



Exact solutions of embedding the four-dimensional perfect fluid in a five- or higher-dimensional Einstein spacetime and the cosmological interpretations

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Abstract

We investigate an exact solution that describes the embedding of the four-dimensional (4D) perfect fluid in a five-dimensional (5D) Einstein spacetime. The effective metric of the 4D perfect fluid as a hypersurface with induced matter is equivalent to the Robertson–Walker metric of cosmology. This general solution shows interconnections among many 5D solutions, such as the solution in the braneworld scenario and the topological black hole with cosmological constant. If the 5D cosmological constant is positive, the metric periodically depends on the extra dimension. Thus we can compactify the extra dimension on S^1 and study the phenomenological issues. We also generalize the metric ansatz to the higher-dimensional case, in which the 4D part of the Einstein equations can be reduced to a linear equation.

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1. Introduction

Exact solutions play an important role in gravitational physics and cosmology. The definition of Einstein spacetime is that the Einstein tensor is zero or proportional to its metric tensor, i.e., $G_{\mu\nu} = -\Lambda g_{\mu\nu}$, where Λ is the cosmological constant. Our Universe is not an Einstein spacetime because it contains matter. In the standard Friedmann–Robertson–Walker (FRW) framework, the content of our Universe is assumed to be perfect fluid in consistent with the cosmological principle. Therefore, we are interested in exact solutions that can describe the embedding of the perfect fluid in a higher-dimensional Einstein spacetime, as a generalization of the embedding only

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between Einstein spacetimes [1,2]. The Cambell–Magaard theorem and its generalized versions [3] indicate that our four-dimensional (4D) Universe can be locally embedded in a five-dimensional (5D) Einstein spacetime. Liu–Mashhoon–Wesson (LMW) solution [4] is a basic solution in the Space–Time–Matter (STM) theory [5], in which a 4D hypersurface in a 5D Ricci-flat spacetime is chosen as our Universe.

The braneworld scenario has been proposed to describe our Universe with extra dimension(s) and new physics. In the Randall–Sundrum (RS) model [6], a famous solution that describes the 4D brane in a 5D bulk with a negative cosmological constant was found in Ref. [7]. As an induced matter scenario, the STM model employs the LMW solution to show its properties. Regardless of the physical meanings of different solutions, these 5D solutions share some common features: They describe the embedding of a 4D submanifold as our Universe to a 5D manifold. The relation between the STM model and the RS model is shown in Ref. [8], and the relation between the

STM model and the Dvali–Gabadadze–Porrati (DGP) model [9] is shown in Ref. [10]. The relation between the RS model and the 5D Schwarzschild–AdS black hole has been analyzed in Refs. [11,12]. It turned out that the LMW solution is locally isometric to a topological black hole (TBH) in Ricci-flat spacetime [13]. This implies that there should also exist the cosmological counterpart of the TBH with cosmological constant.

To generalize the LMW solution, we solve the 5D Einstein equations with cosmological constant and obtain an exact solution, which contains two arbitrary functions and three arbitrary constants. By given the specific forms of the arbitrary functions and constants, this solution can describe many well-known solutions in a unified way. We explicitly show the correspondence between this solution and the one in RS model. This solution is locally isometric to a 5D TBH with cosmological constant. We find that the metric periodically depends on the extra dimension if the 5D cosmological constant is positive. Thus we can compactify the extra dimension on S^1 and then construct a new model, which is different from the RS, DGP, and STM models, and also distinguishable from the ordinary Kaluza–Klein cosmology. Many phenomenological issues are worthy further studying. We also generalize the metric ansatz to the cases with more extra dimensions and find a linear structure, but the general solution cannot be obtained.

The Letter is organized as follows. In Section 2 we obtain the 5D solution and propose two cosmological interpretations. In Section 3 we show that our solution can correspond to the RS model and the TBH. And we give the interconnections between our solution and other 5D solutions. In Section 4 we generalize the 5D ansatz to the case with more extra dimensions and study its features. In the last section we present the conclusion and discuss some future subjects.

