class and $R_{\mathscr{T}} = S$.

4.11. COROLLARY. *Let $\mathscr{T}$ be the torsion class*

$$\operatorname{Ker}\{\mathscr{D}(\mathscr{X})\text{-}\mathbf{mod} \xrightarrow{\mathscr{D}_{\mathscr{X}} \otimes -} \mathscr{D}_{\mathscr{X}}\text{-}\mathbf{mod}\}.$$

Then $\mathscr{T}$ is hereditary and $\mathscr{T} = \langle V \rangle$ where V is the simple left $\mathscr{D}(\mathscr{X})/I(\mathscr{X})$-module.

Proof. By Theorem 3.15(b), $\mathscr{D}_{\mathscr{X}} \otimes_{\mathscr{D}(\mathscr{X})} -$ is exact and so its kernel does form an hereditary torsion class. By Corollary 4.4, $I(\mathscr{X}) = \text{l-ann}\, H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is a maximal ideal, and so the result follows immediately from Theorem 3.15(d).

Remark. (i) In Section 7 we will prove that $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is a simple $\mathscr{D}(\mathscr{X})$-module and hence that $\mathscr{T}$ simply consists of direct sums of copies of $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$.

(ii) The localisation $\mathscr{D}(\mathscr{X})_{\mathscr{T}}$ is uninteresting—$\mathscr{D}(\mathscr{X})_{\mathscr{T}} = \mathscr{D}(\mathscr{X})$. To see this note that $\mathscr{T} \subseteq \operatorname{Ker}\{\mathscr{D}(\mathscr{X})\text{-}\mathbf{mod} \to \mathscr{D}(U)\text{-}\mathbf{mod}\}$ for any open affine subset $U \subset \mathscr{X}$. By Theorem 3.15(e) and Proposition 4.10, this implies that $\mathscr{D}(\mathscr{X})_{\mathscr{T}} \subseteq \bigcap \mathscr{D}(U) = \mathscr{D}(\mathscr{X})$. Curiously, if one regards $V = \mathscr{D}(\mathscr{X})/I(\mathscr{X})$ as a right $\mathscr{D}(\mathscr{X})$-module, then it follows from Corollary 4.4 that $\mathscr{D}(\mathscr{X})_{\mathscr{T}} = S(\mathscr{X})$.

4.12. Given a curve $\mathscr{X}$ with injective normalisation $\mathbb{P}^1$ and $U = \mathscr{X} \setminus p$, for some point$p$, then Theorem 3.15(e) and Proposition 4.10 imply that $\mathscr{D}(U)$ is the localisation of $\mathscr{D}(\mathscr{X})$ at $\mathscr{T}_p$, the torsion class of modules killed by $\mathscr{D}(U) \otimes -$. We begin by showing that, for $\mathscr{X} = \mathbb{P}^1$, it is easy to characterise $\mathscr{T}_p$ using cohomology with supports.

Let $\mathscr{Y}$ be a variety, $\mathscr{Z}$ a closed subset, and $\mathscr{M}$ a sheaf of $\mathscr{D}_{\mathscr{Y}}$-modules. Then one defines the sections of $\mathscr{M}$ with supports in $\mathscr{Z}$ by

$$\Gamma_{\mathscr{Z}}(\mathscr{Y}, \mathscr{M}) = \{s \in \Gamma(\mathscr{Y}, \mathscr{M}) : s_p \neq 0 \Rightarrow p \in \mathscr{Z}\}.$$

It is easy to see that $\Gamma_{\mathscr{Z}}(\mathscr{Y}, -)$ is left exact with right derived functors $H^i_{\mathscr{Z}}(\mathscr{Y}, -)$, the cohomology with supports in $\mathscr{Z}$.

(4.12.1) [Iv, Proposition II.9.2, p. 123]. If $U = \mathscr{Y} \setminus \mathscr{Z}$ there is a long exact sequence

$$\cdots \to H^i_{\mathscr{Z}}(\mathscr{Y}, \mathscr{M}) \to H^i(\mathscr{Y}, \mathscr{M}) \to H^i(U, \mathscr{M}|_U) \to H^{i+1}_{\mathscr{Z}}(\mathscr{Y}, \mathscr{M}) \to \cdots.$$

It follows from the construction that this is a sequence of $\mathscr{D}(\mathscr{Y})$-modules.

(4.12.2) [Iv, II.9.6, p. 125]. If V is an open subset of $\mathscr{Y}$ with $\mathscr{X} \subseteq V$ then there is an excision isomorphism $H^i_{\mathscr{X}}(\mathscr{Y}, \mathscr{M}) \cong H^i_{\mathscr{X}}(V, \mathscr{M}|_V)$. Again, it follows from the construction that it is a $\mathscr{D}(\mathscr{Y})$-module isomorphism.

4.13. Proposition. *Let $p \in \mathbb{P}^1$. Then $H^1_p(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1})$ is a simple $\mathscr{D}(\mathbb{P}^1)$-module and $\mathscr{T}_p = \langle H^1_p(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1})\rangle$.*

Proof. Clearly, we may suppose that $p = 0$. Now apply (4.12.1) and (4.12.2) with V some open affine neighbourhood of 0. Then $H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \cong \mathscr{O}(V\backslash 0)/\mathscr{O}(V)$. Thus $\mathscr{D}(V) \otimes H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \cong \mathscr{O}(V\backslash 0)/\mathscr{O}(V)$. But if $q \neq 0$ is another point of $\mathbb{P}^1$ then $\mathscr{O}_q \otimes H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 0$. It follows that $\mathscr{D}_{\mathbb{P}^1} \otimes H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1})$ is concentrated at 0. Thus $H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \in \mathscr{T}_0$. On the other hand, clearly $I = \mathscr{D}(\mathbb{P}^1)(t\partial + 1) + \mathscr{D}(\mathbb{P}^1)\, t^2\partial$ kills the generator t^{-1} of $H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \cong k[t, t^{-1}]/k[t]$. By the left hand version of Lemma 2.7, I is a maximal left ideal of $\mathscr{D}(\mathbb{P}^1)$ and so $H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \cong \mathscr{D}(\mathbb{P}^1)/I$ is simple. Now let $\mathscr{S} = \langle H^1_0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1})\rangle$. It follows that $t^{-1} \in \mathscr{D}(\mathbb{P}^1)_{\mathscr{S}}$ and hence that $\mathscr{D}(\mathbb{P}^1)_{\mathscr{S}} \supseteq \mathscr{D}(\mathbb{P}^1\backslash 0)$. Thus $\mathscr{S} = \mathscr{T}_0$.

4.14. *Remark.* (a) The last result gives another proof that if p and q are distinct points of $\mathbb{P}^1$ then $\mathscr{T}_p \cap \mathscr{T}_q = 0$ and so $\mathscr{D}(\mathbb{P}^1\backslash p) \oplus \mathscr{D}(\mathbb{P}^1\backslash q)$ is a faithfully flat right $\mathscr{D}(\mathbb{P}^1)$-module.

(b) It follows from [BoB, Corollary 3.7] that $H^1_p(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1})$ is a Verma module, for an appropriate choice of Borel subalgebra.

4.15. In order to investigate the analogue of Proposition 4.13 for singular curves, we use K-theory.

Lemma. *Let p be a point of $\mathscr{X}$. Then $\mathscr{T}_p$ contains precisely two non-isomorphic simple modules.*

Proof. By [Ga, Corollaire V.2.2] and the first paragraph of 4.12, $\mathscr{D}(U)$-**mod** is a quotient category of $\mathscr{D}(\mathscr{X})$-**mod**. It is easy to see that this restricts to the full sub-categories of finitely generated modules. The associated torsion category is $\mathscr{T}'_p$, the full sub-category of $\mathscr{D}(\mathscr{X})$-**mod** with objects the finitely generated modules in $\mathscr{T}_p$. Thus Quillen's localisation Theorem [Q, Theorem 5.5, p. 113] yields a long exact sequence

$$K_1(\mathscr{D}(\mathscr{X})) \to K_1(\mathscr{D}(U)) \to K_0(\mathscr{T}'_P) \to K_0(\mathscr{D}(\mathscr{X})) \to K_0(\mathscr{D}(U)) \to 0.$$

Since $\mathscr{D}(U)$ is Morita equivalent to A_1 and $K_0(\mathscr{D}(\mathscr{X})) \cong \mathbb{Z}^3$, by Theorem 4.9, this tells us that $K_0(\mathscr{T}'_p) \cong \mathbb{Z}^2$. On the other hand, by devissage [Q, Corollary 5.1, p. 112], $K_0(\mathscr{T}'_p)$ is free with basis the non-isomorphic simple modules of $\mathscr{T}'_p$. Thus $\mathscr{T}_p$ contains precisely two non-isomorphic simple modules.

4.16. Proposition. *Let $\mathscr{X}$ be a singular projective curve with injective normalisation $\pi: \mathbb{P}^1 \to \mathscr{X}$ and let p be a point of $\mathscr{X}$. Then $\mathscr{T}_p = \langle H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})\rangle$.*

Proof. By (4.12.1) and (4.12.2), if U is an open affine subset of $\mathscr{X}$ containing p, then $H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \cong \mathcal{O}(U \setminus p)/\mathcal{O}(U)$. Thus a similar argument to the proof of Theorem 4.13 shows that if q is a point of $\mathscr{X}$ then $\mathscr{D}_q \otimes H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \neq 0$ if and only if $q = p$. Thus $\langle H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})\rangle \subseteq \mathscr{T}_p$. In particular, (4.12.1) implies that $H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ is a factor of $H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ and so $H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \in \mathscr{T}_p$. Now, $I(\mathscr{X})\, H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) = 0$, yet $\mathscr{D}_{\mathscr{X}} \otimes H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \neq 0$. Thus, since every module in $\mathscr{T}_P$ is the direct limit of its finite length submodules, by Theorem A, we have $\mathscr{T}_P = \langle H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})\rangle$.

Remark. In Section 7 it is shown that $H_p^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ is a non-split extension of length two.

5. The Cuspidal Plane Cubic Curve

5.1. In this section we study $\mathscr{D}(\mathscr{X}_1)$ where $\mathscr{X}_1$ is the cubic curve with a cusp at (0, 0, 1) defined by $y^2 z = x^3$ in $\mathbb{P}^2$. The main reason for this is that various properties of $\mathscr{D}(\mathscr{X}_1)$ are easy to prove directly. In Section 7 we will use Morita theory to show that these same properties then hold over $\mathscr{D}(\mathscr{Y})$, for a more general curve $\mathscr{Y}$. However, this example may also serve as an illustration of, or introduction to, the methods and results of this paper and so as far as possible we have given direct proofs of the various properties of $\mathscr{D}(\mathscr{X}_1)$.

5.2. Regard $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ and let t be a coordinate function on $\mathbb{A}^1$. Set $\tilde{U}_0 = \mathbb{A}^1$ and $\tilde{U}_1 = \mathbb{P}^1 \setminus \{0\}$ for the standard open affine cover of $\mathbb{P}^1$. In this notation it is clear that the normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}_1$ is defined by the embeddings

$$\mathcal{O}(U_0) = k[t^2, t^3] \subset k[t] = \mathcal{O}(\tilde{U}_0) \qquad \text{and} \qquad \mathcal{O}(U_1) = k[t^{-1}] = \mathcal{O}(\tilde{U}_1).$$

As usual, we set $A_1 = \mathscr{D}(\tilde{U}_0) = k[t, \partial]$ and $\mathscr{D}(\tilde{U}_1) = k[t^{-1}, t^2\partial]$. Observe that $K(\mathscr{X}_1) = k(t)$. Clearly $\mathscr{D}(U_1) = \mathscr{D}(\tilde{U}_1)$ while the structure of $\mathscr{D}(U_0)$ is easily determined.

Lemma. *In the above notation*

(a) $\mathscr{D}(\tilde{U}_0, U_0) = t^2 A_1 + (t\partial - 1)A_1$ *while* $\mathscr{D}(U_0, \tilde{U}_0) = A_1 + A_1 t^{-1}\partial$;

(b) $\mathscr{D}(U_0) = k[t^2, t^3, t\partial, t^2\partial, t\partial^2 - \partial, \partial^2 - 2t^{-1}\partial, \partial^3 - 3t^{-1}\partial^2 + 3t^{-2}\partial]$.

Proof. That $\mathscr{D}(\tilde{U}_0, U_0)$ and $\mathscr{D}(U_0)$ have the specified form follows from [SS, (3.8) and (3.12)]. An easy computation shows that $A_1 + A_1 t^{-1}\partial \subseteq \mathscr{D}(U_0, \tilde{U}_0)$. Conversely, since $\mathscr{D}(\tilde{U}_0, U_0)$ is a maximal right ideal of A_1 and

$\mathscr{D}(U_0, \tilde{U}_0) \subseteq \mathscr{D}(\tilde{U}_0, U_0)^*$, $\mathscr{D}(U_0, \tilde{U}_0)/A_1$ can be at worst a simple left A_1-module. Thus $\mathscr{D}(U_0, \tilde{U}_0) = A_1 + A_1 t^{-1}\partial$.

5.3. Lemma. *In the notation of* (5.2):

(a) $\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1) = t^2\partial\mathscr{D}(\mathbb{P}^1) + (t\partial - 1)\,\mathscr{D}(\mathbb{P}^1)$;

(b) $\mathscr{D}(\mathscr{X}_1, \mathbb{P}^1) = \mathscr{D}(\mathbb{P}^1) + \mathscr{D}(\mathbb{P}^1)\, t^{-1}\partial$;

(c) $\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)^* \supseteq \mathscr{D}(\mathscr{X}_1, \mathbb{P}^1) + \mathscr{D}(\mathbb{P}^1)(\partial^{-1}t^{-2} + t^{-1}) \supset \mathscr{D}(\mathscr{X}_1, \mathbb{P}^1)$;

(d) $\mathscr{D}(\mathscr{X}_1) = k[t\partial, t^2\partial, t\partial^2 - \partial, \partial^2 - 2t^{-1}\partial, \partial^3 - 3t^{-1}\partial^2 + 3t^{-2}\partial]$;

(e) $\mathrm{End}(\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)_{\mathscr{D}(\mathbb{P}^1)}) \supseteq \mathscr{D}(\mathscr{X}_1) + k(\partial^{-1}t^{-2} - \partial + t^{-1}) \supset \mathscr{D}(\mathscr{X}_1)$.

Proof. (a) This follows from the equation

$$\begin{aligned}\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1) &= \mathscr{D}(\tilde{U}_0, U_0) \cap \mathscr{D}(\tilde{U}_1, U_1) \cap \mathscr{D}(\mathbb{P}^1) \\ &= (t^2 A_1 + (t\partial - 1)A_1) \cap \mathscr{D}(\mathbb{P}^1) = t^2\partial\mathscr{D}(\mathbb{P}^1) + (t\partial - 1)\,\mathscr{D}(\mathbb{P}^1),\end{aligned}$$

where the final equality is due to Lemma 2.7.

(b) $\mathscr{D}(\mathscr{X}_1, \mathbb{P}^1) = \mathscr{D}(U_0, \tilde{U}_0) \cap \mathscr{D}(U_1, \tilde{U}_1) = (A_1 + A_1 t^{-1}\partial) \cap k[t^{-1}, t^2\partial]$. Thus, certainly $\mathscr{D}(\mathscr{X}_1, \mathbb{P}^1) \supseteq \mathscr{D}(\mathbb{P}^1) + \mathscr{D}(\mathbb{P}^1)\, t^{-1}\partial$. The other inclusion can either be proved by a direct computation or by observing that, as the elements 1 and $t^{-1}\partial$ generate the local sections $\mathscr{D}(U_i, \tilde{U}_i)$, Theorem 3.2 can be used to show that $\mathscr{D}(\mathbb{P}^1) + \mathscr{D}(\mathbb{P}^1)\, t^{-1}\partial = \mathscr{D}(\mathscr{X}_1, \mathbb{P}^1)$.

(d) Recall that $\mathscr{D}(\mathscr{X}_1) = \mathscr{D}(U_0) \cap k[t^{-1}, t^2\partial]$. Thus, by Lemma 5.2, the asserted generators for $\mathscr{D}(\mathscr{X}_1)$ do indeed lie in $\mathscr{D}(\mathscr{X}_1)$. In order to prove the other inclusion use a graded argument giving ∂ degree -1 and t degree $+1$.

(c) and (e) Direct computations show that $\partial^{-1}t^{-2} + t^{-1} \in \mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)^*$ and that

$$\begin{aligned}\partial^{-1}t^{-2} - \partial + t^{-1} &= -(t\partial - 1)(\partial^{-1}t^{-2} + t^{-1}) \in \mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)\,\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)^* \\ &= S(\mathscr{X}_1),\end{aligned}$$

where $S(\mathscr{X}_1) = \mathrm{End}(\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)_{\mathscr{D}(\mathbb{P}^1)})$. Observe that neither of these elements lie in $\mathscr{D}(K(\mathscr{X}_1)) = k(t)[\partial]$. Thus $\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)^* \supset \mathscr{D}(\mathscr{X}_1, \mathbb{P}^1)$ and $\mathrm{End}\,\mathscr{D}(\mathscr{X}_1, \mathbb{P}^1) \supset \mathscr{D}(\mathscr{X}_1)$.

