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Duality theorems for crossed products over rings

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Abstract

In this paper we improve and extend duality theorems for crossed products obtained by M. Koppinen (C. Chen) from the case of base fields (Dedekind domains) to the case of arbitrary Noetherian commutative ground rings under fairly weak conditions. In particular we extend an improved version of the celebrated Blattner–Montgomery duality theorem to the case of arbitrary Noetherian commutative ground rings.

Keywords: Hopf algebras; Crossed product; Smash product; Duality theorems

Introduction

Crossed products in the theory of Hopf algebras were presented independently by R. Blattner, M. Cohen, S. Montgomery [10] and Y. Doi, M. Takeuchi [20]. The so-called duality theorems for crossed products have their roots in the theory of group rings (e.g., *Cohen–Montgomery duality theorems* [13]).

In [9] R. Blattner and S. Montgomery extended Cohen–Montgomery duality theorems to the case of a Hopf R-algebra with bijective antipode acting on an R-algebra, where R is a base field, providing an infinite version of the finite one achieved independently by M. Van den Bergh [28]. The celebrated *Blattner–Montgomery* duality theorem was

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extended by C. Chen and W. Nichols [14] to the case of Dedekind domains. In a joint paper with J. Gómez-Torrecillas and F. Lobillo [5, Theorem 3.2] that result was extended to the case of arbitrary Noetherian ground rings.

In the case of a Hopf R-algebra (with a not necessarily bijective antipode) over a base field, M. Koppinen introduced in [23, Theorem 4.2] duality theorems for a right *H-crossed product* $A\,\#_\sigma\,H$ with invertible cocycle and a left H-module subalgebra $U \subseteq H^*$. For a Hopf R-algebra with bijective antipode and an R-subbialgebra $U \subseteq H^\circ$, [23, Corollary 5.4] provided an improved version of Blattner–Montgomery duality theorem, dropping the assumption that $U \subseteq H^\circ$ is a Hopf R-subalgebra with bijective antipode.

Inspired by the work of M. Koppinen, C. Chen presented in [12] duality theorems for right H-crossed products $A\,\#_\sigma\,H$ with invertible cocycle. Although his main results were formulated for arbitrary ground rings, the main applications he gave were limited to the case of a base field [12, Corollaries 4, 9] or a Dedekind domain [12, Corollaries 5, 10].

The main objective of this note is unify these duality theorems and their proofs as well as to generalize them to the case of arbitrary Noetherian ground rings under fairly weak conditions. Another improvement is weakening the assumption that the antipode of the Hopf algebra H is bijective by replacing it with the weaker condition that H has a *twisted antipode*, i.e., H^{op} has an antipode $\overline{S}$.

In the first section we present the needed definitions and lemmata. In the second section we present the main result (Theorem 2.9) for a Hopf R-algebra with twisted antipode, a right H-crossed product $A\,\#_\sigma\,H$ with invertible cocycle and a right H-module R-subalgebra $U \subseteq H^*$, where R is an arbitrary Noetherian ground ring. In case ${}_R H$ is locally projective we introduce a right H-submodule $H^\upsilon \subseteq H^*$, such that (H^υ, H) satisfies the modified RL-condition (12) with respect to H and use it to present results analogous to those of M. Koppinen [23] (Theorem 2.16 and Corollary 3.13).

As a corollary, with σ trivial, Theorem 2.21 generalizes Koppinen's version of the Blattner–Montgomery duality theorem [23, Corollary 5.4] to the case of arbitrary Noetherian ground rings (this improves also [5, Theorem 3.2]). Corollary 2.22 extends [25, Corollary 9.4.11] to the case of arbitrary QF ground rings (for an arbitrary right H-crossed product see Corollary 2.11). Given a Hopf R-algebra H with twisted antipode and a right H-crossed product $A\,\#_\sigma\,H$ with invertible cocycle, Theorem 2.25 provides a version Theorem 2.9 formulated for the *cleft H-extension* $(A\,\#_\sigma\,H)/A$.

The third section deals with the case of an arbitrary Hopf R-algebra (not necessarily with twisted antipode). There we generalize results of C. Chen [12] from the case of a base field or a Dedekind domain to the case of arbitrary Noetherian ring. For a locally projective Hopf R-algebra H, we consider the R-subalgebra $H^\omega \subseteq H^*$ presented by M. Koppinen and prove his main duality theorem [23, Theorem 4.2] over arbitrary Noetherian ground rings. We also generalize several corollaries of [23, Section 5] to the case of arbitrary Noetherian ground rings.

By R we denote a commutative ring with $1_R \neq 0_R$. The category of unital R-(bi)modules will be denoted by $\mathcal{M}_R$. Any unadorned tensor product is understood to be over R. We consider R as a linear topological ring with the discrete topology. For R-modules M, N we say an R-submodule $K \subset M$ is *N-pure*, if the canonical map $\mathrm{id}_K \otimes \iota_N : K \otimes_R N \to M \otimes_R N$ is injective. If $K \subset M$ is N-pure for every R-module N, then we say $K \subset M$ is *pure* (in the sense of Cohn). For R-modules M, N we denote by $\tau : M \otimes_R N \to$

$N \otimes_R M$ the canonical *twist* isomorphism. Let A be an R-algebra. The category of *unital* left (respectively right) A-modules will be denoted by ${}_A\mathcal{M}$ (respectively $\mathcal{M}_A$). For an A-module M we call an A-submodule $K \subset M$ *R-cofinite*, if M/K is finitely generated in $\mathcal{M}_R$. For any two R-modules M and N, we consider $\mathrm{Hom}_R(M, N) \hookrightarrow N^M$ with the finite topology induced from the product topology on N^M (N is considered with the discrete topology). See [7] for more details.

We assume the reader is familiar with the theory of Hopf R-algebras. For the basic definitions and concepts we refer to [27] and [25]. For an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ we call a *pure* R-submodule $\widetilde{C} \subseteq C$ an R-coalgebra provided $\Delta_C(\widetilde{C}) \subseteq \widetilde{C} \otimes_R \widetilde{C}$. For an R-coalgebra C and an R-algebra A, we consider $\mathrm{Hom}_R(C, A)$ as an R-algebra under the so-called *convolution product* $(f \star g)(c) := \sum f(c_1)g(c_2)$ and unity $\eta_A \circ \varepsilon_C$.

For an R-coalgebra C and a right C-comodule (M, ϱ_M) we denote by $\mathrm{Cf}(M) \subseteq C$ the R-submodule generated by $\{m_{\langle 0\rangle} \mid m \in M,\ \varrho_M(m) = \sum m_{\langle 0\rangle} \otimes m_{\langle 1\rangle}\}$. For an R-bialgebra H and a right H-comodule M, we set $M^{\mathrm{co}H} = \{m \in M \mid \varrho_M(m) = m \otimes 1_H\}$.

1. Preliminaries

In this section we introduce some needed definitions and results.

1.1. Measuring R-pairings

Let C be an R-coalgebra and A be an R-algebra with a morphism of R-algebras $\beta : A \to C^*$, $a \mapsto [c \mapsto \langle a, c\rangle]$. Then we call $P := (A, C)$ a *measuring R-pairing* (the terminology is inspired by [27, p. 139]). In this case C is an A-bimodule through the left and the right A-actions

$$a \rightharpoonup c := \sum c_1\langle a, c_2\rangle \quad \text{and} \quad c \leftharpoonup a := \sum \langle a, c_1\rangle c_2 \quad \text{for all } a \in A,\ c \in C. \tag{1}$$

1.2. The α-condition

Let V, W be R-modules with an R-linear map $\beta : V \to W^*$. We say the *R-pairing* $P := (V, W)$ satisfies the *α-condition* (or P is an *α-pairing*), if for every R-module M the following map is injective:

$$\alpha_M^P : M \otimes_R W \to \mathrm{Hom}_R(V, M), \quad \sum m_i \otimes w_i \mapsto \Big[v \mapsto \sum m_i\langle v, w_i\rangle\Big]. \tag{2}$$

We say an R-module W satisfies the *α-condition*, if the canonical R-pairing (W^*, W) satisfies the α-condition (equivalently, if ${}_RW$ is locally projective in the sense of B. Zimmermann-Huisgen ([30, Theorem 2.1], [21, Theorem 3.2])). If ${}_RW$ is locally projective, then ${}_RW$ is flat and R-cogenerated (e.g., [4, Bemerkung 2.1.5]).

1.3. The C-adic topology

Let $P = (A, C)$ be a measuring R-pairing and consider C as a left A-module with the induced left A-action in (1). Then A becomes a left linear topological R-algebra under the so-called C-*adic topology* $\mathcal{T}_{C-}(A)$ with neighbourhood basis of 0_A:

$$\mathcal{B}_{C-}(0_A) = \{(0_C : W) \mid W \subset C \text{ is a finite subset}\}.$$

The category of *discrete* left $(A, \mathcal{T}_{C-}(A))$-modules is denoted by $\sigma[{}_AC]$. In fact $\sigma[{}_AC]$ is the *smallest* Grothendieck full subcategory of ${}_A\mathcal{M}$ that contains C. The reader is referred to [7,11] for more investigation of this topology and to [29] for the well-developed theory of categories of type $\sigma[M]$.

1.4. Let $P = (A, C)$ be a measuring α-pairing. Let M be a left A-module and consider the canonical A-linear map $\rho_M : M \to \mathrm{Hom}_R(A, M)$. We set $\mathrm{Rat}^C({}_AM) := \rho_M^{-1}(M \otimes_R C)$ and call M C-*rational*, if $\mathrm{Rat}^C({}_AM) = M$. If ${}_AM$ is C-rational, then we have an R-linear map $\varrho_M := (\alpha_M^P)^{-1} \circ \rho_M : M \to M \otimes_R C$. The class of C-rational left A-modules build a *full* subcategory of ${}_A\mathcal{M}$, which we denote with $\mathrm{Rat}^C({}_A\mathcal{M})$ (see [4, Lemma 2.2.7]).

Theorem 1.5 [2, Theorems 1.14, 1.15]. *For a measuring R-pairing $P = (A, C)$ the following are equivalent*:

(1) *P satisfies the α-condition*;
(2) *${}_RC$ is locally projective and $\beta_P(A) \subseteq C^*$ is dense.*

If these equivalent conditions are satisfied, then we have isomorphisms of categories

$$\mathcal{M}^C \simeq \sigma[{}_AC] = \mathrm{Rat}^C({}_A\mathcal{M}) \simeq \sigma[{}_{C^*}C] = \mathrm{Rat}^C({}_{C^*}\mathcal{M}).$$

1.6. ([5, Remark 2.14, Proposition 2.15], [6]) Assume R to be Noetherian. Let A be an R-algebra and consider A^* as an A-bimodule through the regular left and right actions

$$(af)(b) = f(ba) \text{ and } (fa)(b) = f(ab) \quad \text{for } a, b \in A \text{ and } f \in H^*. \tag{3}$$

We set

$$\begin{aligned} A^\circ &:= \{f \in A^* \mid AfA \text{ is finitely generated in } \mathcal{M}_R\} \\ &= \{f \in A^* \mid \mathrm{Ke}(f) \text{ contains an } R\text{-cofinite } A\text{-ideal}\}. \end{aligned}$$

Then (A, A°) is a measuring α-pairing if and only if $A^\circ \subset R^A$ is pure. In this case A° is a locally projective R-coalgebra and for every R-subcoalgebra $\widetilde{C} \subseteq A^\circ$, the induced R-pairing $(A, \widetilde{C})$ is a measuring α-pairing.

An R-algebra (respectively an R-bialgebra, a Hopf R-algebra) A with $A^\circ \subset R^A$ pure will be called an α-*algebra* (respectively an α-*bialgebra*, a *Hopf α-algebra*). If H is an α-bialgebra (respectively a Hopf α-algebra), then H° is an R-bialgebra (respectively a Hopf R-algebra).

1.7. [10,20] Let H be an R-bialgebra and A an R-algebra. A *weak left H-action* on A is an R-linear map $w : H \otimes_R A \to A$, $h \otimes a \mapsto ha$, such that the induced R-linear map $\beta : A \to \mathrm{Hom}_R(H, A)$, $a \mapsto [h \mapsto ha]$ is an R-algebra morphism and $1_H \rightharpoonup a = a$ for all $a \in A$.

