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Relative cohomology of finite groups and polynomial growth*

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Abstract

This article investigates the cohomology of a finite group relative to a collection of subgroups. In particular a new spectral sequence abutting to relative cohomology is given and is used to deduce that relative cohomology has polynomial growth.

1. Introduction

In the following all groups will be assumed finite.

Between 1964 and 1965 Snapper defined the cohomology of a group relative to permutation of that group [5–9]. In [9] he used this cohomology theory to give a proof of the Frobenius theorem. Later Harris simplified many of his proofs by defining the cohomology of a group relative to a collection of subgroups of that group [4]. The cohomology of a group relative to a permutation representation is then realized as the cohomology relative to the collection of stabiliser subgroups of the permutation representation.

In [3] Blowers discussed a relatively projective resolution of a field k for the join of two permutation representations. Given two permutation representations (G, X) and (L, Y) their join, denoted $(G, X)*(L, Y)$, is given by the permutation representation $(G \times L, X \amalg Y)$ with the natural action of $G \times L$ on $X \amalg Y$. Alternatively if $\mathscr{H}_1$ and $\mathscr{H}_2$ are the stabiliser subgroups of (G, X) and (L, Y) respectively then Blowers' resolution is relatively projective for the stabiliser set $\{G \times H_2,\ H_1 \times L \mid H_i \in \mathscr{H}_i\}$ Using this resolution he gave examples of relative cohomology rings in which all of the cup products are zero.

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In this paper I will generalize Blowers resolution to the situation of a single group G with two collections of subgroups. The resolution will be relatively projective for the union of the two collections. Then I will use it to derive a spectral sequence abutting to relative cohomology which gives some insight into the structure of the relative cohomology ring under cup product. In particular it immediately gives Blowers' results on cup products. As an application of this spectral sequence I will also prove that the relative cohomology ring has polynomial growth.

If M is an RG module and $H \leq G$ then $M \downarrow_H$ will denote the restriction of M to an RH module. Likewise if N is an RH module for then $N \uparrow^G$ will denote the induced module. All modules will be assumed to be finitely generated.

2. Relative cohomology

Given $H \leq G$ recall the definition of a relatively H projective module.

Definition 2.1. An RG module M is said to be *relatively H projective* if whenever we are given modules M_1 and M_2, RG module maps $\lambda: M \to M_1$, and $\mu: M_2 \to M_1$, μ a surjection such that there exists a map $v: M \to M_2$ of RH modules with $\mu \circ v = \lambda$ as RH module maps then there exists a map of RG modules $v': M \to \mathrm{M}_2$ with $\lambda = \mu \circ v'$. A short exact sequence is said to be *H split* if it splits on restriction to H.

An RG module M is said to be relatively $\mathscr{H}$ projective for a collection of subgroups $\mathscr{H}$ of G if each direct summand of M is a relatively H projective module for some $H \in \mathscr{H}$. A short exact sequence is said to be $\mathscr{H}$ split if it splits on restriction to all subgroups $H \in \mathscr{H}$.

A contracting H homotopy of an RG module chain complex is an RH module map of degree $+1$ which is a contracting homotopy on the restriction of the chain complex to H. A contracting $\mathscr{H}$ homotopy is a collection of contracting H homotopies one for each $H \in \mathscr{H}$. If a contracting $\mathscr{H}$ homotopy of a resolution $\mathscr{X}$ exists we say that $\mathscr{X}$ is $\mathscr{H}$ split.

Definition 2.2. A *relatively $\mathscr{H}$ projective resolution* of an RG module M is a long exact sequence $\mathscr{X}$,

$$\cdots \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0,$$

of RG modules such that

(i) $X_0/\mathrm{Im}(\partial_1) \cong M$,

(ii) each X_i is relatively $\mathscr{H}$ projective,

(iii) $\mathscr{X}$ is $\mathscr{H}$ split.

Remarks. (i) Since $\bigoplus_{H\in\mathscr{H}} M\downarrow_H\uparrow^G \to M$ is always $\mathscr{H}$ split, relatively $\mathscr{H}$ projective resolutions always exist.

(ii) Since an RG module N is relatively $\mathscr{H}$ projective if its indecomposable summands are relatively H projective for some $H\in\mathscr{H}$ but there is not necessarily a summand for every $H\in\mathscr{H}$, relatively $\mathscr{H}$ projective resolutions depend only on the maximal non-conjugate subgroups H contained in $\mathscr{H}$ and not on all of the subgroups.

To make this a useful construction we need the following theorem.

Theorem 2.3 (The relative comparison theorem). *Given a map of modules $M\to M'$ and relatively $\mathscr{H}$ projective resolutions $\mathscr{X}$ and $\mathscr{X}'$ of M and M' respectively, we can extend to a map of chain complexes $\{f_n\}: X_n\to X'_n$ and given any two such maps, $\{f_n\}$ and $\{f'_n\}$ there is a contracting chain homotopy $h_n: X_n\to X'_{n+1}$ such that $f_n - f'_n = \partial_{n+1}\circ h_n + h_{n+1}\circ\partial_n$.* □

As in the proof of the non-relative comparison theorem the proof of the relative comparison theorem only depends upon the relative projectivity of the X_n and the $\mathscr{H}$ splitting of $\mathscr{X}'$. This gives us enough machinery to set up relative $\mathscr{H}$ cohomology. If M' is an RG module and

$$\cdots \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0$$

is a relatively $\mathscr{H}$ projective resolution of M, define

$$\mathrm{Ext}^n_{G,\mathscr{H}}(M, M') = H^n(\mathrm{Hom}_{RG}(\mathscr{X}, M'), \delta^*)$$

and the relative $\mathscr{H}$ cohomology of G

$$H^n(\mathscr{H}; G, M) = \mathrm{Ext}^n_{G,\mathscr{H}}(R, M).$$

As in the non-relative case we can define cup product in the usual way via a diagonal map. This gives a associative graded commutative multiplication,

$$\mathrm{Ext}^m_{G,\mathscr{H}}(M', M'')\times \mathrm{Ext}^n_{G,\mathscr{H}}(M, M') \to \mathrm{Ext}^{n+m}_{G,\mathscr{H}}(M, M'').$$

For more information about relative cohomology and cup products see [4–6].

