On the Fujita exponent for a semilinear heat equation with a potential term

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Abstract
We consider the existence and nonexistence of positive global solutions for the Cauchy problem,
\[
\begin{aligned}
\partial_t u &= \Delta u - V(x)u + u^p \quad \text{in } \mathbb{R}^N \times (0, \infty), \\
u(x, 0) &= \phi(x) \geq 0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]
where \( p > 1 \) and \( V \) behaves like \( \omega |x|^{-2}(1 + o(1)) \) with \( \omega > 0 \), as \( |x| \to \infty \). In this paper we determine the so-called Fujita exponent \( p_* \) for this Cauchy problem. Furthermore, for the critical case \( p = p_* \), we prove that the Cauchy problem has no global positive solutions.

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1. Introduction

We consider the Cauchy problem for a semilinear heat equation with a potential,
\[
\begin{aligned}
\partial_t u &= \Delta u - V(x)u + u^p \quad \text{in } \mathbb{R}^N \times (0, \infty), \\
u(x, 0) &= \phi(x) \geq 0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]
where \( p > 1 \), \( N \geq 2 \), \( \partial_t = \partial/\partial t \), \( \phi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \), and the potential \( V \) is nonnegative and behaves like \( \omega |x|^{-2}(1 + o(1)) \) with \( \omega > 0 \), as \( |x| \to \infty \).

In 1966, Fujita [3] considered the Cauchy problem
\[
\begin{aligned}
\partial_t u &= \Delta u + u^p \quad \text{in } \mathbb{R}^N \times (0, \infty), \\
u(x, 0) &= \phi(x) \geq 0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]
and proved that
(A) if $1 < p < p_*$, the problem (1.2) has no positive global solutions;
(B) if $p > p_*$, the problem (1.2) has a positive global solution for some initial data $\phi$,

where $p_* = 1 + 2/N$. We call this critical number $p_* = 1 + 2/N$ the Fujita exponent. The statement (A) also holds for the case $p = p_*$, which was proved by Hayakawa [4], Kobayashi, Sirao, and Tanaka [7] and alternative proofs were given by Aronson and Weinberger [1] and Weissler [11]. Subsequently the Fujita result has been extended by many mathematicians in several directions. For the details, see two survey papers [2] and [8] to this problem and references therein.

As stated in [10], for the Cauchy problem (1.1), the potential $V$ has a strong influence on the Fujita exponent. Zhang [10] considered the problem of the existence and nonexistence of global positive solutions for the Cauchy problem (1.1) on an $N(\geq 3)$-dimensional complete noncompact Riemannian manifold $M$. He studied the relation between the Fujita exponent and the potentials $V$ behaving like $\omega/(1 + d(x)^a)$, by using global bounds for the fundamental solutions of the heat equations with a potential. Here $\omega \in \mathbb{R}$, $a > 0$, and $d(x)$ is the distance between a point $x \in M$ and a reference point $O \in M$. In particular, for the case $M = \mathbb{R}^N$ with $N \geq 3$ and $a \neq 2$, he proved that the Fujita exponent $p_*$ for the Cauchy problem (1.1) is $1 + 2/N$ if $a > 2$, $\infty$ if $1 < a < 2$ and $\omega < 0$, and $1$ if $1 < a < 2$ and $\omega > 0$. Furthermore, for the case $a = 2$, he also proved that $1 < p_* \leq 1 + 2/N$ if $\omega > 0$ and $p_* \geq 1 + 2/N$ if $\omega < 0$. In particular, the case $a = 2$ is a border line case where the Fujita exponent may vary from $1$ to $\infty$, and it would be interesting to study the relation between the Fujita exponent $p_*$ and the constant $\omega$.

In this paper we study the existence and nonexistence of global positive solutions of (1.1) with a potential $V$ behaving like $\omega|x|^{-2}(1 + o(1))$ with $\omega > 0$, as $t \to \infty$, and give the Fujita exponent $p_*$ for the Cauchy problem (1.1), explicitly. Furthermore we prove that the problem (1.1) for the critical case $p = p_*$ has no positive global solutions.

