A unified approach to geometrically nonlinear analysis of tapered bonded joints and doublers

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Abstract

A unified approach for approximating the adhesive stresses in a bond line of a tapered bonded joint or doubler is delineated within the framework of a geometrically nonlinear analysis. The approach follows the Goland–Reissner solution method for a single-lap joint and involves a two-step analysis procedure. The approach also allows for the analysis of a tapered bonded joint and doubler with non-identical adherends. In the first step of the procedure, the two adherends are assumed to be rigidly bonded, and the nonlinear moment distribution along the joint is determined. Since the bending moment solution in this step is simple, it will be derived in closed-form using elementary functions. In the second step analysis, only the overlapped area of the joint is considered with the nonlinear bending moments obtained from the first step at the end of the overlap prescribed as one of its boundary conditions. This latter problem is then solved by using the multi-segment method of integration [Kalnins, A., 1964. Analysis of shell of revolutions subjected to symmetrical and non-symmetrical loads. Journal of Applied Mechanics 31, 1355–1365]. In contrast to the original Goland–Reissner solution method [Goland, M., Reissner, E., 1944. The stresses in cemented joints. Journal of Applied Mechanics 11, A17–A27], the second step analysis can be conducted within both geometrically linear theory and an approximate geometrically nonlinear theory.

Keywords: Bonded joints; Tapered joints; Bonded repairs

1. Introduction

Bonded joints and doublers were found in many engineering applications in aerospace and automotive, as well as the wood and plastic industries because of their efficient load transfer, low cost use, high

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corrosion and fatigue resistance, and crack retardance characteristic. These bonded joints and doublers have been studied extensively over five decades with most of the analytical and theoretical work focused on predicting stresses within the thin adhesive layer. Early theoretical works by Volkensen (1938), De Bruyne (1944), Goland and Reissner (1944) and Hart-Smith (1973a,b,c) provided the basic formulations for analyzing double and single lap shear joints with various degrees of modeling complexity. Single sided bonded doublers and single strap joints are also recently analyzed by Hart-Smith (2004), accounting for the effect of thermal mismatch. Recent works by Oplinger (1994), Suhir (1994), Tsai et al. (1998) refined the above earlier models by removing deficient assumptions and accounting for additional important effects such as adherend shear deformation or large adherend deflection. However, except for the Hart-Smith step joint model (1973c), all of these theoretical works were limited to untapered adherends and also were usually restricted to joints of identical (balanced) adherends.

The adhesive stresses in bonded joints and doublers were normally peaked at the end of the overlap, which can cause failure of the adhesive and compromise the performance of the bonded joints and doublers. To reduce the severity of these peaks, bonded joints and doublers are usually tapered at their overlapped ends. Due to the mathematical complexity, to the author’s knowledge, no closed-form solution is available in the literature for these tapered joints. Numerical solutions are therefore sought for the tapered bonded joints and doublers. There are two basic approaches for numerical analysis of tapered bonded joints and doublers. In the first approach, a set of differential equations and boundary conditions is formulated with adherends modeled as classical beams, and the solutions of these differential equations are obtained by direct numerical integrations. On the other hand, the second approach is by the finite element method. Even though a finite element method may provide accurate solution for tapered joints and doublers with arbitrary geometries and loading conditions, it is an elaborate and computationally demanding task. Moreover, due to singularity at the termini of the joints and doublers, the adhesive stresses obtained from the finite element analysis will be higher and higher as the mesh is further refined so that if a given mesh size is used to model the areas near the end of the overlap for determining the adhesive strength allowable from test via test-analysis correlation, then that same mesh size must be used in all subsequent failure predictive analyses. Since literature on finite element analyses of bonded joints and doublers is vast and these works are beyond the scope of the present paper, no reference on this subject will be cited here. Readers who are interested in that subject should refer to a book by Tong and Steven (1999).

In the practical design environment, there is a specific need for analysis and design tools that can provide accurate stress results for tapered bonded joints and doublers with little computational effort involved. Such tools would be very useful for preliminary design purposes, i.e., in the stages of design where fast estimates of adhesive stresses are needed and where parametric study on the variations of design parameters on joint strength is conducted. In that sense, the direct numerical integration method mentioned in the preceding paragraph seems to be more versatile and robust than the finite element approach. In a recent series of papers by Thomsen (1992), Thomsen et al. (1996), Mortensen and Thomsen (1997, 2002), a unified approach for analyzing the variety of bonded joints and doublers with and without tapered edges, and related problems using numerical integration method was given. This unified approach was also accounted for the effect of a (materially) nonlinear adhesive. However, it was limited to a geometrically linear analysis. The main objective of the present paper is therefore to extend this unified approach to include geometrically nonlinear analyses of the tapered joints and doublers, following the Goland–Reissner two-step solution method.

2. Mathematical formulation and two-step solution method

Even though the approach presented here is so generic that it can be applied to a variety of tapered bonded joints and doublers, however, for simplicity, only the formulation for a single sided doubler and
a single lap joint will be given here. These two joint configurations are of interest since the former configuration represents a one-dimensional model of a bonded repair while the latter is commonly found in structural composite joints. Consider a single-sided tapered doubler and a single lap joint in Figs. 1 and 2, respectively, under the in-plane tensile loads. The adherends in these two configurations are not necessarily identical (balanced) and they can be a general laminate with different ply materials, lay-ups, and thicknesses. In order to assure that the formulation for a single lap joint will follow closely with the corresponding single sided doubler problem, only the outer (upper) adherend of the single lap joint will be tapered at its joining end as shown in Fig. 2. It is worthy to note from static equilibrium consideration that the two supported ends of the single lap joint are reacted by two vertical forces which are of equal magnitude but acting in opposite direction.

Following the work of Goland–Reissner, the solution method for these two joint configurations will also involve a two-step procedure. In the first step of the procedure, the two adherends are assumed to be rigidly bonded, and the nonlinear moment distribution along the joint is determined. Since the moment solution in this step is simple, it will be derived in closed-form using elementary functions. In the second step analysis, only the overlapped region of the joint is considered with the nonlinear bending moments obtained from the first step at the end of the overlap prescribed as one of its boundary conditions. This latter problem is then solved by using multi-segment method of integration. However, in contrast to the original Goland–Reissner solution method, the second step analysis can be conducted within both geometrically linear and nonlinear theory as detailed later.

2.1. Solution for nonlinear moment distribution along the joint

The equations governing the adhesive stresses require knowledge of the bending moment at the ends of the overlap as boundary conditions. Therefore, the analysis starts with a solution for the nonlinear moment distribution along the joint. The solution of a single sided doubler will be delineated first. Due to symmetry, only half of the doubler configuration will be considered. Fig. 3a shows schematics of the analyzing model with separate coordinate system used for each segment of the doubler configuration. The length of each segment is denoted by \( \ell_i \).

From moment equilibrium consideration, the moment distribution in each segment along the joint is related to loads and displacements by:

\[
M_i = -T \cdot \bar{w}_i - T(e_i - e_0),
\]

where \( i = 0, \ldots, N \); \( N \) is the number of segments (steps) in the overlapped region; segment 0 (which corresponds to \( i = 0 \)) is outside the overlap and consists of only the bottom adherend; \( M \) is the bending moment;
This is the axial tensile load applied at the ends of the adherends; $w$ is the transverse deflection; $e_i$ is the $z$-coordinate of the neutral axis of a beam cross section of the segment $i$ measured from the bottom surface of the bottom adherend. Because of the rigid bond assumption, the two adherends in the overlapped region will be treated as a single composite beam in this analysis step. It is worthy to note that the first term of Eq. (1) represents the moment due to large adherend deflections or "beam column" effect while the second term denotes the moment associated with a load path eccentricity due to the variation of the vertical position of the neutral axis along the length of a doubler. For a laminated beam, $e_i$ and $M_i$ are given respectively by

$$e_i = \frac{1}{2} \sum_{k=1}^{n_{\text{ply}}} (C_{11,k})_i \left\{ \frac{z_{k,i}^2 - z_{k-1,i}^2}{(C_{11,k})_i (z_{k,i} - z_{k-1,i})} \right\},$$

$$M_i = -D_i \dot{w}_i''(x_i),$$

$$D_i = \frac{1}{3} \sum_{k=1}^{n_{\text{ply}}} (C_{11,k})_i \{(z_{k,i} - e_i)^3 - (z_{k-1,i} - e_i)^3\},$$

where $(C_{11,k})_i$ is the "$(1,1)$ element" of the stiffness matrix as obtained from the classical laminate theory along the length of a beam for the $k$th ply of the $i$th segment, $D$ is the flexural rigidity, $z_k$ and $z_{k-1}$ are $z$-coordinates of the top and bottom surface of the $k$th ply of the laminate, $n_{\text{ply}}$ is the number of plies, and the prime denotes the differentiation with respect to the coordinate $x$. For a homogeneous and isotropic segment, $e_i = \frac{t_i}{2}$ and $D_i = \frac{E_i t_i^3}{12}$ where $t_i$ is the thickness of $i$th segment. It should be noted that contributions from both bottom adherend and doubler must be accounted for in the calculations of $D$ and $e$ for any segment inside the overlap.

