Neighborly Polytopes and Oriented Matroids

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The theory of oriented matroids is applied to the class of neighborly convex polytopes. After giving shortened and purely combinatorial proofs for various known properties of cyclic and neighborly polytopes, we focus our attention on a very interesting property of neighborly chirotopes. We establish the combinatorial analogue to a theorem of I. Shemer: The chirotope of a neighborly 2k-polytope P is rigid, i.e. the entire internal structure of P is uniquely determined by its boundary complex.

As the main new result we give a negative answer to a question of M. A. Perles: The property to be rigid does not characterize the neighborly 2k-polytopes among all simplicial polytopes.

1. INTRODUCTION

In the present paper the theory of oriented matroids is applied to study a very interesting class of convex polytopes. The neighborly polytopes have in the past two decades received much attention in combinatorial convex geometry due to their connection with certain extremal problems. Most important in its applications is the Upper Bound Theorem which was established by P. McMullen (see [13]). This theorem says that the number $f_j(P)$ of j-dimensional faces of a k-polytope with n vertices is maximal for the cyclic k-polytope $C(n, k)$ with n vertices.

This paper is the third one in a series on convex polytopes and oriented matroids, and the results established here make use of the terminology and ideas that have been developed in [16] and [17]. In particular, we shall employ the concept of rigidity for chirotopes (oriented matroids) [16, Section 3], and, in order to represent higher-dimensional objects, the technique of affine Gale diagrams [17, Definition 2.6] will be used. In our terminology on convex polytopes we follow Grünbaum [10].

Cyclic polytopes are the paradigms for neighborly polytopes, and, in a sense to be made precise below, they form the basic building blocs for all neighborly polytopes. In Section 2 we continue the work of Cordovil and Duchet [9] by studying the basic properties of cyclic polytopes in terms of their chirotopes. Under the well-known correspondence between chirotopes and oriented matroids in their classical axiomatization, our cyclic chirotopes are equal to the alternating matroids in [4]. We give a shortened proof for the surprising fact that the duals of cyclic chirotopes are up to reorientation again cyclic.

For a long time the cyclic polytopes, first discovered by Caratheodory in 1907 (see [10]), were the only known neighborly 2k-polytopes, and Gale proved that for every $n \leq 2k + 3$ every neighborly 2k-polytope is indeed cyclic [10, Theorem 7.2.3]. B. Grünbaum [10, Theorem 7.2.4] showed that for all $k \geq 2$ there exist non-cyclic neighborly 2k-polytopes with $2k + 4$ vertices. Moreover, by 'sewing' new vertices on suitable face towers I. Shemer proved that the number $g(2k + \beta, 2k)$ of combinatorial types of neighborly 2k-polytopes with $2k + \beta$ vertices grows superexponentially as $\beta \to \infty$ ($k \geq 2$ fixed) and as $k \to \infty$ ($\beta \geq 4$ fixed) [14, Theorem 6.1].

In Section 3 we establish a characterization of neighborly polytopes in terms of their chirotopes, and, using the technique of affine Gale diagrams we will give new, very elementary proofs for both Gale's result and the existence of non-cyclic neighborly polytopes in all dimensions $\geq 3$. 
A second result of Shemer's article shows that the even-dimensional neighborly polytopes have another very interesting structural property: A combinatorial equivalence between two neighborly $2k$-polytopes $P$ and $P'$ induces a combinatorial equivalence between all pairs of corresponding subpolytopes of $P$ and $P'$ [14, Theorem 2.12]. Translated into the language of chirotopes, this means that all realizable neighborly $(2k + 1)$-chirotopes are rigid. We generalize this theorem to neighborly chirotopes, thereby establishing a straightforward and purely combinatorial proof for Shemer's geometrical result. The rigidity of neighborly chirotopes was conjectured by J. Bokowski, who made use of this property in the combinatorial part of the algorithm (see [6]) which led to the completion of Altshuler's classification of all neighborly 4-polytopes with 10 vertices [1].

Since prisms and pyramids over rigid chirotopes are rigid again, it is not difficult to see that there are rigid $d$-chirotopes with $n$ vertices for all $d$ and $n$. In the case of simplicial polytopes, however, this property is very strong, and M. A. Perles posed the problem whether the even-dimensional neighborly polytopes are characterized among all simplicial polytopes by the rigidity of the corresponding chirotope.

As major new result of this paper we answer this question to the negative. In Section 5 we prove the existence of a simplicial non-neighborly 8-polytope with 12 vertices whose chirotope is rigid. Using the technique of affine Gale diagrams the proof reduces to reading off certain properties from a configuration of 12 points in the plane.

