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Permutations with forbidden subsequences and a generalized Schröder number

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Abstract

Using the technique of generating trees, we prove that there are exactly 10 classes of pattern avoiding permutations enumerated by the large Schröder numbers. For each integer, $m \geqslant 1$, a sequence which generalizes the Schröder and Catalan numbers is shown to enumerate $\binom{m+2}{2}$ classes of pattern avoiding permutations. Combinatorial interpretations in terms of binary trees and polyominoes and a generating function for these sequences are given. © 2000 Elsevier Science B.V. All rights reserved.

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1. Permutations with forbidden subsequences

Let S_n denote the symmetric group on $[n]=\{1,2,\dots,n\}$. For $\pi \in S_{n-1}$ and $j \in [n]$, let π_j be the permutation in S_n obtained from π by inserting n into the jth position. That is,

$$\begin{pmatrix} i \\ \pi_j(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(j-1) & n & \pi(j) & \cdots & \pi(n-1) \end{pmatrix}.$$

Throughout this paper, we will write only the bottom line of the above two-line notation for permutations.

Permutations which avoid certain patterns or subsequences have been widely studied. Much of the background information related to this paper can be found in [18,19].

Definition 1. Let $\gamma \in S_k$. A permutation $\pi \in S_n$, is said to be *γ-avoiding* if there is no sequence of integers $i_1, i_2, \dots, i_k$ such that $1 \leqslant i_{\gamma(1)} < i_{\gamma(2)} < \cdots < i_{\gamma(k)} \leqslant n$ and

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$\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. The subsequence $\{\pi(i_{\gamma(j)})\}_{j=1}^{k}$ is said to have *type* γ. We write $S_n(\gamma)$ for the set of γ-avoiding permutation of length n. More generally, if $\Gamma \subset S_k$, let $S_n(\Gamma) = \bigcap_{\gamma\in\Gamma} S_n(\gamma)$ be the set of all permutations in S_n which avoid every $\gamma \in \Gamma$. If $\pi \in S_{n-1}(\Gamma)$, then we call j an *active site* if insertion of n into the jth position of π yields an element of $S_n(\Gamma)$.

For example, if $\Gamma = \{132, 231\}$, then $S_n(\{132, 231\})$ is the set of permutations in S_n, none having a three-element subsequence in which the middle element is the largest. $S_4(\{132, 231\}) = \{4321, 3214, 4213, 2134, 4312, 3124, 4123, 1234\}$, and $|S_4(\{132, 231\})| = 8$. The active sites (indicated by arrows) of $^{\downarrow}213^{\downarrow}$ are sites 1 and 4. It was shown by Schmidt and Simion [15], and also by West [21] that $|S_n(\{132, 231\})| = 2^{n-1}$. In the same reference, Schmidt and Simion enumerate $S_n(\Gamma)$ for all $\Gamma \subset S_3$. In the case where Γ consists of a single permutation in S_4, the reader is referred to work of Babson and West [1], Bona [3,4], Regev [12], Stankova [16], and West [18–20].

It will be convenient to define the *reversal*, $\bar{\sigma} \in S_n$ of a permutation $\sigma \in S_n$ as $\bar{\sigma}(i) = \sigma(n+1-i)$, and the *complement*, $\sigma^* \in S_n$ of σ as $\sigma^* = n+1-\sigma(i)$. Then the following lemma (taken from Simion and Schmidt [15]) limits the number of cases which need to be enumerated. In [20], Lemma 2 is formulated in terms of an action of the dihedral group D_4 on permutation matrices.

