Lipschitz Stability of Nonlinear Systems of Differential Equations. II. Liapunov Functions

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INTRODUCTION

In [5], the authors introduced the notion of Lipschitz stability in differential equations. This notion lies somewhere between uniform stability on one side and the notions of asymptotic stability in variation [3] and uniform stability in variation [4] on the other side. However, Lipschitz stability is new only as a nonliner phenomenon, since it coincides with uniform stability in linear systems [5]. An important feature of Lipschitz stability is that, unlike uniform stability, the linearized system inherits the property of Lipschitz stability from the original nonlinear system [5].

In this paper we pursue the study of Lipschitz stability that started in [5] using essentially the techniques of Liapunov functions. Then we give sufficient conditions for the Lipschitz stability of certain nonlinearly perturbed nonlinear systems. Such systems include, among other equations, certain integrodifferential and functional differential equations. Then we give an example which can be investigated successfully using our results but cannot be handled by any previous techniques or results [9].

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Consider the differential system

\[ x' = f(t, x), \]  

(1.1)

where \( f \in C[J \times \mathbb{R}^n, \mathbb{R}^n] \), \( J = [0, \infty) \), \( f(t, 0) = 0 \), and \( x(t, t_0, x_0) \equiv x(t) \) is the solution of (1) with \( x(t_0, t_0, x_0) = x_0 \), where \( t_0 \geq 0 \).

**Definition 1.1** [5]. The zero solution of (1.1) is said to be uniformly Lipschitz stable if there exists \( M > 1 \) and \( \delta > 0 \) such that \( |x(t, t_0, x_0)| \leq M |x_0| \) for \( |x_0| < \delta \) and \( t \geq t_0 \geq 0 \). The constant \( M \) is called the Lipschitz constant.

Consider a continuous function \( V: \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R} \). Then corresponding to \( V \) we define the function

\[ V_{(1.1)}(t, x) = \lim_{h \to 0^+} \frac{1}{h} \{ V(t+h, x+hf(t, x)) - V(t, x) \}. \]

(1.2)

We denote by \( V'(t, x(t)) \) the upper right hand derivative of \( V(t, x(t)) \), i.e.,

\[ V'(t, x(t)) = \lim_{h \to 0^+} \frac{1}{h} \{ V(t+h, x(t+h)) - V(t, x(t)) \}. \]

If \( V \) is locally Lipschitz with respect to \( x \), then \( V_{(1.1)}(t, x) = V'(t, x) \) [10].

**Theorem 1.2.** Suppose that \( f(t, x) \) in (1.1) is locally Lipschitz in \( x \) uniformly in \( t \). Then the zero solution of (1.1) is uniformly Lipschitz stable iff there exists a continuous function \( V(t, x) \) defined for \( t \geq 0 \) and \( |x| < \delta \), such that

1. \( |x| \leq V(t, x) \leq L |x| \), for some constant \( L \).
2. \( |V(t, x) - V(t, y)| \leq |x - y| \) for all \( t \geq 0, x, y \in \mathbb{R}^n \), with \( |x| < \delta \), \( |y| < \delta \).
3. \( V_{(1.1)}(t, x) \leq 0 \).

**Proof.** Sufficiency. Assume conditions (1), (2), and (3). Then

\[ |x(t, t_0, x_0)| \leq V(t, x(t, t_0, x_0)) \]

\[ \leq V(t_0, x_0) \]

\[ \leq L |x_0|, \quad \text{for } |x_0| < \delta. \]

Necessity. Assume that the zero solution of (1.1) is uniformly Lipschitz stable. Let

\[ V(t, x) = \sup_{\tau \geq 0} |x(t + \tau, t, x)|/(1 + e^{-\tau - t}). \]
Then
\[ |x| = |x(t, t, x)| \leq |x(t, t, x)|(1 + e^{-1}) \]
\[ \leq V(t, x) \]
\[ \leq M \sup_{\tau \geq 0} |x|(1 + e^{-\tau - t}) \]
\[ \leq 2M |x| = L |x| . \]

This establishes (1).