2. The 5D solution and two cosmological interpretations

To embed the FRW framework into a 5D spacetime, we use the metric ansatz

$$ds^2 = -B^2(t, y) dt^2 + A^2(t, y) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right) + dy^2, \tag{1}$$

where k is the curvature of the 3D space, and Ω_2 is a 2D solid angle. This metric is written in the so-called Gaussian normal coordinate system. We solve the Einstein equations with a cosmological constant Λ :

$$R_{MN} - \frac{1}{2}g_{MN}R + \Lambda g_{MN} = 0, \tag{2}$$

where the indices M and N run from 0 to 4. For solving the Einstein equations, we regard the 5D cosmological constant as an arbitrary parameter. When we talk about the physical interpretations of the solution, we may specify that Λ is positive or negative. The ty component of Einstein equations gives

$$B = \frac{\dot{A}}{\mu(t)}, \tag{3}$$

where a dot denotes a derivative with respect to the time t , and $\mu(t)$ is an arbitrary function of t . Then by inserting Eq. (3) to the metric ansatz, the tt component of Einstein equations gives a linear equation of A^2 ,

$$(\partial_y^2 + \lambda^2)A^2 = 2(\mu^2 + k), \tag{4}$$

where $\lambda = \sqrt{2\Lambda/3}$. Here Λ can be both positive and negative.

Combining Eq. (4) with other equations, we obtain an exact solution of Einstein equations. If we require the $\lambda \rightarrow 0$ limit to be finite, this general solution is

$$A^2(t, y) = \frac{2}{\lambda^2} [\mu^2 + k + \lambda v \sin \lambda y - \sqrt{(\mu^2 + k)^2 - \lambda^2(v^2 + K)} \cos \lambda y], \tag{5}$$

where $\mu \equiv \mu(t)$ and $v \equiv v(t)$ are arbitrary functions of t , and K is constant. If $\Lambda < 0$, by defining $\lambda = \sqrt{2|\Lambda|/3}$, Eq. (5) can be rewritten as

$$A^2(t, y) = \frac{2}{\lambda^2} [-\mu^2 - k + \lambda v \sinh \lambda y + \sqrt{(\mu^2 + k)^2 + \lambda^2(v^2 + K)} \cosh \lambda y]. \tag{6}$$

The $\lambda \rightarrow 0$ limit of Eq. (5) is the LMW solution

$$A^2(t, y) = (\mu^2 + k)y^2 + 2vy + \frac{v^2 + K}{\mu^2 + k}, \tag{7}$$

which was found by Liu and Mashhoon [5], and restudied by Wesson. Liu has solved the Einstein equations in the 5D bulk and obtained a solution [14], but that one is apparently divergent in the $\Lambda \rightarrow 0$ limit. The solution (5) has been essentially obtained in our previous work [15].

We should give physical interpretations of our solution. The first interpretation is the induced matter scenario with extra dimension unnecessarily compactified. We choose a 4D hypersurface $y = 0$, in which the effective metric is

$$ds^2 = -B^2(t, 0) dt^2 + A^2(t, 0) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right). \tag{8}$$

By using this 4D metric, the Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. This 4D hypersurface is not an Einstein spacetime generally, i.e., $G_{\mu\nu} + \Lambda_4 g_{\mu\nu} \neq 0$, where Λ_4 is the 4D cosmological constant. Thus we can define $8\pi G_N T_{\mu\nu} \equiv G_{\mu\nu} + \Lambda_4 g_{\mu\nu}$, where G_N is the 4D Newton’s constant and $T_{\mu\nu}$ is the energy–momentum tensor in the $y = 0$ hypersurface. This means that the matter is induced from the extra dimension. Mathematically, the Λ_4 and G_N are two new parameters. Physically, they are related to the tension of the brane in the braneworld scenario [16]. The Einstein tensor can be calculated as

$${}^{(4)}G_0^0 = \frac{3(\mu^2 + k)}{A^2}, \tag{9}$$

$${}^{(4)}G_1^1 = {}^{(4)}G_2^2 = {}^{(4)}G_3^3 = \frac{2\mu\dot{\mu}}{A\dot{A}} + \frac{\mu^2 + k}{A^2} \tag{10}$$

where A takes the value in $y = 0$ hypersurface. The energy–momentum tensor is consistent with the perfect fluid. In the following we will show that this scenario contains the FRW

framework with the scale factor $a(t)$ for specific forms of the functions $\mu(t)$ and $\nu(t)$.