Throughout this paper we assume that
\begin{equation}
V \in C^1(\mathbb{R}^N), \quad V \geq 0 \text{ in } \mathbb{R}^N.
\end{equation}
We say that $u$ is a solution of (1.1) if $u$ satisfies (1.1) in the classical sense and $\|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty$ for each time $t > 0$. If the solution $u$ does not exist globally in time, then
\[
\limsup_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty
\]
for some $T > 0$, and we say that the solution $u$ blows-up at the time $T$. For any $\omega > 0$, let $\alpha = \alpha(\omega)$ be the positive root of the algebraic equation $\alpha(\alpha + N - 2) = \omega$, that is,
\[
\alpha(\omega) = \frac{-(N - 2) + \sqrt{(N - 2)^2 + 4\omega}}{2} > 0,
\]
and put
\[
p_*(\omega) \equiv 1 + \frac{2}{N + \alpha(\omega)}.
\]
Then we have the following results, which are the main results of this paper.

**Theorem 1.1.** Assume (1.3) and that there exist positive constants $\omega$, $\theta$, and $R$ such that
\begin{equation}
V(x) \geq \omega|x|^{-2}(1 - |x|^{-\theta}) \quad (1.4)
\end{equation}
for all $x \in \mathbb{R}^N$ with $|x| \geq R$. Then the Cauchy problem (1.1) has a global positive solution for some initial data $\phi$ if $p > p_*(\omega)$.

**Theorem 1.2.** Assume (1.3) and that there exist positive constants $\omega$, $\theta$, and $R$ such that
\begin{equation}
V(x) \leq \omega|x|^{-2}(1 + |x|^{-\theta}) \quad (1.5)
\end{equation}
for all $x \in \mathbb{R}^N$ with $|x| \geq R$. Then the Cauchy problem (1.1) has no global positive solution if $1 < p \leq p_*(\omega)$.

By Theorems 1.1 and 1.2, if $V \geq 0$ in $\mathbb{R}^N$ and
\begin{equation}
V(x) = \omega|x|^{-2}(1 + O(|x|^{-\theta})) \quad \text{as } |x| \to \infty \quad (1.6)
\end{equation}
for some $\theta > 0$, the Cauchy problem (1.1) has a global positive solution for some initial data if and only if $p > p_*(\omega)$. 

Our proofs of Theorems 1.1 and 1.2 depend on a result of the author and Kabeya [5], which is related to the large time behavior of solutions of the heat equation with a potential. Here we assume that the potential \( V \) is a nonnegative, smooth, and radially symmetric function behaving like (1.6), and explain the ideas of the proofs of Theorems 1.1 and 1.2. Then we may take a positive radially symmetric solution \( U \) of
\[
\Delta U - V(x) U = 0 \quad \text{in } \mathbb{R}^N
\]
such that \( U(|x|) = |x|^\alpha(1 + o(1)) \) as \( |x| \to \infty \). Furthermore the \( L^\infty \)-norm of the solution \( v \) of the linear heat equation \( \partial_t v = \Delta v - V(x)v \) behaves like \( CMt^{-(N+\alpha)/2} \) as \( t \to \infty \), where \( C \) is a constant independent of the initial data and
\[
M = \int_{\mathbb{R}^N} v(x, 0) U(|x|) \, dx
\]
(see Proposition 2.1). Then this asymptotic behavior of \( v \) together with the similar argument as in [6] gives upper bounds of the solution \( u \) of (1.1), and prove Theorem 1.1. In the proof of Theorem 1.2, we assume that there exists a global positive solution \( u \) of (1.1) as some initial data, and apply the same argument as in [9] to the solution \( u \). Then the asymptotic behavior of the solution \( v \) of \( \partial_t v = \Delta v - V(x)v \) gives Theorem 1.2 for the case \( 1 < p < p_*(\omega) \). For the critical case \( p = p_*(\omega) \), we consider the asymptotic behavior of the quantity
\[
M(t) \equiv \int_{\mathbb{R}^N} u(x, t) U(|x|) \, dx,
\]
which is a crucial ingredient in the proof for the case \( p = p_*(\omega) \). We prove unboundedness of \( M(t) \), and this unboundedness of \( M(t) \) together with the same argument as in the case \( 1 < p < p_*(\omega) \) gives Theorem 1.2.