Substituting Eq. (3) into Eq. (1) for $M_i$ yields the following differential equations for each segment $i$:

$$\ddot{w}_i - \frac{T}{D_i} \dot{w}_i = \frac{T(e_i - e_0)}{D_i}.$$
The solution of Eq. (4) is given by
\[ \hat{w}_i(x_i) = W_1 \cosh(\xi_i x_i) + W_2 \sinh(\xi_i x_i) - (e_i - e_0), \] (5)
where
\[ \xi_i = \sqrt{\frac{T}{D_i}}. \] (6)

\( W_1 \) and \( W_2 \) are unknown constants which must be determined from the boundary conditions, and the first two terms of Eq. (5) represent the homogeneous solution while the last term is the particular solution. The displacement boundary condition at the right supported end and the symmetry condition at the middle of the doubler configuration require that
\[ \hat{w} = 0 \text{ at } x = 0, \quad \text{or} \quad \hat{w}_0 = 0 \text{ at } x_0 = 0, \therefore W_{10} = 0, \] (7)
and the slope \( \hat{w}' = 0 \) at \( x = \ell_0 + \ell_1 + \cdots + \ell_N \) or \( x_N = \ell_N \), thus,
\[ \xi_N W_{1N} \sinh(\xi_N \ell_N) + \xi_N W_{2N} \cosh(\xi_N \ell_N) = 0. \] (8)

The displacement and slope continuity conditions at each segment junction also require that
\[ W_{1i} \cosh(\xi_i \ell_i) + W_{2i} \sinh(\xi_i \ell_i) - W_{1,i+1} = e_i - e_{i+1}, \]
\[ \ell_i W_{1i} \sinh(\xi_i \ell_i) + \ell_i W_{2i} \cosh(\xi_i \ell_i) - \ell_{i+1} W_{2,i+1} = 0. \] (9)

Thus, Eqs. (7)–(9) provide a system of linear algebraic equations for determining the unknown constants \( W_{1i} \) and \( W_{2i} \) \((i = 0, 1, \ldots, N)\). Once these constants are determined, the nonlinear bending moment at the end of the overlap is found to be
\[ M_L = M_0(x_0 = \ell_0) = -\xi_0^2 D_0 \{ W_{10} \cosh(\xi_0 \ell_0) + W_{20} \sinh(\xi_0 \ell_0) \}. \] (10)

The first step analysis for a tapered bonded doubler will be concluded with the following remarks:

(a) Firstly, in this analysis step, the bending moments \( M_i \) \((i = 0, 1, \ldots, N)\) and thus \( M_L \) are defined with respect to the neutral axis of the composite beam section, which varies along the joint due to the presence of the doubler and its multiple steps.
(b) Secondly, since the thin layer of adhesive will be modeled for in the second analysis step, which thickness may not be an order of magnitude smaller than the adherend thickness, especially in a composite joint, therefore, it will be necessarily to account for the effect of the adhesive layer in the calculations of beam section properties such as \( e_i \) and \( D_i \) by including a small gap between the two adherends in the overlapped region in these calculations.
(c) Finally, since the lengths of the first \( N - 1 \) steps of the doubler, i.e., \( \ell_1, \ldots, \ell_{N-1} \), are normally small relative to \( \ell_0 \) and \( \ell_N \), and the evaluations of hyperbolic functions in Eq. (5) will result in a large exponential number for a certain large combinations of \( \xi_0 \ell_0 \) and \( \xi_N \ell_N \), which will cause an ill condition when solving the system of the algebraic equations for the unknowns \( W_{1i} \) and \( W_{2i} \), a special caution therefore must be taken in dealing with those cases. For instance, by using the transformed variables \( W'_{10} = W_{10} \cdot e^{-\xi_0 \ell_0} \) and \( W'_{20} = W_{20} \cdot e^{-\xi_0 \ell_0} \) for \( W_{10} \) and \( W_{20} \), and expressing Eq. (5) for the first segment in terms of these new variables, the mentioned ill condition can be eliminated.

It remains now to delineate the bending moment solution for a single lap joint considered in Fig. 2. Due to the lack of symmetry in the single lap joint, the whole joint must be analyzed. In contrast to a doubler problem, due to the load path eccentricity, as mentioned in the beginning of Section 2, there are two vertical reactions applying at the left and right end supports, which are equal of magnitude but in an opposite direc-
tions, which include (i) the deflections are zero at previously defined.

upper adherend and represents the length of the upper adherend outside the overlap area; and the rest are equilibrium equation for each segment requires

\[ M_0 = -T \cdot \dot{w}_0 + Px_0, \]

\[ M_i = -T \cdot \dot{w}_i - T(e_i - e_0) + P(x_i + \ell_i - \ell_{i-1}), \quad i = 1, \ldots, N + 1, \quad (11) \]

or

\[ \dot{w}_0'' - \frac{T}{D_0} \dot{w}_0 = Px_0, \]

\[ \dot{w}_i'' - \frac{T}{D_i} \dot{w}_i = \frac{T(e_i - e_0)}{D_i} - \frac{P}{D_i}(x_i + \ell_i - \ell_{i-1}), \quad i = 1, \ldots, N + 1, \quad (12) \]

where \( P \) is the vertical reaction as shown in Fig. 3b and equals to \( T(e_{N+1} + t_a - e_0) \) from consideration of an overall moment equilibrium; \( t_a \) is the thickness of the adhesive layer; segment \( N + 1 \) consists of the only upper adherend and represents the length of the upper adherend outside the overlap area; and the rest are previously defined.

Similar to before, the solution to the differential equation (12) is given by

\[ \dot{w}_0(x_0) = W_{10} \cosh(\xi_0 x_0) + W_{20} \sinh(\xi_0 x_0) + \frac{P}{T} x_0, \]

\[ \dot{w}_i(x_i) = W_{1i} \cosh(\xi_i x_i) + W_{2i} \sinh(\xi_i x_i) - (e_i - e_0) + \frac{P}{T}(x_i + \ell_i - \ell_{i-1}) \quad i = 1, 2, \ldots, N + 1, \quad (13) \]

\( W_{1i} \) and \( W_{2i} \), \( i = 0, \ldots, N + 1 \), again are the unknown constants to be determined by the boundary conditions, which include (i) the deflections are zero at \( x_0 = 0 \) and \( x_{N+1} = \ell_{N+1} \), and (ii) the slope and the deflection are continuous across the segments. Once these unknown constants are solved, the nonlinear bending moments at the right and left ends of the overlap are determined respectively as

\[ M_k = M_0(x_0 = \ell_0) = -\xi_0^2 D_0 \{ W_{10} \cosh(\xi_0 \ell_0) + W_{20} \sinh(\xi_0 \ell_0) \}, \]

\[ M_{li} = M_{N+1}(x_{N+1} = 0) = -\xi_{N+1}^2 D_{N+1} W_{1,N+1}. \quad (14) \]