3. Two relatively projective resolutions

Assume from now that R is a hereditary ring of coefficients.

For the following let $\mathscr{C}$ be a relatively $\mathscr{H}_1$ projective resolution of M_1 with boundary map δ_C, and $\mathscr{D}$ be a relatively $\mathscr{H}_2$ projective resolution of M_2 with differential δ_D where $\mathscr{H}_1$ and $\mathscr{H}_2$ are collection of subgroups of a group G and M_1 and M_2 are RG modules. Assume also that $\mathscr{H}_1$ and $\mathscr{H}_2$ are closed under conjugation and use ε to denote the augmentations for both $\mathscr{C}$ and $\mathscr{D}$. To prove the relative projectivity of the resolutions we need the following:

Lemma 3.1. *If M is a relatively $\mathscr{H}_1$ projective module and N is a relatively $\mathscr{H}_2$ projective module then $M \otimes N$ is a relatively $\mathscr{H} = \{\mathscr{H}_1 \cap {}^gH_2 \mid H_1 \in \mathscr{H}_1, H_2 \in \mathscr{H}_2, g \in G\}$ projective module.*

Proof. Use Mackey's tensor product theorem. □

Proposition 3.2. *If $\mathscr{H} = \{H_1 \cap {}^gH_2 \mid H_1 \in \mathscr{H}_1, H_2, \in \mathscr{H}_2, g \in G\}$ then $\mathscr{C} \otimes \mathscr{D}$ is a relatively $\mathscr{H}$ projective resolution of $M_1 \otimes M_2$.*

Proof. By the above lemma $\mathscr{C} \otimes \mathscr{D}$ is an exact sequence of relatively $\mathscr{H}$ projective modules and as in the group cohomology case, $\mathscr{C} \otimes \mathscr{D}$ resolves $M_1 \otimes M_2$. What remains to be shown is that $\mathscr{C} \otimes \mathscr{D}$ is $\mathscr{H}$ split. Let $H_1 \in \mathscr{H}_1$, $H_2 \in \mathscr{H}_2$ (as $\mathscr{H}_2$ can be assumed to be closed under conjugation we can allow $g = 1$). We know that $\mathscr{C} \otimes M_2$ is H_1 homotopy equivalent to $M_1 \otimes M_2$ and likewise $\mathscr{C} \otimes \mathscr{D}$ is H_2 homotopy equivalent to $\mathscr{C} \otimes M_2$. Therefore $\mathscr{C} \otimes \mathscr{D}$ is $H_1 \cap H_2$ homotopy equivalent to $M_1 \otimes M_2$ or equivalently $\mathscr{C} \otimes \mathscr{D}$ is $H_1 \cap H_2$ split. □

The second resolution is a generalization of a resolution of Blowers. Given an augmented chain complex $\mathscr{X}$ over M let $s(\mathscr{X})$ denote the suspension of $\mathscr{X}$ with $s(\mathscr{X})_0 = M$. Define the join of two augmented chain complexes $\mathscr{X}$ and $\mathscr{Y}$, written $\mathscr{X} * \mathscr{Y}$, by $s(\mathscr{X} * \mathscr{Y}) = s(\mathscr{X}) \otimes s(\mathscr{Y})$. Notice for $\mathscr{C}$ and $\mathscr{D}$ as above we can write $\mathscr{C} * \mathscr{D}$ as

$$(\mathscr{C} * \mathscr{D})_n = \mathscr{C}_n \otimes M_2 \oplus M_1 \otimes \mathscr{D}_n \oplus s(\mathscr{C} \otimes \mathscr{D})_n$$

for $n > 0$. I will use the sign convention $\partial(c_i \otimes d_j) = \delta_C c_i \otimes d_j + (-1)^{i+1} c_i \otimes \delta_D d_j$ for $c_i \otimes d_j \in C_i \otimes D_j$. Elements of M_i are considered as having degree -1.

Proposition 3.3. *$\mathscr{C} * \mathscr{D}$ is a relatively $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2$ projective resolution of $M_1 \otimes M_2$.*

Proof. (The following is a generalization of the proof of Blowers result [3].)

$\mathscr{C} * \mathscr{D}$ is an exact sequence resolving $M_1 \otimes M_2$. The relative projectivity follows from the fact that if U is an RH module and V is an RG module then $(U \uparrow^G \otimes V) \cong (U \otimes V \downarrow_H) \uparrow^G$. Therefore we only need to show that $\mathscr{C} * \mathscr{D}$ is $\mathscr{H}$ split.