The rest of this paper is organized as follows: In Section 2 we give some notation and recall one proposition, which is related to the large time behavior of the linear heat equation with a potential. In Sections 3 and 4, we prove Theorems 1.1 and 1.2, respectively.

2. Preliminaries

In this section we introduce some notation and recall a result of [5], which is related to the large time behavior of solutions of the heat equation with a potential.

For \( R > 0 \), let \( B(0, R) = \{ x \in \mathbb{R}^N : |x| \leq R \} \). Let \( \phi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) such that \( \phi \geq 0 \) in \( \mathbb{R}^N \). We denote by \( S(t)\phi \) the bounded solution of (1.1) with the initial data \( \phi \).

Assume the same conditions as in Theorem 1.1. Let \( V^* \) be a smooth function on \([0, \infty)\) such that
\[
0 \leq V^*(|x|) \leq V(x) \quad \text{on } B(0, R + 1),
\]
\[
V^*(r) = \omega(r + 1)^{-2}(1 - |x|^{-\theta}) \quad \text{on } [R + 1, \infty).
\]
Then we denote by \( S^*(t)\phi \) the solution of
\[
\begin{cases}
\partial_t u = \Delta u - V^*(|x|)u + u^p & \text{in } \mathbb{R}^N \times (0, T), \\
u(x, 0) = \phi(x) & \text{in } \mathbb{R}^N,
\end{cases}
\]
satisfying \( \|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty \) for all \( 0 < t < T \), where \( \phi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) and \( 0 < T \leq \infty \). Since \( V(x) \geq V^*(|x|) \) in \( \mathbb{R}^N \), by the comparison principle, we see that
\[
0 \leq (S(t)\phi)(x) \leq (S^*(t)\phi)(x)
\]
(2.2)
for all \( x \in \mathbb{R}^N \) and \( 0 < t < T \).

Next we assume the same conditions as in Theorem 1.2, instead of Theorem 1.1. Let \( V_* \) be a smooth function on \([0, \infty)\) such that
\[
V_*(|x|) \geq V(x) \quad \text{on } B(0, R + 1),
\]
\[
V_*(r) = \omega(r - 1)^{-2}(1 + |x|^{-\theta}) \quad \text{on } [R + 1, \infty).
\]
Then we denote by $S_a(t)\phi$ the solution of
\[
\begin{cases}
\partial_t u = \Delta u - V_a(|x|)u + u^p & \text{in } \mathbb{R}^N \times (0, T), \\
u(x, 0) = \phi(x) \geq 0 & \text{in } \mathbb{R}^N, 
\end{cases}
\tag{2.3}
\]
satisfying $\|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty$ for all $0 < t < T$, where $\phi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $0 < T \leq \infty$. Since $V(x) \leq V_a(|x|)$ in $\mathbb{R}^N$, by the comparison principle, we see that
\[
0 \leq (S_a(t)\phi)(x) \leq (S(t)\phi)(x)
\tag{2.4}
\]
for all $x \in \mathbb{R}^N$ and $0 < t < T$. In order to prove Theorems 1.1 and 1.2, we consider the Cauchy problems (2.1) and (2.3), instead of (1.1).