Since \( f(t, x) \) in (1.1) is locally Lipschitz in \( x \) uniformly in \( t \), there is a constant \( K = K(M, \delta) \) such that
\[ |x(t + \tau, t, x) - x(t + \tau, t, y)| \leq e^{K\tau} |x - y| \]
for \( |x| < \delta, |y| < \delta, \tau \geq 0 \). Note here that \( |x(t + \tau, t, x)| \leq M |x| < M\delta \) and
\[ |x(t + \tau, t, y)| \leq |x(t + \tau, t, x)| < M\delta . \]

Let \( T > 0 \) be such that \( M = e^T \). Then \( \sup_{\tau \geq 0} |x(t + \tau, t, x)|(1 + e^{-\tau - t}) \leq \sup_{\tau \geq 0} |x|(e^T + e^{T - (t + \tau)}) \). Hence the above sup is realized if \( 0 \leq t + \tau \leq T \); in case \( t < T \) and if \( \tau = 0 \) in case \( t \geq T \). Thus for \( |x| < \delta, |y| < \delta \),

\[ |V(t, x) - V(t, y)| \]
\[ \leq \sup \{ |x(t + \tau, t, x) - x(t + \tau, t, y)|(1 + e^{-\tau - t})| \]
\[ 0 \leq \tau + t \leq T \text{ if } t < T \text{ and } \tau = 0 \text{ if } t \geq T . \]
\[ \leq \sup_{\tau} e^K |x - y|(1 + e^{-\tau - t}) , \quad \text{with the above restrictions on } \tau \]
\[ \leq L |x - y| , \quad \text{which establishes (2)} . \]

Now
\[ V'_{(1.1)}(t, x) = \lim_{h \to 0^+} \frac{1}{h} \{ V(t + h, x(t + h, t, x)) - V(t, x) \} \]
\[ = \lim_{h \to 0^+} \frac{1}{h} \{ \sup_{\tau < 0} |x(t + h + \tau, t, x)|(1 + e^{-t - \tau - h}) \]
\[ - \sup_{\tau \geq 0} |x(t + \tau, t, x)|(1 + e^{-t - \tau})\}
\[ = \lim_{h \to 0^+} \frac{1}{h} \{ \sup_{\tau \geq h} |x(t + \tau, t, x)|(1 + e^{t - \tau - h}) \]
\[ - \sup_{\tau \geq h} |x(t + \tau, t, x)|(1 + e^{-t - \tau})\}
\[ \leq 0 , \quad \text{which establishes (3)} . \]
To prove $V(t, x)$ is continuous we observe that
\[
|V(t + h, x + y) - V(t, x)| \\
\leq |V(t + h, x + y) - V(t + h, x(t + h, t, x + y))| \\
+ |V(t + h, x(t + h, t, x + y)) - V(t, x + y)| \\
+ |V(t, x + y) - V(t, x)|.
\]

The fact that the first and third terms can be made small if $h$ and $y$ are small follows from the Lipschitzian property of $V$ and the continuity of the solutions of (1.1). The second term can be seen to be small if $h$ is small by an argument similar to that used in showing that $V';(1.1)(t, x)$ exists. This completes the proof of the theorem.

**Example 1.3.** Consider the differential equation
\[
x' = -e^tx^3, \quad x(t_0) = x_0.
\]

The solution of (1.3) is given by
\[
x(t, t_0, x_0) = x_0 \left[ 1 + 2x_0^2(e^t - e^{t_0}) \right]^{-1/2}, \quad t \geq t_0 \geq 0.
\]

Applying the techniques of Theorem 1.2, we let $V(t, x) = |x|$. Then $V$ satisfies the sufficiency conditions of Theorem 1.2. Hence the zero solution of (1.3) is uniformly Lipschitz stable. Now $V';(1.3)(t, x) = -e^tx^3 \leq |x|^3$. Hence by Theorem 8.3 in [10], the zero solution of (1.3) is uniformly asymptotically stable.

The following example will show that uniform asymptotic stability does not imply uniform Lipschitz stability.