5.4. Proposition. (a) *$H^1(\mathscr{X}_1, \mathscr{O}_{\mathscr{X}_1}) \cong k$ is a simple left $\mathscr{D}(\mathscr{X}_1)$-module.*

(b) *In the notation of Theorem* 4.3, $F(\mathscr{X}_1) = \mathscr{D}(\mathscr{X}_1)/J(\mathscr{X}_1) \cong \left(\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right)$.

Proof. Čech cohomology shows that

$$\begin{aligned}H^1(\mathscr{X}_1, \mathscr{O}_{\mathscr{X}_1}) &\cong \mathscr{O}(U_0 \cap U_1)/\mathscr{O}(U_0) + \mathscr{O}(U_1) \\ &\cong k[t, t^{-1}]/\{k + t^2k[t] + k[t^{-1}]\} \cong k.\end{aligned}$$

Thus $H^1(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$ is necessarily a simple $\mathscr{D}(\mathscr{X}_1)$-module. In the notation of Corollary 4.4, $I(\mathscr{X}_1) = \operatorname{ann}_{\mathscr{D}(\mathscr{X}_1)} H^1(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$ and hence $\mathscr{D}(\mathscr{X}_1)/I(\mathscr{X}_1) \cong k$. But in the notation of Corollary 4.5, $\mathscr{D}(\mathscr{X}_1)/I(\mathscr{X}_1) \cong M_t(k)$. Thus $t = 1$ and Corollary 4.5 implies that

$$F(\mathscr{X}_1) \cong \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}.$$

5.5. PROPOSITION. *Let $p \in \mathscr{X}_1$. Then $H^1_p(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$ is non-split of length two as a $\mathscr{D}(\mathscr{X})$-module. It has socle $H^0(\mathscr{X}\backslash p, \mathcal{O}_{\mathscr{X}_1\backslash p})/H^0(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$ and factor $H^1(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$.*

Proof. Suppose that $p \neq \pi(0)$. (The case when $p = \pi(0)$ is easy and is left to the reader.) Then we may identify p with an element α of k and by excision (4.12.2) we have

$$H^1_p(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1}) \cong k[t^{-1}, (t^{-1} - \alpha)^{-1}]/k[t^{-1}] = k[y, y^{-1}]/k[y],$$

where $y = t^{-1} - \alpha$. Let us write down some elements of $\mathscr{D}(\mathscr{X}_1)$ in terms of the coordinate y and $\theta = \partial/\partial y$. Clearly $\theta = -t^2\partial \in \mathscr{D}(\mathscr{X}_1)$ and $y\theta = -t\partial + \alpha t^2\partial \in \mathscr{D}(\mathscr{X}_1)$. Now,

$$\mathscr{D}(\mathscr{X}_1) \ni t^{-1}(t\partial)(t\partial - 2) = (y + \alpha)^3\theta^2 + 3(y + \alpha)^2\theta.$$

It follows that $\Delta_i = y^3\theta^2 + 3\alpha^2 y\theta^2 + 3y^2\theta + 3\alpha^2(i+1)\theta \in \mathscr{D}(\mathscr{X}_1)$. Now, $\Delta_i \circ y^{-i} = i(i-2)\, y^{-i+1}$ and so either $H^1_p(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$ is simple or is non-split of length two with socle $\mathscr{D}(\mathscr{X}_1) \circ y^{-2}$. The first possibility cannot occur, either by appealing to Proposition 4.16, or by observing that the following sequence is exact, by (4.12.1):

$$0 \to H^0(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1}) \to H^0(\mathscr{X}_1\backslash p, \mathcal{O}_{\mathscr{X}_1\backslash p}) \to H^1_p(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1}) \to H^1(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1}) \to 0$$

Finally, this sequence also gives the factors of $H^1_p(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$.

6. MORITA EQUIVALENCE OF PULL-BACKS

6.1. In this section we digress from our main aim and consider (Milnor) pull-backs $R = (U, H, \bar{U})$ of the form

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & \bar{U} \\ \uparrow & & \uparrow \pi \\ R & \dashrightarrow & U \end{array} \qquad (6.1.1)$$

with π surjective. Thus $R=\{(u,h)\in U\oplus H: \pi(u)=\sigma(h)\}$. In this situation, any finitely generated, projective R-module T is easily described in terms of projective U- and H-modules. The main result of this section describes the endomorphism ring of such a module T and, as a consequence, determines when two such pull-backs are Morita equivalent.

The reason for our interest in this question is due to Corollary 4.7; if $\mathscr{X}$ and $\mathscr{Y}$ are projective, rational singular curves with injective normalisation maps, then $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$ can be described as pull-backs. It is then a trivial consequence of Corollary 6.12, below, that $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$ will be Morita equivalent.

6.2. Let R be as in (6.1.1). Then the projective, finitely generated R-modules can be classified as follows. Let A and B be finitely generated, projective right modules over U, respectively H, and suppose that there exists a $\bar{U}$-module isomorphism $\alpha: A\otimes_U \bar{U} \xrightarrow{\sim} B\otimes_H \bar{U}$. Then

$$T=(A,B,\alpha)=\{(a,b)\in A\oplus B: \alpha(a\otimes_U 1)=b\otimes_H 1\}$$

is a projective, finitely generated right R-module, and any finitely generated, projective R-module may be obtained in this way (combine [Si, Proposition 59, p. 155] with [Si, Proposition 60, p. 157]). We might remark that these results do require that π be surjective and this is the reason why we have required that hypothesis in (6.1.1).

6.3. Fix a pull-back $R=(U,H,\bar{U})$ as in (6.1.1) and a finitely generated, projective right R-module $T=(A,B,\alpha)$, as in (6.2). We will shortly describe $\operatorname{End}(T_R)$ as a pull-back, but before doing so we need some notation. First, as π is surjective, it is routine to check that the induced map $\pi_1: R\to H$ given by $\pi_1(u,h)=h$, for $(u,h)\in R\subseteq U\oplus H$, is surjective. Similarly, write σ_1 for the induced map $\sigma_1: R\to U$. By [Si, Proposition 61, p. 158], $T\otimes_R U\cong A$ and $T\otimes_R H\cong B$. Thus, if $\mathbf{p}=\ker \pi_1$, then $B\cong T/T\mathbf{p}$. Secondly, if $\mathbf{m}=\ker\pi$ and $\bar{A}=A/A\mathbf{m}$, then the projectivity of A_U ensures that the natural map $\chi: \operatorname{End}(A_U)\to \operatorname{End}(\bar{A}_{\bar{U}})$ is a surjection.

PROPOSITION. *Keep the notation as above. Then $S=\operatorname{End}(T_R)$ identifies naturally with the pull-back*

$$\begin{array}{ccc} \operatorname{End}(B_H) & \xrightarrow{\ \tau\ } & \operatorname{End}(\bar{A}_{\bar{U}}) \\ \uparrow & & \uparrow \chi \\ S & \dashrightarrow & \operatorname{End}(A_U) \end{array}$$

The map τ will be described in the course of the proof.

Remark. The analogous result describing the endomorphism ring of an injective R-module E has been proved in [FV, Lemma 11] and our proof is essentially the dual of theirs.

Proof. Regard H and U as R-modules via the homomorphisms π_1 and σ_1. This gives $\bar{U}$ two natural structures as an R-module, via the homomorphisms σ and π, but as (6.1.1) is a commutative diagram, they are identical. Thus there are natural R-module maps $\theta: T \to T \otimes_R U \cong A$ and $\psi: T \to T \otimes_R H \cong B$ and $\phi: B \to B \otimes_H \bar{U} \xrightarrow{\alpha^{-1}} \bar{A}$. Therefore, the following diagram is commutative, with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T\mathbf{p} & \longrightarrow & T & \xrightarrow{\psi} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & A\mathbf{m} & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & 0 \end{array}$$

Here θ_1 is the map induced from θ. Since T_R is projective the functor $\operatorname{Hom}_R(T, -)$ is exact. Thus applying this functor to the above diagram yields the following commutative diagram, again with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(T, T\mathbf{p}) & \longrightarrow & \operatorname{End}_R(T) & \longrightarrow & \operatorname{Hom}_R(T, B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(T, A\mathbf{m}) & \longrightarrow & \operatorname{Hom}_R(T, A) & \longrightarrow & \operatorname{Hom}_R(T, \bar{A}) & \longrightarrow & 0 \end{array} \tag{6.3.1}$$

Now, one has natural isomorphisms

$$\operatorname{Hom}_R(T, A) \cong \operatorname{Hom}_R(T, T \otimes_R U) \cong \operatorname{Hom}_U(T \otimes_R U, T \otimes_R U),$$

etc. Thus (6.3.1) can be rewritten as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(T, T\mathbf{p}) & \longrightarrow & \operatorname{End}_R(T) & \xrightarrow{\chi_1} & \operatorname{End}_H(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(T, A\mathbf{m}) & \longrightarrow & \operatorname{End}_U(A) & \xrightarrow{\chi} & \operatorname{End}_{\bar{U}}(\bar{A}) & \longrightarrow & 0 \end{array} \tag{6.3.2}$$

It is easy to check that each of the maps in the right hand square of (6.3.2) is a ring homomorphism. (The reason for this is that, for example, τ is the natural ring homomorphism $\operatorname{End}_H B \to \operatorname{End}_{\bar{U}}(\bar{A})$ induced by the isomorphism $B \otimes_H \bar{U} \cong \bar{A}$.) Since $T \subseteq A \oplus B$, certainly

$$\operatorname{End}_R(T) \subseteq \operatorname{Hom}_R(T, A \oplus B) \cong \operatorname{End}_U(A) \oplus \operatorname{End}_H(B).$$

Thus the commutativity of (6.3.2) ensures that End(T) embeds into the pull-back

$$\begin{array}{ccc} \mathrm{End}_H(B) & \xrightarrow{\tau} & \mathrm{End}_{\bar{U}}(\bar{A}) \\ \uparrow & & \uparrow \chi \\ V & \xrightarrow{\tau_1} & \mathrm{End}_U(A) \end{array}$$

It remains to prove the reverse inclusion. First, note that, as R is a pull-back,

$$\mathbf{p} = \ker \pi_1 = (\mathbf{m}, 0) \subset R \subseteq U \oplus H.$$

Since T is the pull-back (A, B, α), it follows that the induced map $\theta_1 : T\mathbf{p} \to A\mathbf{m}$ is an isomorphism. Thus the map $\tau_2 : \mathrm{Hom}_R(T, T\mathbf{p}) \to \mathrm{Hom}_R(T, A\mathbf{m})$ in (6.3.2) is also an isomorphism. Since χ_1 is surjective, a diagram chase ensures that $V \subseteq \mathrm{End}_R(T)$; as required.

6.4. We now turn to the problem of determining when two pull-backs are Morita equivalent. Given the description of End(T) in Proposition 6.3, this amounts to determining when the projective module $T = (A, B, \alpha)$ is a progenerator.

LEMMA. *Let $R = (U, H, \bar{U})$ be a pull-back, as in* (6.1.1) *and suppose that $T = (A, B, \alpha)$ is a finitely generated, projective right R-module. Then T_R is a progenerator if and only if both A_U and B_H are progenerators.*

Proof. If T_R is a progenerator, then a routine argument shows that $A \cong T \otimes_R U$ and $B \cong T \otimes_R H$ are progenerators over U, respectively H.

Conversely, suppose that both A_U and B_H are progenerators. Write $\mathbf{n} = \ker \sigma$ and $\mathbf{m} = \ker \pi$, in the notation of (6.1.1). Let $\pi_1 : R \to U$ be the induced map and set $\mathbf{p} = \ker \pi_1$, as in (6.3). Since A_U is a progenerator, there exists a surjective U-module homomorphism $\phi : A^{(r)} \twoheadrightarrow \mathbf{m}$, for some integer r. Since $A \cong T \otimes_R U$, this induces an R-module homomorphism

$$\psi : T^{(r)} \longrightarrow T^{(r)} \otimes_R U \xrightarrow{\phi} \mathbf{m} \cong (\mathbf{m}, 0) \subset R.$$

Thus $\psi(T^{(r)}) \subseteq T^*T$. Moreover, since $T\mathbf{p} \cong A\mathbf{m}$ (see the proof of Proposition 6.3) one has

$$\psi(T^{(r)}\mathbf{p}) = \phi(A^{(r)}\mathbf{m}) = (\mathbf{m}^2, 0) \subseteq T^*T. \tag{6.4.1}$$

Next, since B_H is a progenerator, there exists a surjection $\bar{\omega} : B^{(u)} \twoheadrightarrow H$. As in (6.3), $\mathbf{p} = \ker \pi_1 = (\mathbf{m}, 0)$ and $R/\mathbf{p} \cong H$, while $B \cong T \otimes_R U \cong T/T\mathbf{p}$. Since T_R is projective, we may therefore pull $\bar{\omega}$ back to a homomorphism $\omega : T^{(u)} \to R$ satisfying $\omega(T^{(u)}) + \mathbf{p} = R$. Combined with (6.4.1), this implies that $T^*T \supseteq \omega(T^{(u)}) + \mathbf{p}^2$. This in turn implies that $T^*T \supseteq \omega(T^{(u)})\mathbf{p} + \mathbf{p}^2 = \mathbf{p}$ whence $T^*T = R$; as required.

6.5. Combining the last two results gives:

THEOREM. *Let R be the pull-back*

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & \bar{U} \\ \uparrow & & \uparrow \pi \\ R & \dashrightarrow & U \end{array}$$

Set $\mathbf{m} = \ker \pi$ and let S be a second ring. Then the following are equivalent:

(a) *R and S are Morita equivalent*;

(b) *there exists a projective, finitely generated right R-module $T = (A, B, \alpha)$ with the following properties*: (i) *A_U and B_H are progenerators and* (ii) *S is isomorphic to the following pull-back S_1*:

$$\begin{array}{ccc} \mathrm{End}_H B & \xrightarrow{\tau} & \mathrm{End}_{\bar{U}}(A/A\mathbf{m}) \\ \uparrow & & \uparrow \chi \\ S_1 & \dashrightarrow & \mathrm{End}_U(A) \end{array}$$

Here τ is defined as in the proof of Proposition 6.3.

6.6. *Remark.* It is natural to ask whether one can obtain a criterion for two pull-backs to be Morita equivalent in terms of their descriptions as pull-backs. Unfortunately, it seems that this cannot be done in any sensible way. The problem is that two pull-backs can easily be isomorphic as rings without their being isomorphic as pull-backs. To take a simple example, let S be the pull-back

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & \bar{U} \\ \uparrow & & \uparrow \pi \\ S & \dashrightarrow & U \end{array}$$

Let V be any subring of U such that $\ker \pi \subset V$ and $\pi(V) \supseteq \sigma(H)$. Then one may write S as the pull-back

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & \pi(V) \\ \uparrow & & \uparrow \pi \\ S & \dashrightarrow & V \end{array}$$

6.7. As an application of Theorem 6.5 we consider an example that will be relevant to the study of rings of differential operators. For $i=1, 2$, fix rings U_i with maximal ideals $\mathbf{m}_i \subset U_i$ such that $\bar{U}_i = U_i/\mathbf{m}_i \cong M_{n(i)}(k)$ for some integers $n(i) \geqslant 2$. Write $n(i) = a(i) + b(i)$ for some integers $a(i), b(i) \geqslant 1$ and define

$$H_i = \begin{pmatrix} M_{a(i)}(k) & M_{a(i),b(i)}(k) \\ 0 & M_{b(i)}(k) \end{pmatrix} \xhookrightarrow{\sigma_i} M_{n(i)}(k).$$

Here $M_{a,b}(k)$ denotes the set of all $a \times b$ matrices with entries from k, and the ring structure of H_i is that induced from $M_{n(i)}(k)$. Let R_i be the pull-back $(U_i, H_i, \bar{U}_i)$. Thus, $\mathbf{m}_i \subset R_i \subset U_i$ with $R_i/\mathbf{m}_i = H_i$. Next, suppose that A is a progenerator, as a right U_1-module, such that (i) $U_2 = \operatorname{End} A$ and (ii) $A\mathbf{m}_1 A^* = \mathbf{m}_2$. (For example, one could take $A = U_1 = U_2$.) Since A is a progenerator,

$$\operatorname{End}(A/A\mathbf{m}_1) \cong U_2/A\mathbf{m}_1 A^* = U_2/\mathbf{m}_2 \cong M_{n(2)}(k).$$

Equivalently, if S_i is the unique, simple right $U_i/\mathbf{m}_i$-module, then $A/A\mathbf{m}_1 \cong S_1^{(n(2))}$.

6.8. Proposition. *Keep the notation of* (6.7). *Assume further that*:

(6.8.1) *Every k-algebra automorphism of* $\bar{U}_2 = M_{n(2)}(k)$ *is induced from a ring automorphism of* U_2.