Next, in order to give a result on the large time behaviors of the solutions of the linear heat equations $\partial_t v = \Delta v - V^*v$ and $\partial_t v = \Delta v - V_e v$, we introduce the condition $(V_\omega)$ on the radially symmetric potential. We say that the radial function $\hat{V} = \hat{V}(r)$ satisfies the condition $(V_\omega)$ for some $\omega > 0$ if there exists a positive constant $\theta$ such that
\[
(V_\omega)
\begin{align*}
\text{(i)} & \quad \hat{V}(|x|) \in C^1(\mathbb{R}^N), \\
\text{(ii)} & \quad \hat{V}(r) \geq 0 \quad \text{on } [0, \infty), \\
\text{(iii)} & \quad \sup_{r \geq 1} \frac{r^2}{2+\theta} \frac{d}{dr} \hat{V}(r) - \omega \frac{\omega}{r^2} < \infty, \\
\text{(iv)} & \quad \sup_{r \geq 1} \frac{r^3}{d} \frac{d}{dr} \hat{V}(r) < \infty.
\end{align*}
\]
Under the condition $(V_\omega)$, there exists a unique positive solution $U_{\hat{V}}$ of the ordinary differential equation
\[
(O) \quad U'' + \frac{N-1}{r} U' - \hat{V}(r) U = 0 \quad \text{in } (0, \infty)
\]
with
\[
\lim_{r \to 0} \sup |U(r)| < \infty, \quad \lim_{r \to \infty} r^{-\alpha(\omega)} U(r) = 1
\tag{2.5}
\]
(see [5]). Let $L_{\hat{V}}(t)\phi$ be the bounded solution of the Cauchy problem
\[
\begin{cases}
\partial_t v = \Delta v - \hat{V}(|x|)v & \text{in } \mathbb{R}^N \times (0, \infty), \\
v(x, 0) = \phi(x) & \text{in } \mathbb{R}^N, 
\end{cases}
\tag{2.6}
\]
where $\phi \in L^2(\mathbb{R}^N, e^{\frac{|x|^2}{4}} dx)$. Then, under the condition $(V_\omega)$, we have the following proposition (see Theorem 1.1 in [5]).

**Proposition 2.1.** Let $\phi \in L^2(\mathbb{R}^N, e^{\frac{|x|^2}{4}} dx)$ and consider the Cauchy problem (2.6) under the condition $(V_\omega)$ for some $\omega > 0$. Then the solution $v = L_{\hat{V}}(t)\phi$ of (2.6) satisfies
\[
\lim_{t \to \infty} (1 + t)^{\frac{N+\alpha(\omega)}{2}} v((1 + t)^{\frac{1}{2}} y, t) = cM |y|^\alpha(\omega) e^{-\frac{|y|^2}{4}}
\]
in $L^2(\mathbb{R}^N, e^{\frac{|y|^2}{4}} dy)$ and $L^\infty(\mathbb{R}^N)$, where
\[
M \equiv \int_{\mathbb{R}^N} \phi(x) U_{\hat{V}}(|x|) dx, \quad c = \left( \int_{\mathbb{R}^N} |x|^{2\alpha(\omega)} e^{-\frac{|x|^2}{4}} dx \right)^{-1}.
\]

Since the functions $V^*$ and $V_e$ satisfy the condition $(V_\omega)$, we may define the functions $U_{V^*}$ and $U_{V_e}$ satisfying (2.5), and see that Proposition 2.1 holds with $\hat{V}$ replaced by $V^*$ or $V_e$. 
3. Proof of Theorem 1.1

Let \( p > p_* (\omega) \) and \( \epsilon \) be a sufficiently small constant to be chosen later such that \( 0 < \epsilon < 1 \). Let \( \phi \in C_0 (\mathbb{R}^N) \) such that \( \phi \geq 0 \) and \( \phi \neq 0 \) in \( \mathbb{R}^N \), and put \( u^* (t) = S^* (t) (\epsilon \phi) \). Following the argument in [6], we define

\[
T = \sup \{ t > 0 : \| u^* (t) \|_{L^\infty (\mathbb{R}^N)} \leq (1 + \tau)^{- \frac{N + q (\omega)}{2}} \text{ for all } \tau \in (0, t) \} > 0,
\]

and prove \( T = \infty \). For this aim, we assume \( T < \infty \). Then we have

\[
\| u^* (T) \|_{L^\infty (\mathbb{R}^N)} = (1 + T)^{- \frac{N + q (\omega)}{2}} \tag{3.1}
\]

and \( u^* \) satisfies

\[
\partial_t u^* = \Delta u^* - V^* (|x|) u^* + (u^*)^p \leq \Delta u^* - V^* (|x|) u^* + (1 + t)^{- \frac{N + q (\omega)}{2}} (p - 1) u^* \tag{3.2}
\]

for all \( (x, t) \in \mathbb{R}^N \times (0, T) \).