Let A have a weak left H-action and $\sigma : H \otimes_R H \to A$ an R-linear map. Then $A \#_\sigma H := A \otimes_R H$ is a (*not necessarily associative*) R-algebra under the multiplication

$$(a \#_\sigma h)(\tilde{a} \#_\sigma \tilde{h}) := \sum a(h_1\tilde{a})\sigma(h_2 \otimes \tilde{h}_1) \#_\sigma h_3\tilde{h}_2 \tag{4}$$

and has in general no unity. If $A \#_\sigma H$ is an associative R-algebra with unity $1_A \#_\sigma 1_H$, then $A \#_\sigma H$ is called a *right H-crossed product*. In this case $(A \#_\sigma H, \mathrm{id} \otimes \Delta_H)$ is a right H-comodule algebra with $(A \#_\sigma H)^{\mathrm{co}H} = A$. We say σ in invertible, if it is invertible in $(\mathrm{Hom}_R(H \otimes_R H, A), \star)$.

Lemma 1.8 ([10], [20, Lemma 10]). *Let H be an R-bialgebra, A an R-algebra with a weak left H-action and $\sigma \in \mathrm{Hom}_R(H \otimes_R H, A)$.*

(1) $1 \#_\sigma 1$ *is a unity for $A \#_\sigma H$ if and only if σ is* normal, *i.e.,*

$$\sigma(h \otimes 1_H) = \varepsilon(h)1_A = \sigma(1_H \otimes h) \quad \text{for all } h \in H. \tag{5}$$

(2) *Assume σ to be normal. Then $A \#_\sigma H$ is an associative R-algebra if and only if σ is a* cocycle, *i.e.,*

$$\begin{aligned}&\sum [h_1\sigma(k_1 \otimes l_1)]\sigma(h_2 \otimes k_2l_2)\\ &\quad = \sum \sigma(h_1 \otimes k_1)\sigma(h_2k_2 \otimes l) \quad \text{for all } h, k, l \in H,\end{aligned} \tag{6}$$

and satisfies the twisted module condition

$$\sum [h_1[k_1a]]\sigma(h_2 \otimes k_2) = \sum \sigma(h_1 \otimes k_1)[(h_2k_2)a] \quad \text{for all } h, k \in H,\ a \in A. \tag{7}$$

1.9. Left smash product

Let H be an R-bialgebra and A a left H-module algebra. Then

$$\sigma : H \otimes_R H \to A, \quad h \otimes k \mapsto \varepsilon(h)\varepsilon(k)1_A$$

is a *trivial* normal cocycle and satisfies the twisted module condition (7). By Lemma 1.8 $A \# H := A \#_\sigma H$ is an associative R-algebra with multiplication

$$(a \# h) \bullet (\tilde{a} \# \tilde{h}) = \sum a(h_1\tilde{a}) \# h_2\tilde{h} \tag{8}$$

and unity $1_A \# 1_H$. If the left H-action on A is also *trivial*, then $A \# H = A \otimes_R H$ as R-algebras. The R-algebra $A \# H$ was presented by M. Sweedler [27, pp. 155–156].

2. The main duality theorem

In this section we present the main result in this note, namely Theorem 2.9. For the convention of the reader we begin with some definitions.

2.1. ([18], [16, p. 375]) Let H be an R-bialgebra and B a right H-comodule algebra. Then $\#(H, B) := (\mathrm{Hom}_R(H, B), \hat{\star})$ is an associative R-algebra with multiplication

$$(f \mathbin{\hat{\star}} g)(h) = \sum f\big(g(h_2)_{\langle 1\rangle} h_1\big) g(h_2)_{\langle 0\rangle} \quad \text{for all } f, g \in \mathrm{Hom}_R(H, B),\ h \in H, \tag{9}$$

and unity $\eta_B \circ \varepsilon_H$. If $U \subseteq H^*$ is a *right* H-module subalgebra (with $\varepsilon_H \in U$), then $B \# U := B \otimes_R U$ is an associative R-algebra with multiplication

$$(b \# f)\big(\tilde{b} \# \tilde{f}\big) = \sum b\tilde{b}_{\langle 0\rangle} \# \big(f \tilde{b}_{\langle 1\rangle}\big) \star \tilde{f} \quad \text{for all } b, \tilde{b} \in B,\ f, \tilde{f} \in U \tag{10}$$

(and unity $1_B \# \varepsilon_H$).

Remark 2.2. Let R be Noetherian, H an α-bialgebra, $U \subseteq H^\circ$ an R-subbialgebra and consider the α-pairing $P := (H, U)$. Since H is a left U-module algebra under the action $f \rightharpoonup h := \sum h_1 f(h_2)$, we can endow $H \otimes_R U$ with the structure of a *left smash algebra* under the multiplication (8). On the other hand H is a right H-comodule algebra under Δ_H, $U \subseteq H^*$ is a right H-module subalgebra under the right regular H-action (3) and $H \otimes_R U$ can be endowed with the structure of a *right smash algebra* under the multiplication (10). It can be easily seen that the two R-algebras are isomorphic. In fact we have for arbitrary $h, \tilde{h} \in H$, $f, \tilde{f} \in U$ and all $k \in H$:

$$\begin{aligned}
\alpha_H^P\big((h \# f) \bullet \big(\tilde{h} \# \tilde{f}\big)\big)(k) &= \alpha_H^P\Big(\sum h\big(f_1 \rightharpoonup \tilde{h}\big) \# f_2 \star \tilde{f}\Big)(k) \\
&= \sum h\big(f_1 \rightharpoonup \tilde{h}\big)\big(f_2 \star \tilde{f}\big)(k) \\
&= \sum h\tilde{h}_1 f_1\big(\tilde{h}_2\big) f_2(k_1) \tilde{f}(k_2) \\
&= \sum h\tilde{h}_1 f\big(\tilde{h}_2 k_1\big) \tilde{f}(k_2) \\
&= \sum h\tilde{h}_1 \big(f\tilde{h}_2\big)(k_1) \tilde{f}(k_2) \\
&= \alpha_H^P\Big(\sum h\tilde{h}_1 \# \big(f\tilde{h}_2\big) \star \tilde{f}\Big)(k) \\
&= \alpha_H^P\big((h \# f)\big(\tilde{h} \# \tilde{f}\big)\big)(k).
\end{aligned}$$

Since α_H^P is injective, we get $(h \# f) \bullet (\tilde{h} \# \tilde{f}) = (h \# f)(\tilde{h} \# \tilde{f})$ and we are done.

The following definition provides a generalization of the *RL-condition* suggested by [9]:

Definition 2.3. Let H be an R-bialgebra, $U \subseteq H^*$ a right H-module subalgebra under the right regular H-action, $V \subseteq H^*$ an R-submodule and consider the R-linear maps

$$\lambda : H \# U \to \mathrm{End}_R(H), \quad \sum h_j \# g_j \mapsto \left[\tilde{k} \mapsto h_j\left(g_j \rightharpoonup \tilde{k}\right)\right],$$
$$\rho : V \to \mathrm{End}_R(H), \quad g \mapsto \left[\tilde{k} \mapsto \tilde{k} \leftharpoonup g\right]. \tag{11}$$

We say (V, U) satisfies the *RL-condition* with respect to H, provided $\rho(V) \subseteq \lambda(H \# U)$, i.e., if

$$\text{for every } g \in V, \quad \exists\left\{(h_j, g_j)\right\} \subset H \times U,$$
$$\text{s.t.} \quad \tilde{k} \leftharpoonup g = \sum h_j\left(g_j \rightharpoonup \tilde{k}\right) \quad \text{for all } \tilde{k} \in H. \tag{12}$$

We say U satisfies the RL-condition with respect to H, if (U, U) satisfies the RL-condition with respect to H.

Lemma 2.4. *Let H be an R-bialgebra, $U \subseteq H^*$ a right H-module subalgebra and consider H as a right H-comodule algebra through Δ_H. Let $\#(H, H)$ and $H \# U$ be the R-algebras defined in* 2.1 *and consider the canonical R-algebra morphism $\beta : H \# U \to \#(H, H)$.*

(1) *If ${}_R H$ is finitely generated projective, then $H \# H^* \overset{\beta}{\simeq} \#(H, H)$ as R-algebras.*
(2) *If H is a Hopf R-algebra with twisted antipode, then $\#(H, H) \simeq \mathrm{End}_R(H)$ as R-algebras.*
(3) *Let H be a finitely generated projective Hopf R-algebra. Then $\lambda : H \# H^* \to \mathrm{End}_R(H)$, defined in* (11)*, is an R-algebra isomorphism. In particular H^* satisfies the RL-condition* (12) *with respect to H.*
(4) *If ${}_R H$ is locally projective and $U \subseteq H^*$ is dense, then $\beta(H \# U) \subseteq \#(H, H)$ is a dense R-subalgebra. If moreover H is a Hopf R-algebra with twisted antipode and ${}_R H$ is projective, then*

$$H \# U \overset{\lambda}{\hookrightarrow} \mathrm{End}_R(H)$$

is a dense R-subalgebra.

Proof. (1) Since ${}_R H$ is finitely generated projective, β is bijective.

(2) Let H be a Hopf R-algebra with twisted antipode $\overline{S}$ and consider the R-linear maps

$$\phi_1 : \#(H, H) \to \mathrm{End}_R(H), \quad f \mapsto \left[h \mapsto \sum f(h_2)h_1\right],$$
$$\phi_2 : \mathrm{End}_R(H) \to \#(H, H), \quad g \mapsto \left[k \mapsto \sum g(k_2)\overline{S}(k_1)\right].$$

For arbitrary $f, g \in \#(H, H)$ and $h \in H$ we have

$$\begin{aligned}\phi_1(f \mathbin{\hat{\star}} g)(h) &= \sum (f \mathbin{\hat{\star}} g)(h_2)h_1 = \sum f\left(g(h_3)_2 h_2\right) g(h_3)_1 h_1 \\ &= \sum f\left(g(h_2)_2 h_{12}\right) g(h_2)_1 h_{11} = \phi_1(f)\left(\sum g(h_2)h_1\right) \\ &= \left(\phi_1(f) \circ \phi_1(g)\right)(h),\end{aligned}$$

i.e., ϕ_1 is an R-algebra morphism. For all R-linear maps $f, g : H \to H$ we have

$$\begin{aligned}(\phi_1 \circ \phi_2)(g)(h) &= \sum \phi_2(g)(h_2)h_1 = \sum g(h_3)\overline{S}(h_2)h_1 \\ &= \sum g(h_2)\varepsilon(h_1) = g(h), \\ (\phi_2 \circ \phi_1)(f)(h) &= \sum \phi_1(f)(h_2)\overline{S}(h_1) = \sum f(h_3)h_2\overline{S}(h_1) \\ &= \sum f(h_2)\varepsilon(h_1) = f(h).\end{aligned}$$

Hence ϕ_1 is an R-algebra isomorphism with inverse ϕ_2.

(3) Let H be a finitely generated projective Hopf R-algebra. By (1) and (2) we have R-algebra isomorphisms

$$H \# U \overset{\beta}{\simeq} \#(H, H) \overset{\phi_1}{\simeq} \mathrm{End}_R(H)$$

(recall that the antipode of a finitely generated projective Hopf R-algebra is bijective by [26, Proposition 4], hence H has a twisted antipode $\overline{S} := S^{-1}$). So $\lambda = \phi_1 \circ \beta : H \# U \to \mathrm{End}_R(H)$ is an R-algebra isomorphisms. In particular $\rho(H^*) \subseteq \mathrm{End}_R(H) = \lambda(H \# H^*)$, i.e., H^* satisfies the RL-condition (12) with respect to H.