**Example 1.4.** Consider the Lienard's equation
\[
x'' + f(x) x' + g(x) = 0, \quad (1.4)
\]
where
\[
f(x) = 3x^2, \quad g(x) = x^3.
\]

Let
\[
G(x) = \int_0^x g(y) \, dy = \frac{x^4}{4}.
\]

Then
\[
xg(x) > 0, x \neq 0, f(x) > 0, x \neq 0, G(x) \to \infty \text{ as } |x| \to \infty.
\]
The above Lienard's equation is equivalent to
\begin{align}
  x' &= y \\
  y' &= -x^3 - 3x^2y.  
\end{align}  \tag{1.5}

The zero solution of (1.5) is uniformly asymptotically stable [7]. The linearized system corresponding to (1.5) is given by
\begin{align}
  x' &= y \\
  y' &= 0.  
\end{align}  \tag{1.6}

It is clear that the zero solution of (1.6) is unstable. This implies that the zero solution of (1.5) is not uniformly Lipschitz stable. This is due to the fact that if the zero solution of (1.5) is uniformly Lipschitz stable, so is the zero solution of (1.6) [5, Theorem 3.4].

Let $\Phi(t, t_0)$ be the fundamental matrix solution of the linearized system
\[ y' = f_x(t, 0) y, \]
where $f_x$ denotes $\partial f/\partial x$. Define the matrix $G(t)$ as follows: $G(t) = \int_t^{\infty} \Phi^T(s, t) \Phi(s, t) ds$, where $\Phi^T$ is the transpose of $\Phi$. Then clearly $G(t)$ is symmetric.

**Theorem 1.5.** Assume that for some constant $L$ and a nondecreasing function $\alpha(t)$,
\begin{enumerate}
  \item $\alpha(t)|x|^2 \leq \langle G(t)x, x \rangle \leq L\alpha(t)|x|^2$ where $\langle \cdot, \cdot \rangle$ is the inner product,
  \item $|G(t)F(t, x)| \leq |x|$, $\beta > 1$.
\end{enumerate}

Then the zero solution of (1.1) is uniformly Lipschitz stable.

**Proof.** Let $V(t, x) = \langle G(t)x, x \rangle$. Let us write $f(t, x)$ in (1.1) as $f(t, x) = f_x(t, 0)x + F(t, x)$. Now
\begin{align}
  V_{1.1}(t, x(t)) &= \langle G'(t)x, x \rangle + \langle (G + G^T)f_x(t, 0)x + F(t, x), x \rangle \\
  &= \langle [G'(t) + 2G(t)f_x(t, 0)]x, x \rangle + \langle 2G(t)F(t, x), x \rangle.  
\end{align}  \tag{1.7}

Claim that
\begin{align}
  \langle [G'(t) + 2G(t)f_x(t, 0)]x, x \rangle &= \langle x, x \rangle.  
\end{align}  \tag{1.8}
To prove the claim we note first that
\[
\frac{\partial \Phi(s, t)}{\partial t} = -\Phi(s, t) f_x(t, 0) \tag{1.9}
\]
and
\[
\frac{\partial \Phi^T(s, t)}{\partial t} = -f_x^T(t, 0) \Phi^T(s, t). \tag{1.10}
\]
Hence
\[
G'(t) = -I + \int_t^\infty \left[ \frac{\partial \Phi^T(s, t)}{\partial t} \Phi(s, t) + \Phi^T(s, t) \frac{\partial \Phi(s, t)}{\partial t} \right] ds. \tag{1.11}
\]
Substituting from (1.9) and (1.10) in (1.11), one obtains
\[
G'(t) = -I - f_x^T(t, 0) G(t) - G(t) f_x(t, 0).
\]
Hence \(\langle [G'(t) + 2G(t) f_x(t, 0)] x, x \rangle = -\langle x, x \rangle\) and the proof of the claim is now complete.

Substituting from (1.8) into (1.7) we have
\[
V_{(1.1)}(t, x(t)) = -\langle x, x \rangle + 2\langle G(t) F(t, x), x \rangle \\
\leq |x|^2 + |x|^2 \\
\leq 0.
\]
Hence
\[
V(t, x(t, t_0, x_0)) \leq V(t_0, x_0). \tag{1.12}
\]
From (i), (ii), and (1.12), one obtains
\[
\alpha(t)|x(t, t_0, x_0)|^2 \leq V(t, x(t, t_0, x_0)) \leq V(t_0, x_0) \leq L\alpha(t_0)|x_0|. 
\]
Hence \(|x(t, t_0, x_0)|^2 \leq L\alpha^{-1}(t_0) \alpha(t_0)|x_0| \leq M^2 |x_0|^2\) and the proof of the theorem is now complete.