On the other hand, let \( v = L^* (t) (\epsilon \phi) \) and put

\[
w (x, t) = v (x, t) \exp \left[ - \int_0^t (1 + \tau)^{- \frac{N + q (\omega)}{2}} (p - 1) d \tau \right].
\]

Then \( w \) satisfies

\[
\begin{aligned}
\partial_t w &= \Delta w - V^* (|x|) w + (1 + t)^{- \frac{N + q (\omega)}{2}} (p - 1) w & \text{in } \mathbb{R}^N \times (0, \infty), \\
w (x, 0) &= \epsilon \phi (x) & \text{in } \mathbb{R}^N.
\end{aligned} \tag{3.3}
\]

Furthermore, by Proposition 2.1 and \( p > p_* (\omega) \), there exist constants \( C_1 \) and \( C_2 \), independent of \( \epsilon \), such that

\[
0 \leq w (x, t) \leq C_1 v (x, t) \leq C_2 \epsilon (1 + t)^{- \frac{N + q (\omega)}{2}} \int_{\mathbb{R}^N} \phi (x) U V^* (|x|) \, dx < \infty
\]

for all \( (x, t) \in \mathbb{R}^N \times (0, \infty) \). So we may take a sufficiently small \( \epsilon \) so that

\[
0 \leq w (x, t) \leq \frac{1}{2} (1 + t)^{- \frac{N + q (\omega)}{2}} \tag{3.4}
\]

for all \( (x, t) \in \mathbb{R}^N \times (0, \infty) \). Therefore, by the comparison principle (3.2)–(3.4), we have

\[
0 \leq u^* (x, t) \leq w (x, t) \leq \frac{1}{2} (1 + t)^{- \frac{N + q (\omega)}{2}} \text{ in } \mathbb{R}^N \times (0, T).
\]

This contradicts (3.1), and we have \( T = \infty \), that is, \( u^* \) is a positive global solution of the Cauchy problem (2.1). Therefore, by (2.2), the Cauchy problem (1.1) has a positive global solution, and the proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

We assume that there exists a global positive solution of (1.1) for some initial data \( \phi \). By the comparison principle, we may assume, without loss of generality, that \( \phi \) has a compact support in \( \mathbb{R}^N \). Furthermore, since \( \phi \geq 0 \) and \( \phi \neq 0 \) in \( \Omega_2 \), we see that

\[
M_* \equiv \int_{\mathbb{R}^N} \phi (x) U V_* (|x|) \, dx > 0. \tag{4.1}
\]

Let \( \lambda \) be the first eigenvalue of the problem

\[
- \Delta \psi = \lambda \psi \text{ in } D, \quad \psi = 0 \text{ on } \partial D \equiv \{ x \in \mathbb{R}^N : 1 < |x| < 2 \}
\]

and \( \psi \) the eigenfunction corresponding to \( \lambda \) such that \( \psi \geq 0 \) in \( D \) and \( \| \psi \|_{L^1 (D)} = 1 \). Put \( \psi_n (x) = n^{-N} \psi (n^{-1} x) \).
Then \( \psi_n \) satisfies
\[
-\Delta \psi_n = \frac{\lambda}{n^2} \psi_n \quad \text{in} \quad D_n = nD, \quad \psi_n = 0 \quad \text{on} \quad \partial D_n, \quad \| \psi_n \|_{L^1(D_n)} = 1.
\]

Let \( u_*(t) = S_*(t) \phi \) and put
\[
F_n(t) = \int_{D_n} u_*(x, t + n^2) \psi_n(x) \, dx.
\]