(4) By [3, Corollary 3.20] $\beta(H \# U) \subseteq \#(H, H)$ is a dense R-subalgebra. If H is a Hopf R-algebra with twisted antipode then

$$\#(H, H) \overset{\phi_1}{\simeq} \mathrm{End}_R(H)$$

as R-algebras by (2) and we are done (notice that β is an embedding, if ${}_R H$ is projective). □

Lemma 2.5. *Let H be a Hopf R-algebra with twisted antipode, A an R-algebra, $U \subseteq H^*$ an R-*submodule *and consider the R-pairing $P := (H, U)$. Then the canonical R-linear map $\alpha := \alpha^P_{A \otimes_R H} : (A \otimes_R H) \otimes_R U \to \mathrm{Hom}_R(H, A \otimes_R H)$ is injective if and only if the following map is injective*

$$\begin{aligned}\chi : A \otimes_R (H \otimes_R U) &\to \mathrm{End}_{-A}(H \otimes_R A), \\ a \otimes (h \otimes f) &\mapsto \big[(k \otimes \tilde{a}) \mapsto h(f \rightharpoonup k) \otimes a\tilde{a}\big].\end{aligned} \tag{13}$$

Proof. Assume H to have a twisted antipode $\overline{S}$. We show first that the R-linear map

$$\epsilon : \mathrm{Hom}_R(H, A \otimes_R H) \to \mathrm{End}_{-A}(H \otimes_R A), \quad g \mapsto \big[k \otimes \tilde{a} \mapsto \tau\big(g(k_2)\big)\big(k_1 \otimes \tilde{a}\big)\big]$$

is bijective with inverse

$$\begin{aligned}\epsilon^{-1} : \mathrm{End}_{-A}(H \otimes_R A) &\to \mathrm{Hom}_R(H, A \otimes_R H), \\ f &\mapsto \big[k \mapsto \tau\big(f(k_2 \otimes 1_A)\big)\big(1_A \otimes \overline{S}(k_1)\big)\big].\end{aligned}$$

In fact we have for all $f \in \mathrm{End}_{-A}(H \otimes_R A)$, $k \in H$, $\tilde{a} \in A$:

$$\begin{aligned}
\epsilon\big(\epsilon^{-1}(f)\big)\big(k \otimes \tilde{a}\big) &= \sum \tau\big[\epsilon^{-1}(f)(k_2)\big]\big(k_1 \otimes \tilde{a}\big) \\
&= \sum \tau\big[\tau\big(f(k_{22} \otimes 1_A)\big)\big(1_A \otimes \overline{S}(k_{21})\big)\big]\big(k_1 \otimes \tilde{a}\big) \\
&= \sum f(k_{22} \otimes 1_A)\big(\overline{S}(k_{21}) \otimes 1_A\big)\big(k_1 \otimes \tilde{a}\big) \\
&= \sum f(k_2 \otimes 1_A)\big(\overline{S}(k_{12})k_{11} \otimes \tilde{a}\big) \\
&= \sum f(k_2 \otimes 1_A)\big(\varepsilon_H(k_1)1_H \otimes \tilde{a}\big) \\
&= f\big(k \otimes \tilde{a}\big)
\end{aligned}$$

and for all $g \in \mathrm{Hom}_R(H, A \otimes_R H)$, $k \in H$:

$$\begin{aligned}
\epsilon^{-1}\big(\epsilon(g)\big)(k) &= \sum \tau\big[\epsilon(g)(k_2 \otimes 1_A)\big]\big(1_A \otimes \overline{S}(k_1)\big) \\
&= \sum \tau\big[\tau\big(g(k_{22})\big)(k_{21} \otimes 1_A)\big]\big(1_A \otimes \overline{S}(k_1)\big) \\
&= \sum g(k_{22})\big(1_A \otimes k_{21}\overline{S}(k_1)\big) \\
&= \sum g(k_2)\big(1_A \otimes k_{12}\overline{S}(k_{11})\big) \\
&= \sum g(k_2)\big(1_A \otimes \varepsilon_H(k_1)\big) \\
&= g(k).
\end{aligned}$$

Moreover we have for all $a \in A$, $h \in H$, $f \in U$ and $k \in H$:

$$\begin{aligned}
(\epsilon \circ \alpha)\big(a \otimes (h \otimes f)\big)\big(k \otimes \tilde{a}\big) &= \tau\big(\alpha\big(a \otimes (h \otimes f)\big)(k_2)\big)\big(k_1 \otimes \tilde{a}\big) \\
&= \sum f(k_2)(h \otimes a)\big(k_1 \otimes \tilde{a}\big) \\
&= \sum hf(k_2)k_1 \otimes a\tilde{a} \\
&= h(f \rightharpoonup k) \otimes a\tilde{a} \\
&= \chi\big(a \otimes (h \otimes f)\big)\big(k \otimes \tilde{a}\big),
\end{aligned}$$

i.e., $\chi = \epsilon \circ \alpha$. Consequently χ is injective if and only if α is so. $\square$

2.6. Let H be a Hopf R-algebra with twisted antipode, $A \#_\sigma H$ a right H-crossed product with invertible cocycle and consider the R-linear maps $\varphi, \psi : H \otimes_R A \to \mathrm{Hom}_R(H, A)$ defined as:

$$\begin{aligned}
\varphi(h \otimes a)\big(\tilde{h}\big) &= \sum \big[\overline{S}\big(\tilde{h}_2\big)a\big]\sigma\big(\overline{S}\big(\tilde{h}_1\big) \otimes h\big), \\
\psi(h \otimes a)\big(\tilde{h}\big) &= \sum \sigma^{-1}\big(\tilde{h}_3 \otimes \overline{S}\big(\tilde{h}_2\big)\big)\big[\tilde{h}_4 a\big]\sigma\big(\tilde{h}_5 \otimes \overline{S}\big(\tilde{h}_1\big)h\big).
\end{aligned}$$

Let $U \subseteq H^*$ be a *right* H-module subalgebra, $V \subseteq H^*$ an R-submodule and consider the canonical R-linear map $J : A \otimes_R V \to \mathrm{Hom}_R(H, A)$. We say (V, U) is *compatible*, if the following conditions are satisfied:

(1) $\varphi(H \otimes_R A),\ \psi(H \otimes_R A) \subseteq J(A \otimes_R V)$;
(2) (V, U) satisfies the *RL-condition* (12) with respect to H.

In the light of Lemma 2.5 and the modified RL-condition (12) we introduce an improved version of [12, Theorem 3, Corollary 4] over arbitrary commutative ground rings:

Proposition 2.7. *Let H be a Hopf R-algebra with twisted antipode $\overline{S}$, $A \#_\sigma H$ a right H-crossed product with invertible cocycle, $U \subseteq H^*$ a right H-module subalgebra and consider the R-pairing $P := (H, U)$. Assume there exists a* right H-submodule $V \subseteq H^*$, *such that (V, U) is compatible. If the canonical R-linear map $\alpha := \alpha^P_{A \otimes_R H} : (A \otimes_R H) \otimes_R U \to \mathrm{Hom}_R(H, A \otimes_R H)$ is injective, then there exists an R-algebra isomorphism*

$$(A \#_\sigma H) \# U \simeq A \otimes_R (H \# U).$$

Proof. Replacing the inverse of the antipode in [12, Lemma 2] with the twisted antipode $\overline{S}$ we have a commutative diagram of R-algebra morphisms

$$\begin{tikzcd}
 & (A \#_\sigma H) \# U \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\
\#(H, A \#_\sigma H) \arrow[rr, "\pi"] & & \mathrm{End}_{-A}(H \otimes_R A) \\
 & A \otimes_R (H \# U) \arrow[ul, "\delta"] \arrow[ur, "\chi"'] &
\end{tikzcd} \tag{14}$$

where

$$\begin{aligned}
\alpha\big(a \# (h \# f)\big)(k) &= (a \# h) f(k), \\
\chi\big(a \otimes (h \# f)\big)\big(k \otimes \tilde{a}\big) &= h(f \rightharpoonup k) \otimes a\tilde{a}, \\
\gamma\big((a \# h) \# f\big)\big(k \otimes \tilde{a}\big) &= \sum h_4(f \rightharpoonup k_3) \otimes \big[\overline{S}(h_3k_2)a\big]\sigma\big(\overline{S}(h_2k_1) \otimes h_1\big)\tilde{a}, \\
\delta\big(a \otimes (h \# f)\big)(k) &= \sum \sigma^{-1}\big(h_2k_4 \otimes \overline{S}(h_1k_3)\big)\big[(h_3k_5)a\big]\sigma\big(h_4k_6 \otimes \overline{S}(k_2)\big) \\
&\quad \# h_5(f \rightharpoonup k_7)\overline{S}(k_1), \\
\pi(g)\big(k \otimes \tilde{a}\big) &= \nu\Big(\sum g(k_5)\big(\sigma^{-1}\big(k_2 \otimes \overline{S}(k_1)\big)\big(k_3 \rightharpoonup \tilde{a}\big) \# k_4\big)\Big),
\end{aligned}$$

and

$$\nu : A \#_\sigma H \to H \otimes_R A, \quad a \#_\sigma h \mapsto \sum h_4 \otimes \big[\overline{S}(h_3)a\big]\sigma\big(\overline{S}(h_2) \otimes h_1\big).$$

Analogously to the proof of [12, Corollary 4] (and replacing the inverse of the antipode by the twisted antipode $\overline{S}$) can be shown that the compatibility of (V, U) implies $\mathrm{Im}(\gamma) \subseteq \mathrm{Im}(\chi)$ and $\mathrm{Im}(\delta) \subseteq \mathrm{Im}(\alpha)$. Assume now that $\alpha := \alpha^P_{A\otimes_R H}$ is injective. Then χ is injective by Lemma 2.5 and consequently δ is injective. Analogously to [12, Lemma 1, p. 2890] π is an R-algebra isomorphism, hence γ is injective and we are done. $\square$

Lemma 2.8. *Let R be Noetherian, W an R-module, $U \subseteq W^*$ an R-submodule and consider the R-pairing $P := (W, U)$. Then the canonical map $\alpha^P_M : M \otimes_R U \to \mathrm{Hom}_R(W, M)$ is injective for an R-module M if and only if $U \subset R^W$ is M-pure. Consequently P satisfies the α-condition if and only if $U \subset R^W$ is pure.*

Proof. Let M be an R-module and consider the commutative diagram

$$\begin{tikzcd} M \otimes_R U \arrow[r, "\alpha^P_M"] \arrow[d, "\mathrm{id}_M \otimes \iota_U"'] & \mathrm{Hom}_R(W, M) \arrow[d, hook] \\ M \otimes_R R^W \arrow[r, "\varpi"'] & M^W \end{tikzcd}$$

where $\varpi(m \otimes f)(w) = mf(w)$. Write $M = \varinjlim_I M_i$ as a direct limit of its finitely generated R-submodules. Since M_i is f.p. in $\mathcal{M}_R$ we have for every $i \in I$ the isomorphism of R-modules

$$\varpi_i : M_i \otimes R^W \to M_i^W, \quad m \otimes f \mapsto \big[w \mapsto mf(w)\big].$$

Moreover for every $i \in I$ the restriction of ϖ on M_i coincides with ϖ_i, hence

$$\varpi = \varinjlim \varpi_{M_i} : \varinjlim M_i \otimes R^W \to \varinjlim M_i^W \subset M^W$$

is injective. It's obvious then that α^P_M is injective iff $\mathrm{id}_M \otimes \iota_U$ is injective iff $U \subset R^W$ is M-pure. $\square$

We are ready now to present the main duality theorem in this note, which generalizes [12, Corollary 4] (respectively [12, Corollary 5]) from the case of a base field (respectively a Dedekind domain) to the case of an arbitrary Noetherian ring:

Theorem 2.9. *Let R be Noetherian, H a Hopf R-algebra with twisted antipode, $A \#_\sigma H$ a right H-crossed product with invertible cocycle, $U \subseteq H^*$ a right H-module subalgebra and consider the R-pairing $P := (H, U)$. Assume there exists a right H-submodule $V \subseteq H^*$, such that (V, U) is compatible. If $U \subset R^H$ is $A \otimes_R H$-pure* (*e.g., H is a Hopf α-algebra and $U \subseteq H^\circ$ is an R-subbialgebra*), *then we have an R-algebra isomorphism*

$$(A \#_\sigma H) \# U \simeq A \otimes_R (H \# U).$$

Proof. By Proposition 2.7 it remains to show that $\alpha^P_{A\otimes_R H}$ is injective. If $U \subset R^H$ is $A \otimes_R H$-pure, then $\alpha^P_{A\otimes_R H}$ is injective by Lemma 2.8. If H is a Hopf α-algebra, then $H^\circ \subset R^H$ is pure and for every R-subbialgebra $U \subseteq H^\circ$, $U \subset R^H$ is pure (since by convention $U \subseteq H^\circ$ is pure), hence $\alpha^P_{A\otimes_R H}$ is injective. □

Definition 2.10. Let R be Noetherian. After [25] we call an R-algebra A *residually finite* (called in other references *proper*), if the canonical map $A \to A^{\circ *}$ is injective (equivalently, if $\bigcap\{\mathrm{Ke}(f) \mid f \in A^\circ\} = 0$).

Corollary 2.11. *Let H be a Hopf R-algebra with twisted antipode and ${}_R H$ projective, $A \#_\sigma H$ a right H-crossed product with invertible cocycle, $U \subseteq H^*$ a right H-module subalgebra and consider the R-paring $P := (H, U)$. Assume there exists a right H-submodule $V \subseteq H^*$, such that (V, U) is compatible. If $U \subseteq H^*$ is dense and the canonical R-linear map $\alpha^P_{A\otimes_R H}$ is injective* (*e.g., R is Noetherian and $U \subseteq R^H$ is A-pure*), *then there exists a dense R-subalgebra $\mathcal{L} \subseteq \mathrm{End}_R(H)$ and an R-algebra isomorphism*

$$(A \#_\sigma H) \# U \simeq A \otimes_R \mathcal{L}$$

This is the case in particular, if R is a QF ring, H is a residually finite Hopf α-algebra and $U \subseteq H^\circ$ is a dense R-subbialgebra.