Lemma 2. *For any set Γ of permutations in S_k, let $\bar{\Gamma} = \{\bar{\gamma}\colon \gamma \in \Gamma\}$, $\Gamma^* = \{\gamma^*\colon \gamma \in \Gamma\}$, and $\Gamma^{-1} = \{\gamma^{-1}\colon \gamma \in \Gamma\}$. Then,*

$$\begin{aligned} &\pi \in S_n(\Gamma) \text{ iff} \\ &\bar{\pi} \in S_n(\bar{\Gamma}) \text{ iff} \\ &\pi^* \in S_n(\Gamma^*) \text{ iff} \\ &\pi^{-1} \in S_n(\Gamma^{-1}). \end{aligned}$$

Thus, $|S_n(\Gamma) = |S_n(\bar{\Gamma})| = |S_n(\Gamma^*)| = |S_n(\Gamma^{-1})|$.

For the purpose of enumerating $S_n(\Gamma)$, we say that the sets Γ, $\bar{\Gamma}$, Γ^*, and Γ^{-1} as defined in Lemma 2 are *equivalent*.

For example, if Γ consists of a single permutation in S_3, this lemma gives that $|S_n(132)| = |S_n(231)| = |S_n(312)| = |S_n(213)|$ and $|S_n(123)| = |S_n(321)|$. In this case, it is well known that $|S_n(\Gamma)| = c_n = 1/(n+1)\binom{2n}{n}$, the nth Catalan number. The first explicit enumeration of 321-avoiding permutations seems to be due to Hammersley [9]. The first published proof is a combinatorial proof given by Rogers [14]. Knuth [10] first proved that $a_n(213) = c_n$. Bijective proofs that $|S_n(321)| = |S_n(213)|$ have been given by Simion and Schmidt [15], West [18], and Richards [13].

In this paper, we consider the following 10 pairs of permutations $(\gamma^1, \gamma^2) \in S_4 \times S_4$: I. (1234, 2134) II. (1324, 2314) III. (1342, 2341) IV. (3124, 3214) V. (3142, 3214) VI. (3412, 3421) VII. (1324, 2134) VIII. (3124, 2314) IX. (2134, 3124) and X. (2413, 3142).

Gire [8] and West [18] showed that for three of these pairs, $|S_n(\gamma^1,\gamma^2)| = r_{n-1}$, the $(n-1)$st large Schröder number. The generating function $\sum_{n\geqslant 1} r_n x^n = (1+x-\sqrt{1-6x+x^2}/2)$, with $r_0 = 1$, defines r_n. It is referred to as 'large' in order to distinguish it from the Schröder number s_n, where $r_n = 2s_n$, for $n > 0$. The first few terms of the sequence are $(r_0, r_1, r_2, r_3, \ldots) = (1, 2, 6, 22, 90, 394, 1806, \ldots)$. Gire showed that $|S_n(2341, 3241)| = r_{n-1}$. (By Lemma 2, (2341,3241) is equivalent to pair IV on our list.) West showed that $S_n(2413, 3142)$((2413, 3142) is pair X) and $S_n(4132, 4231)$ ((4132,4231) is equivalent to pair II) are also counted by r_{n-1}.

Based on computer-generated data for $|S_n(\gamma^1,\gamma^2)|$, where (γ^1,γ^2) is any pair in $S_4\times S_4$, and $n\leqslant 7$, Richard Stanley (private communication) conjectured that these 10 pairs are all enumerated by the Schröder numbers. A result of Stanley's computations is that there are at most 10 pairs which show that no other pair satisfies $|S_7(\gamma^1,\gamma^2)| = 1806$. In Section 3 we verify Stanley's conjecture:

Theorem 3. *There are exactly* 10 *pairs* (*inequivalent in the sense of Lemma* 2), $(\gamma^1,\gamma^2)\in S_4\times S_4$ *such that the number of permutations avoiding both* γ^1 *and* γ^2 *is the* $(n-1)$*st large Schröder number.*

In Theorem 8, we generalize West's [18] results concerning the Catalan and Schröder numbers to enumerate $S_n(\Gamma)$ where Γ is a certain set of $m!$ permutations in S_{m+2}. Taking $m = 1$, we reproduce the Catalan result. Taking $m = 2$ we confirm Stanley's conjecture, Theorem 3, for the first six pairs on the above list. For each $m > 2$, we get a sequence $\{a_n(m)\}$ which generalizes the Schröder and Catalan numbers, and $\binom{m+2}{2}$ classes of permutations counted by $\{a_n(m)\}$. In Proposition 11 we verify Theorem 3 for the remaining three pairs.