Then R_1 *and* R_2 *are Morita equivalent.*

Proof. We first need to construct an appropriate H_1-module B. Let F and G be the top row, respectively the bottom row of H_1, regarded as right H_1-modules. Thus $H_1 \cong F^{(a(1))} \oplus G^{(b(1))}$ and so certainly F and G are projective right H_1-modules. Thus

$$B = F^{(a(2))} \oplus G^{(b(2))}$$

is a progenerator (this is where we need the $a(i)$ and $b(j)$ to be positive). Routine computations show that:

(i) $B \otimes_{H_1} \bar{U}_1 \cong S_1^{(n(2))} \cong A/A\mathbf{m}_1$ as right $\bar{U}_1$-modules. Denote this isomorphism by α^{-1}.

(ii) $$\operatorname{End}_{H_1} B \cong \begin{pmatrix} \operatorname{End}(F^{(a(2))}) & \operatorname{Hom}(G^{(b(2))}, F^{(a(2))}) \\ \operatorname{Hom}(F^{(a(2))}, G^{(b(2))}) & \operatorname{End}(G^{(b(2))}) \end{pmatrix} \cong H_2.$$

Now apply Theorem 6.5. Thus $T = (A, B, \alpha)$ is a progenerator as a right R_1-module and $\operatorname{End}_{R_1}(T)$ is isomorphic to the pull-back $S = (\operatorname{End}(A),$

End(B), End($A/A\mathbf{m}_1$)). In more detail, and in the notation of (6.3) and (6.7), S is the pull-back

$$\begin{array}{ccc} H_2 & \stackrel{\tau}{\hookrightarrow} & \bar{U}_2 \\ \uparrow & & \uparrow \\ S & \dashrightarrow & \bar{U}_2 \end{array}$$

Unfortunately, S may not be equal to R_2. The problem is that the embeddings $H_2 \stackrel{\tau}{\hookrightarrow} \bar{U}_2 \cong M_{n(2)}(k)$ and $H_2 \stackrel{\sigma_2}{\hookrightarrow} \bar{U}_2 \cong M_{n(2)}(k)$ could depend upon different presentations of $\bar{U}_2$ as an $n(2) \times n(2)$ matrix ring. This is where the hypothesis (6.8.1) is used. Thus, let $H_2' = \tau(H_2)$ and $H_2'' = \sigma_2(H_2)$. Then there exists a k-algebra isomorphism $\bar{\omega}$ of $\bar{U}_2$ such that $\bar{\omega}(H_2') = H_2''$. By hypothesis, $\bar{\omega}$ is induced from an automorphism ω of U_2. In particular, $\omega(\mathbf{m}_2) = \mathbf{m}_2$. Thus

$$\begin{aligned} S &\cong \{u \in U_2 \colon [u + \mathbf{m}_2] \in H_2'\} \\ &\cong \{\omega(u) \in U_2 \colon [\omega(u) + \mathbf{m}_2] \in \omega(H_2') = H_2''\} = R_2. \end{aligned}$$

Therefore, R_1 is indeed Morita equivalent to R_2.

6.9. *Remark.* The hypothesis (6.8.1) is unfortunate, but presumably necessary. To see this, suppose that U_2 is a Noetherian domain that is a maximal order in its quotient division ring, for which only the identity automorphism of $\bar{U}_2$ is induced from an automorphism of U_2. Next, construct two pull-backs R_2 and R_2' inside U_2, as in (6.7), but corresponding to different presentations of $\bar{U}_2$ as an $n(2) \times n(2)$ matrix ring. Then it is easy to prove that R_2 and R_2' cannot be isomorphic and so, presumably, they will not be Morita equivalent.

6.10. A weaker version of Proposition 6.8 does hold without (6.8.1). Indeed, the proof of that result also proves

COROLLARY. *Keep the notation of* (6.7), *and let* R_1 *be the pull-back* $(U_1, H_1, \bar{U}_1)$. *Then for some choice of the presentation of* $\bar{U}_2$ *as an* $n(2) \times n(2)$ *matrix ring,* R_1 *will be Morita equivalent to the ring* $R_2 = (U_2, H_2, \bar{U}_2)$.

6.11. One situation where the hypothesis (6.8.1) is trivially satisfied is the following. Let V be a k-algebra, and I a maximal ideal of V such that $V/I \cong k$. Set $U = M_n(V)$ and $\mathbf{m} = M_n(I)$. Then any automorphism of $U/\mathbf{m} \cong M_n(k)$ is certainly induced from one of $U \cong M_n(k) \otimes_k V$. A more interesting example is provided by the following result.

LEMMA. *Let U be the enveloping algebra $U(\mathfrak{sl}_2(k))$, where k is now an algebraically closed field of characteristic zero. Then U has a unique ideal $\mathbf{q}$ such that $U/\mathbf{q} \cong M_2(k)$. Moreover, any k-algebra automorphism of $M_2(k)$ is induced from a k-algebra automorphism of U.*

Remark. This lemma can obviously be considerably generalised, but it sufficient to illustrate Proposition 6.8.

Proof. That $\mathbf{q}$ is unique follows immediately from [Di, Corollary 1.8.5, p. 33]. Indeed, $\mathbf{q}$ is the annihilator of the natural two dimensional representation of $\mathfrak{sl}(k)$. Thus one presentation of $U/\mathbf{q} \cong M_2(k)$ is given by identifying the images in $U/\mathbf{q}$ of the generators e, f, and h of U with the generators of $\mathfrak{sl}(k) \subset \mathfrak{gl}_2(k) = M_2(k)$.

Now let $\bar{\omega}$ be a k-algebra automorphism of $M_2(k)$. By the Skolem–Noether Theorem, $\bar{\omega}$ is an inner automorphism; say $\bar{\omega}(A) = gAg^{-1}$ for some $g \in GL_2(k)$. Thus, if $A \in \mathfrak{sl}_2(k)$, then trace $\bar{\omega}(A) =$ trace $A = 0$ and so $\bar{\omega}$ induces a (Lie algebra) automorphism ω_1 of $\mathfrak{sl}_2(k)$. By the universality of $U(\mathfrak{sl}_2(k))$, ω_1 induces a k-algebra automorphism ω of $U(\mathfrak{sl}_2(k))$. The uniqueness of $\mathbf{q}$ implies that $\omega(\mathbf{q}) = \mathbf{q}$ and so we may consider the automorphism ω' of $U/\mathbf{q}$ induced from ω. The construction of ω clearly forces $\omega' = \bar{\omega}$; as required.

6.12. Using Lemma 6.11, one can show that a surprising number of pull-backs are Morita equivalent.

COROLLARY. *Let k be an algebraically closed field of characteristic zero and, for $i = 1, 2$, let V_i be a primitive factor ring of $U(\mathfrak{sl}_2(k))$ that is not simple. Let U_i be any ring Morita equivalent to V_i. Then U_i has a unique, minimal non-zero ideal, say $\mathbf{m}_i$, and $U_i/\mathbf{m}_i \cong M_{n(i)}(k)$ for some integer $n(i) \geqslant 1$. Assume that $n(i) > 1$ for $i = 1, 2$, and write $n(i) = a(i) + b(i)$ for some strictly positive integers $a(i)$ and $b(i)$. Consider the pull-backs*

$$\begin{array}{ccc} H_i = \begin{pmatrix} M_{a(i)}(k) & M_{a(i),b(i)}(k) \\ 0 & M_{b(i)}(k) \end{pmatrix} & \hookrightarrow & U_i/\mathbf{m}_i \cong M_{n(i)}(k) \\ \uparrow & & \uparrow \\ R_i & \dashrightarrow & U_i \end{array}$$

as in (6.7). Then R_1 is Morita equivalent to R_2.

Remark. An analogous result may be obtained by starting with $V_i = U(\mathfrak{sl}_2(k))$, but it is the given formulation of the corollary that will be relevant to the study of differential operators.

Proof. We begin with a number of simplifications. Let Ω be the Casimir element of $U = U(\mathfrak{sl}_2(k))$. Then the primitive, non-simple factor rings of U are precisely the rings $W_{r+1} = U/(\Omega - r^2 - 2r)U$, for $r \in \mathbb{N}$. Moreover, for each r, W_r has exactly one non-zero ideal, say $\mathbf{n}_r$, and $W_r/\mathbf{n}_r \cong M_r(k)$. These facts may all be found in [Di] and are summarised in [St]. Furthermore, by [St, Corollary 3.3], the rings W_r for $r \in \mathbb{N}$ are all Morita equivalent. Thus, in the statement of the corollary, we may assume that $V_1 = V_2 = U_2 = W_2$ and that $\mathbf{m}_2 = \mathbf{n}_2$. Next, observe that any automorphism ω of U must fix $\mathbf{m}_2$. As $(\Omega - 3)U$ is the unique minimal primitive ideal of U contained in $\mathbf{m}_2, \omega$ must also fix $(\Omega - 3)U$. Thus Lemma 6.11 implies that any automorphism of $U_2/\mathbf{m}_2$ is induced from an automorphism of $W_2 = U/(\Omega - 3)U$. Thus, the corollary is indeed a special case of Proposition 6.8.

7. Morita Equivalence of Rings of Differential Operators

7.1. We now return to the study of rings of differential operators. Throughout this section $\mathscr{X}$ and $\mathscr{Y}$ will denote singular, projective curves for which the normalisation maps, $\pi: \mathbb{P}^1 \to \mathscr{X}$ and $\rho: \mathbb{P}^1 \to \mathscr{Y}$, are injective, while $\mathscr{X}_1$ is the plane, cuspidal, curve of Section 5, with (injective) normalisation map $\pi_1: \mathbb{P}^1 \to \mathscr{X}_1$. It follows easily from the results of Section 4, combined with Corollary 6.12, that $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$ are actually Morita equivalent (see Lemma 7.2, below). This raises the question of whether that Morita equivalence is obtained through the bimodule $\mathscr{D}(\mathscr{X}, \mathscr{Y})$, defined in the natural way. The main aim of this section is to show that this is indeed the case. As a corollary, we show that the Morita equivalence carries $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ to $H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}})$ and hence that these modules are simple. This will, in turn, be used to prove that the integer t, of Corollary 4.5, is equal to $\dim_k H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$.

7.2. Lemma. *Let $\mathscr{X}$ and $\mathscr{Y}$ be as in* (7.1). *Then $\mathscr{D}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathscr{Y})$.*

Proof. Corollary 4.7 proves that $\mathscr{D}(\mathscr{X})$ is a pull-back. Moreover, the ring $S = \mathrm{End}_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathbb{P}^1, \mathscr{X})$, of Corollary 4.7, is Morita equivalent to $\mathscr{D}(\mathbb{P}^1) \cong U(\mathfrak{sl}_2(k))/(\Omega)$. Thus $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$ are pull-backs of precisely the form considered in Corollary 6.12 and hence, by that result, $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$ are Morita equivalent.

7.3. Let $\mathscr{X}$ and $\mathscr{Y}$ be as in (7.1). In order to define $\mathscr{D}(\mathscr{X}, \mathscr{Y})$, we need to choose "compatible" open affine covers of $\mathscr{X}$ and $\mathscr{Y}$, which is achieved as follows. Given an open affine subset $\{U_{\mathscr{X}}\}$ of $\mathscr{X}$, set $\tilde{U} = \pi^{-1}(U_{\mathscr{X}})$ and $U_{\mathscr{Y}} = \rho(\tilde{U})$. $U_{\mathscr{Y}}$ is an open affine subset of $\mathscr{Y}$ and, as π and ρ are injective, the process is symmetric. Such a pair $(U_{\mathscr{X}}, U_{\mathscr{Y}})$ will be called *compatible* and we will always denote $\pi^{-1}(U_{\mathscr{X}}) = \rho^{-1}(U_{\mathscr{Y}})$ by $\tilde{U}$. Given a compatible pair $(U_{\mathscr{X}}, U_{\mathscr{Y}})$, set

$$\mathscr{D}(U_{\mathscr{X}}, U_{\mathscr{Y}}) = \{\theta \in \mathscr{D}(K(\mathscr{X})): \theta \circ \mathcal{O}(U_{\mathscr{X}}) \subseteq \mathcal{O}(U_{\mathscr{Y}})\}.$$

This is a $\mathscr{D}(U_{\mathscr{Y}})$-$\mathscr{D}(U_{\mathscr{X}})$-bimodule and, by Proposition 3.7(iv), is non-zero since it is equal to $\mathscr{D}(\tilde{U}, U_{\mathscr{Y}})\, \mathscr{D}(U_{\mathscr{X}}, \tilde{U})$. Thus, in the notation of Section 2, if we form the sheaves

$$\mathscr{P} = \rho_*\{\mathscr{D}(\mathcal{O}_{\mathbb{P}^1}, \rho^{-1}\mathcal{O}_{\mathscr{Y}}) \otimes_{\mathscr{D}_{\mathbb{P}^1}} \mathscr{D}(\pi^{-1}\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathbb{P}^1})\}$$

and

$$\mathscr{Q} = \pi_*\{\mathscr{D}(\mathcal{O}_{\mathbb{P}^1}, \rho^{-1}\mathcal{O}_{\mathscr{Y}}) \otimes_{\mathscr{D}_{\mathbb{P}^1}} \mathscr{D}(\pi^{-1}\mathcal{O}_{\mathscr{X}}, \mathcal{O}_{\mathbb{P}^1})\}$$

of left $\mathscr{D}_{\mathscr{Y}}$-modules (respectively of right $\mathscr{D}_{\mathscr{X}}$-modules), the sections over $U_{\mathscr{Y}}$ (respectively $U_{\mathscr{X}}$) are

$$\Gamma(U_{\mathscr{Y}}, \mathscr{P}) = \mathscr{D}(U_{\mathscr{X}}, U_{\mathscr{Y}}) = \Gamma(U_{\mathscr{X}}, \mathscr{Q}).$$

Finally, set

$$\mathscr{D}(\mathscr{X}, \mathscr{Y}) = \Gamma(\mathscr{Y}, \mathscr{P}) = \Gamma(\mathscr{X}, \mathscr{Q}).$$

Once again, this is clearly a $\mathscr{D}(\mathscr{Y})$-$\mathscr{D}(\mathscr{X})$-bimodle and is non-zero, since it contains $\mathscr{D}(\mathbb{P}^1, \mathscr{Y})\, \mathscr{D}(\mathscr{X}, \mathbb{P}^1) \supseteq \mathscr{D}(\mathbb{P}^1, \mathscr{Y})$.

7.4. Given the Morita equivalence of $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$, it is natural to ask whether this equivalence can be obtained via $\mathscr{D}(\mathscr{X}, \mathscr{Y})$ in the following sense: Is $\mathscr{D}(\mathscr{X}, \mathscr{Y})$ a progenerative right $\mathscr{D}(\mathscr{X})$-module with endomorphism ring isomorphic to $\mathscr{D}(\mathscr{Y})$?

In order to study $\mathscr{D}(\mathscr{X}, \mathscr{Y})$, we begin with the special case when $\mathscr{X} = \mathscr{X}_1$ is the plane cuspidal curve of Section 5. The reason for this is that now, ρ factors through π_1, in the sense that ρ is the composition $\mathbb{P}^1 \xrightarrow{\pi_1} \mathscr{X}_1 \xrightarrow{\psi} \mathscr{Y}$, for an appropriate map ψ. To show this, observe that one may assume that $\mathscr{Y}$ is smooth at $\rho(\infty)$ and has a singularity at $\rho(0)$. Let $\tilde{U}_0 = \mathbb{P}^1 \backslash \{\infty\}$, $\tilde{U}_1 = \mathbb{P}^1 \backslash \{0\}$ be the standard, open affine cover of $\mathbb{P}^1$, and let $U_0 = \pi_1(\tilde{U}_0)$, $U_1 = \pi_1(\tilde{U}_1)$, and $\{V_i = \rho(\tilde{U}_i)\}$ be the corresponding covers of $\mathscr{X}_1$ and $\mathscr{Y}$ (note that they are compatible). Let t be the coordinate function at zero on $\tilde{U}_0$. Then, it follows easily from the fact that ρ is injective that

$$\mathcal{O}(V_0) \subseteq \mathcal{O}(U_0) = k[t^2, t^3] \subset \mathcal{O}(\tilde{U}_0) = k[t] \tag{7.4.1}$$

and $\mathcal{O}(V_1) \subseteq \mathcal{O}(U_1) = \mathcal{O}(\tilde{U}_1) = k[t^{-1}]$. These containments therefore define ψ.

In order to simplify the notation, set $Q(\mathscr{Y}) = \mathscr{D}(\mathbb{P}^1, \mathscr{Y})$, $P(\mathscr{Y}) = \mathscr{D}(\mathscr{Y}, \mathbb{P}^1)$, and $S(\mathscr{Y}) = \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} Q(\mathscr{Y})$, with the corresponding definitions over $\mathscr{X}$ and $\mathscr{X}_1$.

7.5. LEMMA. *In the notation of* (7.4), $Q(\mathscr{X}_1) \not\subseteq \mathscr{O}(\tilde{U}_i) = \mathscr{O}(U_i)$, *for* $i = 1, 2$.