We first consider the Λ cold dark matter (Λ CDM) model as a simple case. For the exact solution of the Λ CDM model, see Appendix A. We solve the scale factor as

$$a(t) = a_0 \left[\cosh\left(\frac{\sqrt{3\Lambda_4}}{2}(t-t_0)\right) + \sqrt{\frac{3}{\Lambda_4}} H_0 \sinh\left(\frac{\sqrt{3\Lambda_4}}{2}(t-t_0)\right) \right]^{2/3}, \quad (11)$$

where Λ_4 is the 4D cosmological constant in our Universe. For the known function $a(t)$ as the solution of the Λ CDM model, if we choose the arbitrary functions μ and ν as the following form:

$$\mu = \dot{a}(t), \quad (12)$$

$$\nu^2 = -\frac{1}{6}\Lambda a^4 + (\dot{a}^2 + k)a^2 - K, \quad (13)$$

then the $y = 0$ hypersurface of Eq. (5) is exactly our Universe described by the Λ CDM model. Any FRW Universe with the scale factor $a(t)$ can be reproduced by the $y = 0$ hypersurface in the 5D spacetime, if we choose the arbitrary functions as Eqs. (12) and (13). We will show that this solution mathematically unifies many 5D cosmological solutions in the next section. We can use the solution (5) or (6) to construct many cosmological models. For example, we can impose a Z_2 symmetry to the spacetime, as in Ref. [17].

The second interpretation is the induced matter scenario with compactified extra dimension. We require a positive cosmological constant in the 5D spacetime. Then the metric periodically depends on the extra dimension y , which implies that the extra dimension can be compactified on S^1 . From the metric, we can see that the scale factor of the extra dimension is constant, which means that the extra dimension is static. If the Universe begins with a small volume, the extra dimension is still small now. Different values of the coordinate y correspond to different 4D spacetimes. We should take a specific value of y and calculate the 4D effective action for the Universe. The $y = 0$ hypersurface is the same as in the first interpretation, but the second one is more physical and may have distinctive signatures. This model is different from the ADD (Arkani-Hamed, Dimopoulos and Dvali) [18], RS, DGP, and STM models, and also different from the ordinary Kaluza–Klein cosmology, in which the 4D effective metric is independent of the extra dimension. We will study the phenomenological implications of this model in our future work. The comparison of the first and the second interpretations of our solution is shown in Fig. 1.

3. Relation to other 5D solutions in a unified way

3.1. Relation to the RS, STM, and DGP models

In the original work of the RS model [6], the following solution of Einstein equations was used to illustrate the main ideas

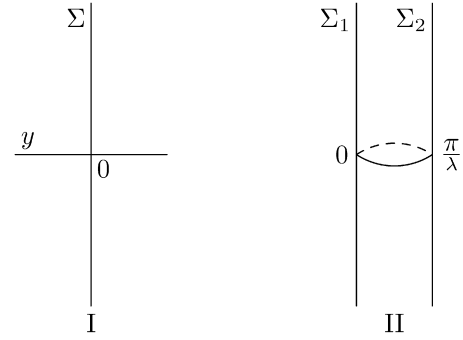


Fig. 1. The physical picture of the two interpretations. In the first one, we take the $y = 0$ hypersurface Σ as our Universe. In the second one, the extra dimension is compactified on S^1 , but different values of y correspond to different hypersurfaces.

of the model:

$$ds^2 = e^{-2kr_c\phi} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 d\phi^2, \quad (14)$$

where k is around the Planck scale, and the extra dimension ϕ is a finite interval whose size is set by r_c . A more general solution with matter in RS model was obtained later [7]. The metric ansatz is the same as Eq. (1). The notations are slightly different in the present work. The energy–momentum tensor in RS model can be written as

$$\tilde{T}^M{}_N = \check{T}^M{}_N|_{\text{bulk}} + T^M{}_N|_{\text{brane}}, \quad (15)$$

$$\check{T}^M{}_N|_{\text{bulk}} = \text{diag}(-\rho_B, -\rho_B, -\rho_B, -\rho_B, -\rho_B), \quad (16)$$

$$T^M{}_N|_{\text{brane}} = \delta(y) \cdot \text{diag}(-\rho_b, p_b, p_b, p_b, 0), \quad (17)$$