2. COMPARISON WITH SCALAR EQUATIONS

Let \(S_\delta = \{x \in \mathbb{R}^n: |x| < \delta\}\) and \(J = [t_0, \infty)\) for \(t_0 > 0\).

**Theorem 2.1.** Suppose that there exist two functions \(V(t, x)\) and \(g(t, u)\) satisfying the following conditions:
(i) \( g(t, u) \in C[J \times \mathbb{R}^+, \mathbb{R}] \) and \( g(t, 0) = 0 \),

(ii) \( V(t, x) \in C[J \times S_\delta \mathbb{R}^+] \), \( V(t, 0) = 0 \), \( V(t, x) \) is locally Lipschitz in \( x \) and satisfies

\[
V(t, x) \geq b(|x|), \quad \text{where} \quad b(r) \in C[[0, \delta], \mathbb{R}^+],
\]

\( b(0) = 0 \), and \( b(r) \) is strictly monotone increasing in \( r \) such that

\[
b^{-1}(ar) \leq q(a) \quad \text{for some function} \quad q,
\]

with \( q(x) \geq 1 \) if \( x \geq 1 \).

(iii) For \((t, x) \in J \times S_\delta\), \( V'(t, x) \leq g(t, V(t, x)) \). If the zero solution of

\[
u' = g(t, u), \quad u(t_0) = u_0 \geq 0
\]

is uniformly Lipschitz stable, then so is the zero solution of (1.1).

**Proof.** Assume that the zero solution of (2.1) is uniformly Lipschitz stable. If \( r(t, t_0, u_0) \) is the maximal solution of (2.1), then \( r(t, t_0, u_0) \leq L u_0 \) for \( |u_0| < \delta \) and some constant \( L \geq 1 \). Using condition (iii), one can apply Theorem 3.1.1 in [8] to obtain

\[
V(t, x(t)) \leq r(t, t_0, u_0).
\]

Choose \( x_0 \) such that \( u_0 = V(t_0, x_0) \) where \( u_0 < \delta \) and \( |x_0| < \delta \).

From (ii) we have

\[
b(|x(t, t_0, x_0)|) = V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0)
\]

\[
\leq L u_0
\]

\[
= L V(t_0, x_0)
\]

\[
\leq LK |x_0|,
\]

where \( K \) is the Lipschitz constant.

Hence

\[
|x(t, t_0, x_0)| \leq b^{-1}(LK |x_0|) \leq |x_0| \leq q(LK) \leq M |x_0|.
\]

**Remark 2.2.** The above theorem remains valid if (ii) is replaced by

(ii)' \( (\lambda_1(t)|x|)^2 \leq V(t, x) \leq (\lambda_2(t)|x|^2 \), where \( \lambda_1(t) \) and \( \lambda_2(t) \) are positive continuous functions with \( \lambda_2(t_0) \geq \lambda_1(t) \).

To prove this remark we follow the steps in the proof of Theorem 2.1
and put \( r_0 = V(t_0, x_0) \). Hence \( (\lambda_1(t) |x|)^2 \leq V(t, x) \leq \lambda_0 = \lambda V(t_0, x_0) \leq L(\lambda_2(t_0) |x_0|)^2 \). Hence

\[
|x(t, t_0, x_0)| \leq L(\lambda_2(t_0)/\lambda_1(t_0)) |x_0| = M |x_0|.
\]

**DEFINITION 2.3** [6]. A function \( W: [0, \infty) \rightarrow [0, \infty) \) is said to belong to the class \( H \) if

\[ (H_1) \quad w(u) \text{ is nondecreasing and continuous for } u \geq 0 \text{ and positive for } u > 0. \]

\[ (H_2) \quad \text{There exists a function } \phi \text{ continuous on } [0, \infty) \text{ with } W(\alpha u) \leq \phi(\alpha) w(u) \text{ for } \alpha > 0 \text{ and } u \geq 0. \]

Consider the scalar differential equation

\[
u' = \lambda(t) w(u), \tag{2.2}
\]

where \( \lambda(t) \) is continuous on \([t_0, \infty)\), \( t_0 \geq 0 \), \( w(u) \in H \), \( w(0) = 0 \), with corresponding multiplier function \( \phi \).

