stochastic processes and their



# Stabilization of partial differential equations by noise

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**Abstract**

We provide an example of a class of partial differential equations being stabilized (in terms of Lyapunov exponents) by noise. In particular, we show that the stability of the heat equation can be improved by adding a stochastic term to the equation. We also give an example of an unstable PDE made stable by noise. 

*Keywords*: Lyapunov exponents; Stochastic partial differential equations; Partial differential equations; Heat equation

## 1. Introduction

The problem of stabilization of ordinary differential equations by noise has been thoroughly studied. There are many examples known, where after adding a stochastic term to an ODE the top Lyapunov exponent becomes smaller, i.e. the stochastic system is more stable than the deterministic one (see for instance Arnold, 1979; Arnold et al., 1983; Arnold and Kloeden, 1989; Pardoux and Wihstutz, 1988; Pardoux and Wihstutz, 1992). Especially interesting are the cases when the top Lyapunov exponent of the deterministic system is greater than zero while the top Lyapunov exponent of the corresponding stochastic system is smaller than zero (i.e. an unstable system is made stable by noise). To our knowledge no such example has been previously known in the case of PDEs. We provide a class of such examples, which have the additional advantage that both the equations and the techniques used in the proofs are simple.

First, we study the Dirichlet problem for the following equation (for a formal setting see Section 2),

$$\frac{\partial u}{\partial t} = \Delta u + \alpha u,$$

where the constant $\alpha$ is arbitrary, with the initial condition $f$. For a class of initial conditions $f$ we calculate the Lyapunov exponents, which are defined as

$$\lambda^u(f) = \limsup_{t\to\infty} \frac{1}{t} \log \|u(t)\|_{L^2}$$

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(for more information about Lyapunov exponents see Arnold and Wihstutz, 1986 and Arnold et al., 1991). Then we consider the Dirichlet problem for the following stochastic partial differential equation

$$\mathrm{d}v(t) = (\Delta v(t) + \beta v(t))\,\mathrm{d}t + \gamma v(t)\,\mathrm{d}W_t,$$

where the constants $\beta$ and $\gamma$ are arbitrary, with the initial condition $f$. We calculate its Lyapunov exponents for the same class of initial conditions $f$. The Lyapunov exponents are defined pathwise, i.e.

$$\lambda^v(f,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log \|v(t,\omega)\|_{L^2}.$$

In this case they exist as limits with probability one and are non-random. We calculate the Lyapunov exponents of the stochastic system as functions of the Lyapunov exponents of the deterministic one – see Theorem 2.

From Theorem 2 it follows that if $\alpha = \beta$ the stability of a deterministic system of the above form can be improved by adding a term with noise ($\gamma \neq 0$) to the equation. More precisely for a fixed initial condition $f$, the Lyapunov exponent of the stochastic system is smaller than the Lyapunov exponent of the deterministic system. In particular, if $\alpha = \beta = 0$ it follows that the exponential stability of the heat equation can be improved. Putting $\alpha = \beta = 2\lambda_0$, where the constant $\lambda_0$ is defined in Section 2, and $\gamma = 2\lambda_0^{1/2}$ we get an example of an exponentially unstable PDE made exponentially stable by noise.

Lyapunov exponents of SPDEs (stochastic partial differential equations) have been studied among others by Carmona and Molchanov (1994) by Flandoli and Schaumlöffel (1990), Schaumlöffel and Flandoli (1991), by Flandoli (1991) and related problems by Brzeźniak and Flandoli (1992). Our approach is, however, different from that in the papers cited above. It is somehow similar to that of Lindemann (1992) mainly because using expansions of elements of a separable Hilbert space with respect to its basis we in fact transfer some problems to the space $l^2$.

## 2. Main results

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$, where $d \leqslant 3$, with $C^\infty$ boundary. We will study the following equation:

$$\frac{\partial u}{\partial t} = \Delta u + \alpha u, \tag{1}$$

where the constant $\alpha$ is arbitrary, $u = u(t,x)$, $t \in \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the interval $[0,\infty)$), $x \in \mathcal{O}$.

We set the initial condition

$$u(0,x) = f(x), \tag{2}$$

and the Dirichlet boundary condition

$$u(t,x) = 0, \quad x \in \partial\mathcal{O}. \tag{3}$$

The function $f$ in (2) takes real values. In this paper, we consider only (deterministic) real-valued initial conditions, $f \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

There exists an orthonormal basis of $L^2(\mathcal{O})$, $\{e_j\}$, $j = 0, 1, 2, \ldots$, satisfying (see Taylor, 1996, p. 304)

$$e_j \in H_0^1(\mathcal{O}) \cap \mathscr{C}^\infty(\bar{\mathcal{O}}), \quad \Delta e_j = -\lambda_j e_j, \quad 0 < \lambda_j \nearrow \infty.$$

If $f \in L^2(\mathcal{O})$, then we can write

$$f = \sum_{j=0}^{\infty} f_j e_j, \quad \text{where } f_j = (f, e_j). \tag{4}$$

Basing on suitable properties of the heat equation (see Taylor, 1996) it is easy to prove that the unique solution to the problem (1)–(3) is given by the formula

$$u(t,x) = \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x), \quad t \geqslant 0. \tag{5}$$

**Theorem 1.** *Let us fix an initial condition $f$, $f \neq 0$. Let $j_0$ be the smallest integer $j \geqslant 0$ in the expansion* (4) *of $f$ such that $f_{j_0} \neq 0$.*