Then, by (2.4), we may define the function \( F_n(t) \) for all \( t \geq 0 \) and \( n = 1, 2, \ldots \). Furthermore, by (1.1), (V), and the Jensen inequality, there exists a positive constant \( C_1 \) such that
\[
F'_n(t) = \int_{D_n} \left[ \Delta u_* - V_*(|x|) u_* + u_*^p \right] \psi_n(x) \, dx \geq -\frac{\lambda}{n^2} F_n(t) - \frac{C_1}{n^2} F_n(t) + F_n(t)^p = -\frac{\lambda + C_1}{n^2} F_n(t) + F_n(t)^p
\]
for all \( t > 0 \) and all sufficiently large \( n \). If
\[
F_n(0)^{p-1} \geq 2 \frac{\lambda + C_1}{n^2},
\]
we have \( F_n(t) \geq F_n(0) \) and
\[
F'_n(t) \geq \frac{1}{2} F_n(t)^p, \quad t \geq 0.
\]

This together with \( p > 1 \) implies that \( F_n(t) \) tends to \( \infty \) in a finite time, and we have a contradiction. Therefore we see that
\[
F_n(0)^{p-1} \leq 2 \frac{\lambda + C_1}{n^2}, \quad (4.2)
\]
for all sufficiently large \( n \).

Let \( v = L_{V_*}(t) \phi \). Then, by Proposition 2.1 and (4.1), there exists a constant \( C_2 = C_2(N) \) such that
\[
\min_{t^{1/2} \leq |x| \leq 2t^{1/2}} v(x, t) \geq C_2 M_0 t^{-\frac{N+\alpha(\omega)}{2}}
\]
for all sufficiently large \( t \). Then, by (2.4), we have
\[
F_n(0) \geq \int_{D_n} v(x, n^2) \psi_n(x) \, dx \geq \min_{n \leq |x| \leq 2n} v(x, n^2) \geq C_2 M_0 n^{-\frac{(N+\alpha(\omega))}{2}}
\]
for all sufficiently large \( n \). Therefore there exists a constant \( L \) such that if either
\[
1 < p < p_* \quad \text{or} \quad p = p_*, \quad M_0 > L,
\]
(4.4) contradicts (4.2). Therefore the proof of Theorem 1.2 for the case \( 1 < p < p_*(\omega) \) is complete. Let \( p = p_*(\omega) \). By (2.4), (2.5) and (4.3), there exists a constant \( C_3 \) such that
\[
\frac{d}{dt} \int_{\mathbb{R}^N} u_*(t) U_{V_*}(|x|) \, dx = \int_{\mathbb{R}^N} u_*(t)^p U_{V_*}(|x|) \, dx \geq \int_{\mathbb{R}^N} v(t)^p U_{V_*}(|x|) \, dx
\]
\[
\geq \int_{\{t^{1/2} \leq |x| \leq 2t^{1/2}\}} v(x, t)^p U_{V_*}(|x|) \, dx
\]
\[
\geq C_3 M_0^p \left( t^{-\frac{N+\alpha(\omega)}{2}} \right)^p + \frac{N}{2} = C_3 M_0^p t^{-1}
\]
for all sufficiently large \( t \). So we obtain
\[
\lim_{t \to \infty} \int_{\mathbb{R}^3} u_*(x, t) U_{V_*}(|x|) \, dx = \infty.
\]
Therefore there exist a constant $T > 0$ and a function $\zeta \in C^\infty_0(\mathbb{R}^N)$ with $0 \leq \zeta \leq 1$ in $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} u_*(x, T)\zeta(x)U_1(|x|) \, dx > L.$$ 

Then $u(T)\zeta \in L^2(\mathbb{R}^N, \rho \, dx)$ and by (4.5), we see that the solution $S_*(t)(u_*(T)\zeta)$ blows-up in a finite time. Since

$$u_*(x, t + T) \geq [S_*(t)(u_*(T)\zeta)](x), \quad x \in \mathbb{R}^N, \quad t > 0,$$

we see that the solution $u_*$ blows-up in a finite time, and have a contradiction for the case $p = p_*(\omega)$. Therefore, for the case $1 < p \leq p_*(\omega)$, there exists no global positive solution of (1.1), and the proof of Theorem 1.2 is complete.

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References