Proof. If $U \subseteq H^*$ is dense, then $\mathcal{L} := H \# U \overset{\lambda}{\hookrightarrow} \mathrm{End}_R(H)$ is a dense R-subalgebra by Lemma 2.4(4) and the isomorphism follows by Theorem 2.9. If R is a QF ring and H is a residually finite Hopf α-algebra, then $H^\circ \subset H^*$ is dense by [4, Proposition 2.4.19]. If moreover $U \subseteq H^\circ$ is a dense R-subbialgebra, then $U \subseteq H^*$ is dense, $\alpha^P_{A\otimes_R H}$ is injective and we are done. □

Corollary 2.12. *Let H be a Hopf R-algebra with twisted antipode, $A \#_\sigma H$ a right H-crossed product with invertible cocycle and consider the R-pairing $P := (H, H^*)$. Then we have an isomorphism of R-algebras*

$$(A \#_\sigma H) \# H^* \simeq A \otimes_R \left(H \# H^*\right)$$

at least when:

(1) *${}_R H$ is finitely generated projective, or*
(2) *${}_R A$ is finitely generated, H is cocommutative and $\alpha^P_{A\otimes_R H}$ is injective* (*e.g., R is Noetherian and $H^* \hookrightarrow R^H$ is $A \otimes_R H$-pure*).

Proof. (1) Since ${}_R H$ is finitely generated projective, the canonical R-linear map $J : A \otimes_R H^* \to \mathrm{Hom}_R(H, A)$ is bijective and H^* satisfies the RL-condition (12) with respect to H by Lemma 2.4(3), hence (H^*, H^*) is compatible. Moreover $P = (H, H^*) \simeq (H^{**}, H^*)$ satisfies the α-condition, since ${}_R H^*$ is finitely generated projective. The result follows now by Proposition 2.7.

(2) Since ${}_R A$ is finitely generated, the canonical R-linear map $J : A \otimes_R H^* \to \mathrm{Hom}_R(H, A)$ is surjective. Since H is cocommutative, H^* satisfies the RL-condition (12) with respect to H, hence (H^*, H^*) is compatible. By assumption $\alpha^P_{A\otimes_R H}$ is injective and we are done by Proposition 2.7. $\square$

Corollary 2.13. *Let H be a free Hopf R-algebra of rank n and $A \#_\sigma H$ a right H-crossed product with invertible cocycle. Then we have an isomorphism of R-algebras*

$$(A \#_\sigma H) \# H^* \simeq A \otimes_R M_n(R) \simeq M_n(A).$$

Proof. By Corollary 2.12 $(A \#_\sigma H) \# H^* \simeq A \otimes_R (H \# H^*)$. Since ${}_R H$ is finitely generated projective, $H \# H^* \simeq \mathrm{End}_R(H)$ by Lemma 2.4(3). But ${}_R H$ is free of rank n, hence $\mathrm{End}_R(H) \simeq M_n(R)$. It is evident that $A \otimes_R M_n(R) \simeq M_n(A)$ and we are done. $\square$

The right H^*-submodule $H^\upsilon \subseteq H^*$

In what follows let H be a *locally projective* Hopf R-algebra with *twisted antipode* and consider the measuring α-pairing $P := (H^*, H)$ (notice that the canonical R-linear map $\alpha^P_R : H \to H^{**}$ is injective).

Lemma 2.14. *Consider H^* with the right H^*-action*

$$(f \leftharpoonup g)(h) := \sum g\big(h_3 \overline{S}(h_1)\big) f(h_2) \quad \textit{for all } f, g \in H^* \textit{ and } h \in H.$$

Then H^* is a right H^*-module and $H^\upsilon := {}^H\mathrm{Rat}(H^*_{H^*})$ is a left H-comodule with structure map $\upsilon : H^\upsilon \to H \otimes_R H^\upsilon$.

Proof. For arbitrary $f, g, \tilde{g} \in H^*$ we have

$$\begin{aligned}
\big(f \leftharpoonup (g \star \tilde{g})\big)(h) &= \sum \big(g \star \tilde{g}\big)\big(h_3 \overline{S}(h_1)\big) f(h_2) \\
&= \sum g\big(h_{31} \overline{S}(h_1)_1\big) \tilde{g}\big(h_{32} \overline{S}(h_1)_2\big) f(h_2) \\
&= \sum g\big(h_{31} \overline{S}(h_{12})\big) \tilde{g}\big(h_{32} \overline{S}(h_{11})\big) f(h_2) \\
&= \sum g\big(h_4 \overline{S}(h_2)\big) \tilde{g}\big(h_5 \overline{S}(h_1)\big) f(h_3) \\
&= \sum \tilde{g}\big(h_3 \overline{S}(h_1)\big) g\big(h_{23} \overline{S}(h_{21})\big) f(h_{22}) \\
&= \sum \tilde{g}\big(h_3 \overline{S}(h_1)\big) (f \leftharpoonup g)(h_2) \\
&= \big((f \leftharpoonup g) \leftharpoonup \tilde{g}\big)(h).
\end{aligned}$$

Since ${}_R H$ is locally projective, analogously to Theorem 1.5 we have that ${}^H\mathrm{Rat}(H^*_{H^*})$ is a left H-comodule. $\square$

Proposition 2.15. *Consider the left H-comodule (H^υ, υ).*

(1) *If $f \in H^\upsilon$, then $\upsilon(f) = \sum f_{\langle -1\rangle} \otimes f_{\langle 0\rangle}$ satisfies the following conditions*:
  (a) $f \star g = \sum g f_{\langle -1\rangle} \star f_{\langle 0\rangle}$ *for all* $g \in H^*$;
  (b) $h \leftharpoonup f = \sum f_{\langle -1\rangle}(f_{\langle 0\rangle} \rightharpoonup h)$ *for all* $h \in H$;
  (c) $\sum h_3 \overline{S}(h_1) f(h_2) = \sum f_{\langle -1\rangle} f_{\langle 0\rangle}(h)$ *for all* $h \in H$.

(2) *Let $f \in H^*$. If there exists $\zeta = \sum f_{\langle -1\rangle} \otimes f_{\langle 0\rangle} \in H \otimes_R H^*$ that satisfies any of the conditions in* (1), *then $f \in H^\upsilon$ and $\upsilon(f) = \zeta$.*

(3) *For all $f, \tilde{f} \in H^\upsilon$ and $g \in H^*$ we have*

$$\big(f \star \tilde{f}\big) \star g = \sum g\big(\tilde{f}_{\langle -1\rangle} f_{\langle -1\rangle}\big) \star \big(f_{\langle 0\rangle} \star \tilde{f}_{\langle 0\rangle}\big).$$

(4) *$H^\upsilon \subseteq H^*$ is a right H-submodule with*

$$\upsilon(fh) = \sum \overline{S}(h_3) f_{\langle -1\rangle} h_1 \otimes f_{\langle 0\rangle} h_2 \quad \textit{for all } h \in H \textit{ and } f \in H^\upsilon.$$

Proof. (1) Let $f \in H^\upsilon$ with $\upsilon(f) = \sum f_{\langle -1\rangle} \otimes f_{\langle 0\rangle}$.

(a) For all $g \in H^*$ and $h \in H$ we have

$$\begin{aligned}
(f \star g)(h) &= \sum f(h_1) g(h_2) = \sum g\big(h_3 \overline{S}(h_{12}) h_{11}\big) f(h_2) \\
&= \sum g\big(h_{23} \overline{S}(h_{21}) h_1\big) f(h_{22}) = \sum (h_1 g)\big(h_{23} \overline{S}(h_{21})\big) f(h_{22}) \\
&= \sum \big(f \leftharpoonup (h_1 g)\big)(h_2) = \sum (h_1 g)(f_{\langle -1\rangle}) f_{\langle 0\rangle}(h_2) \\
&= \sum g(f_{\langle -1\rangle} h_1) f_{\langle 0\rangle}(h_2) = \sum (g f_{\langle -1\rangle})(h_1) f_{\langle 0\rangle}(h_2) \\
&= \Big(\sum (g f_{\langle -1\rangle}) \star f_{\langle 0\rangle}\Big)(h).
\end{aligned}$$

(b) For all $g \in H^*$ and $h \in H$ we have

$$\begin{aligned}
g(h \leftharpoonup f) &= g\Big(\sum f(h_1) h_2\Big) = \sum f(h_1) g(h_2) \\
&= \sum f(h_2) g\big(h_3 \overline{S}(h_{12}) h_{11}\big) = \sum f(h_{22}) g\big(h_{23} \overline{S}(h_{21}) h_1\big) \\
&= \sum f(h_{22}) (h_1 g)\big(h_{23} \overline{S}(h_{21})\big) = \sum \big(f \leftharpoonup (h_1 g)\big)(h_2) \\
&= \sum (h_1 g)(f_{\langle -1\rangle}) f_{\langle 0\rangle}(h_2) = g\Big(\sum f_{\langle -1\rangle} h_1 f_{\langle 0\rangle}(h_2)\Big) \\
&= g\Big(\sum f_{\langle -1\rangle}(f_{\langle 0\rangle} \rightharpoonup h)\Big).
\end{aligned}$$

(c) Trivial.

(2) Let $f \in H^*$ and $\zeta = \sum f_{\langle -1\rangle} \otimes f_{\langle 0\rangle} \in H \otimes_R H^*$. We are done once we have shown that $(f \leftharpoonup g)(h) = \sum g(f_{\langle -1\rangle}) f_{\langle 0\rangle}(h)$ for arbitrary $g \in H^*$ and $h \in H$.

(a) Assume (1)(a) holds. Then we have

$$\begin{aligned}
(f \leftharpoonup g)(h) &= \sum g\big(h_3\overline{S}(h_1)\big) f(h_2)\\
&= \sum \big(\overline{S}(h_1)g\big)(h_{22}) f(h_{21})\\
&= \sum \big(f \star \overline{S}(h_1)g\big)(h_2)\\
&= \sum \big(\overline{S}(h_1)gf_{\langle -1\rangle} \star f_{\langle 0\rangle}\big)(h_2)\\
&= \sum \big(\overline{S}(h_1)gf_{\langle -1\rangle}\big)(h_2) f_{\langle 0\rangle}(h_3)\\
&= \sum g\big(f_{\langle -1\rangle}h_2\overline{S}(h_1)\big) f_{\langle 0\rangle}(h_3)\\
&= \sum g(f_{\langle -1\rangle}) f_{\langle 0\rangle}(h).
\end{aligned}$$

(b) Assume (1)(b) holds. Then we have

$$\begin{aligned}
(f \leftharpoonup g)(h) &= \sum g\big(h_3\overline{S}(h_1)\big) f(h_2)\\
&= \sum \big(\overline{S}(h_1)g\big)(h_2 \leftharpoonup f)\\
&= \sum \big(\overline{S}(h_1)g\big)\big(f_{\langle -1\rangle}(f_{\langle 0\rangle} \rightharpoonup h_2)\big)\\
&= \sum \big(\overline{S}(h_1)g\big)\big(f_{\langle -1\rangle}h_2 f_{\langle 0\rangle}(h_3)\big)\\
&= \sum g\big(f_{\langle -1\rangle}h_2\overline{S}(h_1)\big) f_{\langle 0\rangle}(h_3)\\
&= \sum g(f_{\langle -1\rangle}) f_{\langle 0\rangle}(h).
\end{aligned}$$

(c) Trivial.
(3) Let $f, \tilde{f} \in H^{\upsilon}$. For arbitrary $g \in H^*$ we have by (1)(a):

$$\begin{aligned}
\big(f \star \tilde{f}\big) \star g &= f \star \big(\tilde{f} \star g\big)\\
&= \sum f \star \big(g\tilde{f}_{\langle -1\rangle} \star \tilde{f}_{\langle 0\rangle}\big)\\
&= \sum \big(f \star g\tilde{f}_{\langle -1\rangle}\big) \star \tilde{f}_{\langle 0\rangle}\\
&= \sum \big(g\tilde{f}_{\langle -1\rangle}\big) f_{\langle -1\rangle} \star \big(f_{\langle 0\rangle} \star \tilde{f}_{\langle 0\rangle}\big)\\
&= \sum g\big(\tilde{f}_{\langle -1\rangle} f_{\langle -1\rangle}\big) \star \big(f_{\langle 0\rangle} \star \tilde{f}_{\langle 0\rangle}\big).
\end{aligned}$$