where ρ_B is the energy density in the bulk, and ρ_b and p_b are the energy density and the pressure of the matter in the brane, respectively. Here ρ_B is a negative constant, and ρ_b and p_b are functions only of time. The Einstein equations are $G_{MN} = \kappa^2 \tilde{T}_{MN}$. The relation between A and B is $B = \dot{A}/\dot{a}(t)$, where $a \equiv A(t, 0)$. The solution of A^2 is

$$A^2(t, y) = \frac{1}{2} \left(1 + \frac{\kappa^2 \rho_b^2}{6\rho_B} \right) a^2 + \frac{3C}{\kappa^2 \rho_B a^2} + \left[\frac{1}{2} \left(1 - \frac{\kappa^2 \rho_b^2}{6\rho_B} \right) a^2 - \frac{3C}{\kappa^2 \rho_B a^2} \right] \cosh(\lambda y) - \frac{\kappa \rho_b}{\sqrt{-6\rho_B}} a^2 \sinh(\lambda |y|), \quad (18)$$

where C is constant, and $\lambda = \sqrt{-2\kappa^2 \rho_B/3}$. This λ is the same as the one in Eq. (6), because the 5D cosmological constant $\Lambda = \kappa^2 \rho_B$. The scale factor $a(t)$ in the brane satisfies the modified Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{6} \rho_B + \frac{\kappa^4}{36} \rho_b^2 + \frac{C}{a^4} - \frac{k}{a^2}. \quad (19)$$

Comparing our solution (6) with the solution in RS model, we can see that if we specify some particular form of the arbitrary functions μ and ν , Eq. (6) can be the same as Eq. (18). The correspondence of the functions are

$$\mu^2 + k = \left(\frac{\kappa^2}{6} \rho_B + \frac{\kappa^4}{36} \rho_b^2 \right) a^2 + \frac{C}{a^2}, \quad (20)$$

$$\nu = -\frac{\kappa^2}{6}\rho_b a^2, \tag{21}$$

$$K = C. \tag{22}$$

The integration constant K in Eq. (6) is identified with the integration constant C in Eq. (18). If we require $B = 1$, then $\mu = \dot{a}$ in the $y = 0$ hypersurface. For this choice of μ, ν , and K , Eq. (20) is the same as the modified Friedmann equation (19). The inverse transformation is

$$a^2 = \frac{2}{\lambda^2} \left[-\mu^2 - k + \sqrt{(\mu^2 + k)^2 + \lambda^2(\nu^2 + K)} \right], \tag{23}$$

$$\rho_b = -\frac{6\nu}{\kappa^2 a^2}. \tag{24}$$

The equivalence between the STM model and the RS model has been demonstrated in Ref. [8]. The 4D effective energy–momentum tensor [8,16] is the same in both STM and RS models:

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_N \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu}, \tag{25}$$

where $q_{\mu\nu}$ is the induced metric in the brane, $\tau_{\mu\nu}$ is the energy–momentum tensor in the brane, $\pi_{\mu\nu}$ is the local quadratic correction, and $E_{\mu\nu}$ is the nonlocal Weyl correction. Here Eq. (25) is called Shiromizu–Maeda–Sasaki (SMS) equations [16], which are the effective field equations on the brane (a related work is Ref. [19]). In the present work, we have explicitly shown that our solution can be equivalent to the one in RS model in the $y = 0$ hypersurface. The $\Lambda \rightarrow 0$ limit of Eq. (5) is reduced to the LMW solution in STM model. The correspondence between the STM model and the DGP model is given in Ref. [10]. Therefore, our solution describes many models in a unified way mathematically.

3.2. Relation to the 5D topological black hole

If we replace the solid angle Ω_n in the Schwarzschild solution with an n -dimensional Einstein manifold, the metric remains a solution of Einstein equations. Such a black hole is called a topological black hole (TBH). The equivalence between the LMW solution and the TBH in Ricci-flat spacetime has been demonstrated in Ref. [13]. We will show that the metric (1) with the solution (5) can be obtained by a coordinate transformation of the TBH with cosmological constant. For the details of this method, see Ref. [13]. The metric for a 5D TBH is

$$ds_{\text{TBH}}^2 = -h(R) dT^2 + h^{-1}(R) dR^2 + R^2 d\Omega_{3(k)}^2, \tag{26}$$

where $\Omega_{3(k)}$ is a 3D Einstein manifold with curvature k . After the coordinate transformation

$$R = R(t, y), \quad T = T(t, y), \tag{27}$$

Eq. (26) becomes

$$ds^2 = \left(hT_{,t}^2 - \frac{R_{,t}^2}{h} \right) dt^2 + 2 \left(hT_{,t}T_{,y} - \frac{R_{,t}R_{,y}}{h} \right) dt dy + \left(hT_{,y}^2 - \frac{R_{,y}^2}{h} \right) dy^2 + R^2 d\Omega_{3(k)}^2. \tag{28}$$