Let $\mathscr{H} \in \mathscr{H}_1$ and let t be contracting H homotopy of $\mathscr{C}$ and denote by t' the suspended H homotopy of $s\mathscr{C}$. We shall prove that $t' \otimes 1$ is a contracting H homotopy of $s\mathscr{C} \otimes s\mathscr{D}$. By abuse of notation refer to M_1 as C_{-1}, M_2 as D_{-1} and the augmentation maps by δ_C and δ_D respectively. If $c_i \otimes d_j \in \mathscr{C}_i \otimes D_j$ then

$$\begin{aligned}
\partial(t' \otimes 1)(c_i \otimes d_j) &= \partial(tc_i \otimes d_j)\\
&= \delta_C tc_i \otimes d_j + (-1)^{i+2} tc_i \otimes \delta_D d_j,\\
(t' \otimes 1)\partial(c_i \otimes d_j) &= (t' \otimes 1)(\delta_C c_i \otimes d_j + (-1)^{i+1} c_i \otimes \delta_D d_j)\\
&= t\delta_C c_i \otimes d_j + (-1)^{i+1} tc_i \otimes \delta_D d_j,\\
(\partial(t' \otimes 1) + (t' \otimes 1)\partial)(c_i \otimes d_j) &= t\delta_C c_i \otimes d_j + \delta_C tc_i \otimes d_j\\
&= c_i \otimes d_j.
\end{aligned}$$

Lowering dimension by one gives a contracting H homotopy of $\mathscr{C} * \mathscr{D}$. The case $H \in \mathscr{H}_2$ is similar with more care necessary when choosing signs. □

4. A spectral sequence

In the previous section we described Blowers' construction for the join of two relatively projective resolutions. In this section we will investigate the spectral sequence arising from this in a natural way and reprove Blowers result on the join of two permutation representations using it. Let $\mathscr{C}$ be a relatively $\mathscr{H}_1$ projective of R with boundary map δ_C and $\mathscr{D}$ be a relatively $\mathscr{H}_2$ projective resolution of R with boundary map δ_D. ε will denote the augmentations for both $\mathscr{C}$ and $\mathscr{D}$. Let $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2$. Recall that for $n \neq 0$ we have

$$(\mathscr{C} * \mathscr{D})_n = \mathscr{C}_n \oplus \mathscr{D}_n \oplus s(\mathscr{C} \otimes \mathscr{D})_n.$$

Looking carefully at the boundary map ∂ on $(\mathscr{C} * \mathscr{D})_n$ we see that for $c_n \in \mathscr{C}_n$,

$$\partial(c_n) = \delta_C c_n,$$

for $d_n \in D_n$,

$$\partial(d_n) = \delta_D d_n,$$

and the differential on $c_i \otimes d_j \in C_i \otimes D_j$ is

$$\partial(c_i \otimes d_j) = \begin{cases} \delta_C c_i \otimes d_j + (-1)^{i+1} c_i \otimes \delta_D d_j, & i \neq 0,\ j \neq 0,\\ \varepsilon c_0 d_j - c_0 \otimes \delta_D d_j, & i = 0,\ j \neq 0,\\ \delta_C c_i \otimes d_0 + (-1)^{i+1} \varepsilon d_0 c_i, & i \neq 0,\ j = 0,\\ \varepsilon c_0 d_0 - \varepsilon d_0 c_0, & i = 0,\ j = 0, \end{cases}$$

Define $d_C : (\mathscr{C} \otimes \mathscr{D})_{n-1} \to \mathscr{C}_{n-1}$ and $d_D : (\mathscr{C} \otimes \mathscr{D})_{n-1} \to \mathscr{D}_{n-1}$ by

$$d_C(c_i \otimes d_j) = \begin{cases} 0, & j \neq 0,\\ (-1)^{i+1} \varepsilon d_0 c_i, & j = 0 \end{cases}$$

and

$$d_D(c_i \otimes d_j) = \begin{cases} 0, & i \neq 0, \\ \varepsilon c_0 d_j, & i = 0 \end{cases}$$

respectively. Notice that these are chain maps from $\mathscr{C} \otimes \mathscr{D} \to \mathscr{C}$ and $\mathscr{C} \otimes \mathscr{D} \to \mathscr{D}$ respectively each extending the identity on R (with a nonstandard sign convention on $\mathscr{C} \otimes \mathscr{D}$). This allows us to rewrite the differential on $s(\mathscr{C} \otimes \partial)_n$ as

$$\partial(c_i \otimes d_j) = \delta_{s(C \otimes D)}(c_i \otimes d_j) + d_C(c_i \otimes d_j) + d_D(c_i \otimes d_j).$$

Therefore we can write $\mathscr{C} * \mathscr{D}$ as the total complex of the double complex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle \delta_C + \delta_D} & & \downarrow{\scriptstyle \delta_{s(C \otimes D)}} & & \downarrow \\
C_2 \oplus D_2 & \xleftarrow{d_C + d_D} & s(\mathscr{C} \otimes \mathscr{D})_3 & \longleftarrow & 0 \cdots \\
\downarrow{\scriptstyle \delta_C + \delta_D} & & \downarrow{\scriptstyle \delta_{s(C \otimes D)}} & & \downarrow \\
C_1 \oplus D_1 & \xleftarrow{d_C + d_D} & s(\mathscr{C} \otimes \mathscr{D})_2 & \longleftarrow & 0 \cdots \\
\downarrow{\scriptstyle \delta_C + \delta_D} & & \downarrow{\scriptstyle \delta_{s(C \otimes D)}} & & \downarrow \\
C_0 \oplus D_0 & \xleftarrow{d_C + d_D} & s(\mathscr{C} \otimes \mathscr{D})_1 & \longleftarrow & 0 \cdots
\end{array}$$