**THEOREM 2.4.** Assume that

\[
\int_{t_0}^{t} |\lambda(s)| \, ds \leq K \quad \text{and} \quad \int_{z_0}^{\infty} \frac{ds}{w(s)} = \infty
\]

for all \( t \geq t_0 \). Then the zero solution of (2.2) is uniformly Lipschitz stable, provided that \( \phi(z) \leq Mz \) for \( z < \delta \).

**Proof.** If \( u(t, t_0, u_0) \equiv u(t) \) is the solution of (2.2) with \( u(t_0, t_0, u_0) = u_0 \), then

\[
|u(t)| \leq |u_0| + \int_{t_0}^{t} |\lambda(s)| w(u(s)) \, ds.
\]

Applying Theorem 1 in [9], one obtains

\[
|u(t)| \leq |u_0| W^{-1} \left[ W(1) + \int_{t_0}^{t} |\lambda(s)| \frac{\phi(u_0)}{u_0} \, ds \right], \tag{2.3}
\]

where

\[
W(z) = \int_{z_0}^{z} \frac{ds}{w(s)}, \quad z > 0, \ 1 > z_0 > 0.
\]
Since $\int_{s_0}^{\infty} (ds/w(s)) = \infty$, it follows that

$$W'(1) + \int_{s_0}^{\infty} |\dot{w}(s)| \frac{\phi(u_0)}{u_0} ds \in \text{Dom}(W^{-1}).$$

Furthermore, $W(z)$ is a monotonic increasing function which maps $[t_0, \infty)$ onto $[\infty, \infty)$. Thus $W^{-1}$ is also an increasing function.

From the hypothesis, (2.3) becomes

$$|u(t, t_0, u_0)| \leq u_0 W^{-1}[W(1) + KM] = L |u_0| \text{ for } |u_0| < \delta.$$

3. Perturbed Systems

Consider the perturbed system

$$y' = f(t, y) + g(t, y), \quad y(t_0) = y_0, \quad t_0 \geq 0, \quad (3.1)$$

where $f, g \in C[J \times S_s, \mathbb{R}^n], f(t, 0) = g(t, 0) = 0$.

**Theorem 3.1.** Suppose that the zero solution of (1.1) is uniformly Lipschitz stable with a Lipschitz constant $M$ and there exists a function $w(t, u) \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$, $w(t, 0) = 0$, which is monotonic non-decreasing in $u$ for each $t \in \mathbb{R}^+$ and such that

$$|s(t, u)| \leq w(t, |u|).$$

If the zero solution of the scalar equation

$$u' = Mw(t, u), \quad u(t_0) = u_0 \geq 0 \quad (3.3)$$

is uniformly Lipschitz stable, then so is the zero solution of (3.1).

**Proof.** By Theorem 1.2 there exists a function $V(t, x)$ having the three properties mentioned in the theorem. Using properties (i), (ii), and (iii) in Theorem 1.2 and the assumption on $w$, one obtains

$$V_{(3.1)}(t, y) \leq M |g(t, y)| \leq Mw(t, |y|)$$

$$\leq Mw(t, V(t, y)).$$

Let $r(t, t_0, u_0)$ be the maximum solution of (3.3) with $u_0 = M |y_0|$, $|y_0| < \delta/M$ and $y(t, t_0, y_0)$ be the solution of (3.1) passing through $y_0$ at $t_0$. 


By Theorem 3.1.1 in [8] it follows from (3.4) that \( V(t, y(t, t_0, y_0)) \leq r(t, t_0, u_0) \). From property (i) in Theorem 1.2 we have
\[
|y(t, t_0, y_0)| \leq V(t, y(t, t_0, y_0)) \leq r(t, t_0, u_0) \\
\leq \bar{M} |u_0| = \bar{M} M |y_0| = K |y_0|.
\]

The proof of the theorem is now complete.