The function R can be fixed as $R(t, y) = A(t, y)$. Then by comparing the coefficients of Eqs. (28) and (5), the equations

$$\frac{R_{,t}^2}{\mu^2(t)} = h(R)T_{,t}^2 - \frac{R_{,t}^2}{h(R)}, \tag{29}$$

$$0 = hT_{,t}T_{,y} - \frac{R_{,t}R_{,y}}{h}, \tag{30}$$

$$-1 = h(R)T_{,y}^2 - \frac{R_{,y}^2}{h(R)}, \tag{31}$$

are obtained to solve R and T . Eqs. (29) and (31) can be rewritten as

$$T_{,t} = \frac{R_{,t}}{h(R)} \sqrt{1 + \frac{h(R)}{\mu^2(t)}}, \tag{32}$$

$$T_{,y} = \frac{1}{h(R)} \sqrt{R_{,y}^2 - h(R)}. \tag{33}$$

By substituting these two equations to Eq. (30), R is determined by a single equation

$$R_{,y}^2 = h(R) + \mu^2(t). \tag{34}$$

After $R(t, y)$ is obtained, we can substitute it to Eqs. (32) and (33) to solve $T(t, y)$. The integrable condition is satisfied for these equations.

Starting with Eq. (26) and solving the Einstein equation with a cosmological constant Λ , we can obtain

$$h(R) = k - \frac{K}{R^2} - \frac{1}{6}\Lambda R^2. \tag{35}$$

We solve Eq. (34) and find

$$R^2 = \frac{2}{\lambda^2} \left[\mu^2 + k + \sqrt{(\mu^2 + k)^2 - \lambda^2 K \sin(\lambda y + \varphi)} \right], \tag{36}$$

where $\varphi \equiv \varphi(t)$ is an arbitrary function from the integration. This is consistent with Eq. (5). By applying the identity $a \sin \theta + b \cos \theta = \sqrt{a^2 + b^2} \sin(\theta + \varphi)$, where $\tan \varphi = b/a$ to Eq. (5), and comparing the result with Eq. (36), we obtain

$$\tan \varphi = -\frac{1}{\lambda \nu} \sqrt{(\mu^2 + k)^2 - \lambda^2(\nu^2 + K)}. \tag{37}$$

With the help of the equivalence between our solution and the TBH, we directly obtain the Kretschmann scalar to be

$$I = R_{MNPQ}R^{MNPQ} = \frac{72K^2}{A^8} + \frac{10\Lambda^2}{9}. \tag{38}$$

For the LMW solution, the Kretschmann scalar is $I = R_{MNPQ} \times R^{MNPQ} = 72K^2/A^8$.

In the 4D case, if we attempt to transform the 4D Schwarzschild solution in the same way, we need to solve the equation

$$\frac{\partial R}{\partial y} = \sqrt{1 - \frac{K}{R} + \mu^2(t)}. \tag{39}$$

Although the integration can be evaluated out, we cannot solve R explicitly. In the 6D case, the integration cannot be evaluated out. This implies that the 5D metric (1) is special, e.g., if we modify the Ω_2 to Ω_3 , we cannot have an explicit solution as the counterpart of the 6D TBH. The speciality of 5D

solution has been noticed by Seahra and Wesson [13], but they did not consider the case when the cosmological constant exists. Luckily, Eq. (34) can also be explicitly solved in the more general case with cosmological constant.