Proof. By Lemma 5.3, $Q(\mathscr{X}_1) \ni t\partial - 1$, where $\partial = \partial/\partial t$. Clearly, $(t\partial - 1) \circ t^n = (n-1)t^n \neq 0$, whenever $n \neq 1$. Since $\mathscr{O}(U_0) = k + t^2 k[t]$ and $\mathscr{O}(U_1) = k[t^{-1}]$, this suffices to prove the lemma.

7.6. Keep the notation of (7.4). In attempting to identify $\mathscr{D}(\mathscr{X}_1, \mathscr{Y})$, one obvious object to try is $Z = \mathscr{D}(\mathbb{P}^1, \mathscr{Y})\, \mathscr{D}(\mathscr{X}_1, \mathbb{P}^1)$. Unfortunately, this cannot be a progenerator. For, $\operatorname{End}(Z_{\mathscr{D}(\mathscr{X}_1)})$ contains (and hence is equal to) $\operatorname{End}(Q(\mathscr{Y})_{\mathscr{D}(\mathbb{P}^1)}) = S(\mathscr{Y})$. Thus, by Theorem 3.19, $Z_{\mathscr{D}(\mathscr{X}_1)}$ is not a progenerator. Instead, we begin by studying $L = Q(\mathscr{Y})\{Q(\mathscr{X}_1)_{\mathscr{D}(\mathbb{P}^1)}\}^*$. By Proposition 3.14, $Q(\mathscr{Y})$, and $Q(\mathscr{X}_1)^*$ are progenerators over $\mathscr{D}(\mathbb{P}^1)$, with endomorphism rings $S(\mathscr{Y})$, respectively $S(\mathscr{X}_1)$. Thus $L_{S(\mathscr{X}_1)}$ is a progenerator with endomorphism ring $S(\mathscr{Y})$. (In the notation of Theorem 6.5, $S(\mathscr{X}_1)$ and L will play the role of U and A, and our aim will be to prove that $\mathscr{D}(\mathscr{X}_1, \mathscr{Y})$ is a pull-back $T = (A, B, \alpha)$ for some H-module B. Here $H = \mathscr{D}(\mathscr{X}_1)/\mathfrak{n}$ where $\mathfrak{n} = J(\mathscr{X}_1)$ is the (unique minimal) non-zero ideal of $S(\mathscr{X}_1)$.)

LEMMA. *Let* $K = K(\mathscr{X}_1)$ *and* $M = L \cap \mathscr{D}(K)$. *Then*:

(i) $S(\mathscr{Y}) \cap S(\mathscr{X}_1) \supseteq L \supseteq M \supseteq L\mathfrak{n}$.

(ii) $M = L \cap \mathscr{D}(\mathscr{X}_1)$ *and* $\dim_k L/M = 1$.

Proof. (i) Since ρ factors through π_1, $Q(\mathscr{Y}) \subseteq Q(\mathscr{X}_1)$ and $P(\mathscr{X}_1) \subseteq P(\mathscr{Y})$. Thus $P(\mathscr{X}_1)^{**} = Q(\mathscr{X}_1)^* \subseteq Q(\mathscr{Y})^*$. Therefore,

$$L \subseteq Q(\mathscr{Y})\, Q(\mathscr{Y})^* \cap Q(\mathscr{X}_1)\, Q(\mathscr{X}_1)^* = S(\mathscr{Y}) \cap S(\mathscr{X}_1).$$

Similarly, as $P(\mathscr{X}_1)^{**}/P(\mathscr{X}_1)$ is finite-dimensional,

$$L\mathfrak{n} = Q(\mathscr{Y})\, P(\mathscr{X}_1)^{**}\mathfrak{n} \subseteq Q(\mathscr{Y})\, P(\mathscr{X}_1) \subseteq \mathscr{D}(K)$$

and so $L\mathfrak{n} \subseteq \mathscr{D}(K) \cap L$.

(ii) Suppose that $L \subseteq \mathscr{D}(K)$. By Proposition 3.14(a), $P(\mathscr{Y})\, Q(\mathscr{Y}) = \mathscr{D}(\mathbb{P}^1)$. Thus

$$\mathscr{D}(K) \supseteq P(\mathscr{Y}) L = \mathscr{D}(\mathbb{P}^1)\, P(\mathscr{X}_1)^{**} = P(\mathscr{X}_1)^{**};$$

contradicting Lemma 5.3. Thus $L \not\subseteq \mathscr{D}(K)$ and $M \neq L$. Now, by Proposition 5.4(ii), $S(\mathscr{X}_1)/\mathscr{D}(\mathscr{X}_1)$ is 1-dimensional. Since $L \cap \mathscr{D}(\mathscr{X}_1) \subseteq M \subsetneq$

$L \subseteq S(\mathscr{X}_1)$, this implies that $M = L \cap \mathscr{D}(\mathscr{X}_1)$ and hence that L/M is 1-dimensional.

7.7. LEMMA. *Keep the notation of* (7.6). *Then* $M = \mathscr{D}(\mathscr{X}_1, \mathscr{Y})$.

Proof. Observe that, in the notation of (7.4),

$$\mathscr{D}(\mathscr{X}_1, \mathscr{Y})\, Q(\mathscr{X}_1) \circ \mathscr{O}(\tilde{U}_i) \subseteq \mathscr{D}(\mathscr{X}_1, \mathscr{Y}) \circ \mathscr{O}(U_i) \subseteq \mathscr{O}(V_i),$$

for each i. Thus $\mathscr{D}(\mathscr{X}_1, \mathscr{Y})\, Q(\mathscr{X}_1) \subseteq Q(\mathscr{Y})$. Therefore,

$$\mathscr{D}(\mathscr{X}_1, \mathscr{Y}) \subseteq \mathscr{D}(\mathscr{X}_1, \mathscr{Y})\, S(\mathscr{X}_1) = \mathscr{D}(\mathscr{X}_1, \mathscr{Y})\, Q(\mathscr{X}_1)\{Q(\mathscr{X}_1)^*_{\mathscr{D}(\mathbb{P}^1)}\} \subseteq L.$$

Thus $\mathscr{D}(\mathscr{X}_1, \mathscr{Y}) \subseteq L \cap \mathscr{D}(K) = M$. Conversely, for each i, Lemma 7.5 implies that

$$\begin{aligned} M \circ \mathscr{O}(U_i) &= M \circ (Q(\mathscr{X}_1) \circ \mathscr{O}(\tilde{U}_i)) = MQ(\mathscr{X}_1) \circ \mathscr{O}(\tilde{U}_i) \\ &\subseteq LQ(\mathscr{X}_1) \circ \mathscr{O}(\tilde{U}_i) = Q(\mathscr{Y})\, Q(\mathscr{X}_1)^* Q(\mathscr{X}_1) \circ \mathscr{O}(\tilde{U}_i) \\ &= Q(\mathscr{Y})\, \mathscr{D}(\mathbb{P}^1) \circ \mathscr{O}(\tilde{U}_i) \subseteq \mathscr{O}(V_i). \end{aligned}$$

Thus $M \subseteq \mathscr{D}(\mathscr{X}_1, \mathscr{Y})$.

7.8. We now use the results of Section 6 to prove that $M_{\mathscr{D}(\mathscr{X}_1)}$ is a progenerator with endomorphism ring $\mathscr{D}(\mathscr{Y})$. Let $\mathscr{X}$ be any singular, rational curve with injective normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$. Then, Theorem 4.3 implies that $J(\mathscr{X}) \subseteq \mathscr{D}(\mathscr{X}) \subseteq S(\mathscr{X})$ and

$$F(\mathscr{X}) = \mathscr{D}(\mathscr{X})/J(\mathscr{X}) \cong \begin{pmatrix} M_t(k) & k^{(t)} \\ 0 & k \end{pmatrix} \subseteq M_{t+1}(K) = S(\mathscr{X})/J(\mathscr{X}). \qquad (7.8.1)$$

In particular, $\mathscr{D}(\mathscr{X})$ is a maximal subring of $S(\mathscr{X})$. Write $t(\mathscr{X})$ for the integer t of (7.8.1). Thus $t(\mathscr{X}) \geqslant 1$ while, by Proposition 5.4, $t(\mathscr{X}_1) = 1$. Next, consider the module L of (7.6). If $\mathfrak{n} = J(\mathscr{X}_1)$ and C is the (unique) simple right module over $S(\mathscr{X}_1)/\mathfrak{n} \cong M_2(k)$, then $L/L\mathfrak{n} = C^{(r)}$, for some integer r. Since $L_{S(\mathscr{X}_1)}$ is a progenerator with endomorphism ring $S(\mathscr{Y})$, Morita theory says that

$$S(\mathscr{Y})/J(\mathscr{Y}) = \operatorname{End}(L/L\mathfrak{n})_{S(\mathscr{X}_1)} = M_r(k).$$

Thus $r = t(\mathscr{Y}) + 1 \geqslant 2$. Now consider M. Note that $F(\mathscr{X}_1) = \left(\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right)$ is an hereditary subring of $S(\mathscr{X}_1)/\mathfrak{n}$, with two uniform, projective right modules

$$A = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix} \cong C \qquad \text{and} \qquad B \cong \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.$$

Now, A is an extension of B by a simple $F(\mathscr{X}_1)$-module, not isomorphic to B. Thus, counting composition factors proves that the only $F(\mathscr{X}_1)$-submodules of $L/L\mathbf{n} \cong A^{(r)}$ are those of the form $A^{(x)} \oplus B^{(y)}$, with $x+y \leq r$. Since dim $L/M = 1$, this forces $M/L\mathbf{n} \cong A^{(r-1)} \oplus B$. In particular, and in the notation of Section 6, M is the pull-back $M \cong (L, A^{(r-1)} \oplus B, 1)$ over the pull-back ring $\mathscr{D}(\mathscr{X}_1) \cong (S(\mathscr{X}_1), F(\mathscr{X}_1), S(\mathscr{X}_1)/\mathbf{n})$. Thus, by Theorem 6.5, $M_{\mathscr{D}(\mathscr{X}_1)}$ is a progenerator with endomorphism ring

$$\operatorname{End}_{\mathscr{D}(\mathscr{X}_1)} M = (\operatorname{End} L, \operatorname{End} M/L\mathbf{n}, \operatorname{End} L/L\mathbf{n}) \subsetneq S(\mathscr{Y}).$$

Since $\mathscr{D}(\mathscr{Y}) \subseteq \operatorname{End} M$ and $\mathscr{D}(\mathscr{Y})$ is a maximal subring of $S(\mathscr{Y})$, this forces $\mathscr{D}(\mathscr{Y}) = \operatorname{End} M$. Thus we have proved the first part of

THEOREM. *Let $\mathscr{Y}$ be a singular, projective, rational curve, with injective normalisation map $\rho: \mathbb{P}^1 \to \mathscr{Y}$. Let $\mathscr{X}_1$ be the cuspidal, plane, cubic curve. Then*

(i) $\mathscr{D}(\mathscr{X}_1, \mathscr{Y})_{\mathscr{D}(\mathscr{X}_1)}$ *is a progenerator, with endomorphism ring* $\mathscr{D}(\mathscr{Y})$.

(ii) $\mathscr{D}(\mathscr{X}_1, \mathscr{Y})^*_{\mathscr{D}(\mathscr{X}_1)} = \mathscr{D}(\mathscr{Y}, \mathscr{X}_1)$.

Proof. It remains to prove part (ii). Set $Z = \mathscr{D}(\mathscr{X}_1, \mathscr{Y})^*_{\mathscr{D}(\mathscr{X}_1)}$. Then, by Morita theory, $Z = {}_{\mathscr{D}(\mathscr{Y})}\mathscr{D}(\mathscr{X}_1, \mathscr{Y})^*$. Let the open affine covers $\{U_i\}$ and $\{V_j\}$ be defined as in (7.4). Then, by Theorem 3.15(a), $\mathscr{D}(U_i, V_i) = \mathscr{D}(V_i)\,\mathscr{D}(\mathscr{X}_1, \mathscr{Y})$, for $i = 0, 1$. Thus

$$\mathscr{D}(U_i, V_i)Z = \mathscr{D}(V_i)\,\mathscr{D}(\mathscr{X}_1, \mathscr{Y})Z = \mathscr{D}(V_i)\,\mathscr{D}(\mathscr{Y}) = \mathscr{D}(V_i).$$

By Proposition 3.7, this implies that

$$Z \subseteq \bigcap \{{}_{\mathscr{D}(V_i)}\mathscr{D}(U_i, V_i)^*\} = \bigcap_i \mathscr{D}(V_i, U_i) = \mathscr{D}(\mathscr{Y}, \mathscr{X}_1).$$

Since the containment $\mathscr{D}(\mathscr{Y}, \mathscr{X}_1) \subseteq Z$ is a triviality, this completes the proof.

7.9. *Remark.* Consider what happens when one repeats the above analysis, but with $\mathscr{Y}$ replaced by $\mathbb{P}^1$. Then

$$L = \mathscr{D}(\mathbb{P}^1)\,\mathscr{D}(\mathbb{P}^1, \mathscr{X}_1)^* = Q(\mathscr{X}_1)^* \supset M = P(\mathscr{X}_1).$$

In this case, one still finds that M is a pull-back over $\mathscr{D}(\mathscr{X}_1)$, with endomorphism ring $\mathscr{D}(\mathbb{P}^1)$. However, in the notation of (7.8), $r(\mathbb{P}^1) = 1$ and so $M/L\mathbf{n} \cong B \subset A = L/L\mathbf{n}$. Thus $M/L\mathbf{n}$ is not a generator over $F(\mathscr{X}_1)$—as it cannot be, since $\mathscr{D}(\mathbb{P}^1) = \operatorname{End} M$ is not Morita equivalent to $\mathscr{D}(\mathscr{X}_1)$.

7.10. It is now easy to repeat Theorem 7.8 for an arbitrary pair of curves.

THEOREM. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two singular, rational, projective curves, with injective normalisation maps. Define $\mathscr{D}(\mathscr{X}, \mathscr{Y})$ as in* (7.3). *Then $\mathscr{D}(\mathscr{X}, \mathscr{Y})_{\mathscr{D}(\mathscr{X})}$ is a progenerator with endomorphism ring $\mathscr{D}(\mathscr{Y})$. Moreover, $\mathscr{D}(\mathscr{X}, \mathscr{Y})^* = \mathscr{D}(\mathscr{Y}, \mathscr{X})$, as a module over either $\mathscr{D}(\mathscr{X})$ or $\mathscr{D}(\mathscr{Y})$.*

Proof. By Theorem 7.8, $Z = \mathscr{D}(\mathscr{X}_1, \mathscr{Y})\, \mathscr{D}(\mathscr{X}, \mathscr{X}_1)$ is a progenerator as a right $\mathscr{D}(\mathscr{X})$-module with endomorphism ring isomorphic to $\mathscr{D}(\mathscr{Y})$. Moreover, $Z^* \cong \mathscr{D}(\mathscr{X}_1, \mathscr{X})\, \mathscr{D}(\mathscr{Y}, \mathscr{X}_1)$, as a module over either $\mathscr{D}(\mathscr{X})$ or $\mathscr{D}(\mathscr{Y})$. Thus, in order to prove the theorem, one need only prove that $Z = \mathscr{D}(\mathscr{X}, \mathscr{Y})$. Certainly $Z \subseteq \mathscr{D}(\mathscr{X}, \mathscr{Y})$. If the containment is strict, then Theorem 7.8 implies that

$$\mathscr{D}(\mathscr{Y}, \mathscr{X}_1)\, \mathscr{D}(\mathscr{X}, \mathscr{Y})\, \mathscr{D}(\mathscr{X}_1, \mathscr{X}) \supsetneq \mathscr{D}(\mathscr{Y}, \mathscr{X}_1)\, Z \mathscr{D}(\mathscr{X}_1, \mathscr{X}) = \mathscr{D}(\mathscr{X}_1),$$

which is absurd.

7.11. One should regard Theorem 7.10 as showing that the Morita equivalence between $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathscr{Y})$ is obtained via the "correct" functors. As a consequence, many of the results proved for $\mathscr{D}(\mathscr{X}_1)$, by direct computation, in Section 5 will also hold for $\mathscr{D}(\mathscr{Y})$. The rest of this section will be devoted to such consequences and will therby complete the proof of Theorems B, C, and D of the Introduction.

COROLLARY. *Let $\mathscr{X}$ and $\mathscr{Y}$ be as in Theorem* 7.10. *Let $\{U_{\mathscr{X}}\}$ and $\{U_{\mathscr{Y}}\}$ be compatible open affine subsets of $\mathscr{X}$ and $\mathscr{Y}$, as in* (7.3), *with $\tilde{U}$ the corresponding subset of $\mathbb{P}^1$. Then*:

(i) $\mathscr{D}(\mathscr{X}, \mathscr{Y})\, \mathscr{D}(U_{\mathscr{X}}) = \mathscr{D}(U_{\mathscr{X}}, U_{\mathscr{Y}})$;

(ii) $\mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes_{\mathscr{D}(\mathscr{X})} \mathcal{O}(U_{\mathscr{X}}) \cong \mathcal{O}(U_{\mathscr{Y}})$.

Proof. (i) Use the proof of Corollary 3.17, but with Theorem 7.10 replacing Proposition 3.14(c).