2.2. Solutions for peel and shear stresses in the adhesive

The Goland–Reissner analysis for the adhesive peel and shear stresses in a bonded single-lap joint is based on the linear bending analysis of the overlap area of the joint, using their nonlinear estimate for the bending moments in the adherends just outside the bonded area from the first step rigid bond analysis as the key boundary conditions. This same analysis technique will be used here, but slightly modified to approximately account for the geometrical nonlinearity. As mentioned earlier in Section 1, due to mathematical complexity, this analysis portion will be carried out using the multi-segment method of integration. Again, the formulation and solution for a tapered bonded doubler will be delineated first. With reference to Fig. 4, the equilibrium equations are set up for the bottom adherend and the doubler in each segment of the overlap area. These equilibrium equations for segment \( i \) \( (i = 1, 2, \ldots, N) \) can be written as follows:

- For the bottom adherend:

\[ N_{li}' = -\tau_{ai}, \]

\[ Q_{li}' = -\sigma_{ai}, \]

\[ M_{li}' = Q_{li}' - \tau_{ai} \left( \frac{L_i + L_s}{2} \right) - N_{li} \cdot \dot{w}_i'. \quad (15) \]
For the doubler or upper adherend:

\[
N_{2i} = \tau_{ai}, \quad Q_{2i} = \sigma_{ai}, \quad M_{2i} = Q_{2i} \left( \frac{t_2 + t_a}{2} \right) - N_{2i} \cdot \bar{w}_i',
\]

where \( N \) and \( Q \) are normal stress resultant and vertical shear resultant, respectively; \( M \) again denotes the moment; \( \tau_a \) and \( \sigma_a \) are the adhesive shear and peel stresses; \( t_1, t_2 \) and \( t_a \) are the total thickness of the bottom adherend, the first step of the doubler or upper adherend, and adhesive layer, respectively; \( \bar{w}_i \) is the transverse deflection of a segment \( i \) of the overlap area as if the bottom adherend and doubler act in unison; the subscripts 1 and 2 denote the bottom adherend and the doubler, respectively, and a prime designates a differentiation with respect to the coordinate \( x \). \( \bar{w}_i \) is considered to be the overall transverse deflection of the overlap area and it is already obtained based on rigid bond assumption from

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Fig. 4. Schematic diagrams for calculating adhesive peel and shear stresses: (a) an overlap of a tapered doubler, (b) an overlap of a tapered single lap joint, and (c) stress and moment resultants of a differential element in the overlap.
Section 2.1 as part of the nonlinear bending moment solution. In Section 2.1, \( \hat{\omega}_0 \) \((i = 0)\) corresponds to the overall transverse displacement outside the overlap area while \( \hat{\omega}_i \) \((i = 1, 2 \ldots N)\) is the deflection of the \(i\)th segment inside the overlap. At this point, it is important to point out the difference between the present formulation and that from Goland–Reissner analysis as well as any underlying assumptions for this second step analysis. Firstly, the effect of the geometrical nonlinearity is accounted in the present formulation by including the underlined terms in Eqs. (15) and (16). These terms represent approximately the additional moment in the adherend and doubler due to their large bending deflections. The otherwise geometrically linear analysis will totally omit them. Secondly, for simplification, the bending moment in the present analysis step is defined differently from the first step. The bending moment in the bottom adherend is now defined with respect to the mid-plane of the bottom adherend. In contrast, the bending moment in the doubler or upper adherend is always defined with respect to the mid-plane of the first segment of the doubler for all of its segments (steps). As detailed later, this definition of the doubler’s bending moment is preferred because it will make the doubler moment and shear distribution continuous across the segments. It should be emphasized that the effect of the geometrical nonlinearity has been accounted for in the present formulation approximately since only the “average” bending deflection of the overlap area obtained from Section 2.1 is used in Eqs. (15) and (16) in the underlined terms, rather than the individual bending deflections of the bottom adherend and doubler as would be required in an exact nonlinear analysis. However, by using such approximation, Eqs. (15) and (16) will provide a system of linear differential equations that can be solved by an appropriate numerical method.

The equilibrium equations (15) and (16) do not provide the complete equations for solving the adhesive peel and shear stresses. Additional equations that must be considered are the kinematics and constitutive relationships for the adherends and adhesive, and they are given below:

- For the \(i\)th segment of the bottom adherend:
  \[
  u_{1i} = \bar{u}_{1i} + z\beta_{1i},
  \]
  \[
  \beta_{1i} = -\hat{\omega}_{1i},
  \]
  \[
  \ddot{u}_{1i} = \frac{D_1}{D_1 A_1 - B_1^2} N_{1i} - \frac{B_1}{D_1 A_1 - B_1^2} M_{1i},
  \]
  \[
  \ddot{\beta}_{1i} = -\frac{B_1}{D_1 A_1 - B_1^2} N_{1i} + \frac{A_1}{D_1 A_1 - B_1^2} M_{1i} .
  \]

- For the \(i\)th segment of the doubler or upper adherend:
  \[
  u_{2i} = \bar{u}_{2i} + z\beta_{2i},
  \]
  \[
  \beta_{2i} = -\hat{\omega}_{2i},
  \]
  \[
  \ddot{u}_{2i} = \frac{D_2}{D_2 A_2 - B_2^2} N_{2i} - \frac{B_2}{D_2 A_2 - B_2^2} M_{2i},
  \]
  \[
  \ddot{\beta}_{2i} = -\frac{B_2}{D_2 A_2 - B_2^2} N_{2i} + \frac{A_2}{D_2 A_2 - B_2^2} M_{2i} .
  \]

- For the \(i\)th segment of the adhesive:
  \[
  \sigma_{ai} = \frac{E_a}{t_a} (w_{2i} - w_{1i}),
  \]
  \[
  \tau_{ai} = \frac{G_a}{t_a} \left( u_{2i} - \frac{t_2}{2} \beta_{2i} - \bar{u}_{1i} - \frac{t_1}{2} \beta_{1i} \right) .
  \]
In Eqs. (17) and (18), $A$, $B$ and $D$ are the extensional, coupling and flexural rigidities and they are defined according to the classical laminate theory as

$$A = \sum_{k=1}^{n_{pl}} (C_{11,k}) \{z_k - z_{k-1}\},$$

$$B = \frac{1}{2} \sum_{k=1}^{n_{pl}} (C_{11,k}) \{(z_k - h_0)^2 - (z_{k-1} - h_0)^2\},$$

$$D = \frac{1}{3} \sum_{k=1}^{n_{pl}} (C_{11,k}) \{(z_k - h_0)^3 - (z_{k-1} - h_0)^3\},$$

where $h_0$ is the $z$-coordinate of the reference plane; $u$ and $\bar{u}$ denote the extensional displacements measured at an arbitrary $z$ plane and at a reference plane, respectively; $\beta$ is a rotation; and the rest are previously defined. It is worthy to note that $A_1$, $B_1$ and $D_1$ are constant along the joint due to the uniformity of the bottom adherend so that the subscript $i$ denoting the segment number has been dropped in these quantities for clarity. In contrast, $A_{2i}$, $B_{2i}$ and $D_{2i}$ are expected to have different value within each segment depending on the doubler thickness and its lay-up composition within each segment. For all segments of the bottom adherend and doubler, $h_0$ is always chosen to be equal to $t_{1z}$ for the bottom adherend and $t_{2z}$ for the doubler to be consistent with the definition of bending moments stated earlier in this section for them.