2. Generating trees

The enumerative technique used by West in [18,19], was that of generating trees, first introduced in the study of Baxter permutations [5]. In addition to giving many new enumerative results, this technique has the advantage that it often also gives a natural bijection between equinumerous sets of permutations. We are able to extend West's definition of Catalan and Schröder trees to obtain our results. Thus, in addition to enumerating certain classes of permutations, we obtain combinatorial proofs that some of these classes are equinumerous.

Definition 4. A *generating tree* is a rooted labeled tree with the property that if v_1 and v_2 are any nodes with the same label and l is any label, then v_1 and v_2 have exactly the same number of children with the label l. To specify a generating tree, it suffices to specify

- the label of the root, and

- a set of succession rules defining the number of children a node with label k has and what the label of each child is.

Using the notation of West [18], we are interested in the number of nodes on level n, denoted by Σ_n with the root being on level 1.

For example, the *complete binary generating tree* is defined to have the labels:

- Root: (2).
- Rule: $(2) \rightarrow (2)(2)$.

This notation means that any node with label 2 will have two children, each of which also has label 2. In this generating tree, $\Sigma_n = 2^{n-1}$.

Definition 5. For a set of forbidden permutations, $\Gamma = \{\gamma^1, \gamma^2, \dots, \gamma^l\}$ define the tree $T(\Gamma)$, as the rooted, labeled tree with root $(1) \in S_1(\Gamma)$. The vertices on the nth level are the permutations of $S_n(\Gamma)$. $\pi \in S_n(\Gamma)$ is a child of $\pi' \in S_{n-1}(\Gamma)$ if π can be obtained from π' by inserting n into some position of π'. It is clear that the number of elements on the nth level of $T(\Gamma)$ is $|S_n(\Gamma)|$.

In [19], West showed that $T(\{132, 231\})$ is isomorphic to the complete binary tree. Here, West also used generating trees to give a bijective proof that $|S_n(123)| = |S_n(132)| = c_n$ by showing that $T(123)$ and $T(132)$ are both isomorphic to the Catalan generating tree:

Definition 6. The *Catalan generating tree* is defined by the labels:

- Root: (2).
- Rule: $(k) \rightarrow (2)(3)\cdots(k+1)$.

West showed that $\Sigma_n = c_n$.

The generating function for $\{c_n\}$ is given by $\sum_{n\geqslant 1} c_n x^n = (1 - \sqrt{1-4x})/2x$.

Similarly, in [18] West showed that $|S_n(2413, 3142)| = |S_n(4132, 4231)| = r_{n-1}$ by showing that $T(2413, 3142)$ and $T(4132, 4231)$ are isomorphic to the Schröder generating tree, defined below. Recently, Barcucci et al. [2] showed that $T(4231, 4132)$ ((4231, 4132) is equivalent to pair II. (1324,2314)) is also isomorphic to the Schröder generating tree.

Definition 7. The *Schröder generating tree* is defined by the labels:

- Root: (2).
- Rule: $(k) \rightarrow (3)(4)\cdots(k+1)(k+1)$.

In this case, $\Sigma_{n-1} = r_{n-1}$.

3. Permutations enumerated by the Schröder numbers

In this section, we complete the proof of Theorem 3, that there are exactly 10 pairs of length four patterns enumerated by the large Schröder numbers. Six of the pairs (pairs I–VI on the list found in Section 1), including two of the pairs previously enumerated (pairs II and IV), come as a corollary to Theorem 8 which enumerates permutations in S_n avoiding the set of $m!$ permutations in S_{m+2} in which the positions occupied by $m+1$ and $m+2$ are fixed. Pairs VII, VIII and IX are enumerated in Proposition 11, and pair X is due to West [18].