We now give an example to show the sharpness of the previous theorem. In essence we show that the condition \( |g(t, y)| \leq c |y| \) is not enough to obtain the conclusion of the theorem.

**Example 3.2.** Consider the system
\[
\begin{align*}
x'_1 &= x_2 + x_1 p(t) \\
x'_2 &= -x_1 + x_2 p(t),
\end{align*}
\]  
(3.5)

where \( p(t) \) is a continuous function for \( t \geq 0 \) and \( |p(t)| \leq K \) for some constant \( K > 0 \). The condition \( |g(t, x)| \leq c |x| \) is satisfied for \( C = K \). Furthermore, the zero solution of \( x'(t) = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_2 \end{pmatrix} \) is uniformly Lipschitz stable. From (3.5), it follows that \( d(x_1^2 + x_2^2) = 2(x_1^2 + x_2^2) p(t) \, dt \). Hence \( |x| = |x_0| \exp \int_{t_0}^{t} p(s) \, ds \).

If we put \( p(t) = 1/(1 + t) \), then \( |x| = |x_0| \sqrt{(1 + t)/(1 + t_0)} \). Thus the zero solution of (3.5) is not uniformly Lipschitz stable.

However, if we let \( p(t) = 1/(1 + t^2) \), then \( w(t, u) = u/(1 + t^2) \) satisfies the conditions of the theorem and, consequently, the zero solution of (3.5) is uniformly Lipschitz stable.

**Lemma 3.3.** Let \( u(t) \), \( a(t) \), \( b(t) \), \( c(t) \), \( k(t) \), and \( l(t) \) be nonnegative continuous functions on \( [0, \infty) \) and \( u_0 \) be a nonnegative constant such that
\[
\begin{align*}
u(t) &\leq u_0 + \int_{t_0}^{t} a(s) u(s) \, ds + \int_{t_0}^{t} b(s) \int_{t_0}^{s} c(\tau) u(\tau) \, d\tau \, ds \\
&\quad + \int_{t_0}^{t} k(s) \int_{t_0}^{s} l(\tau) u(\tau) \, d\tau \, ds.
\end{align*}
\]

Then \( u(t) \leq u_0 \exp \left[ \int_{t_0}^{t} [a(s) + b(s) \int_{t_0}^{s} c(\tau) \, d\tau + k(s) \int_{t_0}^{s} l(\tau) \, d\tau] \, ds \right] \), \( 0 \leq t_0 \leq t < \infty \).

**Proof.** Let
\[
R(t) = u_0 + \int_{t_0}^{t} a(s) u(s) \, ds + \int_{t_0}^{t} b(s) \int_{t_0}^{s} c(\tau) u(\tau) \, d\tau \\
+ \int_{t_0}^{t} k(s) \int_{t_0}^{s} l(\tau) u(\tau) \, d\tau \, ds.
\]
Then from the assumption, \( u(t) \leq R(t) \). Furthermore,

\[
R'(t) = a(t) u(t) + b(t) \int_{t_0}^{t} c(s) u(s) \, ds + k(t) \int_{t_0}^{t} l(s) u(s) \, ds
\]

\[
\leq a(t) R(t) + b(t) \int_{t_0}^{t} c(s) R(s) \, ds + k(t) \int_{t_0}^{t} l(s) R(s) \, ds.
\]  

(3.6)

Since \( R(t) \) is nondecreasing (3.6) may be written as

\[
R'(t) \leq R(t) \left[ a(t) + b(t) \int_{t_0}^{t} c(s) \, ds + k(t) \int_{t_0}^{t} l(s) \, ds \right].
\]  

(3.7)

Integrating (3.7) yields the conclusion of the lemma.