Taking Hom's we have a double complex spectral sequence abutting to relative cohomology with zero page:

$$E_0^{0,q} = \mathrm{Hom}_{RG}(C_q \oplus D_q, M),$$

$$E_0^{1,q} = \mathrm{Hom}_{RG}(s(\mathscr{C} \otimes \mathscr{D})_{q+1}, M),$$

$$E_0^{p,q} = 0 \text{ for } p \neq 0 \text{ and } p \neq 1,$$

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow{\scriptstyle (\delta_C + \delta_D)^*} & & \uparrow{\scriptstyle \delta^*_{s(C \otimes D)}} & & \uparrow \\
\mathrm{Hom}_{RG}(C_2 \oplus D_2, M) & \xrightarrow{(d_C + d_D)^*} & \mathrm{Hom}_{RG}(s(C \otimes D)_3, M) & \longrightarrow & 0 \cdots \\
\uparrow{\scriptstyle (\delta_C + \delta_D)^*} & & \uparrow{\scriptstyle \delta^*_{s(C \otimes D)}} & & \uparrow \\
\mathrm{Hom}_{RG}(C_1 \oplus D_1, M) & \xrightarrow{(d_C + d_D)^*} & \mathrm{Hom}_{RG}(s(C \otimes D)_2, M) & \longrightarrow & 0 \cdots \\
\uparrow{\scriptstyle (\delta_C + \delta_D)^*} & & \uparrow{\scriptstyle \delta^*_{s(C \otimes D)}} & & \uparrow \\
\mathrm{Hom}_{RG}(C_0 \oplus D_0, M) & \xrightarrow{(d_C + d_D)^*} & \mathrm{Hom}_{RG}(s(C \otimes D)_1, M) & \longrightarrow & 0 \cdots
\end{array}$$

We define $\mathscr{K} = \{H_1 \cap {}^gH_2 \mid H_1 \in \mathscr{H}_1,\ H_2 \in \mathscr{H}_2,\ g \in G\}$, and by taking the vertical cohomology we calculate the E_1 page:

$$E_1^{0,q} = H^q(\mathscr{H}_1; G, M) \oplus H^q(\mathscr{H}_2; G, M),$$

$$E_1^{1,q} = H^q(\mathscr{K}; G, M),$$

$$E_1^{p,q} = 0 \text{ for } p \neq 0 \text{ and } p \neq 1,$$

$$\begin{array}{ccc} \vdots & & \vdots \qquad \vdots \\ H^2(\mathscr{H}_1; G, M) \oplus H^2(\mathscr{H}_2; G, M) & \xrightarrow{(d_C + d_D)^*} & H^2(\mathscr{K}; G, M) \longrightarrow 0 \\ H^1(\mathscr{H}_1; G, M) \oplus H^1(\mathscr{H}_2; G, M) & \xrightarrow{(d_C + d_D)^*} & H^1(\mathscr{K}; G, M) \longrightarrow 0 \\ H^0(\mathscr{H}_1; G, M) \oplus H^0(\mathscr{H}_2; G, M) & \xrightarrow{(d_C + d_D)^*} & H^0(\mathscr{K}; G, M) \longrightarrow 0 \end{array}$$

As this spectral sequence has just two columns we have that $E_2 \cong E_\infty$. Calculating $H^*(\mathscr{H}; G, M)$ requires the understanding of $H^*(\mathscr{H}_i; G, M)$ for i both one and two, $H^*(\mathscr{K}; G, M)$ and the map between them.

Firstly we will investigate the maps d_C^* and d_D^* in the case $M \cong R$.

Lemma 4.1. *d_C^* and d_D^* are algebra homomorphisms when $M \cong R$.*

Proof. I will just prove the result for d_C^*. We want to show

$$d_C^*(\alpha \cup \beta) = d_C^*(\alpha) \cup d_C^*(\beta)$$

where $\cup$ denotes cup product in both algebras. Let $\hat{\alpha} \in \mathrm{Hom}_{RG}(C_n, R)$ and $\hat{\beta} \in \mathrm{Hom}_{RG}(C_m, R)$ represent elements $\alpha \in H^n(\mathscr{H}_1; G, R)$ and $\beta \in H^m(\mathscr{H}_1; G, R)$ respectively. Also allow Δ^C and Δ^D be diagonal maps for $\mathscr{C}$ and $\mathscr{D}$ respectively. A diagonal map for $\mathscr{C} \otimes \mathscr{D}$ with the shifted sign convention is given by $-\Psi(\Delta^C \otimes \Delta^D)$ where Ψ shuffles the middle two factors with a sign determined by their degrees. Thus for $c_i \in C_i$ and $d_j \in D_j$,

$$d_C^*(\alpha \cup \beta)(c_i \otimes d_j) = d_C^*(\Delta^{C*}(\hat{\alpha} \otimes \hat{\beta}))(c_i \otimes d_j).$$

This is zero unless $i = n + m$ and $j = 0$ in which case

$$\begin{aligned} d_C^*(\alpha \cup \beta)(c_{n+m} \otimes d_0) &= \Delta^{C*}(\hat{\alpha} \otimes \hat{\beta})(d_C(c_{n+m} \otimes d_0)) \\ &= (-1)^{n+m+1} \varepsilon(d_0)(\hat{\alpha} \otimes \hat{\beta})(\Delta^C c_{n+m}). \end{aligned}$$