Theorem 8. *Let m be a positive integer. Let $\Gamma = \Gamma(s,t) = \{\gamma^1, \gamma^2, \ldots, \gamma^{m!}\}$ be the set of $m!$ permutations in S_{m+2} such that for fixed s and t, $1 \leqslant s, t \leqslant m+2$, $\gamma^i(s) = m+1$ and $\gamma^i(t) = m+2$, for all $1 \leqslant i \leqslant m!$. $T(\Gamma)$ is isomorphic to the generating tree which has root (2) and recursive rule*

$$(k) \to \begin{cases} (k+1)^k & \text{if } 2 \leqslant k \leqslant m, \\ (m+1)(m+2)\cdots(k)(k+1)^m & \text{if } k \geqslant m+1. \end{cases} \tag{1}$$

Therefore, for all $(s,t),(s',t') \in [m+2] \times [m+2]$, $T(\Gamma(s,t)) \cong T(\Gamma(s',t'))$, and $|S_n(\Gamma(s,t))| = |S_n(\Gamma(s',t'))|$, for every positive integer n.

For example, when $m=3$, $\Gamma(2,4) = \{14253, 14352, 24153, 24351, 34152, 34251\}$, and $T(\Gamma)$ has rule, for $k \geqslant 4$, $(k) \to (4)(5)\cdots(k)(k+1)(k+1)(k+1)$. The crux of our argument is that when generating $S_n(\Gamma)$ from $S_{n-1}(\Gamma)$, we need only to keep track of the position which the largest element of $\sigma \in S_{n-1}(\Gamma)$ occupies. In this example, $\sigma_1 = {}^{\downarrow}5^{\downarrow}1^{\downarrow}2^{\downarrow}3^{\downarrow}4^{\downarrow}$ and $\sigma_2 = {}^{\downarrow}1^{\downarrow}5^{\downarrow}234^{\downarrow}$ are in $S_5(\Gamma)$. We will argue that σ_1 has six active sites (indicated by arrows) because 5 is too far to the left to play the role of 4 in a forbidden subsequence while σ_2 has only four active sites, since 5 is in the position to play the role of 4 in case 6 is inserted in any of the two inactive sites. More generally, we will show that in a forbidden subsequence, created by the insertion of $n+1$, n will play the role of $m+1$ while $n+1$ will play the role of $m+2$. The reader should be advised that in this proof we will compute the number of elements in $S_{n+1}(\Gamma)$, equivalently, the number of active sites in $S_n(\Gamma)$, relying on information about $S_{n-1}(\Gamma)$.

Proof. If $n \leqslant m$, then it is clear that $|S_n(\Gamma)| = |S_n|$ and $T(\Gamma)$ has recursive rule $(k) \to (k+1)^k$, for $2 \leqslant k \leqslant m$. Assume that $s < t$. If $s > t$, then a symmetric argument will give the tree isomorphism. Let $\pi \in S_{n-1}(\Gamma)$, for $n \geqslant m+1$, such that π has k active sites.

Let $\pi_j \in S_n(\Gamma)$ be the child of π obtained by inserting an n into the jth active site of π, $j = 1, \ldots, k$. For any r, $1 \leqslant r \leqslant n-1$, if the site between $\pi(r)$ and $\pi(r+1)$ is inactive in π, then the site between $\pi(r)$ and $\pi(r+1)$ is also inactive in π_j. There are thus at most $k+1$ active sites in π_j.

We claim that the number of active sites in π_j is determined by j, the position occupied by n. Consider the sequence obtained by inserting $n+1$ into one of the $k+1$ potentially active sites in π_j, say the rth site, $1 \leqslant r \leqslant k+1$. A forbidden subsequence is created by this insertion if and only if this sequence is of the form $v = v_1, v_2, \ldots, v_{m+2}$ with $v_s = n$, $v_t = n+1$, and for $i \neq s, t$; $1 \leqslant v_i < n$. It is clear that $n+1$ must play the role of the largest element, and that a sequence of this form is forbidden. If n does not participate in the new forbidden subsequence, then insertion of n into π would have already created the forbidden subsequence supposedly created by $n+1$, and the rth site of π_j would not be a potentially active site.