By substituting Eq. (19) into Eqs. (15) and (16) for $\sigma_{ai}$ and $\tau_{ai}$, the resulting equations together with (17) and (18) can be rewritten into a system of first ordered differential equations as follows for each segment $i$:

$$\{\varphi'\} = [\Psi]_i \{\varphi\} = \begin{bmatrix} [\Psi_{11}]_i & [\Psi_{12}]_i \\ [\Psi_{21}]_i & [\Psi_{22}]_i \end{bmatrix} \{\varphi\},$$

where

$$\begin{align*}
\{\varphi\} &= \begin{bmatrix} \bar{u}_{1i} \\
\bar{w}_{1i} \\
\beta_{1i} \\
N_{1i} \\
M_{1i} \\
Q_{1i} \\
\bar{u}_{2i} \\
\bar{w}_{2i} \\
\beta_{2i} \\
N_{2i} \\
M_{2i} \\
Q_{2i} \end{bmatrix}, \\
[\Psi]_i &= \begin{bmatrix}
0 & 0 & 0 & \frac{D_1}{D_1 A_1 - B_1^2} & \frac{-B_1}{D_1 A_1 - B_1^2} & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{B_1}{D_1 A_1 - B_1^2} & \frac{A_1}{D_1 A_1 - B_1^2} & 0 \\
G_a & 0 & \frac{G_a}{t_a} \frac{t_1}{2} & 0 & 0 & 0 \\
\frac{G_a}{t_a} \frac{(t_1 + t_2)}{2} & 0 & \frac{G_a t_1}{t_a} (\frac{t_1 + t_2}{4}) & -\bar{w}'_i & 0 & 1 \\
0 & \frac{E_a}{t_a} & 0 & 0 & 0 & 0
\end{bmatrix},
\end{align*}$$
Eq. (21), for \( i = 1, 2, \ldots, N \), provides a system of \( 12 \times N \) first order linear differential equations for \( N \) unknown vectors \( \{ \varphi_i \} \), which is subjected to the following boundary conditions:

(a) At the left end of the overlap area, the doubler is stress and moment free, thus \( N_{21}(0) = M_{21}(0) = Q_{21}(0) = 0 \), noting \( i = 1 \).

(b) In contrast, the stress, moment and shear resultants in the bottom adherend at the left overlap end are given respectively by \( N_{11}(0) = T, M_{11}(0) = M_L + T(e_0 - \frac{t_1}{2}), \) and \( Q_{11}(0) = 0 \), where \( M_L \) is obtained previously from the first step analysis in Section 2.1 and given by Eq. (10). The fact that \( M_{11}(0) \neq M_L \) is because different reference plane has been used in the first and second step analysis in defining the bending moment of the bottom adherend. The reader is reminded that in the first step analysis, the bending moment \( M_L \) is defined with respect to the neutral axis (or neutral plane) of the bottom adherend while \( M_{11} \) in the present analysis is defined with respect to the mid-plane of the bottom adherend. Unless the bottom adherend is a symmetric laminate or isotropic, in general, \( e_0 \neq \frac{t_1}{2} \) and thus \( M_{11}(0) \neq M_L \).
(c) In addition, since the reference plane for bending moment computation is selected to be the same throughout the overlap area for each adherend, the extensional and transverse deflections, the slope, and the normal stress, shear and moment resultants in the doubler and in the bottom adherend must be continuous across the segment junction. Thus,

\[
\{ \varphi(\ell) \}_i = \{ \varphi(0) \}_{i+1},
\]

(d) Finally, the symmetry conditions at the middle of the overlap area require

\[
\begin{align*}
\tilde{u}_{1N}(\ell_N) &= \tilde{u}_{2N}(\ell_N) = 0, \\
\beta_{1N}(\ell_N) &= \beta_{2N}(\ell_N) = 0, \\
Q_{1N}(\ell_N) &= w_{2N}(\ell_N) = 0.
\end{align*}
\]

It is worthy to note that the condition \( w_{2N}(\ell_N) = 0 \) is specified in lieu of \( Q_{2N}(\ell_N) = 0 \) so that the unknown integration constant resulting from numerical integration of the transverse deflection can be uniquely determined. With these boundary conditions stated, the problem of determining adhesive peel and shear stresses in a doubler is completely formulated. Since the formulation of the single lap joint problem will be similar to that of a doubler, it will be presented here first before proceeding to the solution of Eq. (21) with the boundary conditions listed in (a)–(d) above.

The governing equation for a single lap joint is still given by Eq. (21) because of the similarity between the two joint configurations in the overlap area. However, there will be some difference in their boundary conditions. The boundary conditions listed above as (a) and (c) remain applicable to the single lap joint problem. However, the conditions (b) and (d) need to be changed as follows:

(b') At the left overlap end,

\[
N_{11}(0) = T, \quad M_{11}(0) = M_L + T \left( e_0 - \frac{t_1}{2} \right), \quad \text{and} \quad w_{11}(0) = 0.
\]

(d') At the right overlap end (noting that segment \( N \) is no longer represents the segment at the middle of the overlap area)

\[
\begin{align*}
N_{1N}(\ell_N) &= M_{1N}(\ell_N) = Q_{1N}(\ell_N) = 0, \\
w_{2N}(\ell_N) &= \tilde{u}_{2N}(\ell_N) = 0, \\
M_{2N}(\ell_N) &= M_U + T \left( e_N - \frac{t_2}{2} \right).
\end{align*}
\]

It is worthy to note that conditions for transverse deflections of the bottom and upper adherends are specified in lieu of the shear resultant, i.e., \( w_{11}(0) = 0 \) and \( w_{2N}(\ell_N) = 0 \) so that overall equilibrium can be maintained. These two displacement boundary conditions will yield the solutions for the vertical shear resultants in the bottom and upper adherends at the two ends of the overlap, which are of equal magnitude but in opposite direction as required. Furthermore, when the present analysis is conducted within a geometrical linear theory (with the underlined terms in Eqs. (15) and (16) being omitted), these displacement boundary conditions also yield the solution of the shear resultants at the two overlap ends to be

\[
Q_{11}(0) = -Q_{2N}(\ell_N) = \frac{M_U + M_L}{c} - \frac{T}{c} \left( \frac{t_1 + t_2}{2} + t_a \right),
\]

as expected from a static moment equilibrium of the joint overlap as shown in Fig. 4. It should be emphasized that in the geometrical nonlinear case \( Q_{11}(0) \) and \( Q_{2N}(\ell_N) \) cannot be derived by just considering the overall moment equilibrium of the joint overlap without solving the differential equations since transverse deflections of the adherends which are not known a priori would influence the calculation of the overall moment equilibrium.
It remains now to outline a numerical method for solving a set of differential equations given by Eq. (21) along with either boundary conditions listed in (a)–(d) for a doubler configuration or in (a), (b'), (c) and (d') for a single lap joint. It was found in the earlier study by Thomsen (1992) and Thomsen et al. (1996) that the differential equation set (21) are most effectively solved by the so-called multi-segment method of integration. The details of the multi-segment method of integration can be found the references cited above, and its brief description is given in Appendix A.

3. Numerical results and discussion

The present approach will be demonstrated first with numerical examples of untapered joints and doublers with identical adherends. This is because closed-form solutions for these configurations are available in literature for direct comparison with the results obtained from the present approach. In the first example, a single lap joint with two identical isotropic adherends is considered. The geometry and material properties of the identical adherends and of the adhesive as well as the applied load per unit width $P$ are given below (with subscripts 1, 2 and a denoting the bottom adherend, doubler or upper adherend and adhesive, respectively):

- Adherend: isotropic, $E_1 = E_2 = 68.95$ GPa, $v_1 = v_2 = 0.3$, $t_1 = t_2 = 1.27$ mm, where the subscripts 1 and 2 denote the bottom adherend and doubler, respectively.
- Adhesive: isotropic, $E_a = 1.793$ GPa, $G_a = 0.6895$ GPa, $t_a = 0.127$ mm.
- Geometry: $l_0 = 25.4$ cm, overlap length is 6.35 cm.
- Tensile load: $T/t_1 = 137.9$ MPa.

Analytical solutions for this example problem using geometrically linear and nonlinear formulation in step 2 of the present approach are presented in Figs. 5 and 6 for the adhesive peel and shear stresses,

![Adhesive Peel Stress](image-url)

Fig. 5. Distribution of adhesive peel stress in an untapered single lap joint with $t_1 = t_2 = 1.27$ mm.
respectively, and these solutions for brevity will be simply referred to as linear and nonlinear solutions in all discussion below. For comparison purpose, the corresponding Goland–Reissner solutions\(^1\) are also shown in these figures along with the newly obtained solutions. The Goland–Reissner solutions follow closely the present linear solutions as expected since they are based on the same formulation, especially when the involved adherends are identical so that there is no coupling between the peel and shear behavior. Also from Figs. 5 and 6, the differences in the maximum peel and shear stresses between linear and nonlinear solutions are small for this typical lap joint configuration. This is because due to a combination of relatively high bending stiffness of the adherends and a relatively short overlap length to adherend thickness ratio, the nonlinear effect will not have a significant impact on the bending deflection of the joint.