On the other hand,

$$(d_C^*(\alpha) \cup d_C^*(\beta))(c_i \otimes d_j) = -(d_C^*\hat{\alpha} \otimes d_C^*\hat{\beta})\, \Psi(\Delta^C c_i \otimes \Delta^D d_j).$$

But $d_C^*(\alpha)$ and $d_C^*(\beta)$ are zero except on $C_n \otimes D_0$ and $C_m \otimes D_0$ respectively. Hence $(d_C^*(\alpha) \cup d_C^*(\beta))(c_i \otimes d_j) = 0$ unless $i = n + m$ and $j = 0$. In this case,

$$\begin{aligned}(d_C^*(\alpha) \cup d_C^*(\beta))(c_{n+m} \otimes d_0) &= -(d_C^*\hat{\alpha} \otimes d_C^*\hat{\beta})\Psi(\Delta^C_{n,m}c_{n+m} \otimes \Delta^D_{0,0}\,d_0) \\ &= (-1)^{n+m+1}(\varepsilon \otimes \varepsilon)(\Delta^D_{0,0}d_0)(\hat{\alpha} \otimes \hat{\beta})(\Delta^C_{n,m}c_{n+m}) \\ &= (-1)^{n+m+1}\varepsilon(d_0)(\hat{\alpha} \otimes \hat{\beta})(\Delta^C c_{n+m}). \quad \square\end{aligned}$$

For more general modules M these arguments can be extended to show that d_C^* and d_D^* are $H^*(\mathscr{H}_1; G, R)$ and $H^*(\mathscr{H}_2; G, R)$ module maps respectively where the algebra actions are induced by d_C^* and d_D^* respectively. This means that in order to calculate the E_2 page of this spectral sequence it is enough to understand what the horizontal differential does to the generators in the zeroth column.

The last part of this section will be devoted to investigating the cup product in relation to this spectral sequence in the case $M \cong R$. Since the horizontal differential is not a derivation the cup product will not induce a ring structure on the E_0 page of this spectral sequence. The cup product is, however, computable in this spectral sequence.

The following is an explicit diagonal map given by Blowers for $\mathscr{C} * \mathscr{D}$ based on the diagonal maps for $\mathscr{C}$ and $\mathscr{D}$. As above let Δ^C, and Δ^D be diagonal maps for $\mathscr{C}$ and $\mathscr{D}$ respectively. Define

$$\Psi' : (s\mathscr{C} \otimes s\mathscr{C}) \otimes (s\mathscr{D} \otimes s\mathscr{D}) \to (s\mathscr{C} \otimes s\mathscr{D}) \otimes (s\mathscr{C} \otimes sD)$$

to be the map which shuffles the middle factors with a sign determined by their suspended degrees, and let 1 denote the unit in $R \cong (s\mathscr{C})_0 \cong (s\mathscr{D})_0$. Then a diagonal map for $\mathscr{C} * \mathscr{D}$ is given by

$$\Delta_n = \sum_{p+q=n} \Delta_{p,q},$$

$$\Delta_{p,q}(c_r * d_s) = \begin{cases} \Psi'((\Delta^C_{p,r-p}(c_r)) \otimes (1 \otimes d_s)), & p \le r, \\ \Psi'((c_r \otimes 1) \otimes (\Delta^D_{p-r-1,q}(d_s))), & p > r. \end{cases}$$

With this diagonal map we can explicitly calculate the product structure on $H^*(\mathscr{H}; G, R)$ from the product structures on $H^*(\mathscr{H}_1; G, R)$, $H^*(\mathscr{H}_2; G, R)$ and $H^*(\mathscr{K}; G, R)$.

Theorem 4.2. *Given* $\alpha_1, \alpha_2 \in H^*(\mathscr{H}_1; G, R)$, $\beta_1, \beta_2 \in H^*(\mathscr{H}_2; G, R)$, *and* $\gamma_1, \gamma_2 \in H^*(\mathscr{K}; G, R)$ *such that* α_i *and* β_i *are of the same degree and* $(d_C^* + d_D^*)(\alpha_i + \beta_i) = 0$ *(i.e.* $\alpha_i + \beta_i$ *is a cocycle), then:*

$$\begin{aligned}&(\alpha_1 + \beta_1 + \gamma_1) \cup (\alpha_2 + \beta_2 + \gamma_2) \\ &\quad = \alpha_1 \cup_C \alpha_2 + \beta_1 \cup_D \beta_2 + (-1)^{\deg(\alpha_1)+1} d_C^*\alpha_1 \cup_{\mathscr{C}\otimes\mathscr{D}} \gamma_2 + \gamma_1 \cup_{\mathscr{C}\otimes\mathscr{D}} d_D^*\beta_2,\end{aligned}$$

where $\cup$, $\cup_C$, $\cup_D$, *and* $\cup_{\mathscr{C}\otimes\mathscr{D}}$ *represent the cup products in* $H^*(\mathscr{H}; G, R)$, $H^*(\mathscr{H}_1; G, R)$, $H^*(\mathscr{H}_2; G, R)$ *and* $H^*(\mathscr{K}; G, R)$ *respectively and the cocycles written on the right are extended by zero so as to be defined on all of* $\mathscr{C} * \mathscr{D}$.