(ii) One has

$$\begin{aligned}\mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes \mathcal{O}(U_{\mathscr{X}}) &\cong \mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes \mathscr{D}(U_{\mathscr{X}}) \otimes \mathcal{O}(U_{\mathscr{X}}) \\ &\cong \mathscr{D}(U_{\mathscr{X}}, U_{\mathscr{Y}}) \otimes \mathcal{O}(U_{\mathscr{X}}) = \mathcal{O}(U_{\mathscr{Y}}),\end{aligned}$$

where the final steps follow from Theorem 7.10 and Proposition 3.7(v).

7.12. COROLLARY. *Let $\mathscr{X}$ and $\mathscr{Y}$ be as in Theorem* 7.10. *Then*

$$\mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes_{\mathscr{D}(\mathscr{X})} H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \cong H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}).$$

Moreover, $H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$ is a simple $\mathscr{D}(\mathscr{Y})$-module.

Proof. By deleting two points from $\mathscr{X}$ and proceeding as in (7.3), one may choose a compatible pair of open affine covers $\{U_1, U_2\}$ and $\{V_1, V_2\}$ for $\mathscr{X}$ and $\mathscr{Y}$. By a Čech cohomology calculation (see [Ha, pp. 218–222])

$$H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \cong \mathcal{O}(U_1 \cap U_2)/\mathcal{O}(U_1) + \mathcal{O}(U_2).$$

Since $\mathscr{D}(\mathscr{X}, \mathscr{Y})_{\mathscr{D}(\mathscr{X})}$ is projective, Corollary 7.11 implies that

$$\begin{aligned} &\mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \\ &\quad \cong \mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes \mathcal{O}(U_1 \cap U_2)/(\mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes \mathcal{O}(U_1) + \mathscr{D}(\mathscr{X}, \mathscr{Y}) \otimes \mathcal{O}(U_2)) \\ &\quad \cong \mathcal{O}(V_1 \cap V_2)/\mathcal{O}(V_1) + \mathcal{O}(V_2) \cong H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}). \end{aligned} \tag{7.12.1}$$

Now set $\mathscr{X} = \mathscr{X}_1$. By Proposition 5.4, $H^1(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1})$ is a simple $\mathscr{D}(\mathscr{X}_1)$-module. Since $\mathscr{D}(\mathscr{X}_1, \mathscr{Y})$ is a progenerator as a right $\mathscr{D}(\mathscr{X}_1)$-module, (7.12.1) implies that

$$\mathscr{D}(\mathscr{X}_1, \mathscr{Y}) \otimes H^1(\mathscr{X}_1, \mathcal{O}_{\mathscr{X}_1}) \cong H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$$

is also a simple module.

7.13. Corollary. *Let $\mathscr{Y}$ be any singular curve with injective normalisation map $\rho\colon \mathbb{P}^1 \to \mathscr{Y}$. If $J(\mathscr{Y})$ is the minimal non-zero ideal of $\mathscr{D}(\mathscr{Y})$, then*

$$\mathscr{D}(\mathscr{Y})/J(\mathscr{Y}) \cong \begin{pmatrix} M_t(k) & k^{(t)} \\ 0 & k \end{pmatrix},$$

where $t = \dim_k H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$.

Proof. With the exception of showing that $t = \dim_k H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$, this follows from Corollary 4.5. However, by Corollary 7.12, and in the notation of Corollary 4.4, $H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$ is a simple module over $\mathscr{D}(\mathscr{Y})/I(\mathscr{Y}) \cong M_t(k)$. This is only possible if $t = \dim_k H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}})$.

7.14. Corollary. *Let $\mathscr{Y}$ be any singular curve with injective normalisation map $\rho\colon \mathbb{P}^1 \to \mathscr{Y}$. Then there exists a non-split, short exact sequence of left $\mathscr{D}(\mathscr{Y})$-modules*:

$$0 \to H^1(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) \to \mathscr{D}(\mathbb{P}^1, \mathscr{Y}) \otimes_{\mathscr{D}(\mathbb{P}^1)} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) \to 0.$$

Proof. $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ is the unique, finite-dimensional, simple left $\mathscr{D}(\mathbb{P}^1)$-module. Moreover, by Proposition 3.14, $Q(\mathscr{Y}) = \mathscr{D}(\mathbb{P}^1, \mathscr{Y})$ is a progenerator as a right $\mathscr{D}(\mathbb{P}^1)$-module, with endomorphism ring $S(\mathscr{Y})$. Thus $Z = \mathscr{D}(\mathbb{P}^1, \mathscr{Y}) \otimes_{\mathscr{D}(\mathbb{P}^1)} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ is the unique, simple, finite-dimensional, left $S(\mathcal{Y})$-module. Therefore, Z has annihilator $J(\mathcal{Y})$ and may be identified with the right hand column Z_1, of $S(\mathcal{Y}) \cong M_{t+1}(k)$. Finally, Corollary 7.13 implies that, as a left $\mathcal{D}(\mathcal{Y})$-module, Z_1 is a non-split extension of $H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ by $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$.

7.15. COROLLARY. *Let $\mathcal{Y}$ be any singular curve with injective normalisation map $\rho: \mathbb{P}^1 \to \mathcal{Y}$. If $K = K(\mathcal{Y})$, then $\mathcal{D}(\mathcal{Y}) = S(\mathcal{Y}) \cap \mathcal{D}(K)$.*

Proof. Certainly $\mathcal{D}(\mathcal{Y}) \subseteq S(\mathcal{Y}) \cap \mathcal{D}(K)$, while by Theorem 4.3 (or Corollary 7.13) $\mathcal{D}(\mathcal{Y})$ is a maximal subring of $S(\mathcal{Y})$. Thus to prove the corollary, we need only show that $S(\mathcal{Y}) \nsubseteq \mathcal{D}(K)$.

By Theorem 7.8, $\mathcal{D}(\mathcal{X}_1, \mathcal{Y})$ is a progenerator as a left $\mathcal{D}(\mathcal{Y})$-module, with endomorphism ring isomorphic to $\mathcal{D}(\mathcal{X}_1)$. Now, $S(\mathcal{Y})$ is the unique maximal order containing and equivalent to $\mathcal{D}(\mathcal{Y})$ (see Corollary 4.9). Thus the usual Morita theoretical argument implies that $S(\mathcal{X}_1) = \mathcal{D}(\mathcal{Y}, \mathcal{X}_1)$ $S(\mathcal{Y})\, \mathcal{D}(\mathcal{X}_1, \mathcal{Y})$. But, if $S(\mathcal{Y}) \subseteq \mathcal{D}(K)$, then this implies that $S(\mathcal{X}_1) \subseteq \mathcal{D}(K)$; contradicting Lemma 5.3.

7.16. THEOREM. *Let $\mathcal{Y}$ be any singular curve with injective normalisation ρ: $\mathbb{P}^1 \to \mathcal{Y}$ and let p be a point of $\mathcal{Y}$. Then $H^1_p(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a non-split $\mathcal{D}(\mathcal{Y})$-module of length two, with socle $H^0(\mathcal{Y} \backslash p, \mathcal{O}_{\mathcal{Y} \backslash p})/H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and factor $H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$.*

Proof. Let $\pi: \mathbb{P}^1 \to \mathcal{X}_1$ be the normalisation map for the cuspidal cubic curve $\mathcal{X}_1$ and let $q = \pi(\rho^{-1}(p)) \in \mathcal{X}_1$. Then, by excision (4.12.2), $H^1_q(\mathcal{X}_1, \mathcal{O}_{\mathcal{X}_1}) \cong \mathcal{O}(U \backslash q)/\mathcal{O}(U)$, where U is an open affine neighbourhood of q. Now, if $V = \rho(\pi^{-1}(U))$ then $\mathcal{D}(\mathcal{X}_1, \mathcal{Y}) \otimes_{\mathcal{D}(\mathcal{X}_1)} \mathcal{O}(U) \cong \mathcal{O}(V)$. Thus, we have

$$\mathcal{D}(\mathcal{X}_1, \mathcal{Y}) \otimes H^1_q(\mathcal{X}_1, \mathcal{O}_{\mathcal{X}_1}) \cong \mathcal{O}(V \backslash p)/\mathcal{O}(V) \cong H^1_p(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$$

by excision, again. Thus the result follows from the Morita equivalence of Theorem 7.8 combined with the Proposition 5.5.

8. ASSOCIATED GRADED RINGS

8.1. Let $\mathcal{Y}$ be an affine curve with normalisation map $\rho: \tilde{\mathcal{Y}} \to \mathcal{Y}$. In [SS] it was shown that the associated graded ring $\operatorname{gr} \mathcal{D}(\mathcal{Y}) = \bigoplus_{n \geq 0} \mathcal{D}^n(\mathcal{Y})/\mathcal{D}^{n-1}(\mathcal{Y})$ is naturally a subring of $\operatorname{gr} \mathcal{D}(\tilde{\mathcal{Y}})$. Moreover $\operatorname{gr} \mathcal{D}(\mathcal{Y})$ is an affine commutative ring if (and only if) ρ is injective. In this section we show that the same result holds for rational projective curves. The proof follows along the same lines as that given in [SS] and so some of the details will be left to the reader.

8.2. Fix a rational curve $\mathscr{X}$ with normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$. Note that, if we identify $\mathscr{D}(\mathscr{X})$ with a subring of $\mathscr{D}(K)$, for $K = K(\mathscr{X})$, then $\mathscr{D}^n(\mathscr{X}) = \mathscr{D}(\mathscr{X}) \cap \mathscr{D}^n(K)$ for all n. In particular,

$$\operatorname{gr} \mathscr{D}(\mathscr{X}) \cong \bigoplus \{(\mathscr{D}^n(\mathscr{X}) + \mathscr{D}^{n-1}(K))/\mathscr{D}^{n-1}(K)\} \subseteq \operatorname{gr} \mathscr{D}(K). \quad (8.2.1)$$

Throughout this section we will identify $\operatorname{gr} \mathscr{D}(\mathscr{X})$ with its image in $\operatorname{gr} \mathscr{D}(K)$.

LEMMA. $\operatorname{gr} \mathscr{D}(\mathscr{X}) \subseteq \operatorname{gr} \mathscr{D}(\mathbb{P}^1)$ *and* $\operatorname{gr} \mathscr{D}(\mathbb{P}^1) \cong k[a, b, c]/(ac - b^2)$.

Proof. Write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ with the corresponding open affine cover $\{\tilde{V}_1 = \mathbb{A}^1, \tilde{V}_2 = \mathbb{P}^1 \backslash \{0\}\}$ and set $V_i = \pi(\tilde{V}_i)$. Thus, if t is a coordinate function on $\mathbb{A}^1$ then, as usual, $\mathscr{D}(\mathbb{P}^1) = k[\partial, t\partial, t^2\partial]$. It follows routinely that $\operatorname{gr} \mathscr{D}(\mathbb{P}^1) \cong k[a, b, c]/(ac - b^2)$ where a, b, and c are the leading symbols of ∂, $t\partial$, and $t^2\partial$. Finally,

$$\operatorname{gr} \mathscr{D}(\mathscr{X}) \subseteq \operatorname{gr} \mathscr{D}(V_1) \cap \operatorname{gr} \mathscr{D}(V_2) \subseteq \operatorname{gr} \mathscr{D}(\tilde{V}_1) \cap \operatorname{gr} \mathscr{D}(\tilde{V}_2) = \operatorname{gr} \mathscr{D}(\mathbb{P}^1),$$

where the second containment comes from [SS, Proposition 3.11].

8.3. The rest of the section is devoted to proving that, if $\mathscr{X}$ is as in (8.2) with π injective, then $\operatorname{gr} \mathscr{D}(\mathbb{P}^1)$ is a finitely generated $\operatorname{gr} \mathscr{D}(\mathscr{X})$-module. An appeal to Eakin's Theorem will then imply that $\operatorname{gr} \mathscr{D}(\mathscr{X})$ is an affine Noetherian ring. If $\mathscr{Y}$ is an affine curve for which $\rho: \tilde{\mathscr{Y}} \to \mathscr{Y}$ is injective then the starting point in [SS], for proving that $\operatorname{gr} \mathscr{D}(\mathscr{Y})$ is Noetherian, is the observation that $\mathscr{D}(\tilde{\mathscr{Y}}, \mathscr{Y}) \cap \mathscr{O}(\tilde{\mathscr{Y}}) \neq 0$. Of course, this is not true when $\mathscr{X}$ is projective and so we need to find some other canonical element in $\mathscr{D}(\mathscr{X})$.

Thus, fix a singular projective curve $\mathscr{X}$ with an injective normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$. Pick open affine covers $\{U_1, ..., U_r, U_\infty\}$ of $\mathscr{X}$ and $\{\tilde{U}_1, ..., \tilde{U}_r, \tilde{U}_\infty\}$ of $\mathbb{P}^1$ such that

(a) $\pi(\tilde{U}_i) = U_i$ for each i,

(b) U_∞ is smooth,

(c) for $1 \leqslant i \leqslant r$, U_i has exactly one singularity, say at $\pi(\alpha_i)$, and

(d) in the notation of (8.2), $U_i \subseteq V_1$ for $1 \leqslant i \leqslant r$.

As usual, set $\mathscr{O}(\tilde{V}_1) = k[t]$. Since π is injective, this implies that, for $1 \leqslant i \leqslant r$, $\mathscr{O}(U_i) \supset (t - \alpha_i)^{n_i} \mathscr{O}(\tilde{U}_i)$ for some n_i and hence that

$$\mathscr{D}(\tilde{U}_i, U_i) \ni \prod_{j=1}^{n_i} ((t - \alpha_i)\, \partial - j). \quad (8.3.1)$$

The next lemma shows how to use this to obtain a specific element in $\mathscr{D}(\mathscr{X})$—indeed it even finds one in $\mathscr{D}(\mathbb{P}^1, \mathscr{X})$.

LEMMA. *$\mathscr{D}(\mathbb{P}^1, \mathscr{X}) \ni \Theta = g(\alpha_1, n_1, 0)\, g(\alpha_2, n_2, n_1) \cdots g(\alpha_r, n_r, (n_1 + n_2 + \cdots + n_{r-1}))$ where $g(\alpha, \beta, \gamma) = \prod_{j=1}^{\beta} ((t-\alpha)\,\partial - j - \gamma)$.*

Proof. The key observation is that, for any $\alpha, \beta \in k$ and $\rho \in \mathbb{N}$ one has

$$((t-\alpha)\partial - \rho)((t-\beta)\,\partial - \rho - 1) = ((t-\beta)\,\partial - \rho)((t-\alpha)\,\partial - \rho - 1). \tag{8.3.2}$$

Now Θ has the form

$$\Theta = ((t-\beta_1)\,\partial - 1)((t-\beta_2)\,\partial - 2) \cdots ((t-\beta_s)\,\partial - s)$$

for some $\beta_i \in k$ and integer s. Thus the obvious induction using (8.3.2) allows one to swop terms in the expression for Θ. In particular, for any $1 \leqslant u \leqslant r$,

$$\Theta = g(\alpha_u, n_u, 0)\Phi_u \qquad \text{for some} \quad \Phi_u \in k[t\partial, \partial]. \tag{8.3.3}$$

By the construction of the U_i for $1 \leqslant i \leqslant r$, certainly $\mathscr{D}(\tilde{U}_i) \supseteq k[t, \partial]$. Thus (8.3.1) and (8.3.3) combine to show that $\Theta \in \mathscr{D}(\tilde{U}_i, U_i)$. Finally

$$\mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \bigcap_{i=1}^{r} \mathscr{D}(\tilde{U}_i, U_i) \cap \mathscr{D}(\tilde{U}_\infty, U_\infty) \cap \mathscr{D}(\mathbb{P}^1)$$
$$= \bigcap \mathscr{D}(\tilde{U}_i, U_i) \cap \mathscr{D}(\mathbb{P}^1) \ni \Theta.$$

8.4. Next, we relate the "size" of $\operatorname{gr} \mathscr{D}(\mathbb{P}^1)/\operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X})$ to that of $\mathscr{D}(\mathbb{P}^1)/\mathscr{D}(\mathbb{P}^1, \mathscr{X})$ as this will give the required estimate for the size of $\operatorname{gr} \mathscr{D}(\mathbb{P}^1)/\operatorname{gr} \mathscr{D}(\mathscr{X})$. The appropriate invariant is defined as follows. Given a module M over a k-algebra R, define the *long length* of M, written $ll(M)$, to be the maximum integer n such that there exists a chain of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0,$$

such that each M_i is infinite-dimensional (if no such integer exists, set $ll(M) = \infty$). Note that $ll(M)$ is finite in two cases of interest; if M has finite length over any k-algebra R or if M is a finitely generated module of Krull dimension one over any affine commutative k-algebra.