Similarly, the results for the adhesive peel and shear stresses of a doubler configuration with the same adherend geometry and material properties are shown respectively in Figs. 7 and 8. Hart-Smith (2004) recently has developed a closed-form solution for this doubler configuration and his solutions are therefore also plotted along with the present results in Figs. 7 and 8 for comparison. Since his work is still unpublished, a brief description of his solution method is given in the Appendix B for future discussion. It is found from Figs. 7 and 8 that the present linear solution for the peak adhesive peel stress is in better agreement with the corresponding Hart-Smith solution than the nonlinear solution. However, an opposite trend is observed for the peak adhesive shear stress, in which a nonlinear solution is preferred over the linear solution. This abnormality can be explained as follows.

From Appendix B, Hart-Smith solution for the adhesive peel utilizes the moment boundary condition at the end of the overlap and the condition of no net resulting peel force across the interface for determining its unknown constants. The second boundary condition is equivalent to the condition that the vertical shear is zero at the overlap end. Since the adhesive stress analysis in the Hart-Smith second step is geometrically

\(^1\) Actually, the modified Goland–Reissner solutions given by Tsai and Morton (1994) which accounts for the length of the adherend and adhesive thickness on the normalized-edge-moment are used in the comparison. However, for simplicity, the normalized edge moment has been calculated by using Eq. (9) given in the mentioned paper, which assumes zero thickness of adhesive. Thus, the present linear results for the adhesive stresses in a single lap joint are expected to be slightly different from the Goland–Reissner solutions.
linear with all boundary conditions equivalently imposed at the end of the overlap, his peel stress solution therefore should be similar to the present linear result. On the other hand, the adhesive shear stress solution has been derived using the same moment boundary condition at the overlap end as well as the implicitly

Fig. 7. Distribution of adhesive peel stress in an untapered doubler with $t_1 = t_2 = 1.27$ mm.

Fig. 8. Distribution of adhesive shear stress in an untapered doubler with $t_1 = t_2 = 1.27$ mm.
zero bending moment condition at the middle of the doubler as evidence from Eq. (B.14) of Appendix B, and this zero moment condition is obtained from the first step geometrically nonlinear analysis for a doubler configuration with a long overlap length. It should be reminded that because of the zero bending moment at the joint middle, the stresses in the bottom adherend and doubler there will be the same, uniform across their thickness, and for a balanced doubler they are equal to one half of the far field stress (see Appendix B). Even though the bending analysis in the second step for computing adhesive shear is considered to be a geometrically linear analysis, however, as explained later, the use of the second boundary condition at the middle joint based on results of the first step nonlinear analysis will make Hart-Smith’s analysis inconsistent. In other words, a truly linear analysis of Hart-Smith second step will require rather a different boundary condition at the joint middle than the one specified in his analysis.

The second step analysis of the present approach for a doubler configuration always utilized the moment boundary conditions at the overlap end and the symmetry condition at the middle of the joint, regardless of the analysis type, i.e., linear or nonlinear analysis. However, upon solving the governing differential equations, this same symmetry condition will yield different solution for the normal stress and moment resultants at the middle of the doubler, depending on the type of the analysis, as demonstrated in Figs. 9 and 10. This can be seen further by considering an example of a long doubler configuration under a high applied load. For a very long overlap length and for a high applied load, adhesive peel and shear stresses will be decayed to zero near the middle area of the overlap, and the bottom adherend and doubler will act as if they are rigidly bonded there. The bending moment at the middle of the joint, therefore, as predicted by the first step nonlinear analysis, will be zero. If the second step is performed within a linear theory, the present approach will not necessarily yield the same zero bending moment condition there even for this extreme case of a very long overlap length and with high applied load. This is because the condition at the middle of the joint can be determined solely from static moment equilibrium of the overlap area without explicitly solving the differential equations. From static moment equilibrium, there is always a bending moment at the middle of the joint which from Fig. 3 is equal to $T(e_{N} - \epsilon_0)$. On the other hand, a second step analysis performed within a geometrically nonlinear theory will predict the zero bending moment

![Diagram](image_url)

Fig. 9. Distribution of normal stress resultant in an untapered doubler with $t_1 = t_2 = 1.27$ mm.
condition at the joint middle as expected, since (a) the underlined nonlinear terms in the governing equations (15) and (16) make the overlap area statically indeterminate so that the static moment equilibrium condition alone cannot be used to determine the bending moment there, and (b) through inclusion of these nonlinear terms, the effect of the alignment of the neutral plane of the middle area of the overlap with the line of load on the bending moment distribution is correctly accounted for. In light of the above discussion, the Hart-Smith second step analysis for the adhesive shear stress will not be truly a linear analysis since it has used implicitly but inconsistently the zero bending moment at the joint middle. Such inconsistency is believed to make his adhesive shear stress solution being in a closer agreement with the present nonlinear result.

To validate the above claim, the above analysis for the doubler configuration is repeated with a much thinner adherend and doubler \( t_1 = t_2 = 0.254 \text{ mm} \) where the effect of geometrical nonlinearity is more pronounced. It is worthy to note that the chosen thicknesses of the adherend and doubler in the current analysis are unrealistically small for typical joints used in practice, however, they are only used here to demonstrate the extreme difference between linear and nonlinear solution of the adhesive stresses. Adhesive stress results from the new analysis are presented and compared with Hart-Smith solutions in Figs. 11 and 12. Again, these new results follow the same trend when compared with Hart-Smith solutions for this extreme case. For reference, the normal stress and moment resultants in the bottom adherend and in a doubler at their middle are also reported in Table 1. It is clear from Table 1 that the nonlinear solution indicates a nearly zero bending moment at the middle of the joint as expected, but not the linear solution.

So far the solutions of the present approach have been compared with the existing close-form solutions. It is also of interest to compare the new solution with the finite element result. The previous solutions for the case of a single lap joint \( t_1 = t_2 = 1.27 \text{ mm} \) and for the case of extremely thin adherend and doubler \( t_1 = t_2 = 0.254 \text{ mm} \) are compared with the finite element results in Figs. 5, 6 and 11, 12, respectively. In the finite element analysis, the bottom adherend, the doubler or upper adherend and the adhesive are modeled by 2-D isoparametric elements. A typical finite element mesh near one end of the overlap area is shown in Fig. 13 for a single lap joint problem. For a single lap joint configuration, one end of the joint is specified to

![Fig. 10. Distribution of moment resultant in an untapered doubler with \( t_1 = t_2 = 1.27 \text{ mm} \).](image)
be simply supported (fixed transverse displacement, zero bending moment) while the other end is prescribed as pinned end (fixed both longitudinal and transverse displacements). In contrast, due to symmetry, only half of the doubler joint configuration is modeled in the finite element analysis with one end being simply...

Fig. 11. Distribution of adhesive peel stress in an untapered doubler with $t_1 = t_2 = 0.254$ mm.

Fig. 12. Distribution of adhesive shear stress in an untapered doubler with $t_1 = t_2 = 0.254$ mm.
supported and the other end at the vertical center line being imposed with symmetric condition (fixed longitudinal displacement and zero transverse force). The finite element analyses were carried out by using NASTRAN code with geometrically nonlinear solution (Solution 106) and under plane stress condition. In Figs. 5, 6 and 11, 12, the peel and shear stresses from the finite element analyses are computed at the mid-plane of the adhesive layer, which are roughly equal to the average stresses across the adhesive thickness.