Proof. Since the cup product is graded commutative we only need to check the following:

$$\alpha_1 \cup \beta_2 = 0,$$

$$\gamma_1 \cup \gamma_2 = 0,$$

$$\alpha_1 \cup \alpha_2 = \alpha_1 \cup_C \alpha_2,$$

$$(\alpha_1 + \beta_1) \cup \gamma_2 = (-1)^{\deg(\alpha_1)+1} d_C^* \alpha_1 \cup_{\mathscr{C} \otimes \mathscr{D}} \gamma_2.$$

The identity $\alpha_1 \cup \beta_1 = 0$ follows as the image of the diagonal map Δ is never in $(\mathscr{C} \otimes R) \otimes (R \otimes \mathscr{D})$. Likewise since the image is always in either $(\mathscr{C} \otimes R) \otimes (\mathscr{C} * \mathscr{D})$ or $(\mathscr{C} * \mathscr{D}) \otimes (R \otimes \mathscr{D})$ we have $\gamma_1 \cup \gamma_2 = 0$.

To prove the first nontrivial identity let $\hat{\alpha}_1 \in \mathrm{Hom}_{RG}(\mathscr{C}_n, R)$ and $\hat{\alpha}_2 \in \mathrm{Hom}_{RG}(C_m, R)$ represent α_1 and α_2 respectively. For $c_i \in C_i$ and $d_j \in D_j$:

$$\begin{aligned}(\alpha_1 \cup \alpha_2)(c_i * d_j) &= \Delta^*(\hat{\alpha}_1 \otimes \hat{\alpha}_2)(c_i * d_j) \\ &= (\hat{\alpha}_1 \otimes \hat{\alpha}_2)\left(\sum_{p+q=n+m} \Delta_{p,q}(c_i * d_j)\right).\end{aligned}$$

But $\hat{\alpha}_1 \otimes \hat{\alpha}_2$ will be nonzero only when $i = n + m$ and $j = -1$ (here we abuse notation as before and think of $D_{-1} \cong R$). Thus

$$\begin{aligned}(\alpha_1 \cup \alpha_2)(c_{n+m} * 1) &= (\hat{\alpha}_1 \otimes \hat{\alpha}_2)(\Psi'(\Delta_{p,q}^C c_{n+m} \otimes (1 \otimes 1))) \\ &= (\hat{\alpha}_1 \otimes \hat{\alpha}_2)(\Delta^C c_{n+m}) \\ &= (\alpha_1 \cup_C \alpha_2)(c_{n+m}).\end{aligned}$$

To prove $(\alpha_1 + \beta_1) \cup \gamma_2 = (-1)^{\deg(\alpha_1)+1} d_C^* \alpha_1 \cup_{\mathscr{C} \otimes \mathscr{D}} \gamma_2$ we will use the standard resolutions for $\mathscr{C}$ and $\mathscr{D}$. Namely $\mathscr{C}_n = RX^{n+1}$ where (G, X) is a permutation representation associated to $\mathscr{H}_1$ and $D_m = RY^{m+1}$ where (G, Y) is a permutation representation associated to $\mathscr{H}_2$ (See [5] for more details of this construction). In particular the diagonal maps for $\mathscr{C}$ and $\mathscr{D}$ are the Alexander–Whitney maps,

$$\Delta^C(x_0, \ldots, x_n) = \sum_{i=0}^{n} (x_0, \ldots, x_i) \otimes (x_i, \ldots, x_n),$$

and likewise for $\mathscr{D}$.

Let $\hat{\gamma} \in \mathrm{Hom}_{RG}((\mathscr{C} \otimes \mathscr{D})_{m-1}, R)$, $\hat{\alpha} \in \mathrm{Hom}_{RG}(C_n, R)$, and $\hat{\beta} \in \mathrm{Hom}_{RG}(D_n, R)$ represent elements $\gamma \in H^{m-1}(\mathscr{K}; G, R)$, $\alpha \in H^n(\mathscr{H}_1; G, R)$, and $\beta \in H^n(\mathscr{H}_2; G, R)$ such that $(d_C^* + d_D^*)(\alpha + \beta) = 0$. On $\mathscr{C} \otimes R$ and $R \otimes D$, $(\alpha + \beta) \cup \gamma = 0$ and $d_C^* \alpha \cup_{\mathscr{C} \otimes \mathscr{D}} \gamma = 0$ so we need to compare values for $c * d$ only where $c = (c_0, \ldots, c_i) \in C_i$, $d = (d_0, \ldots, d_j) \in D_j$, $i + j = n + m - 1$, and $i, j \neq -1$.

$$((\alpha + \beta) \cup \gamma)(c * d) = ((\hat{\alpha} + \hat{\beta}) \otimes \hat{\gamma})\left(\sum_{p+q=n+m-1} \Delta_{p,q}(c * d)\right)$$

As $\hat{\alpha} + \hat{\beta}$ is only nonzero on $C_n \otimes R$ and $R \otimes D_n$ we have that the cup product is zero unless $p = n \leq i$. Assume for the following that $p = n \leq i$ in which case

$$\begin{aligned}((\alpha + \beta) \cup \gamma)(c * d) &= ((\hat{\alpha} + \hat{\beta}) \otimes \hat{\gamma})(\Delta_{n,m-1}(c * d)) \\ &= ((\hat{\alpha} + \hat{\beta}) \otimes \hat{\gamma})(\Psi'(\Delta^C_{n,i-n} c \otimes (1 \otimes d))) \\ &= ((\hat{\alpha} + \hat{\beta}) \otimes \hat{\gamma})((c_0, \ldots, c_n) \otimes 1 \otimes (c_n, \ldots, c_i) \otimes d) \\ &= \hat{\alpha}(c_0, \ldots, c_n) \cdot \hat{\gamma}(c_n, \ldots, c_i) \otimes d).\end{aligned}$$