(4) Let $f \in H^{\upsilon}$ and $h \in H$. Then we have for all $g \in H^*$ and $k \in H$:

$$\begin{aligned}
\big((fh) \leftharpoonup g\big)(k) &= \sum g\big(k_3\overline{S}(k_1)\big)(fh)(k_2)\\
&= \sum g\big(k_3\overline{S}(k_1)\big) f(hk_2)
\end{aligned}$$

$$
\begin{aligned}
&= \sum g\big(\overline{S}(h_4)h_3k_3\overline{S}(k_1)\overline{S}(h_{12})h_{11}\big) f(h_2k_2)\\
&= \sum g\big(\overline{S}(h_3)h_{23}k_3\overline{S}(k_1)\overline{S}(h_{21})h_1\big) f(h_{22}k_2)\\
&= \sum \big(h_1 g\overline{S}(h_3)\big)\big(h_{23}k_3\overline{S}(k_1)\overline{S}(h_{21})\big) f(h_{22}k_2)\\
&= \sum \big(h_1 g\overline{S}(h_3)\big)\big(h_{23}k_3\overline{S}(h_{21}k_1)\big) f(h_{22}k_2)\\
&= \sum \big(h_1 g\overline{S}(h_3)\big)\big((h_2k)_3\overline{S}\big((h_2k)_1\big)\big) f\big((h_2k)_2\big)\\
&= \sum \big(f \leftharpoonup \big(h_1 g\overline{S}(h_3)\big)\big)(h_2k)\\
&= \sum \big(h_1 g\overline{S}(h_3)\big)(f_{\langle -1\rangle}) f_{\langle 0\rangle}(h_2k)\\
&= \sum g\big(\overline{S}(h_3) f_{\langle -1\rangle} h_1\big)(f_{\langle 0\rangle}h_2)(k). \qquad \square
\end{aligned}
$$

As a consequence of Proposition 2.7 and Theorem 2.9 and analogous to [23, Theorem 4.2] we get

Theorem 2.16. *Let H be a locally projective Hopf R-algebra with twisted antipode, $U \subseteq H^*$ a right H-module subalgebra, $P := (H, U)$ the induced R-pairing and assume that $\varphi(H \otimes_R A)$, $\psi(H \otimes_R A) \subseteq J(A \otimes_R \upsilon^{-1}(H \otimes_R U))$. If $\alpha^P_{A\otimes_R H}$ is injective* (*e.g., R is Noetherian and $U \subset R^H$ is A-pure*), *then there is an R-algebra isomorphism*

$$(A \#_\sigma H) \# U \simeq A \otimes_R (H \# U).$$

Proof. It follows from Proposition 2.15(4) that $V := \upsilon^{-1}(H \otimes_R U) \subset H^*$ is a right H-submodule. Since $V \subseteq H^\upsilon$, it follows by Proposition 2.15(1)(b) that (V, U) satisfies the RL-condition (12) with respect to H. Consequently (V, U) is compatible. If $\alpha^P_{A\otimes_R H}$ is injective, then the result follows by Proposition 2.7. $\square$

Corollary 2.17. *Let H be a locally projective Hopf R-algebra with twisted antipode, $U \subseteq H^\upsilon$ a right H-module subalgebra of H^*, $P := (H, U)$ the induced R-pairing and assume that $\upsilon(U) \subseteq H \otimes_R U$ and $\bar{\varphi}(H \otimes_R A)$, $\bar{\psi}(H \otimes_R A) \subseteq J(A \otimes_R U)$. If $\alpha^P_{A\otimes_R H}$ is injective* (*e.g., R is Noetherian and $U \subset R^H$ is A-pure*), *then there is an R-algebra isomorphism*

$$(A \#_\sigma H) \# U \simeq A \otimes_R (H \# U).$$

Blattner–Montgomery duality theorem revisited

The following definition is suggested by [9, Definition 1.3]:

Definition 2.18. Let R be Noetherian, H an R-bialgebra, $U \subseteq H^\circ$ an R-submodule and A a left H-module algebra. Then A will be called *U-locally finite* if and only if for every $a \in A$ there exists a finite subset $\{f_1, \ldots, f_k\} \subset U$, such that $\bigcap_{i=1}^k \mathrm{Ke}(f_i) \subseteq (0_A : a)$.

Lemma 2.19 [1, Proposition 3.3]. *Let R be Noetherian, H an α-bialgebra, $U \subseteq H^\circ$ an R-subbialgebra and consider the measuring α-pairing (H, U).*

(1) *If A is a right (a left) U-comodule algebra, then A is a left (a right) H-module algebra.*
(2) *If A is a left (a right) H-module algebra, then $\mathrm{Rat}^U({}_H A)$ is a right (a left) U-comodule algebra.*

The following result generalizes [9, Lemma 1.5] from the case of a base field to the case of an arbitrary Noetherian ground ring.

Lemma 2.20. *Let R be Noetherian, A an R-algebra, H an α-bialgebra and $U \subseteq H^\circ$ an R-subbialgebra. Then A is a U-locally finite left H-module algebra if and only if A is a right U-comodule algebra.*

Proof. Consider the measuring α-pairing (H, U). Assume A to be a right U-comodule algebra. Then A is a left H-module algebra by Lemma 2.19(1). Moreover for every $a \in A$ with $\varrho(a) = \sum_{j=1}^n a_j \otimes g_j \in A \otimes_R U$ we have $\bigcap_{j=1}^n \mathrm{Ke}(g_i) \subseteq (0_A : a)$, i.e., ${}_H A$ is U-locally finite. On the other hand, assume A to be a U-locally finite left H-module algebra and consider H with the left U-adic topology $\mathcal{T}_{U-}(H)$ (see 1.3). By Lemma 2.19(2) $\mathrm{Rat}^U({}_H A)$ is a right U-comodule algebra and we are done once we have shown that $A = \mathrm{Rat}^U({}_H A)$. By assumption there exists for every $a \in A$ a subset $W = \{f_1, \dots, f_k\} \subset U$, such that $\bigcap_{i=1}^k \mathrm{Ke}(f_i) \subseteq (0_A : a)$. If $h \in (0_U : W)$, then $f_i(h) = (hf_i)(1_H) = 0$ for $i = 1, \dots, k$ and so $(0_U : W) \subseteq \bigcap_{i=1}^k \mathrm{Ke}(f_i) \subseteq (0_A : a)$, i.e., A is a discrete left $(H, \mathcal{T}_{U-}(H))$-module (see 1.3). Consequently $A \in \sigma[{}_H U] = \mathrm{Rat}^U({}_H \mathcal{M})$ (see Theorem 1.5), i.e., $A = \mathrm{Rat}^U({}_H A)$. □

The following result provides an improved version of Blattner–Montgomery duality theorem for the case of arbitrary Noetherian base rings, replacing the assumption "$U \subseteq H^\circ$ is a Hopf R-subalgebra with bijective antipode" in the original version [9, Theorem 2.1] and in [5, 3.2] with "$U \subseteq H^\circ$ is any R-subbialgebra" (as suggested by M. Koppinen [23, Corollary 5.4]); and replacing the assumption that H has a bijective antipode with the weaker condition that H has a twisted antipode $\overline{S}$.

Corollary 2.21. *Let R be Noetherian, H a Hopf α-algebra with twisted antipode and $U \subseteq H^\circ$ an R-subbialgebra. Let A be a U-locally finite left H-module algebra and consider A with the induced right H-comodule structure. If there exists a right H-submodule $V \subseteq H^*$, such that $\mathrm{Cf}(A) \cup \overline{S}^*(\mathrm{Cf}(A)) \subseteq V$ and (V, U) satisfies the RL-condition* (12) *with respect to H, then we have an isomorphism of R-algebras*

$$(A \# H) \# U \simeq A \otimes_R (H \# U).$$

Proof. For the trivial cocycle $\sigma(h \otimes k) := \varepsilon(h)\varepsilon(k)1_A$ we have $A \#_\sigma H = A \# H$. Consider the canonical R-linear map $J : A \otimes_R V \to \mathrm{Hom}_R(H, A)$. For every $h \in H$ and $a \in A$ we have

$$\varphi(h\otimes a)(\tilde{h}) = [\bar{S}(\tilde{h})a]\varepsilon(h) = \sum a_{\langle 0\rangle}\langle \bar{S}(\tilde{h}), a_{\langle 1\rangle}\rangle \varepsilon(h)$$
$$= \sum a_{\langle 0\rangle}\bar{S}^*(a_{\langle 1\rangle})(\tilde{h})\varepsilon(h) = J\Big(\sum a_{\langle 0\rangle}\varepsilon(h)\otimes \bar{S}^*(a_{\langle 1\rangle})\Big)(\tilde{h}),$$

i.e., $\varphi(H\otimes_R A)\subseteq J(A\otimes_R V)$. On the other hand we have for all $h,\tilde{h}\in H$ and $a\in A$:

$$\psi(h\otimes a)(\tilde{h}) = [\tilde{h}a]\varepsilon(h) = \sum a_{\langle 0\rangle}\langle \tilde{h}, a_{\langle 1\rangle}\rangle \varepsilon(h) = J\Big(\sum a_{\langle 0\rangle}\varepsilon(h)\otimes a_{\langle 1\rangle}\Big)(\tilde{h}),$$

i.e., $\psi(H\otimes_R A)\subseteq J(A\otimes_R U)$. By assumption (V,U) satisfies the RL-condition (12) with respect to H, hence (V,U) is compatible and the result follows then by Theorem 2.9 (notice that $P=(H,U)$ is an α-pairing). □

As a consequence of Corollaries 2.11 and 2.21 we get

Corollary 2.22. *Let R be Noetherian, H a projective Hopf α-algebra with twisted antipode and $U\subseteq H^\circ$ an R-subbialgebra. Let A be a U-locally finite left H-module algebra and consider A with the induced right H-comodule structure. Assume there exists a right H-submodule $V\subseteq H^*$, such that* $\mathrm{Cf}(A)\cup\bar{S}^*(\mathrm{Cf}(A))\subseteq V$ *and (V,U) satisfies the RL-condition* (12) *with respect to H. If $U\subseteq H^*$ is dense, then there exists a dense R-subalgebra $\mathcal{L}\subseteq \mathrm{End}_R(H)$ and an R-algebra isomorphism*

$$(A\,\#\,H)\,\#\,U \simeq A\otimes_R \mathcal{L}.$$

In particular this holds, if R is a QF ring, H is residually finite and $U\subseteq H^\circ$ is dense.

Cleft H-extensions

Hopf–Galois extensions were presented by S. Chase and M. Sweedler [15] for a *commutative* R-algebra acting on a Hopf R-algebra and are considered as generalization of the classical Galois extensions over fields (e.g., [25, 8.1.2]). In [24] H. Kreimer and M. Takeuchi extended these to the *noncommutative* case.

2.23. H-extensions [17]

Let H be an R-bialgebra, B a right H-comodule algebra and consider the R-algebra $A:=B^{\mathrm{co}H}=\{a\in B\mid \varrho(a)=a\otimes 1_H\}$. The algebra extension $A\hookrightarrow B$ is called a *right H-extension*. A (*total*) *integral* for B is an *H-colinear* map $\theta: H\to B$ (with $\theta(1_H)=1_B$). If B admits an integral, which is invertible in $(\mathrm{Hom}_R(H,B),\star)$, then the right H-extension $A\hookrightarrow B$ is called *cleft.*

Lemma 2.24 ([20, Theorems 9, 11], [8, Theorem 1.18], [19, 1.1.]). *Let H be an R-bialgebra.*

(1) *If B/A is a* cleft *right H-extension with total invertible integral $\theta: H \to B$, then A is a left H-module algebra through*

$$ha = \sum \theta(h_1) a \theta^{-1}(h_2) \quad \textit{for all } h \in H \textit{ and } a \in A$$

and $A \#_\sigma H$ is a right H-crossed product with invertible cocycle

$$\sigma(h \otimes k) = \sum \theta(h_1)\theta(k_1)\theta^{-1}(h_2 k_2),$$
$$\textit{where} \quad \sigma^{-1}(h \otimes k) = \sum \theta(h_1 k_1)\theta^{-1}(k_2)\theta^{-1}(h_2).$$

Moreover $B \simeq A \#_\sigma H$ as right H-comodule algebras.