In addition, it has shown that the following Fukul–Seahra–Wesson (FSW) solution [20] is equivalent to LMW solution after Wick rotations:

$$ds_{\text{FSW}}^2 = -d\tau^2 + b^2(\tau, w) d\Omega_{3(k)}^2 + \frac{b_{,\tau}^2(\tau, w)}{\zeta^2(w)} dw^2, \quad (40)$$

where

$$b^2(\tau, w) = [\zeta^2(w) - k]\tau^2 + 2\chi(w) + \frac{\chi^2(w) - K}{\zeta^2(w) - k}. \quad (41)$$

Similarly, we can easily generalize the FSW solution to the case with cosmological constant from Eq. (5).

3.3. Relations to other solutions

The following Lemaître–Tolman–Bondi (LTB) metric [21] has been used to describe the inhomogeneous Universe:

$$ds^2 = -dt^2 + \frac{A'(r, t)^2}{1 + f(r)} dr^2 + A(r, t)^2 d\Omega_2^3, \quad (42)$$

where the unknown functions $A(t, r)$ and $f(r)$ are to be solved with the energy–momentum tensor. The metric (1) is essentially a 5D LTB metric

$$ds^2 = -dt^2 + \frac{A'(r, t)^2}{1 + f(r)} dr^2 + A(r, t)^2 d\Omega_{3(k)}^3, \quad (43)$$

which can be transformed to the same form as Eq. (1) by a double Wick rotation

$$y \rightarrow it, \quad t \rightarrow ir, \quad r \rightarrow \tilde{r}. \quad (44)$$

This implies that the 5D LTB metric can be explicitly solved in Einstein spacetime. However, the 4D LTB metric cannot be explicitly solved (it can be solved in parametric form). It is a coincidence that A^2 satisfies a linear equation in 5D case, while A^2 satisfies a nonlinear equation in 4D or other cases generally. This confirms that the 5D TBH and the 5D LTB metric are special. In fact, the linear structure for the ansatz (1) will be more clear in the case of more extra dimensions, as the next section shows.

The metric ansatz can be written as the Weyl type

$$ds^2 = e^{2\beta(t,y)}(-dt^2 + dy^2) + e^{2\alpha(t,y)} g_{ij} dx^i dx^j, \quad (45)$$

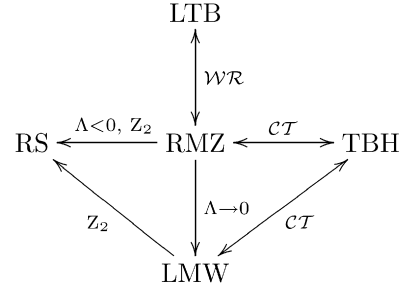
which was used in Ref. [22]. For the generalized Weyl solution [23], see Appendix A. The solution for Eq. (45) in Ricci-flat spacetime can be directly obtained as a generalized Weyl solution. However, the generalized Weyl solution is only applicable to the vacuum Einstein equations. We must simplify the ansatz (45) to solve Einstein equations with cosmological constant. To make the ansatz (1) as general as Eq. (45), we should add another unknown function before dy^2 . The physical meaning of the simplification as Eq. (1) is setting the extra dimension to be static. Mathematically, this simplification enhances the symmetry of the spacetime. The metric (45) contains

Table 1
Three cases of the 5D cosmological constant

Cosmological constant	Physical model
$\Lambda < 0$	Randall–Sundrum model (Type II)
$\Lambda = 0$	Dvali–Gabadadze–Porrati model
$\Lambda > 0$	The second interpretation

three Killing vectors, while Eq. (1) contains four Killing vectors essentially, because it can be transformed to a 5D TBH.

The relations between the solutions are as follows:



Here \mathcal{WR} denotes a double Wick rotation, and \mathcal{CT} denotes a coordinate transformation. The related solutions are as follows:

- LTB denotes the 5D LTB metric, Eq. (43).
- RS denotes a solution in the RS model, Eq. (18), which was found by Binétruy, Deffayet, Ellwanger, and Langlois.
- RMZ denotes our solution, Eq. (5).
- TBH denotes the topological black hole, Eq. (26).
- LMW denotes the LMW solution, Eq. (7).

The generality of our solution originates from two arbitrary functions and three arbitrary constants that it contains. An important problem is whether we can obtain some new physics from this solution. We can look the relation between these solutions in the physical point of view. The braneworld scenarios describe a brane in a 5D bulk with a cosmological constant Λ . If $\Lambda < 0$, we have the RS model, and if $\Lambda = 0$, we have the DGP model. Both of these two models have rich phenomenological issues. Consequently, there seems a vacancy in the $\Lambda > 0$ case. The second interpretation of our solution may correspond to a new model to fill in this vacancy, as shown in Table 1.