On the other hand, let $\Psi: (\mathscr{C} \otimes \mathscr{C}) \otimes (\mathscr{D} \otimes \mathscr{D}) \to (\mathscr{C} \otimes \mathscr{D}) \otimes (\mathscr{C} \otimes \mathscr{D})$ be the map as before which shuffles the middle factors with a sign determined by their degrees. As $i, j \neq -1$ we can regard $c * d$ as an element of $s(\mathscr{C} \otimes \mathscr{D})$ and shall denote it by $c \otimes d$. Thus:

$$(d^*_C \alpha \cup_{\mathscr{C} \otimes \mathscr{D}} \gamma)(c \otimes d) = -(d^*_C \hat{\alpha} \otimes \hat{\gamma})(\Psi(\Delta^C c \otimes \Delta^D d).$$

But $(d^*_C \hat{\alpha} \otimes \hat{\gamma})$ is nonzero only on $(\mathscr{C}_n \otimes \mathscr{D}_0) \otimes (\mathscr{C} \otimes \mathscr{D})_{m-1}$ which implies that $(d^*_C \alpha \cup_{\mathscr{C} \otimes \mathscr{D}} \gamma)(c \otimes d) = 0$ unless $p = n \leq i$ and $q = 0$ in which case

$$\begin{aligned}(d^*_C \alpha \cup_{\mathscr{C} \otimes \mathscr{D}} \gamma)(c \otimes d) &= -(d^*_C \hat{\alpha} \otimes \hat{\gamma})((c_0, \ldots, c_n) \otimes (d_0) \otimes (c_n, \ldots, c_i) \otimes d) \\ &= (-1)^{n+1} \varepsilon(d_0) \hat{\alpha}(c_0, \ldots, c_n) \cdot \hat{\gamma}((c_n, \ldots, c_i) \otimes d) \\ &= (-1)^{n+1}((\alpha + \beta) \cup \gamma)(c * d).\end{aligned}$$

As an application of this spectral sequence, given two permutation representations (G, X) and (L, Y) define their join as $(G, X) * (L, Y) = (G \times L, X \amalg Y)$. If $\mathscr{H}_1$ and $\mathscr{H}_2$ denote the collection of stabilisers for $(G \times L, X)$ and $(G \times L, Y)$ respectively then $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2$ is the collection of stabilisers for their join.

Theorem 4.3 (Blowers). *If (G, X) and (L, Y) are permutation representations and if $\alpha, \beta \in H^*(\mathscr{H}; G \times L, R)$ with $\deg \alpha > 0$ and $\deg \beta > 0$ then $\alpha \cup \beta = 0$.*

Proof. Define $\mathscr{K} = \{H_1 \cap {}^g H_2 \mid H_1 \in \mathscr{H}_1, H_2 \in \mathscr{H}_2, g \in G\}$. Via the Künneth Theorem,

$$H^n(\mathscr{K}; G \times L, R) \cong \bigoplus_{i+j=n} H^i(\mathscr{H}_1; G \times L, R) \otimes H^j(\mathscr{H}_2; G \times L, R),$$

and as L is a normal subgroup of $G \times L$ we have

$$H^i(\mathscr{H}_1; G \times L, R) \cong H^i(\mathscr{H}_1; G, R),$$

where we abuse notation and use $\mathscr{H}_1$ to denote the stabilisers for (G, X). The E_1 page of the spectral sequence is therefore

$$E_1^{0,n} = H^n(\mathscr{H}_1; G, R) \oplus H^n(\mathscr{H}_2; L, R),$$

$$E_1^{1,n} = \bigoplus_{i+j=n} H^i(\mathscr{H}_1; G, R) \otimes H^j(\mathscr{H}_2; L, R),$$

$$E_1^{p,n} = 0 \text{ for } p > 1.$$

For $n > 0$ the map from $E_1^{0,n} \to E_1^{0,n}$ is just the obvious injection and therefore the E_2 page has nonzero entries in $E_2^{0,0}$ and in the first column. Therefore all cup products are trivial. □

5. Polynomial growth of $H^n(\mathscr{H}; G, M)$

In this section let k be a field of characteristic p.

As shown by the above example the relative cohomology ring of a group is no longer necessarily finitely generated as it is in the non-relative case. The following says that the dimension of $H^n(\mathscr{H}; G, M)$ grows polynomially as a function of n.

Theorem 5.1. *Given M a finitely generated kG module, there exists a polynomial f such that*

$$\dim_k H^n(\mathscr{H}; G, M) < f(n) \quad \text{for all } n \geq 0.$$

In this situation we will say that $H^*(\mathscr{H}; G, M)$ has polynomial growth.

To prove the theorem first we reduce as in the non-relative case to the Sylow p-subgroups of G and then proceed by triple induction: first on the order of the p-group, secondly on the order of the largest subgroup in $\mathscr{H}$ and lastly on the number of classes of non-conjugate maximal subgroups in $\mathscr{H}$. As we are only trying to prove polynomial growth we only need to exhibit a spectral sequence abutting to the relative cohomology which has a page whose total complex has polynomial growth.

Proof. First we reduce to the Sylow p subgroup of G. To do so we use a Corollary of Snapper [5].

Corollary 5.2. *Let M be a kG module, K a subgroup of G, and $\mathscr{H}$ a collection of subgroups of G. Then*

$$\mathrm{Tr}_{K,G}\, \mathrm{res}_{G,K} : H^n(\mathscr{H}; G, M) \to H^n(\mathscr{H}; G, M)$$

consists of multiplying the elements of $H^n(\mathscr{H}; G, M)$ by the index $|G:K|$ for all n. In particular for $\alpha \in H^n(\mathscr{H}; G, M)$ with $n > 0$, $|G| \cdot \alpha = 0$.