(2) *Let H be a Hopf R-algebra. If $B := A \#_\sigma H$ is a right H-crossed product with invertible cocycle $\sigma \in \mathrm{Hom}_R(H \otimes_R H, A)$, then B/A is a* cleft *right H-extension with invertible total integral*

$$\theta: H \to A \#_\sigma H, \quad \theta(h) = 1_A \# h,$$
$$\textit{where} \quad \theta^{-1}(h) = \sum \sigma^{-1}\big(S(h_2) \otimes h_3\big) \#_\sigma S(h_1).$$

Let H be a Hopf R-algebra with twisted antipode, B/A a cleft right H-extension with invertible total integral $\theta: H \to B$ and consider the R-linear maps $\tilde{\varphi}, \tilde{\psi}: A \otimes_R H \to \mathrm{Hom}_R(H, A)$ defined as:

$$\tilde{\varphi}(h \otimes a)\big(\tilde{h}\big) = \sum \theta\big(\overline{S}\big(\tilde{h}_2\big)\big) a \theta(h_1) \theta^{-1}\big(\overline{S}\big(\tilde{h}_1\big) h_2\big), \tag{15}$$

$$\tilde{\psi}(h \otimes a)\big(\tilde{h}\big) = \sum \theta^{-1}\big(\overline{S}\big(\tilde{h}_3\big)\big) a \theta\big(\overline{S}\big(\tilde{h}_2\big) h_1\big) \theta^{-1}\big(\tilde{h}_4 \overline{S}\big(\tilde{h}_1\big) h_2\big). \tag{16}$$

With the help of Lemma 2.24 one can easily derive the following version of Theorem 2.9 and Corollary 2.11 for *cleft* right H-extensions:

Theorem 2.25. *Let R be Noetherian, H a Hopf R-algebra with twisted antipode, B/A a* cleft *right H-extension with invertible total integral $\theta: H \to B$, $U \subseteq H^*$ a right H-module subalgebra and consider the R-pairing $P := (H, U)$. Assume there exists a right H-submodule $V \subseteq H^*$, such that*:

(1) $\tilde{\varphi}(H \otimes_R A),\ \tilde{\psi}(H \otimes_R A) \subseteq J(A \otimes_R V)$;
(2) (V, U) *satisfies the* RL-condition (12) *with respect to H.*

If $U \subset R^H$ is $A \otimes_R H$-pure (*e.g., H is a Hopf α-algebra and $U \subseteq H^\circ$ is an R-subbialgebra*)*, then there is an R-algebra isomorphism*

$$B \# U \simeq A \otimes_R (H \# U).$$

If moreover ${}_R H$ is projective and $U \subseteq H^*$ is dense (*e.g., R is a QF ring, H is residually finite and $U \subseteq H^\circ$ is dense*)*, then $B \# U \simeq A \otimes_R \mathcal{L}$ for a* dense *R-subalgebra $\mathcal{L} \subseteq \mathrm{End}_R(H)$.*

3. Koppinen duality theorem

In this section we prove an improved version of Koppinen's duality theorem presented in [23] over arbitrary Noetherian ground rings under fairly weak conditions. In fact the results in this section are similar to those in the second section with a main advantage, that they are evident for arbitrary Hopf R-algebras (not necessarily with twisted antipodes).

3.1. Let H be an R-bialgebra and B a right H-comodule algebra. Then $\#^{\mathrm{op}}(H,B) = \mathrm{Hom}_R(H,B)$ is an associative R-algebra with multiplication

$$(f \,\tilde{\star}\, g)(h) = \sum f(h_2)_{\langle 0\rangle} g\big(h_1 f(h_2)_{\langle 1\rangle}\big) \quad \text{for all } f,g \in \mathrm{Hom}_R(H,B),\ h \in H \tag{17}$$

and unity $\eta_B \circ \varepsilon_H$. If $U \subseteq H^*$ is a *left* H-module subalgebra (with $\varepsilon_H \in U$), then $B \,\#^{\mathrm{op}}\, U = B \otimes_R U$ is an associative R-algebra with multiplication

$$(b \,\#\, f)\big(\tilde{b} \,\#\, \tilde{f}\big) = \sum b_{\langle 0\rangle}\tilde{b} \,\#\, \big(\big(b_{\langle 1\rangle}\tilde{f}\big) \star f\big) \quad \text{for all } b,\tilde{b} \in B,\ f,\tilde{f} \in U \tag{18}$$

(and unity $1_B \,\#\, \varepsilon_H$).

Definition 3.2. Let H be an R-bialgebra, $U \subseteq H^*$ a left H-module subalgebra under the left regular H-action, $V \subseteq H^*$ an R-submodule and consider the R-linear maps

$$\begin{aligned}\bar{\lambda} : H \,\#^{\mathrm{op}}\, U \to \mathrm{End}_R(H), &\quad \sum h_j \otimes g_j \mapsto \Big[\tilde{k} \mapsto \sum \big(g_j \rightharpoonup \tilde{k}\big)h_j\Big],\\ \bar{\rho} : V \to \mathrm{End}_R(H), &\quad g \mapsto \big[\tilde{k} \mapsto \tilde{k} \leftharpoonup g\big].\end{aligned} \tag{19}$$

We say (V,U) satisfies the *RL-condition with respect to H*, if $\bar{\rho}(V) \subseteq \bar{\lambda}(H \,\#^{\mathrm{op}}\, U)$, i.e., if

$$\begin{aligned}&\text{for every } g \in V, \quad \exists\big\{(h_j,g_j)\big\} \subset H \times U\\ &\text{s.t.}\quad \tilde{k} \leftharpoonup g = \sum \big(g_j \rightharpoonup \tilde{k}\big)h_j \quad \text{for every } \tilde{k} \in H.\end{aligned} \tag{20}$$

Lemma 3.3. *Let H be an R-bialgebra, $U \subseteq H^*$ a left H-module subalgebra and consider H as a right H-comodule algebra through Δ_H. Let $\#^{\mathrm{op}}(H,H)$ and $H \,\#^{\mathrm{op}}\, U$ be the R-algebras defined in* 3.1 *and consider the canonical R-algebra morphism $\bar{\beta} : H \,\#^{\mathrm{op}}\, U \to \#^{\mathrm{op}}(H,H)$.*

(1) *If ${}_R H$ is finitely generated projective, then $H \,\#^{\mathrm{op}}\, H^* \overset{\bar{\beta}}{\simeq} \#^{\mathrm{op}}(H,H)$ as R-algebras.*
(2) *If H is a Hopf R-algebra, then $\#^{\mathrm{op}}(H,H) \simeq \mathrm{End}_R(H)^{\mathrm{op}}$ as R-algebras.*
(3) *Let H be a finitely generated projective Hopf R-algebra. Then $\bar{\lambda} : H \,\#^{\mathrm{op}}\, H^* \to \mathrm{End}_R(H)^{\mathrm{op}}$, defined in* (19)*, is an R-algebra isomorphism. In particular H^* satisfies the RL-condition* (20) *with respect to H.*

(4) *If* ${}_R H$ *is locally projective and* $U \subseteq H^*$ *is* dense, *then* $\bar{\beta}(H \#^{\mathrm{op}} U) \subseteq \#^{\mathrm{op}}(H, H)$ *is a dense* R*-subalgebra. If moreover* H *is a projective Hopf* R*-algebra, then*

$$H \#^{\mathrm{op}} U \overset{\bar{\lambda}}{\hookrightarrow} \mathrm{End}_R(H)^{\mathrm{op}}$$

is a dense R*-subalgebra.*

Proof. (1) Since ${}_R H$ is finitely generated projective, $\bar{\beta}$ is bijective.

(2) Let H be a Hopf R-algebra and consider the R-linear maps

$$\bar{\phi}_1 : \#^{\mathrm{op}}(H, H) \to \mathrm{End}_R(H)^{\mathrm{op}}, \quad f \mapsto \Big[h \mapsto \sum h_1 f(h_2)\Big],$$
$$\bar{\phi}_2 : \mathrm{End}_R(H)^{\mathrm{op}} \to \#^{\mathrm{op}}(H, H), \quad g \mapsto \Big[k \mapsto \sum S(k_1) g(k_2)\Big].$$

For arbitrary $f, g \in \#^{\mathrm{op}}(H, H)$ and $h \in H$ we have

$$\begin{aligned}\bar{\phi}_1(f \,\tilde{\star}\, g)(h) &= \sum h_1 (f \,\tilde{\star}\, g)(h_2) = \sum h_1 f(h_3)_1 g\big(h_2 f(h_3)_2\big)\\ &= \sum h_{11} f\big(h_2)_1 g\big(h_{12} f(h_2)_2\big) = \bar{\phi}_1(g)\Big(\sum h_1 f(h_2)\Big)\\ &= \big(\bar{\phi}_1(g) \circ \bar{\phi}_1(f)\big)(h),\end{aligned}$$

i.e., $\bar{\phi}_1$ is an R-algebra morphism. For all R-linear maps $f, g : H \to H$ and $h \in H$ we have

$$\begin{aligned}\big(\bar{\phi}_1 \circ \bar{\phi}_2\big)(g)(h) &= \sum h_1 \bar{\phi}_2(g)(h_2) = \sum h_1 S(h_2) g(h_3)\\ &= \sum \varepsilon(h_1) g(h_2) = g(h),\\ \big(\bar{\phi}_2 \circ \bar{\phi}_1\big)(f)(h) &= \sum S(h_1) \bar{\phi}_1(f)(h_2) = \sum S(h_1) h_2 f(h_3)\\ &= \sum \varepsilon(h_1) f(h_2) = f(h).\end{aligned}$$

Hence $\bar{\phi}_1$ is an R-algebra isomorphism with inverse $\bar{\phi}_2$.

(3) Let H be a finitely generated projective Hopf R-algebra. By (1) and (2)

$$H \#^{\mathrm{op}} H^* \overset{\bar{\beta}}{\simeq} \#^{\mathrm{op}}(H, H) \overset{\bar{\phi}_1}{\simeq} \mathrm{End}_R(H)^{\mathrm{op}}$$

as R-algebras. Hence $\bar{\lambda} = \bar{\phi}_1 \circ \bar{\beta} : H \#^{\mathrm{op}} H^* \to \mathrm{End}_R(H)^{\mathrm{op}}$ is an R-algebra isomorphism. In particular $\bar{\rho}(H^*) \subseteq \mathrm{End}_R(H)^{\mathrm{op}} = \bar{\lambda}(H \#^{\mathrm{op}} H^*)$, i.e., H^* satisfies the RL-condition (20) with respect to H.

(4) By [3, Theorem 3.18(2)] $\bar{\beta}(H \#^{\mathrm{op}} U) \subseteq \#^{\mathrm{op}}(H, H)$ is a dense R-subalgebra. If H is a Hopf R-algebra, then

$$\#(H, H) \overset{\bar{\phi}_1}{\simeq} \mathrm{End}_R(H)^{\mathrm{op}}$$

as R-algebras by (2) and we are done (notice that $\bar{\beta}$ is an embedding, if ${}_R H$ is projective). $\square$

Lemma 3.4. *Let H be a Hopf R-algebra, A an R-algebra, $U \subseteq H^*$ a left H-submodule and consider the R-paring $\overline{P} := (H, U)$. Then the canonical R-linear map $\bar{\alpha} := \alpha^{\overline{P}}_{A \otimes_R H} : (A \otimes_R H) \otimes_R U \to \mathrm{Hom}_R(H, A \otimes_R H)$ is injective if and only if the following map is injective*

$$\begin{aligned} &\bar{\chi} : A \otimes_R (H \otimes_R U) \to \mathrm{End}_{A-}(A \otimes_R H), \\ &a \otimes (h \otimes f) \mapsto \big[(\tilde{a} \otimes k) \mapsto \tilde{a}a \otimes (f \rightharpoonup k)h\big]. \end{aligned} \tag{21}$$

Proof. First we show that the R-linear map

$$\bar{\epsilon} : \mathrm{Hom}_R(H, A \otimes_R H) \to \mathrm{End}_{A-}(A \otimes_R H), \quad g \mapsto \big[\tilde{a} \otimes k \mapsto (\tilde{a} \otimes k_1)g(k_2)\big]$$

is bijective with inverse

$$\bar{\epsilon}^{-1} : \mathrm{End}_{A-}(A \otimes_R H) \to \mathrm{Hom}_R(H, A \otimes_R H), \quad f \mapsto \big[k \mapsto (1_A \otimes S(k_1))f(1_A \otimes k_2)\big].$$