8.5. LEMMA. *Write $\mathscr{D}(\mathbb{P}^1) = k[\partial, t\partial, t^2\partial]$ as in (8.2). Then*

(i) *If $X = \mathscr{D}(\mathbb{P}^1)/((t-\alpha)\partial - r)\,\mathscr{D}(\mathbb{P}^1)$ for some $\alpha \in k$ and integer r, then $ll(X) = 2$.*

(ii) *If $R = \operatorname{gr} \mathscr{D}(\mathbb{P}^1) = k[a, b, c]/(ac - b^2)$ then $ll(R/bR) = 2$.*

Proof. By the change of variable $t \to t + \alpha$, we may suppose that $\alpha = 0$ in part (i). Next, observe that, if X is given the induced filtration

$$X_n = (\mathscr{D}^n(\mathbb{P}^1) + (t\partial - r)\,\mathscr{D}(\mathbb{P}^1))/(t\partial - r)\,\mathscr{D}(\mathbb{P}^1),$$

then gr $X = \bigoplus X_n/X_{n-1} \cong R/bR$. Thus in order to prove the lemma it suffices to show that $ll(X) \geqslant 2$ and $ll(R/bR) \leqslant 2$. Since $R = bR \oplus k[a] \oplus ck[c]$ this latter requirement is obvious. Now consider X and suppose that $r \geqslant 0$. Then

$$\mathscr{D}(\mathbb{P}^1) = (t\partial - r)\,\mathscr{D}(\mathbb{P}^1) \oplus \partial k[\partial] \oplus k[t^2\partial]. \tag{8.5.1}$$

Set $f = t^2\partial$ and write $I = (t\partial - r)\,\mathscr{D}(\mathbb{P}^1) \subset J = I + f^r\,\mathscr{D}(\mathbb{P}^1)$. Then it is immediate from (8.5.1) that J/I is infinite-dimensional. Observe that $f(t\partial + 1) = t\partial f$ and hence that $f^r t\partial = (t\partial - r) f^r$. Similarly, $f^r\partial = f^{r-1}t\partial(t\partial - 1) = \cdots = (t\partial - r) f^{r-1} t\partial$. Thus J/I is a homomorphic image of (and hence by Lemma 2.7 equal to) $\mathscr{D}(\mathbb{P}^1)/(t\partial\mathscr{D}(\mathbb{P}^1) + \partial\mathscr{D}(\mathbb{P}^1))$. It follows from (8.5.1) that $\mathscr{D}(\mathbb{P}^1)/J$ must be infinite-dimensional. If $r < 0$ then the above argument still works, provided that one replaces f by $e = \partial$.

8.6. If M is a finitely generated right (or left) $\mathscr{D}(\mathbb{P}^1)$-submodule of $\mathscr{D}(K)$ filter M by $\{M_n = M \cap \mathscr{D}^n(K)\}$ and set gr $M = \bigoplus M_i/M_{i-1}$. As in (8.2) we may identify gr M with $\bigoplus (M_i + \mathscr{D}^{i-1}(K))/\mathscr{D}^{i-1}(K)$. In particular, if $N \subseteq M$ then gr $N \subseteq$ gr M.

LEMMA. (i) *Write $\mathscr{D}(\mathbb{P}^1) = k[\partial, t\partial, t^2\partial]$ and suppose that $\Theta \in k[t\partial, \partial]$ is defined as in Lemma* 8.3. *If V and W are right ideals of $\mathscr{D}(\mathbb{P}^1)$ such that $\Theta\mathscr{D}(\mathbb{P}^1) \subseteq V \subset W \subseteq \mathscr{D}(\mathbb{P}^1)$, then $ll(W/V) = ll(\text{gr } W/\text{gr } V)$.*

(ii) *Let $\mathscr{X}$ be a singular projective curve such that the normalisation map $\pi\colon \mathbb{P}^1 \to \mathscr{X}$ is injective. Then $ll(\mathscr{D}(\mathbb{P}^1)/\mathscr{D}(\mathbb{P}^1, \mathscr{X})) = ll(\text{gr } \mathscr{D}(\mathbb{P}^1)/\text{gr } \mathscr{D}(\mathbb{P}^1, \mathscr{X}))$.*

(iii) *Similarly, $ll(\mathscr{D}(\mathscr{X}, \mathbb{P}^1)/\mathscr{D}(\mathbb{P}^1)) \leqslant ll(\text{gr } \mathscr{D}(\mathscr{X}, \mathbb{P}^1)/\text{gr } \mathscr{D}(\mathbb{P}^1))$.*

Proof. (i) Given right ideals $I \subset J$ of $\mathscr{D}(\mathbb{P}^1)$ then it is an easy exercise to show that J/I is infinite dimensional if and only if gr J/gr I is. Thus $ll(J/I) \leqslant ll(\text{gr } J/\text{gr } I)$. Since Lemma 8.5 implies that $ll(\mathscr{D}(\mathbb{P}^1)/\Theta\mathscr{D}(\mathbb{P}^1)) = ll(\text{gr } \mathscr{D}(\mathbb{P}^1)/\text{gr } \Theta\mathscr{D}(\mathbb{P}^1))$ and long length is additive on short exact sequences, the result follows easily.

(ii) This follows immediately from part (i) combined with Lemma 8.3.

(iii) This is clear. (In fact the proof of part (ii) can be easily modified to prove that one has equality here.)

8.7. THEOREM. *Let $\mathscr{X}$ be a projective curve such that the normalisation map $\pi\colon \mathbb{P}^1 \to \mathscr{X}$ is injective. Then, under the identifications of* (8.2):

(i) gr $\mathscr{D}(\mathscr{X}) \subseteq$ gr $\mathscr{D}(\mathbb{P}^1)$;

(ii) gr $\mathscr{D}(\mathbb{P}^1)$/gr $\mathscr{D}(\mathscr{X})$ *is a finite-dimensional k-vector space*;

(iii) gr $\mathscr{D}(\mathscr{X})$ *is an affine, Noetherian, commutative domain.*

Proof. Part (i) is just Lemma 8.2. Since gr $\mathscr{D}(\mathbb{P}^1)$ is affine, part (iii) follows from part (ii) combined with Eakin's Theorem [Ma, Theorem 3.7]. Thus it remains to prove part (ii). This is very similar to the proof of [SS, Theorem 3.12], and so some of the details will be left to the reader.

By Corollary 3.6, $\mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \{_{\mathscr{D}(\mathbb{P}^1)}\mathscr{D}(\mathscr{X}, \mathbb{P}^1)\}^*$ and so Lemma 1.4 implies that $\mathscr{D}(\mathbb{P}^1, \mathscr{X})^*/\mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ is finite-dimensional. Also, if I and J are right ideals of $\mathscr{D}(\mathbb{P}^1)$ such that $\mathscr{D}(\mathbb{P}^1, \mathscr{X}) \subseteq I \subset J \subseteq \mathscr{D}(\mathbb{P}^1)$ with J/I infinite-dimensional, then Lemma 1.4 implies that I^*/J^* is also infinite-dimensional. Thus if $I^\dagger = I^* \cap \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$, etc., then $I^\dagger/J^\dagger$ is infinite-dimensional. Hence

$$m = ll(\mathscr{D}(\mathscr{X}, \mathbb{P}^1)/\mathscr{D}(\mathbb{P}^1)) \geqslant ll(\mathscr{D}(\mathbb{P}^1)/\mathscr{D}(\mathbb{P}^1, \mathscr{X})) \tag{8.7.1}$$

(and by a dual argument one actually has equality).

By Lemma 8.6(iii) we may therefore choose a chain of gr $\mathscr{D}(\mathbb{P}^1)$-modules

$$\operatorname{gr} \mathscr{D}(\mathbb{P}^1) = I_0 \subset I_1 \subset \cdots \subset I_m = \operatorname{gr} \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$$

such that I_i/I_{i-1} is infinite-dimensional for each i. But if $J_1 \subset J_2$ are non-zero ideals of gr $\mathscr{D}(\mathbb{P}^1)$ with J_2/J_1 infinite-dimensional, then NJ_2/NJ_1 is also infinite-dimensional for any non-zero ideal N. Thus

$$\begin{aligned} \operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X}) = \operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X})\, I_0 \subset \cdots \subset \operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X})\, I_m \\ = \operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X}) \cdot \operatorname{gr} \mathscr{D}(\mathscr{X}, \mathbb{P}^1) \end{aligned} \tag{8.7.2}$$

is also a chain where each factor is infinite-dimensional. Now, (8.7.1) and Lemma 8.6(ii) combine to prove that $ll(\operatorname{gr} \mathscr{D}(\mathbb{P}^1)/\operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X})) \leqslant m$. Therefore, (8.7.2) forces $\operatorname{gr} \mathscr{D}(\mathbb{P}^1)/\operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X}) \cdot \operatorname{gr} \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$ to be finite-dimensional. Finally, as

$$\operatorname{gr} \mathscr{D}(\mathbb{P}^1, \mathscr{X}) \cdot \operatorname{gr} \mathscr{D}(\mathscr{X}, \mathbb{P}^1) \subseteq \operatorname{gr}(\mathscr{D}(\mathbb{P}^1, \mathscr{X})\, \mathscr{D}(\mathscr{X}, \mathbb{P}^1)) \subseteq \operatorname{gr} \mathscr{D}(\mathscr{X}) \subseteq \operatorname{gr} \mathscr{D}(\mathbb{P}^1),$$

this in turn implies that $\operatorname{gr} \mathscr{D}(\mathbb{P}^1)/\operatorname{gr} \mathscr{D}(\mathscr{X})$ is finite-dimensional.

8.8. COROLLARY. *Let $\mathscr{X}$ be a singular projective curve with injective normalisation $\pi: \mathbb{P}^1 \to \mathscr{X}$ and write $\mathscr{D}(\mathbb{P}^1) = k[\partial, t\partial, t^2\partial]$. Then for some integer $n \geqslant 1$, $\mathscr{D}(\mathscr{X})$ contains an element of the form $\theta = \partial^n + \phi$ where $\phi \in \mathscr{D}^{n-1}(K)$.*

Proof. Write $\operatorname{gr} \mathscr{D}(\mathbb{P}^1) \cong k[a, b, c]/(ac - b^2)$, where a is the image of ∂. Then Theorem 8.7 implies that $\operatorname{gr} \mathscr{D}(\mathscr{X})$ contains some polynomial $f(a) \in k[a]$ of degree, say n. Now take θ to be any preimage in $\mathscr{D}(\mathscr{X})$ of $f(a)$.

Remark. We know of no direct way of computing the element θ given by the above corollary.

9. Twisted Differential Operators

9.1. In this section we investigate differential operators on rational, projective curves with coefficients in a line bundle. Some of the arguments are only slightly more general than those used earlier, and so will be omitted. If $\mathscr{L}$ is an invertible sheaf, over a rational, projective curve $\mathscr{X}$, then we define $\mathscr{D}_{\mathscr{L}}$, the sheaf of differential operators with coefficients in $\mathscr{L}$, by $\mathscr{D}_{\mathscr{L}} = \mathscr{L} \otimes_{\mathcal{O}} \mathscr{D}_{\mathscr{X}} \otimes_{\mathcal{O}} \mathscr{L}^{-1}$. Here $\mathscr{L}^{-1} = \mathscr{H}om_{\mathcal{O}}(\mathscr{L}, \mathcal{O})$, is the dual sheaf of $\mathscr{L}$. If U is an open affine subset then $\mathscr{D}_{\mathscr{L}}(U) = \mathscr{L}(U)\,\mathscr{D}(U)\,\mathscr{L}^{-1}(U)$ and so $\mathscr{D}_{\mathscr{L}}$ is clearly a sheaf of rings. Moreover, the obvious containment, $\mathcal{O}_{\mathscr{X}} \subset \mathscr{D}_{\mathscr{L}}$, makes $\mathscr{D}_{\mathscr{L}}$ a quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module and we denote by $\mathscr{D}_{\mathscr{L}}$-**mod** the category of sheaves of $\mathscr{D}_{\mathscr{L}}$-modules quasi-coherent over $\mathscr{X}$. Write $\mathscr{D}_{\mathscr{L}}(\mathscr{X}) = \Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{L}})$, for the ring of globally defined operators. Notice that, since $\mathscr{L}$ is naturally a $\mathscr{D}_{\mathscr{L}}$-module, $\Gamma(\mathscr{X}, \mathscr{L})$ is a left $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-module. Moreover, the injection $\mathscr{L} \hookrightarrow K(\mathscr{X})$, shows that $H^1(\mathscr{X}, \mathscr{L})$ is a left $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-module.

In this section it is shown that $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Noetherian. When the normalisation map, $\pi: \mathbb{P}^1 \to \mathscr{X}$, is injective the list of all Morita equivalence classes in which $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ occurs is given and conditions on $\mathscr{L}$ are found under which $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$. These conditions also imply a Beilinson–Bernstein-type equivalence of categories and show that $\Gamma(\mathscr{X}, \mathscr{L})$ is a simple $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-module.

9.2. Example. The invertible sheaves over $\mathbb{P}^1$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n)$, for $n \in \mathbb{Z}$ (see [Ha, II.6.17], for example). If W_0, W_1 is the usual cover of $\mathbb{P}^1$, with $\mathcal{O}(W_0) = k[t]$, $\mathcal{O}(W_1) = k[t^{-1}]$, then

$$\Gamma(W_0, \mathcal{O}(n)) = k[t], \qquad \Gamma(W_1, \mathcal{O}(n)) = t^n k[t^{-1}].$$

Thus, writing $\mathscr{D}_n = \mathscr{D}_{\mathcal{O}(n)}$, we obtain

$$\mathscr{D}_n(W_0) = \mathscr{D}(W_0) = k[t, \partial]$$

and

$$\mathscr{D}_n(W_1) = t^n \mathscr{D}(W_1) t^{-n} = k[t^{-1}, t(t\partial - n)].$$

An easy calculation shows that $\mathscr{D}_n(\mathbb{P}^1) = k[\partial, t\partial, t(t\partial - n)]$ and hence that $\mathscr{D}_n(\mathbb{P}^1)$ is isomorphic to a primitive factor ring of $U(\mathfrak{sl}_2(k))$, namely $U(\mathfrak{sl}_2)/(\Omega - n^2 - 2n)$, where Ω is the Casimir element. We remark that all these rings are Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$ except for $\mathscr{D}_{-1}(\mathbb{P}^1)$, which is the simple ring of infinite global dimension in Remark 1.5 (see [St, Proposition 3.5]). It is well known that $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$ and $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is a simple $\mathscr{D}_n(\mathbb{P}^1)$-module, for $n \geqslant 0$, whereas $H^0(\mathbb{P}^1, \mathcal{O}(n)) = 0$ and

$H^1(\mathbb{P}^1, \mathcal{O}(n))$ is simple, for $n \leqslant -2$. Of course $H^0(\mathbb{P}^1, \mathcal{O}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. See [Di] and [Ha, Theorem III.5.1].

9.3. For the remainder of this section fix a projective curve $\mathscr{X}$, with normalisation $\pi: \mathbb{P}^1 \to \mathscr{X}$. Let $\mathscr{L}$ be an invertible sheaf over $\mathscr{X}$. The inverse image sheaf $\pi^*\mathscr{L} = \mathcal{O}_{\mathbb{P}^1} \otimes \mathscr{L}$ is isomorphic to $\mathcal{O}(n)$, for some $n \in \mathbb{Z}$, and so, by replacing $\mathscr{L}$ with an isomorphic subsheaf of $K(\mathscr{X})$, we may assume that $\pi^*\mathscr{L} = \mathcal{O}(n)$. Our first aim is to prove that $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Noetherian. Just as in the non-twisted case, our analysis of $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is carried out through certain bimodules, admitting a comparison between the structure of $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ and that of $\mathscr{D}_n(\mathbb{P}^1)$. The direct image functor, π_*, makes $\pi_*\mathscr{D}_n$ into a sheaf of rings quasi-coherent over $\mathcal{O}_{\mathscr{X}}$. Now define $\mathscr{D}_{\mathscr{L}}(\pi_*\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathscr{X}})$, to be the sheaf of $\mathscr{D}_{\mathscr{L}}$-$\pi_*\mathscr{D}_n$-bimodules with sections $\mathscr{D}_{\mathscr{L}}(\tilde{U}, U) = \mathscr{L}(U)\, \mathscr{D}(\tilde{U}, U)\, \mathscr{L}^{-1}(U)$, on an open affine subset U of $\mathscr{X}$. Here $\tilde{U} = \pi^{-1}(U)$, as usual. Finally define

$$\mathscr{D}_{\mathscr{L}}(\mathbb{P}^1, \mathscr{X}) = \Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{L}}(\pi_*\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathscr{X}})).$$

9.4. LEMMA. *$\mathscr{D}_{\mathscr{L}}(\mathbb{P}^1, \mathscr{X})$ is a non-zero left ideal of $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$, and a right ideal of $\mathscr{D}_n(\mathbb{P}^1)$.*

Proof. Let U be an open affine subset of $\mathscr{X}$ with the property that its pre-image $\tilde{U}$ is either contained in W_0 or W_1 (in the notation of Example 9.2). By the argument at the beginning of the proof of Proposition 2.3, it is enough to show that $\mathscr{D}_{\mathscr{L}}(\tilde{U}, U) \cap \mathscr{D}_n(\mathbb{P}^1) \neq 0$. Let t be the coordinate function on W_0. Then $\tilde{U} \subset W_0$ (respectively, $\tilde{U} \subset W_1$) and so $\mathcal{O}(\tilde{U}) = k[t]_f$, for some $0 \neq f \in k[t]$ (respectively, $\mathcal{O}(\tilde{U}) = k[t^{-1}]_g$). Now there exists an element $0 \neq h \in \mathcal{O}(\tilde{U}) \cap \mathscr{D}(\tilde{U}, U)$, and hence $h \in \mathscr{D}_{\mathscr{L}}(\tilde{U}, U)$. Moreover, we may suppose that $h \in k[t]$ (respectively, $h \in k[t^{-1}]$). Finally, the identities

$$t^p \prod_{i=0}^{p-1} (t\partial + i - n) = (t(t\partial - n))^p$$

$$\left(\text{respectively } t^{-p} \prod_{i=0}^{p-1} (t\partial + i) = \partial^p\right)$$

make it clear that $\mathscr{D}_{\mathscr{L}}(\tilde{U}, U) \cap \mathscr{D}_n(\mathbb{P}^1) \neq 0$, as required.