Proof. Here $\mathrm{Tr}_{K,G}$ is seen as a map from $H^n(\{K \cap {}^gH \mid H \in \mathscr{H},\ g \in G\}, (K, M)$ to $H^n(\mathscr{H}, G, M)$ and the proof is the same as in the non-relative case. □

Define

$$H^+(\mathscr{H}; G, M) = \bigoplus_{n>0} H^n(\mathscr{H}; G, M).$$

Snapper's result implies for any element $\alpha \in H^+(\mathscr{H}; G, M)$, $|G| \cdot \alpha = 0$. Therefore set $H^+(\mathscr{H}; G, M)_{(p)}$ to consist of all elements of $H^+(\mathscr{H}; G, M)$ annihilated by a power of p,

$$H^+(\mathscr{H}; G, M) = \bigoplus_{p \text{ prime}} H^+(\mathscr{H}; \mathrm{G}, \mathrm{M})_{(p)}.$$

Also, as a corollary of Snapper's result

$$\mathrm{res}_{G,P} : H^+(\mathscr{H}; G, M)_{(p)} \to H^*(\{{}^gH \cap P \mid H \in \mathscr{H}, g \in G\}; P, M)$$

is injective for P a Sylow p-subgroup of G.

Now we may assume that G is a p group and we will use induction on its order. The case $|G| = p$ is already proved because there are no non-trivial subgroups and therefore the relative group cohomology is either trivial or just ordinary group cohomology.

Assume by induction that we have polynomial growth for all p groups of order less than p^n and for arbitrary collections of subgroups of such a p group. This means that for any p group G, $|G| < p^n$ and $\mathscr{H}$ a collection of subgroups of G there exists a polynomial f such that

$$\dim_k H^n(\mathscr{H}; G, M) \leq f(n).$$

To prove the general result we will use induction first on the order of the largest subgroup in the collection $\mathscr{H}$ and then on the number of classes of maximal non-conjugate subgroups.

First assume that the largest subgroup in the collection $\mathscr{H}$ is of order p and that there is only one class of maximal subgroups in $\mathscr{H}$ with representative H. Let N be a maximal normal subgroup of G which contains H. The extension

$$1 \to N \to G \to G/N \to 1$$

gives a Lyndon–Hochschild–Serre spectral sequence with

$$E_1^{p,q} \cong H^*(C_p, H^*(\mathscr{K}; N, M))$$

where C_p is the cyclic group of order p and $\mathscr{K} = \{{}^gH \cap N \mid g \in G\}$ is the collection of subgroups, $\mathscr{H}$, restricted to N. Notice this collection will in general have more than one class of non-conjugate maximal subgroups (see [5] for more details of this spectral sequence). As $|N| < p^n$ we know that $H^*(\mathscr{K}; N, M)$ has polynomial growth. This implies that $H^*(C_p, \mathscr{H}^*(\mathscr{K}; N, M))$ has polynomial growth. Therefore we have a spectral sequence whose total complex on the E_2 page has polynomial growth which implies that $H^*(H; G, M)$ has polynomial growth.

Continuing with the case that the largest subgroup in $\mathscr{H}$ has order p, assume that if $\mathscr{H}$ has less than m classes of non-conjugate maximal subgroups than $H^*(\mathscr{H}; G, M)$ has polynomial growth. Let $\mathscr{H}$ have m classes of maximal subgroups with representatives H_i for $1 \leq i \leq n$. Let $\mathscr{H}_1 = \{H_1\}$ and $\mathscr{H}_2 = \{H_2, \ldots, H_m\}$. Then the spectral sequence from the previous section has E_1 page

$$E_1^{0,q} = H^q(\mathscr{H}_1; G, M) \oplus H^q(\mathscr{H}_2; G, M),$$

$$E_1^{1,q} = H^q(G, M),$$

$$E_1^{p,q} = 0 \text{ for } p > 1.$$

As each column has polynomial growth we can bound the dimension of the total complex by a polynomial which implies $H^*(\mathscr{H}; G, M)$ has polynomial growth.

By induction if the order of the largest subgroup in $\mathscr{H}$ is less than p^n then assume we have polynomial growth. The case when there is only one conjugacy class of maximal subgroups in $\mathscr{H}$ with representative of order p^n is proved using a Lyndon–Hochschild–Serre spectral sequence exactly as it was when that subgroup had order p.

Finally assume that $\mathscr{H}$ has m conjugacy classes of maximal subgroups and the maximal order of a subgroup in $\mathscr{H}$ is p^n. Let H_i for $1 \leq i \leq n$ be representatives of the classes and assume H_1 has order p^n. As above define $\mathscr{H}_1 = \{H_1\}$ and $\mathscr{H}_2 = \{H_2, \ldots, H_m\}$. Then, the spectral sequence from the previous section has an E_1 page

$$E_1^{0,q} = H^q(\mathscr{H}_1; G, M) \oplus H^q(\mathscr{H}_2; G, M),$$

$$E_1^{1,q} = H^q(\{H_1 \cap {}^gH_i \mid g \in G\}; (G, M),$$

$$E_1^{p,q} = 0 \quad \text{for } p > 1.$$

Again as each column has polynomial growth, $H^*(\mathscr{H}; G, M)$ has polynomial growth. □

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