In fact we have for all $f \in \mathrm{End}_{A-}(A \otimes_R H)$, $k \in H$, $\tilde{a} \in A$:

$$\begin{aligned} \bar{\epsilon}\big(\bar{\epsilon}^{-1}(f)\big)\big(\tilde{a} \otimes k\big) &= \sum (\tilde{a} \otimes k_1)\bar{\epsilon}^{-1}(f)(k_2) \\ &= \sum (\tilde{a} \otimes k_1)\big(1_A \otimes S(k_2)\big)f(1_A \otimes k_3) \\ &= \sum \big(\tilde{a} \otimes k_1 S(k_2)\big)f(1_A \otimes k_3) \\ &= \sum \big(\tilde{a} \otimes \varepsilon_H(k_1)1_H\big)f(1_A \otimes k_2) \\ &= \sum (\tilde{a} \otimes 1_H)f(1_A \otimes k) \\ &= f(\tilde{a} \otimes k) \end{aligned}$$

and for all $g \in \mathrm{Hom}_R(H, A \otimes_R H)$, $k \in H$:

$$\begin{aligned} \bar{\epsilon}^{-1}\big(\bar{\epsilon}(g)\big)(k) &= \sum \big(1_A \otimes S(k_1)\big)\bar{\epsilon}(g)(1_A \otimes k_2) \\ &= \sum \big(1_A \otimes S(k_1)\big)(1 \otimes k_2)g(k_3) \\ &= \sum \big(1_A \otimes S(k_1)k_2\big)g(k_3) \\ &= \sum \big(1_A \otimes \varepsilon_H(k_1)1_H\big)g(k_2) \\ &= (1_A \otimes 1_H)g(k) \\ &= g(k). \end{aligned}$$

Moreover we have for all $a \in A$, $h \in H$, $f \in U$ and $k \in H$:

$$\begin{aligned} (\bar{\epsilon} \circ \bar{\alpha})\big(a \otimes (h \otimes f)\big)\big(\tilde{a} \otimes k\big) &= \sum (\tilde{a} \otimes k_1)\alpha^{\overline{P}}_{A \otimes_R H}\big(a \otimes (h \otimes f)\big)(k_2) \\ &= \sum (\tilde{a} \otimes k_1)(a \otimes h)f(k_2) \\ &= \sum \tilde{a}a \otimes k_1 f(k_2)h \end{aligned}$$

$$
\begin{aligned}
&= \tilde{a}a \otimes (f \rightharpoonup k)h \\
&= \bar{\chi}\big(a \otimes (h \otimes f)\big)\big(\tilde{a} \otimes k\big),
\end{aligned}
$$

i.e., $\bar{\chi} = \bar{\epsilon} \circ \bar{\alpha}$. Consequently $\bar{\chi}$ is injective iff $\bar{\alpha}$ is so. □

3.5. Let H be a Hopf R-algebra, $A \#_\sigma H$ a right H-crossed product with invertible cocycle and consider the R-linear maps $\bar{\varphi}, \bar{\psi} : H \otimes_R A \to \mathrm{Hom}_R(H, A)$ defined as

$$
\begin{aligned}
\bar{\varphi}(h \otimes a)\big(\tilde{h}\big) &= \sum \big[\tilde{h}_1 a\big]\sigma\big(\tilde{h}_2 \otimes h\big), \\
\bar{\psi}(h \otimes a)\big(\tilde{h}\big) &= \sum \sigma^{-1}\big(S\big(\tilde{h}_3\big) \otimes \tilde{h}_4\big)\big[S\big(\tilde{h}_2\big)a\big]\sigma\big(S\big(\tilde{h}_1\big) \otimes \tilde{h}_5 h\big).
\end{aligned}
$$

Let $U \subseteq H^*$ be a *left* H-module subalgebra, $V \subseteq H^*$ an R-submodule and consider the R-linear map $J : A \otimes_R V \to \mathrm{Hom}_R(H, A)$. We say (V, U) is *compatible*, if the following conditions are satisfied:

(1) $\bar{\varphi}(H \otimes_R A),\ \bar{\psi}(H \otimes_R A) \subseteq J(A \otimes_R V)$;
(2) (V, U) satisfies the *RL-condition* (20) with respect to H.

Analogously to Proposition 2.7 and in the light of Lemma 3.4 and the modified RL-condition (20) we restate [12, Theorem 8, Corollary 9] for the case of an arbitrary commutative ground ring:

Proposition 3.6. *Let H be a Hopf R-algebra, $A \#_\sigma H$ be a right H-crossed product with invertible cocycle, $U \subseteq H^*$ a left H-module subalgebra and consider the R-pairing $\overline{P} := (H, U)$. Assume there exists an R-*submodule *$V \subseteq H^*$, such that (V, U) is compatible. If the canonical R-linear map*

$$
\bar{\alpha} := \alpha^{\overline{P}}_{A \otimes_R H} : (A \otimes_R H) \otimes_R U \to \mathrm{Hom}_R(H, A \otimes_R H)
$$

is injective, then there exists an R-algebra isomorphism

$$
(A \#_\sigma H) \#^{\mathrm{op}} U \simeq A \otimes_R \big(H \#^{\mathrm{op}} U\big).
$$

Proof. By [12, Lemma 7] we have a commutative diagram of R-algebra morphisms

$$
\begin{tikzcd}
 & (A \#_\sigma H) \#^{\mathrm{op}} U \arrow[dl, "\bar{\alpha}"'] \arrow[dr, "\bar{\gamma}"] & \\
\#^{\mathrm{op}}(H, A \#_\sigma H) \arrow[rr, "\bar{\pi}"] & & \mathrm{End}_{A-}(A \otimes_R H)^{\mathrm{op}} \\
 & A \otimes_R (H \#^{\mathrm{op}} U) \arrow[ul, "\bar{\delta}"] \arrow[ur, "\bar{\chi}"'] &
\end{tikzcd}
\tag{22}
$$

where

$$\begin{aligned}
\bar{\alpha}\big((a\,\#\,h)\,\#^{\mathrm{op}}\,f\big)(k) &= (a\,\#\,h)f(k),\\
\bar{\chi}\big(a\otimes\big(h\,\#^{\mathrm{op}}\,f\big)\big)\big(\tilde{a}\otimes k\big) &= \tilde{a}a\otimes(f\rightharpoonup k)h,\\
\bar{\gamma}\big((a\,\#\,h)\,\#^{\mathrm{op}}\,f\big)\big(\tilde{a}\otimes k\big) &= \sum \tilde{a}[k_1 a]\sigma(k_2\otimes h_1)\otimes(f\rightharpoonup k_3)h_2,\\
\bar{\delta}\big(a\otimes\big(h\,\#^{\mathrm{op}}\,f\big)\big)(k) &= \sum \sigma^{-1}\big(S(k_4)\otimes k_5\big)\big[S(k_3)a\big]\sigma\big(S(k_2)\otimes k_6 h_1\big)\\
&\quad\#\,S(k_1)(f\rightharpoonup k_7)h_2,\\
\bar{\pi}(g)\big(\tilde{a}\otimes k\big) &= \sum\big(\tilde{a}\,\#_\sigma\,k_1\big)g(k_2).
\end{aligned}$$

By assumption $\bar{\alpha} := \alpha^{\overline{P}}_{A\otimes_R H}$ is injective, hence $\bar{\chi}$ is by Lemma 3.4 injective and consequently $\bar{\delta}$ is injective. Moreover $\bar{\pi}$ is an R-algebra isomorphism by [23, Proposition 4.1], hence $\bar{\gamma}$ is injective. It remains then to show that $\mathrm{Im}(\bar{\gamma}) \subseteq \mathrm{Im}(\bar{\chi})$ and $\mathrm{Im}(\bar{\delta}) \subseteq \mathrm{Im}(\bar{\alpha})$. For arbitrary $a\otimes h \in A\otimes_R H$, there exists $\sum a_u\otimes g_u \in A\otimes_R V$ such that $\bar{\varphi}(h_1\otimes a) = J(\sum a_u\otimes g_u)$ and moreover there exists $\sum h_{uj}\,\#^{\mathrm{op}}\,g_{uj} \in H\otimes_R U$ with $\bar{\rho}(g_u) = \bar{\lambda}(\sum h_{uj}\,\#^{\mathrm{op}}\,g_{uj})$. So for all $a, \tilde{a}\in A$, $h, k\in H$ and $f\in U$:

$$\begin{aligned}
\bar{\gamma}\big((a\,\#\,h)\,\#^{\mathrm{op}}\,f\big)\big(\tilde{a}\otimes k\big) &= \sum \tilde{a}[k_1 a]\sigma(k_2\otimes h_1)\otimes(f\rightharpoonup k_3)h_2\\
&= \sum \tilde{a}[k_{11} a]\sigma(k_{12}\otimes h_1)\otimes(f\rightharpoonup k_2)h_2\\
&= \sum \tilde{a}\bar{\varphi}(h_1\otimes a)(k_1)\otimes(f\rightharpoonup k_2)h_2\\
&= \sum \tilde{a}J\Big(\sum a_u\otimes g_u\Big)(k_1)\otimes(f\rightharpoonup k_2)h_2\\
&= \sum \tilde{a}a_u g_u(k_1)\otimes(f\rightharpoonup k_2)h_2\\
&= \sum \tilde{a}a_u\otimes g_u(k_1)k_2 f(k_3)h_2\\
&= \sum \tilde{a}a_u\otimes(k_1\leftharpoonup g_u)f(k_2)h_2\\
&= \sum \tilde{a}a_u\otimes(g_{u,j}\rightharpoonup k_1)h_{u,j}f(k_2)h_2\\
&= \sum \tilde{a}a_u\otimes k_1 g_{u,j}(k_2)f(k_3)h_{u,j}h_2\\
&= \sum \tilde{a}a_u\otimes\big((g_{u,j}\star f)\rightharpoonup k\big)h_{u,j}h_2\\
&= \bar{\chi}\big(a_u\otimes\big(h_{u,j}h_2\,\#^{\mathrm{op}}\,(g_{u,j}\star f)\big)\big)\big(\tilde{a}\otimes k\big),
\end{aligned}$$

i.e., $\mathrm{Im}(\bar{\gamma}) \subseteq \mathrm{Im}(\bar{\chi})$. For arbitrary $a\otimes h\in A\otimes_R H$, there exists $\sum a_w\otimes g_w\in A\otimes_R V$ such that $\bar{\varphi}(h_1\otimes a) = J(\sum a_w\otimes g_w)$ and moreover there exists $\sum h_{wj}\,\#^{\mathrm{op}}\,g_{wj}\in H\otimes_R U$ with $\bar{\rho}(g_u) = \bar{\lambda}(\sum h_{wj}\,\#^{\mathrm{op}}\,g_{wj})$. So we have for all $a\in A$, $h, k\in H$ and $f\in U$:

$$\begin{aligned}
\bar{\delta}\big(a\otimes\big(h\,\#^{\mathrm{op}}\,f\big)\big)(k) &= \sum \sigma^{-1}\big(S(k_4)\otimes k_5\big)\big[S(k_3)a\big]\sigma\big(S(k_2)\otimes k_6 h_1\big)\,\#\,S(k_1)(f\rightharpoonup k_7)h_2\\
&= \sum \sigma^{-1}\big(S(k_{23})\otimes k_{24}\big)\big[S(k_{22})a\big]\sigma\big(S(k_{21})\otimes k_{25}h_1\big)\\
&\quad\#\,S(k_1)(f\rightharpoonup k_3)h_2
\end{aligned}$$

$$
\begin{aligned}
&= \sum \bar{\psi}(h_1 \otimes a)(k_2) \# S(k_1)(f \rightharpoonup k_3)h_2 \\
&= \sum J\Bigl(\sum a_w \otimes g_w\Bigr)(k_2) \# S(k_1)(f \rightharpoonup k_3)h_2 \\
&= \sum a_w g_w(k_2) \# S(k_1)(f \rightharpoonup k_3)h_2 \\
&= \sum a_w \# S(k_1) g_w(k_2) k_3 f(k_4) h_2 \\
&= \sum a_w \# S(k_1)(k_2 \leftharpoonup g_w) f(k_3) h_2 \\
&= \sum a_w \# S(k_1)(g_{w,j} \rightharpoonup k_2) h_{w,j} f(k_3) h_2 \\
&= \sum a_w \# S(k_1) k_2 g_{w,j}(k_3) f(k_4) h_{w,j} h_2 \\
&= \sum a_w \# g_{w,j}(k_1) f(k_2) h_{w,j} h_2 \\
&= \sum a_w \# (g_{w,j} \star f)(k) h_{w,j} h_2 \\
&= \bar{\alpha}\bigl((a_w \# h_{w,j} h_2) \#^{\mathrm{op}} g_{w,j} \star f\bigr)(k),
\end{aligned}
$$

i.e., $\mathrm{Im}(\bar{\delta}) \subseteq \mathrm{Im}(\bar{\alpha})$ and we are done. $\square$