9.5. We can now prove the analogue of Theorem A for twisted differential operators.

THEOREM. *Let $\mathscr{X}$ be a rational, projective curve and $\mathscr{L}$ an invertible sheaf over $\mathscr{X}$. Then*

(a) *$\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is a Noetherian domain of left and right Krull dimension one.*

(b) *$\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is a finitely generated k-algebra.*

(c) *If M is a $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-module of finite length, then* $\operatorname{End}_{\mathscr{D}_{\mathscr{L}}(\mathscr{X})} M$ *is a finite-dimensional k-vector space.*

(d) *$\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ has a unique, minimal non-zero ideal $J_{\mathscr{L}}(\mathscr{X})$. Moreover $F_{\mathscr{L}}(\mathscr{X}) = \mathscr{D}_{\mathscr{L}}(\mathscr{X})/J_{\mathscr{L}}(\mathscr{X})$ is finite-dimensional over k.*

Proof. By Lemma 9.4, $\mathscr{D}_{\mathscr{L}}(\mathbb{P}^1, \mathscr{X})$ is a non-zero right ideal of $\mathscr{D}_n(\mathbb{P}^1)$ for which

$$\mathscr{D}_{\mathscr{L}}(\mathbb{P}^1, \mathscr{X}) \subseteq \mathscr{D}_{\mathscr{L}}(\mathscr{X}) \subseteq \operatorname{End}_{\mathscr{D}_n(\mathbb{P}^1)} \mathscr{D}_{\mathscr{L}}(\mathbb{P}^1, \mathscr{X}).$$

Now apply Theorem 1.7.

9.6. For the remainder of this section we assume that the normalisation map $\pi: \mathbb{P}^1 \to \mathscr{X}$ is injective, and that $\mathscr{L}$ is an invertible sheaf on $\mathscr{X}$. For the results below we shall need to compare $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ with $\mathscr{D}(\mathbb{P}^1)$ and so, first, we must define appropriate bimodules. Denote by $\mathscr{D}(\pi_* \mathscr{O}_{\mathbb{P}^1}, \mathscr{L})$ the sheaf of $\mathscr{D}_{\mathscr{L}}$-$\pi_* \mathscr{D}_{\mathbb{P}^1}$-bimodules with sections $\mathscr{L}(U)\, \mathscr{D}(\tilde{U}, U)$ on an open affine subset U of $\mathscr{X}$. Similarly, $\mathscr{D}(\mathscr{L}, \pi_* \mathscr{O}_{\mathbb{P}^1})$ is a sheaf of $\pi_* \mathscr{D}_{\mathbb{P}^1}$-$\mathscr{D}_{\mathscr{L}}$-bimodules and has sections $\mathscr{D}(U, \tilde{U})\, \mathscr{L}^{-1}(U)$ over U. We write

$$\mathscr{D}(\mathbb{P}^1, \mathscr{L}) = \Gamma(\mathscr{X}, \mathscr{D}(\pi_* \mathscr{O}_{\mathbb{P}^1}, \mathscr{L}))$$

and

$$\mathscr{D}(\mathscr{L}, \mathbb{P}^1) = \Gamma(\mathscr{X}, \mathscr{D}(\mathscr{L}, \pi_* \mathscr{O}_{\mathbb{P}^1})).$$

Notice that $\mathscr{D}(\mathbb{P}^1, \mathscr{L})$ is a $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-$\mathscr{D}(\mathbb{P}^1)$-bimodule and $\mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ is a $\mathscr{D}(\mathbb{P}^1)$-$\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-bimodule. In general neither of these modules will be contained in $\mathscr{D}(\mathbb{P}^1)$, which is why they were not used to prove Theorem 9.5.

9.7. LEMMA. (a) *$\mathscr{D}(\mathbb{P}^1, \mathscr{L})$ is a non-zero, finitely generated, fractional right $\mathscr{D}(\mathbb{P}^1)$-ideal and a finitely generated, fractional left $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-ideal.*

(b) *$\mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ is a non-zero, finitely generated, fractional right $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-ideal and a finitely generated, fractional left $\mathscr{D}(\mathbb{P}^1)$-ideal.*

(c) $\mathscr{D}(\mathscr{L}, \mathbb{P}^1) \subseteq [\mathscr{D}(\mathbb{P}^1, \mathscr{L})_{\mathscr{D}(\mathbb{P}^1)}]^*$.

(d) $\mathscr{D}(\mathbb{P}^1, \mathscr{L}) = [{}_{\mathscr{D}(\mathbb{P}^1)}\mathscr{D}(\mathscr{L}, \mathbb{P}^1)]^*$ *and so is a projective right $\mathscr{D}(\mathbb{P}^1)$-module.*

(e) $\operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} \mathscr{D}(\mathscr{L}, \mathbb{P}^1) = \mathscr{D}_{\mathscr{L}}(\mathscr{X})$.

Proof. Write $M = \mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ and $N = \mathscr{D}(\mathbb{P}^1, \mathscr{L})$. If $\mathscr{M}$ is the unique quasi-coherent $\mathscr{D}_{\mathbb{P}^1}$-module with $\pi_* \mathscr{M} = \mathscr{D}(\mathscr{L}, \pi_* \mathscr{O}_{\mathbb{P}^1})$, then

$$\mathscr{D}(\mathscr{L}, \mathbb{P}^1) = \Gamma(\mathscr{X}, \mathscr{D}(\mathscr{L}, \pi_* \mathscr{O}_{\mathbb{P}^1})) = \Gamma(\mathbb{P}^1, \mathscr{M}) \neq 0,$$

where the non-equality follows from Theorem 3.2. Parts (a), (b), and (c) of the lemma and the fact that $\mathscr{D}(\mathbb{P}^1, \mathscr{L}) \subseteq \mathscr{D}(\mathscr{L}, \mathbb{P}^1)^*$ follow routinely from this observation combined with Theorem 9.5 (see, for example, the proof of Corollary 2.5).

By Theorem 3.2, $\mathscr{M}$ is generated by its global sections, $\mathscr{D}(\mathscr{L}, \mathbb{P}^1)$. Thus, for any open affine subset U of $\mathscr{X}$ with $\tilde{U} = \pi^{-1}(U)$, one has $\mathscr{D}(\tilde{U})\, \mathscr{D}(\mathscr{L}, \mathbb{P}^1) = \mathscr{M}(\tilde{U}) = \mathscr{D}(U, \tilde{U})\, \mathscr{L}^{-1}(U)$. Thus, by Proposition 3.7, $\operatorname{End}_{\mathscr{D}(\tilde{U})} \mathscr{M}(\tilde{U}) = \mathscr{L}(U)\, \mathscr{D}(U)\, \mathscr{L}^{-1}(U) = \mathscr{D}_{\mathscr{L}}(U)$. The argument of Corollary 3.8 can now be used to prove the rest of the lemma.

9.8. We can now determine the possibilities for the Morita equivalence class of $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$.

THEOREM. *Let $\mathscr{X}$ be a curve with injective normalisation $\mathbb{P}^1 \to \mathscr{X}$ and let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$. Then $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Morita equivalent to one of the following three rings: $\mathscr{D}(\mathscr{X})$, $\mathscr{D}(\mathbb{P}^1)$, or $\mathscr{D}_{-1}(\mathbb{P}^1) \cong U(\mathfrak{sl}_2)/(\Omega + 1)$.*

Remark. The first of these possibilities occurs, of course, if $\mathscr{L} = \mathscr{O}_{\mathscr{X}}$. Later we will show that the other two possibilities also arise.

Proof. Put $M = \mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ and $N = \mathscr{D}(\mathbb{P}^1, \mathscr{L})$. If W is a $\mathscr{D}(\mathbb{P}^1)$-module then W^* will denote the dual, $\operatorname{Hom}_{\mathscr{D}(\mathbb{P}^1)}(W, \mathscr{D}(\mathbb{P}^1))$. Thus, by the last result, $N = M^*$ is a projective $\mathscr{D}(\mathbb{P}^1)$-module and $\operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} M = \mathscr{D}_{\mathscr{L}}(\mathscr{X})$. We shall consider four cases.

Case A. $N_{\mathscr{D}(\mathbb{P}^1)}$ is not a generator.

As N is not a generator, $\operatorname{Trace}(N) = N^*N \neq \mathscr{D}(\mathbb{P}^1)$. Thus, $\operatorname{Trace}(N) = \mathbf{m}$ which, since N is projective, implies that $N\mathbf{m} = N$. But, by 1.4(d), $\mathbf{m}N^* = \mathbf{m}M^{**} \subseteq M \subseteq N^* = \mathbf{m}N^*$. Thus $M = N^*$ and $M = \mathbf{m}M$. It follows that $\mathscr{D}_{\mathscr{L}}(\mathscr{X}) \cong \operatorname{End}_{\mathscr{D}(\mathbb{P}^1)} M$ is simple and hence is Morita equivalent to $\mathscr{D}_{-1}(\mathbb{P}^1)$.

Case B. $N_{\mathscr{D}(\mathbb{P}^1)}$ is a generator and $M = N^*$.

In this case N and M are progenerators and so End M is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$.

Case C. $N_{\mathscr{D}(\mathbb{P}^1)}$ is a generator and ${}_{\mathscr{D}(\mathbb{P}^1)}M$ is not a generator.

In this case $M = \mathbf{m}N^*$. As End N^* is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$, it is a maximal order, and so End M = End $\mathbf{m}N^*$ = End N^* is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$.

Case D. ${}_{\mathscr{D}(\mathbb{P}^1)}M$ is a generator and $M \neq N^*$.

The proof of Theorem 4.3 shows that End M shares its unique minimal ideal, J, with End N and hence is a pull-back

$$\begin{array}{ccccc} \begin{pmatrix} M_p(k) & M_{p,q}(k) \\ 0 & M_q(k) \end{pmatrix} = & H & \hookrightarrow & \text{End } N/J = M_{p+q}(k) \\ & \uparrow & & \uparrow \\ & \text{End } M & \longrightarrow & \text{End } N \end{array}$$

for some positive integers p and q. Now Corollary 6.12 implies that End M is Morita equivalent to $\mathscr{D}(\mathscr{X})$.

A routine verification shows that Cases A–D exhaust all possibilities for M and N and this completes the proof of the theorem.

9.9. By analogy with Example 9.2, one expects that the possibilities for the Morita class of $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ are characterised by (non-)vanishing of the cohomology of $\mathscr{L}$, and this suggests the following conjecture.

Conjecture. (a) $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathscr{X})$ if and only if $H^0(\mathscr{X}, \mathscr{L})$ and $H^1(\mathscr{X}, \mathscr{L})$ are non-zero.

(b) $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$ if and only if exactly one of $H^0(\mathscr{X}, \mathscr{L})$ and $H^1(\mathscr{X}, \mathscr{L})$ is non-zero

(c) $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}_{-1}(\mathbb{P}^1)$ if and only if $H^0(\mathscr{X}, \mathscr{L}) = H^1(\mathscr{X}, \mathscr{L}) = 0$.

(d) Whenever $H^i(\mathscr{X}, \mathscr{L})$ is non-zero, it is a simple $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-module.

9.10. Next it is shown that "asymptotically" $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-**mod** is equivalent to $\mathscr{D}(\mathbb{P}^1)$-**mod** and hence $\mathscr{D}_{\mathscr{L}}$-**mod**. Firstly define

$$\mathscr{D}(\mathscr{L}, \mathscr{X}) = \Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{X}} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathscr{L}^{-1})$$

and

$$\mathscr{D}(\mathscr{X}, \mathscr{L}) = \Gamma(\mathscr{X}, \mathscr{L} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathscr{D}_{\mathscr{X}}).$$

LEMMA. (a) *$\mathscr{D}(\mathscr{L}, \mathscr{X})$ is a non-zero, finitely generated, fractional right $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-ideal and a finitely generated, fractional left $\mathscr{D}(\mathscr{X})$-ideal.*

(b) *$\mathscr{D}(\mathscr{X}, \mathscr{L})$ is a non-zero, finitely generated, fractional right $\mathscr{D}(\mathscr{X})$-ideal and a finitely generated, fractional left $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-ideal.*

(c) *$\mathscr{D}(\mathscr{X}, \mathscr{L}) = [{}_{\mathscr{D}(\mathscr{X})}\mathscr{D}(\mathscr{L}, \mathscr{X})]^*$ and hence is a projective right $\mathscr{D}(\mathscr{X})$-module.*

(d) $\text{End}_{\mathscr{D}(\mathscr{X})} \mathscr{D}(\mathscr{L}, \mathscr{X}) = \mathscr{D}_{\mathscr{L}}(\mathscr{X})$.

Proof. By Theorem 3.15(a), $\mathscr{D}(\mathscr{L}, \mathscr{X})$ generates $\mathscr{D}_{\mathscr{X}} \otimes \mathscr{L}^{-1}$. Now use the proof of Lemma 9.7.

9.11. LEMMA. *If $\mathscr{L}$ is generated by global sections then*

(a) $\mathscr{D}(\mathscr{L}, \mathscr{X}) = [\mathscr{D}(\mathscr{X}, \mathscr{L})_{\mathscr{D}(\mathscr{X})}]^*$ *and hence is a projective left $\mathscr{D}(\mathscr{X})$-module.*

(b) $\operatorname{End}(\mathscr{D}(\mathscr{X}, \mathscr{L})_{\mathscr{D}(\mathscr{X})}) = \mathscr{D}_{\mathscr{L}}(\mathscr{X})$.

Proof. Since $\mathscr{L}$ is generated by global sections, then so is $\mathscr{L} \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{D}$. Thus, once again, the proof is analogous to that of Lemma 9.7.

9.12. PROPOSITION. *Let $I(\mathscr{X})$ be the ideal defined in Corollary 4.4. If $\mathscr{L}$ is generated by global sections and $H^1(\mathscr{X}, \mathscr{L}) = 0$ then $\mathscr{D}(\mathscr{L}, \mathscr{X})\, \mathscr{D}(\mathscr{X}, \mathscr{L}) = I(\mathscr{X})$.*

Proof. Let U be an open affine subset of $\mathscr{X}$. Then

$$\begin{aligned}\mathscr{D}(\mathscr{X}, \mathscr{L}) \otimes_{\mathscr{D}(\mathscr{X})} \mathscr{O}(U) &\cong \mathscr{D}(\mathscr{X}, \mathscr{L}) \otimes_{\mathscr{D}(\mathscr{X})} \mathscr{D}(U) \otimes_{\mathscr{D}(U)} \mathscr{O}(U)\\ &\cong \mathscr{L}(U) \otimes_{\mathscr{O}(U)} \mathscr{D}(U) \otimes_{\mathscr{D}(U)} \mathscr{O}(U)\\ &\qquad \text{since } \mathscr{L} \text{ is generated by its global sections}\\ &\cong \mathscr{L}(U).\end{aligned}$$

Since $\mathscr{D}(\mathscr{X}, \mathscr{L})_{\mathscr{D}(\mathscr{X})}$ is projective, the Čech cohomology argument used in the proof of Corollary 7.12 shows that

$$\mathscr{D}(\mathscr{X}, \mathscr{L}) \otimes_{\mathscr{D}(\mathscr{X})} H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) \cong H^1(\mathscr{X}, \mathscr{L}) = 0.$$

Thus, by Lemmas 9.10(c) and 9.11(a),

$$\operatorname{Trace}(\mathscr{D}(\mathscr{X}, \mathscr{L})) = \mathscr{D}(\mathscr{L}, \mathscr{X})\, \mathscr{D}(\mathscr{X}, \mathscr{L}) \subseteq \text{l-ann}\, H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) = I(\mathscr{X}),$$

where the final equality is Corollary 4.4. A similar argument implies that

$$\mathscr{D}(\mathscr{X}, \mathscr{L}) \otimes_{\mathscr{D}(\mathscr{X})} H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) \cong H^0(\mathscr{X}, \mathscr{L}) \neq 0$$

and so $\operatorname{Trace}(\mathscr{D}(\mathscr{X}, \mathscr{L})) \nsubseteq L(\mathscr{X})$. Thus Corollary 4.4 forces $\operatorname{Trace}(\mathscr{D}(\mathscr{X}, \mathscr{L})) = I(\mathscr{X})$, as required.