The number of active sites in π_j is determined by j, s, and t as follows. The $t-1$ sites at the beginning of π_j and the $(m+2)-(t)$ sites at the end of π_j are always active, since $m+2$ has $t-1$ numbers to its left and $(m+2)-(t)$ numbers to its right in each element of Γ. This leaves at most $(k+1)-(m+1) = k-m$ of the potentially active sites to check. Call these *middle* sites. These are located between the (t)th and the $[(k+1)-(m+2-t)]$th potentially active sites. In the example preceding the proof, the first three and the last site are always active. There are three cases:

(1) If $j < s$, then since all of the sites to the left of t are active and $t > s > j$, n is too far to the left in π_j to create a forbidden subsequence, so π_j has $(k+1)$ active sites. (σ_1 of the example preceding this proof is an instance of this case.)

(2) If $j \geqslant (k+1) - [(m+2)-(s+1)] = s + (k-m)$, then since $m+2-s > m+2-t$, all of the sites to the right of j are active, making n too far to the right in π_j to create a forbidden subsequence, so π_j has $(k+1)$ active sites. ($\sigma_4 = 12354$ and $\sigma_5 = 12345$ are examples.)

(3) If $s \leqslant j \leqslant s + (k-m)$ then $j - s$ of the $k-m$ middle sites will be active in π_j. A forbidden subsequence is created whenever $n+1$ is inserted at least $t-s$ sites to the right of j. If $j > t$, the number of active middle sites to the right of j plus the number of active middle sites to the left of j is $((t-s)+(j-t)) = (j-s)$.

If $j \leqslant t$, then the number of active middle sites to the right of j minus the number of these active sites which are not in the middle is $((t-s)-(t-j)) = (j-s)$. (σ_2 has no active middle sites, for example, while $\sigma_3 = 12534$ has 1.)

Summarizing, (1) and (2) give that a node with label $k \geqslant m+1$ will have m children with label $(k+1)$. Case (3) gives $k-m$ children with labels $(m+1), (m+2), \ldots, (k)$, respectively.

Corollary 9. $T(1234, 2134) \cong T(1324, 2314) \cong T(1342, 2341) \cong T(3124, 3214) \cong T(3142, 3241) \cong T(3412, 3421)$. *And for all positive integers*, n, $|S_n(1234, 2134)| = |S_n(1324, 2314)| = |S_n(1342, 2341)| = |S_n(3124, 3214)| = |S_n(3142, 3241)| = |S_n(3412, 3421)| = r_{n-1}$, *the* $(n-1)$*st large Schröder number.*

Proof. Let $m = 2$ in Theorem 8. $\square$

Corollary 10. *Let* $\sigma \in S_3$. $T(\sigma)$ *is isomorphic to the Catalan generating tree, and for all* $n \geqslant 1$, $|S_n(\sigma)| = c_n$, *the nth Catalan number.*

Proof. Let $m = 1$ in Theorem 8. $\square$

The proof of Theorem 3 is completed in Proposition 11. The proof of Proposition 11 is similar to that of Theorem 8. Since the position of 3 is not fixed in the remaining Schröder pairs, we must keep track of the position of the largest element of each permutation in $S_n(\Gamma)$.