It is well known that the adhesive stresses in the joint at the corner of its overlap ends are singular (Tong and Steven, 1999). The adhesive stresses therefore become mesh dependent there. In that regard, a comparison between the present solution and the finite element result needs some physical interpretation. Recent work by Wang and Rose (2000) suggested that the failure in the adhesive can be predicted by using a stress intensity parameter of that corner singularity. Similar to the conventional linear elastic fracture mechanics approach, the Wang and Rose approach assumes that the adhesive will fail if its corner stress intensity factor reaches a critical value determined from the test. Through their boundary layer type analysis and using a finite element method, they further showed that these stress intensity factors can be estimated from the outer boundary layer adhesive peel and shear stresses via certain explicit formulas. The outer boundary layer adhesive peel and shear stresses are those obtained from closed-form methods. Thus, within the corner singularity context, the present solutions for the peak adhesive peel and shear stresses probably should be compared with the corresponding finite element results near the overlap end but outside the corner singularity dominant region where the mesh dependency poses an interpretation problem.

Table 1
Normal stress and moment resultants in bottom adherend and doubler at their middle

<table>
<thead>
<tr>
<th>Analysis</th>
<th>Normal stress resultant in bottom adherend (N/mm)</th>
<th>Normal stress resultant in doubler (N/mm)</th>
<th>Moment resultant in bottom adherend (N mm/mm)</th>
<th>Moment resultant in a doubler (N mm/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonlinear</td>
<td>17.72</td>
<td>17.30</td>
<td>−0.0086</td>
<td>−0.0086</td>
</tr>
<tr>
<td>Linear</td>
<td>28.93</td>
<td>6.09</td>
<td>−0.48</td>
<td>−0.48</td>
</tr>
</tbody>
</table>

Fig. 13. A refined mesh near an overlap end of an untapered single lap joint.

The two-dimensional stress distributions in the adhesive near an end of the overlap for the case of a single lap joint and for case of an extremely thin adherend/doubler are plotted in Figs. 14, 15 and 16, 17,
respectively. Stresses in these fringe plots have been scaled by a factor roughly equal to 145, i.e., a stress value of 1000 in these fringe plots corresponds to 6.89 MPa. From Figs. 15 and 17, the adhesive shear stress at the corner appears to attain the same high value over a quite large zone. This corner shear stress value is therefore considered as the outer boundary layer type of stress to be compared with the analytical solution. This stress is found to be around 27.2 MPa for a single lap joint and about 11 MPa for a thin doubler configuration. In contrast, the plots of the adhesive peel stresses from Figs. 14 and 16 show a much steeper stress gradient near the corner. However, by considering the peel stresses at a distance of roughly one half adhesive thickness\(^2\) away from the free edge as the outer boundary layer stresses for comparing with

\(^2\) Since the element size in that meshing area is about one adhesive thickness, adhesive peel stresses to be compared with analytical predictions are calculated closed to the centroid of a most critical element at the free edge.
analytical predictions, these peel stresses are found to be fairly close to the analytical predictions: 33.2 MPa for a single lap joint example and 12.8 MPa for an extremely thin adherend/doubler problem. Another way to compare the two-dimensional finite element results with the analytical predictions is to use the averaged adhesive peel and shear stresses across the thickness of the adhesive in the comparison as had been done in Figs. 5, 6 and 11, 12. It is also interesting to note that the linear predictions of the peak adhesive shear stresses in both cases are in better agreement with finite element (FE) results than the nonlinear predictions (see Figs. 6 and 12). In contrast, the nonlinear predictions tend to agree better with the finite element results on the adhesive shear distribution over entire overlap length as shown in Fig. 12.

Intuitively, the former observation appears to contradict with a normal expectation in which the nonlinear solution for the peak adhesive shear stress where it is thought to be more accurate does not provide a better agreement the finite element result. However, as found in Oplinger (1994) and Tsai et al. (1998), the
bending moment distribution along the joint in the Goland–Reissner approach (1944) for a single lap joint is inconsistent and may be inaccurate when compared with the finite element results for some joint configurations. In contrast, the Goland–Reissner solutions for the adhesive stresses based on a second step linear bending analysis are very adequate. Thus, any error accrued from the first step analysis seems to be negated by the error introduced in the second step. Based on these findings, an improvement of the second step analysis alone without the first by including the nonlinear terms in the formulation may not lead to an overall improvement of adhesive stresses since any error accrued from the first step will remain through the rest of the analysis, assuming that the second step analysis is exact.

Having verified the present solutions for non-tapered joints and doublers with finite element results and with recent Hart-Smith closed-form solutions in the preceding paragraphs, it remains now to demonstrate the solutions of a tapered joint or doubler. Corresponding solutions to the previous first doubler example problem but with a tapered edge doubler (see Fig. 18) are presented in Figs. 19 and 20 for the adhesive peel stress distribution.

![Fig. 18. Geometry of a balanced doubler with a taper ratio of 10:1 and \(t_1 = 1.27\) mm.](image1)

![Fig. 19. Distribution of adhesive peel stress in a balanced doubler with a taper ratio of 10:1 and \(t_1 = 1.27\) mm.](image2)
and shear stress, respectively. In the latter analysis, the doubler is tapered at its edge by a slope of 1:10 with the minimum thickness of 0.127 mm at the tip as shown in Fig. 18. From Figs. 19 and 20, it is clear that the effect of edge tapering is to reduce the peak adhesive stresses significantly, as expected. However, except at the edge, the adhesive shear stresses are not peaked at the beginning of each step of the multiple-step tapered region as predicted by the classical non-bending solutions. In the classical non-bending solution, since the doubler becomes thicker across the segment junction, some of the axial load in the bottom adherend will be attracted to the doubler and causes a sharp load transfer there. Since all load transfer must be taken through the adhesive layer and in the absence of out-of-plane bending mostly through shearing, adhesive shear stress will peak at the beginning of each step. In a classical non-bending analysis, only the first equations of Eqs. (15) and (16) are considered. Thus, the above physical argument is clearly equivalent to the mathematical statement given by the first equation of either Eq. (15) or (16): the shear stress in an adhesive is equal to a rate of change (differential) of the normal stress resultant in a doubler or bottom adherend. It then follows that an adhesive shear stress will always peak at the segment junction where a rate of change in the normal stress resultant is locally highest.

In contrast, when the out-of-plane deflection is considered in the analysis, the load transfer near the beginning of each step of a tapered doubler will be much more complicated. Consider a infinitesimal element of the bottom adherend at one of the segment junctions after a first load transfer at the overlap end as shown in Fig. 21. The stress and moment resultants in the bottom adherend just to the left of the segment junction are denoted by \( N_l, Q_l \) and \( M_l \), respectively. Similarly, stress and moment resultants just to the right of a segment junction are \( N_r, Q_r \) and \( M_r \). Due to an increase in the doubler extensional and bending stiffness, the axial deflection and bending curvature of a bottom adherend in a right segment of a junction will be smaller than those in the left segment because the doubler increasingly resists those deformations. From linear elasticity, a smaller deformation or bending curvature will result in a smaller normal stress resultant or moment resultant. Thus, \( N_r < N_l \) and \( M_r < M_l \).

For discussion purpose, the effect of the step change in a doubler thickness on the axial deflection and out-of-plane bending will be examined here separately as if they are uncoupled. Even though in reality these

![Fig. 20. Distribution of adhesive shear stress in a balanced doubler with a taper ratio of 10:1 and \( t_1 = 1.27 \text{ mm} \).](image-url)
two deformations are coupled, however, such simplification will bring out more clearly the essential features of the load transfer across a segment junction. From force and moment equilibrium considerations, the adhesive shear stresses due to each individual axial and bending deformation, i.e., $\tau_{a,\text{axial}}$ and $\tau_{a,\text{bending}}$, must act in the direction shown in Fig. 21, and they are opposite to each other. It is worthy to note that peel stress does not contribute to the moment equilibrium equation as evidence from a third equation of Eq. (15). Furthermore, since $M_r$ depends only on $Q_l$ (not $Q_r$) and $Q_l$ is dominated by the bending deformation of a segment on the left of the junction, the effect of a change in doubler bending stiffness in reducing the moment resultant from $M_l$ to $M_r$ across a segment junction therefore must be due mostly to the adhesive shear stress. Now, since bending stiffness is a cubic function of an adherend thickness while extensional stiffness is only a linear function of thickness, a small change in a doubler thickness will have a more significant effect on the bending deformation than the axial deflection. As a result, there is a larger change in the moment resultant than that in the normal stress resultant across a junction. Consequently, adhesive shear stresses accompanying with these changes must qualitatively satisfy the following result $|\Delta\tau_{a,\text{axial}}| < |\Delta\tau_{a,\text{bending}}|$. Thus, across a segment junction, a net change in the adhesive shear stress will be in a direction of $\tau_{a,\text{bending}}$. Since the adhesive shear stress during the very first load transfer is always in the same direction as $\tau_{a,\text{axial}}$, and a net change in adhesive shear stress across any subsequent junction is in a direction of $\tau_{a,\text{bending}}$, the adhesive shear stress will drop across a segment junction as shown in Fig. 20.