Consider the perturbed system

\[
y' = f(t, y) + g(t, Ty) + h(t, y, Ly),
\]

(3.8)

where \( T \) and \( L \) are continuous operators which map \( \mathbb{R}^n \) into \( \mathbb{R}^n \),

\[
g \in C[\mathbb{J} \times \mathbb{R}^n, \mathbb{R}^n], \quad h \in C[\mathbb{J} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n],
\]

\[
f \in C[\mathbb{J} \times \mathbb{R}^n, \mathbb{R}^n], \quad f(t, 0) = 0, \quad \text{and} \quad f_x(t, x) \ (\partial f / \partial x) \text{ exists and is continuous on } \mathbb{J} \times \mathbb{R}^n.
\]

The fundamental matrix solution \( \Phi(t, t_0, x_0) \) of

\[
z' = f_x(t, x(t, t_0, x_0)) z
\]

is given by [7]

\[
\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} (x(t, t_0, x_0)).
\]

**Definition 3.4.** [5]. The zero solution of (1.1) is said to be uniformly Lipschitz stable in variation if there exists \( M \geq 1 \) and \( \delta > 0 \) such that \( |\Phi(t, t_0, x_0)| \leq M \) for \( |x_0| < \delta \) and \( t \geq t_0 \geq 0 \).

**Theorem 3.5.** Suppose that the zero solution of (1.1) is uniformly Lipschitz stable in variation and the following conditions are satisfied:

- (i) \( |g(t, Ty)| \leq b(t) \int_{t_0}^{t} c(s) |y(s)| \, ds, \quad b(t), c(t) \in C^+ (\mathbb{J}), \)
- (ii) \( |h(t, y, Ly)| \leq a(t)|y| + k(t) \int_{t_0}^{t} l(s) |y(s)| \, ds, \quad a(t), k(t), l(t) \in C^1 (\mathbb{J}), \)
- (iii) \( \int_{t_0}^{\infty} \left[ a(s) + b(s) \int_{t_0}^{s} c(\tau) \, d\tau + k(s) \int_{t_0}^{s} l(\tau) \, d\tau \right] \, ds < \infty. \)

Then the zero solution of (3.8) is uniformly Lipschitz stable.
Proof. Let \( y(t, t_0, y_0) \) be the solution of (3.8) with \( y(t_0, t_0, y_0) = y_0 \). Then by the nonlinear variation of constants formula of Alekseev [1] we have

\[
y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y)\left[ g(t, Ty(s)) + h(t, y(s), L_y(s)) \right] ds
\]

\[
\leq M |y_0| + M \int_{t_0}^{t} a(s) |y(s)| ds + M \int_{t_0}^{t} b(s) \int_{t_0}^{s} c(\tau) |y(\tau)| d\tau ds
\]

\[
+ M \int_{t_0}^{t} k(s) \int_{t_0}^{s} l(\tau) |y(\tau)| d\tau ds.
\]

Applying Lemma (3.3) and using assumption (iii), one obtains

\[
|y(t, t_0, y_0)| \leq L |y_0| \quad \text{for} \quad |y_0| < \delta.
\]

**Example 3.6.** Consider the equations

\[
x' = -\frac{1}{2} x^3 \tag{3.10}
\]

\[
y' = -\frac{1}{2} y^3 + te^{-t}y + e^{-t} \sin t \int_{0}^{t} h(s) y(s) ds. \tag{3.11}
\]

If we omit the term \( te^{-t}y \) from (3.11), then we have the integrodifferential equation that was considered by Yang [9]. Clearly the zero solution of (3.10) is uniformly Lipschitz stable in variation. If \( |h(t)| \in L_1(0, \infty) \), then Corollary 1 in [9] does apply to ensure that the zero solution of (3.11) is uniformly Lipschitz stable. However, if \( |h(t)| \notin L_1(0, \infty) \) as, for example, when \( h(t) = 1 \), then Yang’s result is not applicable.

Theorem 3.5 is, however, applicable in this case. If we just let \( a(t) = te^{-t} \), \( k(t) = e^{-t} \), \( l(t) = h(t) \), and \( b(t) = 0 \), then assumption (iii) becomes

\[
\int_{t_0}^{t} (te^{-t} + e^{-t}) \int_{t_0}^{s} h(s) ds dt < \infty,
\]

which is satisfied, for example, if \( h(t) \) is a polynomial of degree \( n \) in \( t \).

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**References**