*Then the Lyapunov exponent of* Eqs. (1)–(3) *exists as a limit and is given by*

$$\lambda^u(f) = -\lambda_{j_0} + \alpha.$$

**Proof.** On the one hand,

$$\frac{1}{t} \log \left\| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right\| \leqslant \frac{1}{t} \log \left( \sum_{j=j_0}^{\infty} |\exp(t(-\lambda_{j_0} + \alpha)) f_j|^2 \right)^{1/2}$$
$$= -\lambda_{j_0} + \alpha + \frac{1}{t} \log \|f\|,$$

while on the other hand,

$$\frac{1}{t} \log \left\| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right\|$$
$$\geqslant \frac{1}{t} \log |\exp(t(-\lambda_{j_0} + \alpha)) f_{j_0}| = -\lambda_{j_0} + \alpha + \frac{1}{t} \log |f_{j_0}|.$$

The existence of the limit and the equality follow. □

Let us now consider the following stochastic equation:

$$\mathrm{d}v(t) = (\Delta v(t) + \beta v(t))\,\mathrm{d}t + \gamma v(t)\,\mathrm{d}W_t, \tag{6}$$

where the constants $\beta$ and $\gamma$ are arbitrary and $W_t$ is a real-valued Wiener process, with the initial condition

$$v(0,x,\omega) = f(x), \quad x \in \mathcal{O}, \tag{7}$$

and the Dirichlet boundary condition

$$v(t,x,\omega)=0, \quad x\in\partial\mathcal{O}. \tag{8}$$

Conditions (7) and (8) hold for a.a. $\omega\in\Omega$.

Following Da Prato et al. (1982) by a *strict solution* to the problem (6)–(8), we mean a process $v$ such that with probability 1:

$$v\in\mathscr{C}([0,T];L^2(\mathcal{O})); \quad \forall t\in[0,T]\ v(t)\in H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$$

$$t\to\Delta v(t)+\beta v(t)\in L^1(0,T;L^2(\mathcal{O}))$$

$$t\to\gamma v(t)\in L^2(0,T;L^2(\mathcal{O}))$$

and the problem (6)–(8) is satisfied.

If there exists $v(t,x)$ such that Eq. (6) is satisfied then its expansion with respect to the basis $\{e_j\}$ has the form

$$v(t,x)=\sum_{j=0}^{\infty} y_j(t)e_j(x), \tag{9}$$

where $y_j(t)$, for $j=0,1,2,\ldots,$ satisfy the following stochastic equations:

$$\mathrm{d}y_j(t)=(-\lambda_j+\beta)y_j(t)\,\mathrm{d}t+\gamma y_j(t)\,\mathrm{d}W_t,$$

$$y_j(0)=f_j$$

and, therefore,

$$y_j(t)=\exp(\gamma W_t)\exp((-\lambda_j+\beta-\tfrac{1}{2}\gamma^2)t)f_j. \tag{10}$$

(see for instance Arnold, 1974). We put

$$v(t,x)=\sum_{j=0}^{\infty}\exp(\gamma W_t)\exp((-\lambda_j+\beta-\tfrac{1}{2}\gamma^2)t)f_je_j(x), \quad t\in\mathbb{R}^+. \tag{11}$$

Obviously,

$$v(t,x)=\exp(\gamma W_t)\exp(((\beta-\alpha)-\tfrac{1}{2}\gamma^2)t)u(t,x),$$

where $u(t,x)$ is the solution to the problem (1)–(3).

Let us fix an interval $[0,T]$ for arbitrary $T>0$.

**Proposition 1.** *The process* $v(t)$ *given by formula* (11) *is the unique strict solution to the problem* (6)–(8) *on the interval* $[0,T]$.

**Proof.** We omit the proof of the above proposition. It follows easily from suitable properties of the solution of the heat equation (see Taylor, 1996).

Since $T$ is arbitrary, it makes sense to study the asymptotic properties of $v(t,x)$, in particular its Lyapunov exponents.

We will next, for a given initial condition $f$, calculate the Lyapunov exponent of the stochastic system (6)–(8) as a function of the Lyapunov exponent of the deterministic system (1)–(3).

**Theorem 2.** *Let $f \neq 0$. Then the Lyapunov exponent of the system* (6)–(8) *almost surely exists as a limit, is non-random and the following formula holds*:

$$\lambda^v(f) = \lambda^u(f) + (\beta - \alpha) - \tfrac{1}{2}\gamma^2 \quad \text{a.s.}$$

**Proof.** We compute

$$\lambda^v(f) = \lim_{t\to\infty} \frac{1}{t} \log \|v(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \left\| \exp(\gamma W_t) \exp\left(\left((\beta-\alpha) - \frac{1}{2}\gamma^2\right) t\right) u(t) \right\|$$
$$= \lambda^u(f) + (\beta - \alpha) - \tfrac{1}{2}\gamma^2 \quad \text{a.s.}$$

since by the strong law of large numbers $\lim_{t\to\infty}(W_t/t) = 0$ with probability 1 (see Arnold, 1974, p. 46). □

### 2.1. Applications

Let us consider the case when $\alpha = \beta$. Then it follows from Theorem 2 that for a fixed initial condition $f$, the Lyapunov exponent of the stochastic system (6)–(8), with $\gamma \neq 0$, is smaller than the Lyapunov exponent of the deterministic system (1)–(3).

**Example 1.** Let us take $\alpha = \beta = 0$. Then (1)–(3) is the Dirichlet problem for the heat equation. It follows that the exponential stability of the heat equation can be improved by adding to the equation a term with white noise.

**Example 2.** Let us take $\alpha = \beta = 2\lambda_0$ and $\gamma = 2\lambda_0{}^{1/2}$. The top Lyapunov exponent of the deterministic system is equal to $\lambda_0 > 0$, while the top Lyapunov exponent of the stochastic system is equal to $-\lambda_0 < 0$. Thus, an exponentially unstable system has become exponentially stable.


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