As a consequence of Lemma 2.8 and Proposition 3.6 we get a theorem, analogous to Theorem 2.9, which generalizes [12, Corollary 9] (respectively [12, Corollary 10]) from the case of a base field (respectively a Dedekind domain) to the case of an arbitrary Noetherian ground ring:

Theorem 3.7. *Let R be Noetherian, H a Hopf R-algebra, $A \#_\sigma H$ a right H-crossed product with invertible cocycle, $U \subseteq H^*$ a left H-module subalgebra and consider the R-pairing $\overline{P} := (H, U)$. Assume there exists an R-submodule $V \subseteq H^*$, such that (V, U) is compatible. If $U \subset R^H$ is $A \otimes_R H$-pure* (*e.g., H is a Hopf α-algebra and $U \subseteq H^\circ$ is an R-subbialgebra*)*, then we have an R-algebra isomorphism*

$$(A \#_\sigma H)^{\mathrm{op}} \# U \simeq A \otimes_R \bigl(H \#^{\mathrm{op}} U\bigr).$$

Corollary 3.8. *Let H be a projective Hopf R-algebra, $A \#_\sigma H$ a right H-crossed product with invertible cocycle, $U \subseteq H^*$ a left H-module subalgebra and consider the R-paring $P := (H, U)$. Assume there exists an R-submodule $V \subseteq H^*$, such that (V, U) is compatible. If $U \subseteq H^*$ is dense and the canonical R-linear map $\alpha^P_{A \otimes_R H}$ is injective* (*e.g., R is Noetherian and $U \subseteq R^H$ is A-pure*)*, then there is a dense$R$-subalgebra $\mathcal{L} \subseteq \mathrm{End}_R(H)^{\mathrm{op}}$ and an R-algebra isomorphism*

$$(A \#_\sigma H) \#^{\mathrm{op}} U \simeq A \otimes_R \mathcal{L}.$$

This is the case in particular, if R is a QF ring, H is a residually finite Hopf α-algebra and $U \subseteq H^\circ$ is a dense R-subbialgebra.

Proof. If $U \subseteq H^*$ is dense, then $\mathcal{L} := H \#^{\mathrm{op}} U \overset{\bar{\lambda}}{\hookrightarrow} \mathrm{End}_R(H)^{\mathrm{op}}$ is by Lemma 3.3 a dense R-subalgebra. If $\alpha^P_{A \otimes_R H}$ is injective, then the isomorphism follows by Theorem 3.7. If R is a QF ring and H is a residually finite Hopf α-algebra, then $H^\circ \subset H^*$ is dense by [4, Proposition 2.4.19]. If moreover $U \subseteq H^\circ$ is a dense R-subbialgebra, then $U \subseteq H^*$ is dense, $\alpha^P_{A \otimes_R H}$ is injective and we are done. □

Similar argument to those in the proof of Corollary 2.12 can be used to prove

Corollary 3.9. *Let H be a Hopf R-algebra and $A \#_\sigma H$ a right H-crossed product with invertible cocycle. Then we have an isomorphism of R-algebras*

$$(A \#_\sigma H) \#^{\mathrm{op}} H^* \simeq A \otimes_R \bigl(H \#^{\mathrm{op}} H^*\bigr)$$

at least when

(1) ${}_R H$ *is finitely generated projective, or*
(2) ${}_R A$ *is finitely generated, H is cocommutative and $\alpha^P_{A \otimes_R H}$ is injective* (*e.g.*, *R is Noetherian and $H^* \hookrightarrow R^H$ is $A \otimes_R H$-pure*).

The subalgebra $H^\omega \subseteq H^*$

In what follows let H be a *locally projective* Hopf R-algebra and consider the measuring α-pairing $P := (H^*, H)$ (notice that the canonical R-linear map $\alpha^P_R : H \to H^{**}$ is injective).

3.10. Consider H^* with the right H^*-action

$$(f \leftharpoondown g)(h) := \sum f(h_2) g\bigl(S(h_1) h_3\bigr) \quad \text{for all } f, g \in H^* \text{ and } h \in H.$$

Then H^* is a right H^*-module and $H^\omega := {}^H\mathrm{Rat}(H^*_{H^*})$ is analogously to Theorem 1.5 a left H-comodule with structure map $\omega : H^\omega \to H \otimes_R H^\omega$.

Analogously to [23, Propositions 3.2, 3.3] we have

Proposition 3.11. *Consider the left H-comodule (H^ω, ω).*

(1) *If $f \in H^\omega$, then $\omega(f) = \sum f_{\langle -1 \rangle} \otimes f_{\langle 0 \rangle}$ satisfies the following conditions*:
  (a) $f \star g = \sum f_{\langle -1 \rangle} g \star f_{\langle 0 \rangle}$ *for all $g \in H^*$*;
  (b) $h \leftharpoondown f = \sum (f_{\langle 0 \rangle} \rightharpoondown h) f_{\langle -1 \rangle}$ *for all $h \in H$*;
  (c) $\sum f(h_2) S(h_1) h_3 = \sum f_{\langle 0 \rangle}(h) f_{\langle -1 \rangle}$ *for all $h \in H$.*
(2) *Let $f \in H^*$. If there exists $\zeta = \sum f_{\langle -1 \rangle} \otimes f_{\langle 0 \rangle} \in H \otimes_R H^*$ that satisfies any of the conditions in* (1), *then $f \in H^\omega$ and $\omega(f) = \zeta$.*
(3) *$H^\omega \subseteq H^*$ is an R-subalgebra and moreover a left H-comodule algebra.*

(4) *$H^{\omega} \subseteq H^{*}$ is a left H-module subalgebra with*

$$\omega(hf) = \sum h_1 f_{\langle -1\rangle} S(h_3) \otimes h_2 f_{\langle 0\rangle} \quad \textit{for all } h \in H \textit{ and } f \in H^{\omega}.$$

As a consequence of Proposition 3.6 and Theorem 3.7 we get the following generalization of [23, Theorem 4.2]:

Theorem 3.12. *Let H be a locally projective Hopf R-algebra, $U \subseteq H^{*}$ a left H-module subalgebra, $P := (H, U)$ the induced R-pairing and assume that $\bar{\varphi}(H \otimes_R A)$, $\bar{\psi}(H \otimes_R A) \subseteq J(A \otimes_R \omega^{-1}(H \otimes_R U))$. If $\alpha^{P}_{A \otimes_R H}$ is injective (e.g., R is Noetherian and $U \subset R^{H}$ is A-pure), then there is an R-algebra isomorphism*

$$(A \#_{\sigma} H)^{\mathrm{op}} \# U \simeq A \otimes_R \big(H \#^{\mathrm{op}} U\big).$$

Proof. Consider the R-submodule $V := \omega^{-1}(H \otimes_R U)$. Since $V \subseteq H^{\omega}$, it is clear by Proposition 3.11(1)(b) that (V, U) satisfies the RL-condition (20) with respect to H. Consequently (V, U) is compatible. If $\alpha^{P}_{A \otimes_R H}$ is injective, then we are done by Proposition 3.6. □

Corollary 3.13. *Let H be a locally projective Hopf R-algebra, $U \subseteq H^{\omega}$ a left H-module subalgebra, $P := (H, U)$ the induced R-pairing and assume that $\omega(U) \subseteq H \otimes_R U$ and $\bar{\varphi}(H \otimes_R A)$, $\bar{\psi}(H \otimes_R A) \subseteq J(A \otimes_R U)$. If $\alpha^{P}_{A \otimes_R H}$ is injective (e.g., R is Noetherian and $U \subset R^{H}$ is A-pure), then there is an R-algebra isomorphism*

$$(A \#_{\sigma} H)^{\mathrm{op}} \# U \simeq A \otimes_R \big(H \#^{\mathrm{op}} U\big).$$

Remark 3.14. If the Hopf algebra H has a bijective antipode then it has a twisted antipode, namely $\overline{S} := S^{-1}$. In the proofs (by different authors) of several duality theorems for smash products assuming the bijectivity of the antipode, no use was made of $S \circ S^{-1} = \mathrm{id} = S^{-1} \circ S$; instead there was a heavy use of the main properties of S^{-1}, namely that it is an algebra and coalgebra anti-morphism, and that

$$\sum S^{-1}(h_2) h_1 = \varepsilon(h) 1_H = \sum h_2 S^{-1}(h_1) \quad \text{for every } h \in H.$$

A twisted antipode has also these main properties and this is why the original versions (in [4]) of the results in section two remain true after replacing the bijectivity of the antipode by the weaker condition of the existence of a twisted antipode!!

3.15. (Compare [22, Lemma 5.3].) Let H be a Hopf R-algebra with a twisted antipode $\overline{S}$ and $A \#_{\sigma} H$ a right H-crossed product with invertible cocycle σ. Then $h^{\mathrm{op}} a^{\mathrm{op}} := \overline{S}(h) a$ induces on A^{op} a weak left H^{op}-action and $A^{\mathrm{op}} \#_{\tau} H^{\mathrm{op}}$ is a right H^{op}-crossed product with invertible cocycle

$$\tau : H \otimes_R H \to A, \quad (h, k) \mapsto \sigma^{-1}\big(\overline{S}(h), \overline{S}(k)\big).$$

Moreover $A \#_{\sigma} H \simeq (A^{\mathrm{op}} \#_{\tau} H^{\mathrm{op}})^{\mathrm{op}}$ as right H-comodule algebras.

Remark 3.16. As indicated earlier, the original versions [4] of the main duality theorems for smash products were proved under the assumption of the bijectivity of the antipode of H and it was not clear why such an assumption is not needed in the corresponding results for opposite smash products. Upon suggestion of the referee this condition is replaced in this paper with the weaker condition that H has a twisted antipode which clarifies, to some extent, this issue (notice that the rule of H is played in the third section by H^{op} which has a twisted antipode!!). However, it should be noted that the results in the third section cannot be deduced directly from the corresponding results in the second section, since (in light of 3.15) we have to assume that H has a twisted antipode!!

However, some of duality theorems for smash products can be deduced from the corresponding ones for opposite smash products under the assumption that H has a twisted antipode. In what follows we give one of these results.

3.17. Let R be Noetherian, H a Hopf α-algebra with twisted antipode, $U \subseteq H^\circ$ an R-subbialgebra and consider the R-subbialgebra $U^{\mathrm{cop}} \subseteq (H^{\mathrm{op}})^\circ$. Assume there exists an R-submodule $V \subseteq (H^{\mathrm{op}})^*$, such that

$$\begin{aligned} &\text{for every } g \in V, \quad \text{there exist } \{(h_j, g_j)\} \subset H \times U, \\ &\text{s.t.} \quad \tilde{h} \leftharpoonup g = \sum h_j \big(g_j \rightharpoonup \tilde{h}\big) \quad \text{for all } h \in H \end{aligned} \tag{23}$$

and that for every $(h, a) \in H \times A$ there exist subclasses $\{a_u, g_u\}, \{b_w, g_w\} \subset A \times V$ with

$$\begin{aligned} \sum \sigma^{-1}\big(\overline{S}(\tilde{h}_2) \otimes \overline{S}(h)\big)\big[\overline{S}(\tilde{h}_1)a\big] &= \sum a_u g_u\big(\tilde{h}\big), \\ \sum \sigma^{-1}\big(\tilde{h}_1, \overline{S}(\tilde{h}_5 h)\big)\big[\tilde{h}_2 a\big]\sigma\big(\tilde{h}_3 \otimes \overline{S}(\tilde{h}_4)\big) &= \sum b_w g_w\big(\tilde{h}\big). \end{aligned}$$

Combining [23, Corollary 2.4] and Theorem 3.7 we get the R-algebra isomorphisms

$$\begin{aligned} (A \#_\sigma H) \# U &\simeq \big(\big(A^{\mathrm{op}} \#_\tau H^{\mathrm{op}}\big) \#^{\mathrm{op}} U^{\mathrm{cop}}\big)^{\mathrm{op}} \simeq A \otimes_R \big(H^{\mathrm{op}} \#^{\mathrm{op}} U^{\mathrm{cop}}\big)^{\mathrm{op}} \\ &\simeq \big(A^{\mathrm{op}} \otimes_R \big(H^{\mathrm{op}} \#^{\mathrm{op}} U^{\mathrm{cop}}\big)\big)^{\mathrm{op}} \simeq A \otimes_R (H \# U). \end{aligned}$$

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