4. More extra dimensions

We want to generalize the codimension one scenario to the case with more extra dimensions. The codimension two brane has been studied, such as in Ref. [24]. If the number of extra dimensions is n , we propose a metric ansatz as

$$ds^2 = -B^2(t, Y) dt^2 + A^2(t, Y) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right) + \sum_i dy_i^2, \quad (46)$$

where y_1, y_2, \dots, y_n are coordinates of the extra dimensions, and $Y \equiv (y_1, y_2, \dots, y_n)$. Define $\lambda = \sqrt{2\Lambda/3}$, where Λ is the $(4 + n)$ -dimensional cosmological constant. The relation between A and B is also $B = \dot{A}/\mu(t)$. We find that as along as

the A^2 satisfies the following Helmholtz equation with source term:

$$\left(\sum_i \partial_i^2 + \lambda^2\right)A^2 = 2(\mu^2 + k), \tag{47}$$

the tensor $G_{\mu\nu} + \Lambda g_{\mu\nu}$ will be as the form

$$\begin{pmatrix} 0_{4 \times 4} & 0 \\ 0 & X_{n \times n} \end{pmatrix}.$$

The linear equation (47) guarantees that the 4D part of the Einstein equations is satisfied. But the other parts of Einstein equations are nonlinear and cannot be solved generally. For example, in the Ricci-flat spacetime, other parts of Einstein equations can be written as

$$[(A^2)_{,ij}A^6]_{,i} = 0, \tag{48}$$

for any $i, j = 0, 1, \dots, n$. We can add some other fields in extra dimension to compensate the non-zero components, and other fields may stabilize the extra dimensions.

We can also consider the case that both A and B are independent of time. The 5D ansatz is

$$ds^2 = -B^2(y)dt^2 + A^2(y)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right) + dy^2, \tag{49}$$

which cannot be directly treated as the special case of the ansatz (1), because $B = 0$ if $\dot{A} = 0$. However, if we require $B = 1$ and then $\mu = \dot{A}$, we have $\mu = 0$ when $\dot{A} = 0$. We can directly check that the following A with $B = 1$ is an exact solution:

$$A^2(y) = \frac{2}{\lambda^2}\left[k + \lambda v \sin \lambda y - \sqrt{k^2 - \lambda^2(v^2 + K)} \cos \lambda y\right], \tag{50}$$

where v is constant. The $\lambda \rightarrow 0$ limit is

$$A^2(y) = ky^2 + 2vy + \frac{v^2 + K}{k}, \tag{51}$$

as the LMW solution implies. Here Eq. (50) is a special solution for the ansatz (49). In the Ricci-flat spacetime, the following solution:

$$A^2(y) = ky^2 + c, \quad B^2(y) = \frac{y^2}{ky^2 + c}, \tag{52}$$

where c is constant, is also a special solution for the ansatz (49).

The higher-dimensional ansatz is

$$ds^2 = -B^2(Y)dt^2 + A^2(Y)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right) + \sum_i dy_i^2. \tag{53}$$

The function A also satisfies a linear equation

$$\left(\sum_i \partial_i^2 + \lambda^2\right)A^2 = 2k, \tag{54}$$

where λ is defined as above. As long as A satisfies this equation, the tt component of Einstein equations will be satisfied.

However, other nonlinear equations of A and B cannot be written as a simple form. We can show another special solution. In the Ricci-flat spacetime, the 6D ansatz

$$ds^2 = -B^2 dt^2 + A^2(dr^2 + r^2 d\Omega_2^2) + dy^2 + dz^2, \tag{55}$$

gives a Weyl type solution

$$A^2(y, z) = y^2 - z^2, \quad B^2(y, z) = \frac{y^2 z^2}{y^2 - z^2}. \tag{56}$$

The analysis of this solution is beyond the scope of this Letter.