To examine the effect of different doubler thickness on the adhesive stresses, the above analysis is repeated for $t_{2N}$ (doubler full thickness) = 2.54 mm. Results from that analysis for an unbalanced doubler are presented in Figs. 22 and 23. Compared to the previous results for $t_{2N} = 1.27$ mm, the adhesive stresses are worst for a thicker doubler, especially in a shear component, as expected.
4. Conclusion

A unified approach for approximating the adhesive stresses in a bonded line of a tapered bonded joint or doubler is presented. This approach is proved to be versatile and robust for assessing bonded joints in the...
daily design and analysis environment. Beside those joint and doubler configurations considered in the paper, the approach can be effectively used for analyzing T-joint under both in-plane tensile load and a vertical pull-off load, provided that appropriate boundary conditions are incorporated in the formulation. Even though the effect of the adherend shear deformation has not been considered in the approach, however, such effect can be easily implemented into the formulation as well.

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Appendix A. Multi-segment method of integration

The differential equation set given by Eq. (21) for \( i = 1, 2, \ldots, N \), together with the boundary conditions listed in (a)–(d) or (a), (b'), (c), and (d') constitute a multiple-point boundary value problem which can be expressed in the following general form:

\[
\{ \Phi(x) \} = [A(x)] \cdot \{ \Phi(x) \},
\]

\[
[T_1] \cdot \{ \Theta_1 \} + [T_2] \cdot \{ \Theta_2 \} = [G],
\]

where

\[
\{ \Phi \} = \begin{cases} \{ \varphi_1 \} \\ \{ \varphi_2 \} \\ \vdots \\ \{ \varphi_N \} \end{cases}, \quad [A] = \begin{bmatrix} [\Psi]_1 \\ \vdots \\ [\Psi]_N \end{bmatrix},
\]

\[
\{ \Theta_1 \} = \begin{cases} \{ \varphi_1(0) \} \\ \{ \varphi_2(0) \} \\ \vdots \\ \{ \varphi_N(0) \} \end{cases}, \quad \{ \Theta_2 \} = \begin{cases} \{ \varphi_1(\ell_1) \} \\ \{ \varphi_2(\ell_2) \} \\ \vdots \\ \{ \varphi_N(\ell_N) \} \end{cases},
\]

\([T_1],[T_2]\) and \([G]\) are constant matrices known from the statements of the boundary conditions. It should be emphasized that the elements of matrices \( \{ \Theta_1 \} \) and \( \{ \Theta_2 \} \) are elements of \( \{ \Phi \} \) evaluated respectively at the beginning and end points of each segment \( i \).

In the multi-segment method of integration, the boundary-value problem will be reduced to a series of initial-value problems as follows (Kalnins, 1964). Assuming that the solution of (A.1) can be written as

\[
\{ \Phi(x) \} = [\Omega(x)] \cdot \{ L \},
\]

where vector \( \{ L \} \) represents \( 12 \cdot N \) arbitrary constants, and \( [\Omega(x)] \) is defined as the homogeneous solution of Eq. (A.1) in the form

\[
[\Omega'(x)] = [A(x)] [\Omega(x)].
\]
The initial conditions for determining \([\Omega(x)]\) is
\[
[\Omega(0)] = [I],
\] (A.7)
where \([I]\) is the identity matrix. A numerical integration scheme for obtaining this homogeneous solution will be detailed later. However, for the present discussion purpose, let assume that the initial-value problem has been solved with its homogeneous solution denoted symbolically as \([\Omega(x)]\).

Evaluation of Eq. (A.5) at \(x = 0\) leads to \(\{\Phi(0)\} = [\Omega(0)]\{L\}\), which by virtue of Eq. (A.7) and a first equation of (A.3) and (A.4) yields
\[
\{\Theta_1\} = \{L\}. 
\] (A.8)

Thus Eq. (A.5) can be expressed as
\[
\{\Phi(x)\} = [\Omega(x)]\{\Theta_1\}. 
\] (A.9)

The next step is to relate \(\{\Theta_2\}\) to \(\{\Theta_1\}\). Since elements of matrix \(\{\Theta_2\}\) are elements of \(\{\Phi\}\) evaluated at the end point of each segment \(i\), an evaluation of Eq. (A.9) at these end points will give the desired relationship between \(\{\Theta_2\}\) and \(\{\Theta_1\}\), i.e.,
\[
\{\Theta_2\} = \{\Phi(X)\}_{i=\text{segment endpoint}} = [\Omega(X)]_{i=\text{segment endpoint}}\{\Theta_1\}. 
\] (A.10)

Eq. (A.10) together with Eq. (A.2) provide a system of algebraic equations for solving \(\{\Theta_1\}\) and \(\{\Theta_2\}\). Once \(\{\Theta_1\}\) and \(\{\Theta_2\}\) are solved, the solution at any value of \(x\) is obtained from Eq. (A.9). In summary, the solution of the multiple-point boundary-value problem can be obtained by the following steps: (i) solving the initial-value problem with the governing differential equations given by Eq. (A.6) and the initial conditions of the multiple-point boundary-value problem can be obtained by the following steps: (i) solving the system of algebraic equations for \(\{\Theta_1\}\) and \(\{\Theta_2\}\) using Eqs. (A.10) and (A.2), and finally (iii) evaluating Eq. (A.9) at any point of interest for its solution.

So far the homogeneous solution of the initial-value problem has been assumed to be already solved and denoted symbolically as \([\Omega(x)]\) in the preceding paragraphs. Thus, a brief description of a suitable numerical method for obtaining \([\Omega(x)]\) will be given here. Differential equation (A.6) and initial condition (A.7) is a compact notation of the following set of differential equations and initial condition:
\[
\frac{d}{dx} \{\Omega_j\}_{12Nx1} = [A] \{\Omega_j\}_{12Nx1}, \quad (j = 1, \ldots, 12 \cdot N), 
\] (A.11)
\[
\{\Omega_j(0)\} = \{\delta_j\}, 
\] (A.12)
where \(\{\delta_j\}\) is a vector corresponding to a \(j\)th column of matrix \([\Omega(x)]\), and \(\{\delta_j\}\) is a vector with all of its components null except for the \(j\)th component where it has a value of 1. Differential equation (A.6) in its standard form of Eq. (A.11) can be solved by means of any method of direct numerical integration. In the present analysis the direct integration was performed using adaptive step-size fourth- and fifth-order Runge–Kutta–Fehlberg method. A Fortran subroutine of this integration method is available from the internet and has been obtained for use in the present study.

In Section 2.2 of the main text and in this appendix, the number of segments for solving the adhesive peel and shear stresses so far has been chosen to be identical to that number of steps in the doubler or upper adherend’s overlap length as shown in Fig. 4, i.e., \(i = 1, 2, \ldots, N\). However, as discussed by Kalnins (1964), the solution obtained by the multi-segment method of integration may suffer a complete loss of accuracy at some critical length of the interval. Thus, if the length of the segment in the analysis exceeds this critical length, the obtained solution will be inaccurate. Nevertheless, the loss of accuracy of the solution can be avoided by subdividing the length of each segment into many sub-segments. Since these sub-segments can be treated identically as the original segments, the formulation in that case for adhesive peel and shear stresses will remain very much the same as before. For example, let assume that due to these segment’s subdivisions the total number of segments increases from \(N\) to \(N + m\), \((m > 0)\), the formulation...
outlined in Section 2.2 and the numerical solution procedure given above will remain the same for this latter case except that (a) \( N \) is changed to \( N + m \) in all related equations, and (b) \( A_{2i}, B_{2i} \) and \( D_{2i} \) do not necessarily have different values for different \( i \), and \( l_i \) is now the sub-segment length.