Proposition 11. *If* $\Gamma = \{2134, 1324\}$, $\Gamma = \{3124, 2314\}$, *or* $\Gamma = \{2134, 3124\}$, *then* $T(\Gamma)$ *is isomorphic to the Schröder generating tree. Therefore,* $|S_n(\{2134, 1324\})| = |S_n(\{3124, 2314\})| = |S_n(\{2134, 3124\})| = r_{n-1}$, *for every positive integer n.*

Proof. Let $\pi_j = p_1 \cdots p_n \in S_n(\Gamma)$ be the child of $\pi \in S_{n-1}(\Gamma)$ obtained by inserting an n into the jth active site of π, $j = 1, \ldots, k$. As in the proof of Theorem 8, we observe that:

- There are thus at most $k+1$ active sites in π_j. Furthermore, the $r \leqslant k+1$ active sites are the leftmost r sites in π_j. This is a consequence of the fact that both forbidden subsequences end with a 4.
- If a forbidden subsequence is created by the insertion of $n+1$ into π_j, n must participate and play the role of 3, while $n+1$ plays the role of 4. As a result, all sites to the right of this insertion will be inactive, and all those to the left will be active. Thus, the inactive sites in π_j are those to the right of the end of the first subsequence, $p_{t_1}, p_{t_2}, p_{t_3}$, of type abc where $abc4 \in \Gamma$, and $p_{t_i} = n$, for some $i = 1, 2, 3$.

Let $p_m = \max\{p_i \colon 1 \leqslant i \leqslant k, i \neq j\}$.

Case $T(\{(2134, 1324)\})$:

(1) If $j = 1$, then n is too far to the left to participate in a subsequence of type 13. If $j = m+1$, then any subsequence of type 213 or 132 in π_j where n plays the role of 3 corresponds to a subsequence of the same type in π with p_m playing the role of 3. The sites to the right of this subsequence are inactive in π_j. In both cases, π_j has $k+1$ active sites.

(2) If $1 < j < m$ then $p_{j-1} < p_{j+1} < n$. Otherwise, $p_{j-1}p_{j+1}p_m$ is of type 213, making the sites to the right of p_m inactive in π. This contradicts that $m \leqslant k$. Thus, $p_{j-1}np_{j+1}$ is a subsequence of type 132. In this case, π_j has $j+1$ $(3 \leqslant j+1 \leqslant k)$ active sites, sites $1, 2, \ldots, j+1$.

If $m < j-1$ (so, $3 \leqslant j \leqslant k$), then $p_m p_{j-1} n$ is a subsequence of type 213. In this case, π_j has j active sites, sites $1, 2, \ldots, j$.

As j ranges over the sites $\{2, \ldots, k\} - \{m+1\}$, we get that π has $k-2$ descendants with labels $(3)(4)\cdots(k)$.

Therefore, for $k > 2$, $T(\{(2134, 1324)\})$ satisfies the recursive rule

$$(k) \to (3)(4)\cdots(k)(k+1)(k+1).$$

Case $T(\{(2134, 3124)\})$:

An argument similar to the case $T(\{(2134,1324)\})$ will give that:

(1) If $j = m+1$ or $m-1$, then π_j has $k+1$ active sites, since, in either case, p_m already plays the role of 3 in any subsequence of type 13 or 31, and

(2) As j ranges over the sites $\{1,\dots,k\} - \{m+1, m-1\}$, we get that π has $k-2$ descendants with labels $(3)(4)\cdots(k)$.

If $1 \leqslant j < m-1$, then $np_{j+1}p_{j+2}$ is of type 312. Otherwise, $p_{j+1}p_{j+2}p_m$ is of type 213 contradicting our choice of p_m. In this case, the active sites are $1,2,3,\dots,j+2$.

If $m+1 < j \leqslant k$, then $p_m p_{j-1} n$ is of type 213, and the active sites are $1,2,3,\dots,j$.

Therefore, for $k > 2$, $T(\{(3124,2314)\})$ satisfies the recursive rule

$$(k) \to (3)(4)\cdots(k)(k+1)(k+1).$$

Case $T(\{(3124,2314)\})$:

Here, we get:

(1) If $j = k$ or $m-1$, then π_j has $k+1$ active sites, since no new subsequence of type 31 is formed, and

(2) As j ranges over the sites $\{1,\dots,k-1\} - \{m-1\}$, π has $k-2$ descendants with labels $(3)(4)\cdots(k)$.