5. Conclusion and discussion

We have obtained an exact solution of 5D Einstein equations with cosmological constant and shown the interconnections between this solution and other solutions. Two interpretations to this solution are given. In the first interpretation, we take the $y = 0$ hypersurface as our Universe with induced matter. We have demonstrated the mathematical equivalence of our solution and the solution in RS model and the TBH. In the second interpretation, we require the 5D cosmological constant to be positive and thus compactify the extra dimension on S^1 . This scenario is similar to the Kaluza–Klein cosmology, but the 4D effective metric depends on the coordinate of the extra dimension. We also propose a metric ansatz with more extra dimensions, and find that the 4D part of Einstein equations is reduced to a Helmholtz equation with source term. This linear structure in the 5D case shows that the 5D TBH and the 5D LTB metric are special. We also give some special solutions in the time-independent case.

We shall discuss some possible future developments of our work. The phenomenological implications of our solution, especially the $\Lambda > 0$ case, should be studied in details.

- Different coordinate systems may cover different patches of the whole manifold. In the Penrose diagram of the extended Schwarzschild–(A)dS manifold, which patch does this solution cover?
- If the 5D cosmological constant is positive, is it possible to construct a physical model, which is parallel to the RS and DGP model?
- The corrections to the observable quantities should be calculated. What is the 4D effective action after the dimension reduction? What is the correction to the Newtonian potential in 4D?
- The currently accelerating expansion of our Universe may be due to modified gravity [25] or higher-dimensional effects. Can this solution give some new insight to the 5D or 4D gravitational physics?
- The stabilization of the extra dimension(s). The moduli stabilization may be related to dark energy [26]. Thus, what will happen in our case?
- Can this solution explain inflation and large scale structure of our Universe?

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Appendix A. Mathematical notes

The solution of the Λ CDM model is presented as follows. The FRW metric is

$$ds^2 = -dt^2 + a^2(t)(dr^2 + d\Omega_2^2), \quad (\text{A.1})$$

where $a(t)$ is the scale factor to be solved. The content of the Universe is assumed to be a perfect fluid, whose energy–momentum tensor is $T^\mu_\nu = \text{diag}(-\rho, p, p, p)$. Einstein equations with a cosmological constant Λ are reduced to Friedmann equations,

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \end{aligned} \quad (\text{A.2})$$

Note that this Λ is Λ_4 in Section 2. For the Λ CDM model, the equation of state is $p = 0$. The solution of $a(t)$ is Eq. (11). The H – a relation can be solved as

$$H(a) = H_0^2 \left[\left(1 - \frac{\Lambda}{3H_0^2} \right) \left(\frac{a_0}{a} \right)^3 + \frac{\Lambda}{3H_0^2} \right], \quad (\text{A.3})$$

where H_0 is the Hubble constant, Ω_m is the ratio of the matter density. The relation between Ω_m and Λ is $\Omega_m = 1 - \Lambda/(3H_0^2)$. The linearization of Friedmann equations was studied in Ref. [27], and the physical analysis of some special cases was studied in Ref. [28].

The generalized Weyl solution [23] is a successful application of the integrable theory to general relativity. It is valid for vacuum Einstein equations with a D -dimensional metric that admits $D - 2$ orthogonal Killing vector fields. The metric ansatz is

$$ds^2 = \sum_{i=1}^{D-2} \epsilon_i e^{2U_i} (dx^i)^2 + e^{2C} dZ d\bar{Z}, \quad (\text{A.4})$$

where U_i and C are functions of Z and \bar{Z} only, and $\epsilon_i = \pm 1$. The functions U_i satisfy a Laplace equation

$$\partial_Z \partial_{\bar{Z}} \exp \left(\sum_j U_j \right) = 0. \quad (\text{A.5})$$

Thus the solution can be written as $\sum_j U_j = \log(w(Z) + \tilde{w}(\bar{Z}))$, and the function C is given by $C = \frac{1}{2} \log(\partial_Z w \partial_{\bar{Z}} \tilde{w}) + \nu$, where ν is determined by

$$\partial_Z \nu = -\frac{w + \tilde{w}}{\partial_Z w} \sum_{i < j} \partial_Z U_i \partial_Z U_j, \quad (\text{A.6})$$

$$\partial_{\bar{Z}} \nu = -\frac{w + \tilde{w}}{\partial_{\bar{Z}} \tilde{w}} \sum_{i < j} \partial_{\bar{Z}} U_i \partial_{\bar{Z}} U_j. \quad (\text{A.7})$$

The integrable condition for ν has been satisfied.

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