Appendix B. Hart-Smith solution for an untapered bonded doubler

Hart-Smith solution (2004) for an untapered doubler bonded to a skin is outlined here as it is not yet available in literature. It also follows a two-step analysis procedure. For simplicity, skin and doubler are assumed to be isotropic materials. In the first step, a geometrically nonlinear bending analysis of a rigidly bonded doubler is performed. Following a similar derivation as given in Section 2.2 of the main text, the transverse displacement inside the overlap and the peak bending moment at the end of the overlap are obtained as:

\[
\hat{w}(x) = e \left\{ 1 - \frac{\cosh(\xi_1 x)}{\cosh(\xi_1 c) + \left( \frac{\xi_1}{\xi_0} \right) \sinh(\xi_1 c)} \right\}, \tag{B.1}
\]

\[
M_0 = -\frac{\left( \frac{\xi_1}{\xi_0} \right) \tanh(\xi_1 c)}{1 + \left( \frac{\xi_1}{\xi_0} \right) \tanh(\xi_1 c)} Te, \quad \tag{B.2}
\]

where

\[
e = \frac{E_d t_d \left( t_s + t_d \right)}{2 E_s t_s + E_d t_d} = \frac{S}{1 + S} \left( \frac{t_s + t_d}{2} \right), \quad S = \frac{E_d t_d}{E_s t_s},
\]

\[
\xi_i = \sqrt{\frac{\bar{T}}{D_i}}, \quad D_0 = \frac{E_s t_s^3}{12(1 - v_s^2)},
\]

\[
D_1 = \frac{E_s t_s^3}{12(1 - v_s^2)} + \frac{E_d t_d^3}{12(1 - v_d^2)} + e^2 E_s t_s + \left( \frac{t_s + t_d}{2} - e \right)^2 E_d t_d,
\]

\( c \) is the half doubler length, subscripts \( s \) and \( d \) denote a skin and doubler, respectively, coordinate \( x \) is measured from the doubler center, and the rest are defined similarly as in the main text. As pointed out by Hart-Smith and others, for a long doubler, from Eq. (B.1), the bending moment at the middle of a doubler will be equal to zero, i.e., \( \frac{\partial^2 \hat{w}}{\partial x^2} (x = 0) = 0 \), with the axial load in a doubler given by \( T(x = 0) \). For a balanced doubler (\( S = 1 \)), the axial loads in the skin and doubler are the same and equal to one of the far field load.

Once the bending moment and the transverse displacement are calculated, the peel and shear stresses in the adhesive can be determined following the solution procedure given below.

B.1. Peel stress analysis

The adhesive peel stresses \( \sigma_a \) can be expressed by the differential transverse deflection between the skin and the patch in the following form,

\[
\frac{\sigma_a}{E_a} = \frac{w_s - w_d}{t_a}, \tag{B.4}
\]

where \( w \) again is the transverse deflection, \( E_a \) and \( t_a \) are the elastic modulus and thickness of the adhesive, respectively, while subscript “a” is for adhesive, “s” for the skin, and “d” for the doubler. In contrast, the
adhesive shear strains \( \gamma_a \) and stresses can be established by the differential longitudinal displacements, \( u \), between the skin and the patch,

\[
\gamma_a = \frac{\tau_a}{G_a} = \frac{u_s - u_d}{t_a}, \quad (B.5)
\]

in which \( \tau_a \) and \( G_a \) are the shear stress and shear modulus of the adhesives. Assuming that in each segment the patch and the skin behave as if they were a single plate of combined thickness except for small zones immediately adjacent to the edge of the patch, where the load was transferred from the skin to the patch, one can infer that the adhesive shear stress was zero, and that the axial forces were correspondingly uniform throughout most of the bonded region. In that case, the governing equation for the adhesive peel stress can be expressed approximately as,

\[
\left( \frac{D_0 + D_d}{2} \right) \frac{d^4 (w_s - w_d)}{dx^4} + \frac{2E_a}{t_a} (w_s - w_d) \approx -(D_0 - D_d) \frac{d^4 \hat{w}}{dx^4}, \quad (B.6)
\]

where

\[
D_d = \frac{E_d t_d^3}{12(1 - \nu_d^2)}, \quad (B.7)
\]

\( \hat{w} \) is the average transverse deflection inside the overlap which is already determined from the above geometrically nonlinear bending analysis and the rest is previously defined. For a doubler that is identical to a skin (balanced doubler), the right hand side of Eq. (B.6) is equal to zero.

The general solution to the above equation of a balanced doubler can be approximated by a two-term solution for a region in the immediate proximity of the ends of the doubler as

\[
w_s - w_d \approx e^{-\chi} (A \cos(\chi s) + B \sin(\chi s) ), \quad (B.8)
\]

in which \( \chi \) is defined as,

\[
\chi = \frac{E_a}{t_a (D_0 + D_d)}, \quad (B.9)
\]

and \( s \) is the local coordinate measured from the free edge of the doubler. One boundary condition to establish the unknown coefficients \( A \) and \( B \) is that the integral over the overlap length is zero. The other boundary condition involves the bending moments in the doubler and skin just inside the end of the doubler. They must physically be equal to the values just outside the doubler, with the moment in the doubler equal to zero and that in the skin equal to \( M_0 \). With the above two boundary conditions, one can write the peel stress as following,

\[
\sigma_a = \frac{E_a}{t_a} \frac{M_0}{2\chi^2 D_0} e^{-\chi} \{ \cos(\chi s) - \sin(\chi s) \}. \quad (B.10)
\]

### B.2. Shear stress analysis

The effect of the adhesive peel stresses on the shear stress distribution is neglected in Hart-Smith analysis. The effect of the peel stresses on the shear stress distribution will be small unless there is a strong imbalance between the skin and the patch, then the governing equation for the shear stress have the following form:

\[
\frac{d^3 \tau_a}{dx^3} = 4\chi^2 \frac{d\tau_a}{dx}, \quad (B.11)
\]

where

\[
\chi = \frac{G_a}{t_a} \left( \frac{1}{E_s t_s} + \frac{1}{E_d t_d} \right). \quad (B.12)
\]
As with the preceding analysis for the adhesive peel stresses, it is assumed that the adhesive shear stresses are also localized. Solution to the governing equation is then given by

\[ \tau_a = A_s e^{-2\beta_s} + H_s, \]  

(B.13)

where \( A_s \) and \( H_s \) are the coefficients to be determined by the boundary conditions. Since the derivative of adhesive shear strain can be expressed in terms of both extensional force and the bending moment, \( A_s \) can be found by one boundary condition with the evaluation of the adhesive shear strain at the end of doubler. Furthermore, since the load transferred through the adhesive is equal to the load in the doubler at its middle, this fact provides another boundary condition for evaluating the remaining coefficient \( H_s \). For a long overlap, using the result from the above geometrically nonlinear bending analysis for a rigidly bonded doubler, this condition implies

\[ \int_0^{+c} \tau ds = T \left( \frac{S}{1 + S} \right) = \frac{A_s}{2\lambda} + H_s c. \]  

(B.14)

The adhesive shear stress within the overlap can then be rewritten as following:

\[ \tau_a = \frac{G_a}{2\lambda t_s E_s} \left( \frac{T - 6M_0}{t_s^2} \right) e^{-2\beta_s} + \frac{1}{c} \left[ T \left( \frac{S}{S + 1} \right) - \frac{G_a}{4\lambda^2 t_s E_s} \left( \frac{T - 6M_0}{t_s^2} \right) \right]. \]  

(B.15)

References