If $1 \leqslant j < m-1$, then let $l = \min\{i \neq j\colon\ p_{j+1} < p_i, 1 \leqslant i \leqslant k\}$. If $1 \leqslant l < j$, then $p_l n p_{j+1}$ is of type 231 and the active sites are $1,2,\dots,j+1$.

If $2 \leqslant j+1 < l \leqslant m$, then $np_{j+1}p_l$ is of type 312 and the active sites are $1,2,3,\dots,l$.

If $m < j < k$, then $p_m n p_{j+1}$ is of type 231 and the active sites are $1,2,3\dots,j+1$.

Therefore, for $k > 2$, $T(\{(3124,2314)\})$ satisfies the recursive rule

$$(k) \to (3)(4)\cdots(k)(k+1)(k+1).$$

4. Generating functions

It is well known that the Catalan number, c_n counts the number of full binary plane trees with $n+1$ leaves. A *weighted binary plane tree* with weight set $[m]$ is a full binary plane tree in which each interior node is given a weight equal to $1,2,\dots,m$. A node is said to be *well-weighted* if, whenever it has weight $w > 1$, its right child is not a leaf. A weighted binary plane tree is *well-weighted* if all of its interior nodes are well-weighted. In [7], Foata and Zeilberger, using the combinatorial interpretation of the small Schröder numbers s_n ($r_n = 2s_n$) as the number of well-weighted binary plane trees with n leaves and weight set $\{1,2\}$, proved bijectively the recurrence:

$$3(2n-1)s_n = (n+1)s_{n+1} + (n-2)s_{n-1} \quad \text{for } n \geqslant 2$$

with $s_1 = s_2 = 1$ (see [6, p. 57]). This recurrence is a generalization of the linear recurrence for the Catalan numbers given by

$$2(2n-1)c_n = (n+1)c_{n+1} \quad \text{for } n \geqslant 1$$

with $c_1 = 1$. A straightforward generalization of their proof shows that the number $f_n(m)$ of well-weighted binary plane trees with n leaves and weight set $[m]$, satisfies the recurrence

$$(2n-1)(m+1)f_n(m)=(n+1)f_{n+1}(m)+(n-2)(m-1)^2 f_{n-1}(m)$$

for $n\geqslant 2$ with initial conditions $f_2(m)=m, f_3(m)=m+m^2$.

Sulanke [17] gave a combinatorial proof of this recurrence in terms of parallelogram polyominoes where m colors are available for the columns. He found the generating function for $\{f_{n(m)}\}$ to be

$$\sum_{n\geqslant 2} f_n(m)x^n = \frac{1+(m+1)x-\sqrt{1-2(m+1)x+(m-1)^2x^2}}{2}.$$

In [11], Pergola and Sulanke obtained the Schröder generating tree for polyominoes in which the columns are allowed 2 colors. Allowing m colors for the columns, one obtains the generating tree defined for $m\geqslant 1$ by:

- Root: $(m+1)$.
- Rule: $(k)\rightarrow(m+1)(m+2)\cdots(k+1)^m$.

This tree satisfies $\Sigma_n=f_n(m)$. When $m=1$, we get the Catalan generating tree. When $m=2$, the recursive rule is the same as that for the Schröder generating tree. In fact, $\Sigma_n=s_n$, the small Schröder number. The Schröder tree can be obtained from this one by taking 2 copies of the root at the start.

Let $a_n(m)=|S_n(\Gamma)|$, where Γ and m are as in Theorem 8. Let $T(\Gamma)$ be the corresponding generating tree. Since for $n\geqslant m$, $a_n(m)$ satisfies the same recurrence as $f_n(m)$, and $a_{m-1}(m)=(m-1)!$ we get the generating function

$$\sum_{n\geqslant m} a_n(m)x^n=(m-1)!x^{(m-2)}\left(\frac{1+(m-1)x-\sqrt{1-2(m+1)x+(m-1)^2x^2}}{2}\right).$$

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