9.13. THEOREM. *Let $\mathscr{X}$ be a curve with injective normalisation $\mathbb{P}^1 \to \mathscr{X}$ and suppose that $\mathscr{L}$ is an invertible sheaf over $\mathscr{X}$, generated by global sections, and with $H^1(\mathscr{X}, \mathscr{L}) = 0$. Then $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$.*

Proof. Put $L = \mathscr{D}(\mathscr{L}, \mathscr{X})$, $L^* = \mathscr{D}(\mathscr{X}, \mathscr{L})$, $P = \mathscr{D}(\mathscr{X}, \mathbb{P}^1)$, and $Q = \mathscr{D}(\mathbb{P}^1, \mathscr{X})$. Now consider the fractional right $\mathscr{D}(\mathbb{P}^1)$-ideal $L^*Q \cong$

$L^* \otimes_{\mathscr{D}(\mathscr{X})} Q$. Since $L^*_{\mathscr{D}(\mathscr{X})}$ is projective, by Proposition 9.11(c), and $Q_{\mathscr{D}(\mathbb{P}^1)}$ is projective, by Corollary 3.8, then L^*Q is a projective right $\mathscr{D}(\mathbb{P}^1)$-module. Moreover

$$\begin{aligned}(Q^*L)(L^*Q) &= Q^*I(\mathscr{X})Q && \text{by Lemma 9.11 and Proposition 9.12}\\ &= Q^*Q && \text{since } PQ = \mathscr{D}(\mathbb{P}^1)\text{, by Proposition 3.14}\\ & && \text{and } I(\mathscr{X}) = QP\text{, by Corollary 4.4}\\ &= \mathscr{D}(\mathbb{P}^1) && \text{by Proposition 3.14.}\end{aligned}$$

Thus $L^*Q_{\mathscr{D}(\mathbb{P}^1)}$ is a progenerator. Since $L^*QP = L^*$, by Proposition 9.12,

$$\mathscr{D}_{\mathscr{L}}(\mathscr{X}) \subseteq \operatorname{End}(L^*Q_{\mathscr{D}(\mathbb{P}^1)}) \subseteq \operatorname{End}(L^*_{\mathscr{D}(\mathscr{X})}) = \mathscr{D}_{\mathscr{L}}(\mathscr{X}),$$

where the last equality is Proposition 9.11(b). Thus $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$.

9.14. As a consequence of Theorem 9.13 we can deduce an equivalence of categories à la Beilinson–Bernstein.

COROLLARY. *Let $\mathscr{X}$ and $\mathscr{L}$ be as in the theorem. Then the mutually inverse functors $\Gamma(\mathscr{X}, \_)$ and $\mathscr{D}_{\mathscr{L}} \otimes_{\mathscr{D}_{\mathscr{L}}(\mathscr{X})} \_$ make $\mathscr{D}_{\mathscr{L}}$-**mod** and $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-**mod** equivalent categories.*

Proof. Retain the notation of the previous proof. We must show that the equivalence of categories of Theorem 3.14 is through the "correct" functors. Firstly, observe that it is easy to modify the argument in [SS, Sect. 6.1] to show that the mutually inverse functors $\mathscr{D}(\pi_* \mathscr{O}_{\mathbb{P}^1}, \mathscr{L}) \otimes_{\pi_* \mathscr{D}_{\mathbb{P}^1}} \_$ and $\mathscr{D}(\mathscr{L}, \pi_* \mathscr{O}_{\mathbb{P}^1}) \otimes_{\mathscr{D}_{\mathscr{L}}} \_$ give an equivalence of categories between $\pi_* \mathscr{D}_{\mathbb{P}^1}$-**mod** and $\mathscr{D}_{\mathscr{L}}$-**mod**. Consider then, the chain of equivalences of categories:

$$\mathscr{D}(\mathbb{P}^1)\text{-}\mathbf{mod} \xrightarrow{\mathscr{D}_{\mathbb{P}^1} \otimes \_} \mathscr{D}_{\mathbb{P}^1}\text{-}\mathbf{mod} \xrightarrow{\pi_*} \pi_* \mathscr{D}_{\mathbb{P}^1}\text{-}\mathbf{mod} \xrightarrow{\mathscr{D}(\pi_* \mathscr{O}_{\mathbb{P}^1}, \mathscr{L}) \otimes \_} \mathscr{D}_{\mathscr{L}}\text{-}\mathbf{mod}.$$

A routine verification shows that $\mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ in $\mathscr{D}(\mathbb{P}^1)$-**mod** maps to $\mathscr{D}_{\mathscr{L}}$ in $\mathscr{D}_{\mathscr{L}}$-**mod**. Thus we must show that $\mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ is a progenerator to prove the corollary. Now, in the notation of Theorem 9.13, we have

$$\begin{aligned}\mathscr{D}(\mathscr{L}, \mathbb{P}^1)\, \mathscr{D}(\mathscr{L}, \mathbb{P}^1)^* &= \mathscr{D}(\mathscr{L}, \mathbb{P}^1)\, \mathscr{D}(\mathbb{P}^1, \mathscr{L}) && \text{by Lemma 9.7}\\ &\supseteq (PL)(L^*Q) = PI(\mathscr{X})Q && \text{by Proposition 9.12}\\ &= PQPQ && \text{by Corollary 4.4}\\ &= \mathscr{D}(\mathbb{P}^1) && \text{by Proposition 3.14(a).}\end{aligned}$$

Thus ${}_{\mathscr{D}(\mathbb{P}^1)}\mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ is a generator. But $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ and $\mathscr{D}(\mathbb{P}^1)$ are Morita equivalent. Thus, by Lemma 9.7(e) and 1.3, $\mathscr{D}(\mathscr{L}, \mathbb{P}^1)$ is a progenerator; as required.

9.15. By Theorem 9.13, $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ has a unique finite-dimensional, simple module (up to isomorphism). The last result allows us to identify it.

COROLLARY. *Let $\mathscr{X}$ and $\mathscr{L}$ be as in the theorem. Then $\Gamma(\mathscr{X}, \mathscr{L})$ is the finite-dimensional, simple $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-module.*

Proof. The chain of equivalences at the beginning of the proof of Corollary 9.14 maps $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ to $\mathscr{L}$. Thus $\mathscr{L}$ is a simple object of $\mathscr{D}_{\mathscr{L}}$-**mod**. The equivalence of categories given by the statement of Corollary 9.14 completes the proof.

9.16. We conclude this section with some examples to illustrate the results above. Let $\mathscr{X} = \mathscr{X}_n$ be the rational projective curve obtained by glueing

$$\mathcal{O}(U) = k + t^{n+1}k[t] \qquad \text{and} \qquad \mathcal{O}(V) = k[t^{-1}]$$

along $\mathcal{O}(U \cap V) = k[t, t^{-1}]$. In particular, $\mathscr{X}_1$ is the plane, cuspidal cubic curve of Section 5. For each integer m, define an invertible sheaf $\mathcal{O}(m/2n+1)$, on $\mathscr{X}_n$, by

$$\Gamma(U, \mathcal{O}(m/2n+1)) = \mathcal{O}(U)$$

and

$$\Gamma(V, \mathcal{O}(m/2n+1)) = t^m\mathcal{O}(V) = t^m k[t^{-1}].$$

The reason for this notation is that $\mathcal{O}(1)$ is a very ample sheaf (embedding $\mathscr{X}_n$ in $\mathbb{P}^{n+1}$).

9.17. PROPOSITION. *Let $\mathscr{L} = \mathcal{O}(m/2n+1)$, for some integer m. If $0 \leqslant m \leqslant n-1$ then $\mathscr{D}_{\mathscr{L}}(\mathscr{X}_n)$ is Morita equivalent to $\mathscr{D}(\mathscr{X}_n)$. Otherwise $\mathscr{D}_{\mathscr{L}}$ is Morita equivalent to $\mathscr{D}(\mathbb{P}^1)$. Moreover, Conjecture* 9.9(d) *holds for $\mathscr{L}$.*

Proof. We only consider the case $m \geqslant 0$. We leave $m < 0$ to the reader. It is easy to check that 1, $t^{m+2}\partial \in \mathscr{D}(\mathscr{X}, \mathscr{L})$ and that, moreover, these sections generate $\mathscr{L} \otimes \mathscr{D}_{\mathscr{X}}$. It follows, as in Lemma 9.11, that $\mathscr{D}(\mathscr{L}, \mathscr{X}) = \mathscr{D}(\mathscr{X}, \mathscr{L})^*$ is a projective $\mathscr{D}(\mathscr{X})$-module. Hence, similarly to Corollary 7.12,

$$\mathscr{D}(\mathscr{X}, \mathscr{L}) \otimes_{\mathscr{D}(\mathscr{X})} H^i(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \cong H^i(\mathscr{X}, \mathscr{L}) \qquad \text{for} \quad i = 0, 1. \quad (9.17.1)$$

Now, in the notation of 4.4, the possibilities for the trace ideal of $\mathscr{D}(\mathscr{X}, \mathscr{L})$ are $J(\mathscr{X})$, $I(\mathscr{X})$, $L(\mathscr{X})$, or $\mathscr{D}(\mathscr{X})$. Since $I(\mathscr{X})$ annihilates $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$, $L(\mathscr{X})$ annihilates $H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$, and $J(\mathscr{X})$ annihilates both, we can see which possibility occurs from an explicit Čech cohomology computation:

$$H^0(\mathscr{X}, \mathscr{L}) \neq 0 \qquad m \geqslant 0.$$

$$H^1(\mathscr{X}, \mathscr{L}) \begin{cases} = 0 & m \geqslant n. \\ \neq 0 & n > m \geqslant 0. \end{cases}$$

It follows that

$$\mathrm{Trace}(\mathscr{D}(\mathscr{X}, \mathscr{L})) \begin{cases} = \mathscr{D}(\mathscr{X}) & n > m \geqslant 0. \\ = I(\mathscr{X}) & m \geqslant n. \end{cases}$$

If $\mathrm{Trace}(\mathscr{D}(\mathscr{X}, \mathscr{L})) = \mathscr{D}(\mathscr{X})$, then, by 9.17.1, $H^0(\mathscr{X}, \mathscr{L})$ and $H^1(\mathscr{X}, \mathscr{L})$ are simple $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$-modules. In the other case, the argument of Theorem 9.13 and Corollary 9.15 completes the proof.

9.18. We have already seen that $\mathscr{D}_{\mathscr{L}}(\mathscr{X})$ can be Morita equivalent to $\mathscr{D}(\mathscr{X})$ and $\mathscr{D}(\mathbb{P}^1)$. We now show that the third possibility $\mathscr{D}_{-1}(\mathbb{P}^1)$ can occur.

Let $\mathscr{X}_1$ be the plane, cuspidal, cubic curve of Section 5 and fix $0 \neq \alpha \in k$. Let $\{U, V\}$ be the open affine cover of $\mathscr{X}_1$ defined by

$$\mathscr{O}(U) = k[t^2, t^3, (t^2 - \alpha^2)^{-1}] \qquad \text{and} \qquad \mathscr{O}(V) = k[t^{-1}].$$

We consider the degree zero, invertible sheaf $\mathscr{L}$ with sections

$$\Gamma(U, \mathscr{L}) = (t - \alpha)\, \mathscr{O}(U) \qquad \text{and} \qquad \Gamma(V, \mathscr{L}) = \mathscr{O}(V) = k[t^{-1}].$$

9.19. Lemma. $\mathscr{D}(\mathbb{P}^1, \mathscr{L}) = t^2\partial\mathscr{D}(\mathbb{P}^1) + (\alpha\partial - t\partial - \alpha t\partial^2)\, \mathscr{D}(\mathbb{P}^1)$.

Proof. Write $A = k[t, \partial]$. Then

$$\begin{aligned}\mathscr{D}(\mathbb{P}^1, \mathscr{L}) &= (t - \alpha)(t^2 A_{(t^2 - \alpha^2)} + (t\partial - 1)\, A_{(t^2 - \alpha^2)}) \cap \mathscr{D}(\mathbb{P}^1) \\ &= (t^2 A + (t - \alpha)(t\partial - 1) A) \cap \mathscr{D}(\mathbb{P}^1).\end{aligned}$$

It is an easy exercise to show that $W = t^2\partial\mathscr{D}(\mathbb{P}^1) + (\alpha\partial - t\partial - \alpha t\partial^2)\, \mathscr{D}(\mathbb{P}^1) \subseteq \mathscr{D}(\mathbb{P}^1, \mathscr{L})$. Suppose, for a contradiction, that there exists $w \in \mathscr{D}(\mathbb{P}^1, \mathscr{L}) \backslash W$. Using the explicit generators of W, an easy degree argument allows one to assume that $w \in k(t - \alpha)\partial + k[\partial]$. It is easy to see that this leads to a contradiction.

9.20. PROPOSITION. *Let $\mathscr{L}$ be as in* 9.18. *Then* $\mathscr{D}_{\mathscr{L}}(\mathscr{X}_1)$ *is Morita equivalent to* $\mathscr{D}_{-1}(\mathbb{P}^1)$.

Proof. We must show that Case A of Theorem 9.8 applies and, to show this, it clearly suffices to prove that $\mathscr{D}(\mathbb{P}^1, \mathscr{L})\mathbf{m} = \mathscr{D}(\mathbb{P}^1, \mathscr{L})$. This is a straightforward exercise, using Lemma 9.19, and is left to the reader.

REFERENCES

[B] H. BASS, Finitistic dimension and a homological generalisation of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960), 466–488.

[BB] A. BEILINSON AND J. BERNSTEIN, Localisation de $\mathfrak{g}$-modules, *C. R. Acad. Sci. Paris* **292** (1981), 15–18.

[Bo] A. BOREL, "Algebraic *D*-Modules," Perspectives in Math., Vol. 2, Academic Press, Boston, 1987.

[BoB] W. BORHO AND J.-L. BRYLINSKI, Differential operators on homogeneous spaces, I, *Invent. Math.* **69** (1982), 437–476.

[Di] J. DIXMIER, "Enveloping Algebras," North-Holland, Amsterdam, 1977.

[EGA] A. GROTHENDIECK, "Eléments de géométrie algébrique IV," Publications Mathématique 32, Institut des Hautes Études Scientifiques, Paris, 1967.

[FV] A. FACCHINI AND P. VAMOS, Injective modules over pull-backs, *J. London Math. Soc.* **31** (1985), 425–438.

[Ga] P. GABRIEL, Des catégories Abélinnes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[Ha] R. HARTSHORNE, "Algebraic Geometry," Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York, 1977.

[Ho] T. J. HODGES, *K*-theory of Noetherian rings, *in* "Séminaire d'algèbre Malliavin," Lecture Notes in Mathematics, Vol. 1404, Springer-Verlag, Berlin, 1989.

[HS] T. J. HODGES AND S. P. SMITH, Sheaves of noncommutative algebras and the Beilinson–Berstein equivalence of categories, *Proc. Amer. Math. Soc.* **93** (1985), 379–386.

[Iv] B. IVERSEN, "Cohomology of Sheaves," Universitext, Springer-Verlag, Berlin, 1986.

[Ma] H. MATSUMURA, "Commutative Algebra," Cambridge Univ. Press, Cambridge, 1986.

[Mn] I. MUSSON, Differential operators on some affine cones, *in* "Filtered Rings," Proceedings of Antwerp conference, June 1988, in press.

[MR] J. C. MCCONNELL AND J. C. ROBSON, "Noncommutative Noetherian Rings," Wiley, London, 1987.

[MS] S. M. MONTGOMERY AND L. W. SMALL, Fixed rings of Noetherian rings, *Bull. London. Math. Soc.* **13** (1981), 33–38.

[Mu] J. L. MUHASKY, The differential operator ring of an affine curve, *Trans. Amer. Math. Soc.* **307** (1988), 705–723.

[Q] D. QUILLEN, Higher algebraic *K*-theory, *in* "Algebraic *K*-Theory, I," pp. 85–147, Lecture Notes in Math., Vol. 341, Springer-Verlag, Berlin, 1973.

[RS] J. C. ROBSON AND L. W. SMALL, Orders equivalent to the first Weyl algebra, *Quart. J. Math. Oxford Ser. (2)* **37** (1986), 475–482.

[RS2] J. C. ROBSON AND L. W. SMALL, Another change of rings theorem, *Bull. London Math. Soc.* **20** (1988), 297–301.

[S] STENSTRÖM, "Rings of Quotients," Grundelhren der Math. Wissenschaften, Vol. 217, Springer-Verlag, Berlin, 1975.

[Si] J. R. Silvester, "Introduction to Algebraic K-Theory," Chapman & Hall, London, 1981.

[Sm] S. P. Smith, The primitive factor rings of the enveloping algebra of $sl(2, \mathbb{C})$, *J. London Math. Soc. (2)* **24** (1981), 97–108.

[SS] S. P. Smith and J. T. Stafford, Differential operators on an affine curve, *Proc. London Math. Soc. (3)* **56** (1988), 229–259.

[St] J. T. Stafford, Homological properties of the enveloping algebra $U(SL_2)$, *Math. Proc. Cambridge Philos. Soc.* **91** (1982), 29–37.