2. CYCLIC POLYTOPES AND CHIROTOPES

Cyclic polytopes are the best known examples of neighborly polytopes. We shall study the properties of the cyclic $k$-polytope $C(n, k)$ in terms of the assigned cyclic $(k + 1)$-chirotope $\chi_{n,k+1}^{k}$ with $n$ vertices. In the classical notion of oriented matroids as collection of signed vectors, $\chi_{n,k+1}^{k}$ is the alternating matroid of rank $k + 1$ with $n$ points [4]. In [9] Cordovil and Duchet studied the properties of alternating matroids, and some of the results of this section can be found in a slightly different form in their paper, too. In particular, they established the rigidity of all alternating matroids of odd rank, a result that we prove for the general case of neighborly chirotopes in Section 4.

A curve $\mathcal{C}$ in $\mathbb{R}^k$ is called of order $k$ if $\mathcal{C}$ intersects any affine hyperplane of $\mathbb{R}^k$ in at most $k$ points. In other words: any $k + 1$ points on the curve $\mathcal{C}$ span a (non-degenerate) $k$-simplex in $\mathbb{R}^k$. The cyclic polytope $C(n, k)$ is defined as follows. Take any $n$ distinct points on a $k$-order curve $\mathcal{C}$ and let $C(n, k)$ be their convex hull. We shall see that this definition does not depend on the particular choice of $\mathcal{C}$ and the vertices on $\mathcal{C}$, because the chirotope of any $n$ distinct points on $\mathcal{C}$ is always the cyclic chirotope $\chi^{n,k+1}_k$ as defined below, and by [17, Remark 2.1] the face lattice of a polytope is determined by the affine chirotope of its vertices. In another paper [15] the author proves the converse for even dimensions, i.e. for every $k$-polytope $P \subset \mathbb{R}^k$, $k$ even, which is combinatorially equivalent to $C(n, k)$ there is a $k$-order curve $\mathcal{C}_P \subset \mathbb{R}^k$ such that the vertices of $P$ lie on $\mathcal{C}_P$.

The paradigm for a $k$-order curve is the moment curve $\mathcal{M}_k \subset \mathbb{R}^k$ which is given by the parameterization $x(t) := (t, t^2, \ldots, t^k)$, $t \in \mathbb{R}$. The moment curve $\mathcal{M}_k$ is of order $k$ because the oriented volume of any simplex spanned by $x(t_1), \ldots, x(t_{k+1}) \in \mathcal{M}_k$ equals up to a non-zero constant just the Vandermonde determinant

$$
\begin{vmatrix}
1 & t_1^1 & t_1^2 & \cdots & t_1^k \\
1 & t_2^1 & t_2^2 & \cdots & t_2^k \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & t_{k+1}^1 & t_{k+1}^2 & \cdots & t_{k+1}^k \\
\end{vmatrix}
= \prod_{i<j} (t_j - t_i).
$$
which vanishes only if \( t_i = t_j \) for some \( i \neq j \). Let \( \Lambda(n, d) := \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d | 1 \leq \lambda_1 < \cdots < \lambda_d \leq n\} \).

We define the cyclic chirotope by
\[
\chi_{nd}: \Lambda(n, d) \rightarrow \{-1, 0, +1\}, \quad \lambda \mapsto +1.
\]

**Proposition 2.1.** A curve \( \mathcal{C} := \{x(t) | t \in \mathbb{R}\} \subseteq \mathbb{R}^k \) is of order \( k \) if and only if for any \( n \) points \( x(t_1), \ldots, x(t_n) \in \mathcal{C} \) with \( t_1 < \cdots < t_n \) the affine \((k + 1)\)-chirotope of \( \{x(t_1), \ldots, x(t_n)\} \) is either \( \chi_{n,k+1} \) or \(-\chi_{n,k+1}\).

**Proof.** Let \( T := \{(t_1, t_2, \ldots, t_{k+1}) \in \mathbb{R}^{k+1} | t_1 < \cdots < t_{k+1}\} \). For any curve \( \mathcal{C} \) with parameterization \( x_\varphi: \mathbb{R} \rightarrow \mathbb{R}^k \), the mapping \( D_\varphi: T \rightarrow \mathbb{R} \)
\[
(t_1, t_2, \ldots, t_{k+1}) \mapsto \det \begin{pmatrix}
1 & 1 & \ldots & 1 \\
x_\varphi(t_1) & x_\varphi(t_2) & \ldots & x_\varphi(t_{k+1})
\end{pmatrix}
\]
is continuous. By definition, \( \mathcal{C} \) is of order \( k \) if and only if \( 0 \notin \text{Im}(D_\varphi) \), i.e. 0 is not in the image of \( D_\varphi \). Since \( T \) is a connected topological space this means that \( \mathcal{C} \) is of order \( k \) if and only if either \( \text{Im}(D_\varphi) > 0 \) or \( \text{Im}(D_\varphi) < 0 \), which proves the claim. \( \square \)

Proposition 2.1 shows that the cyclic polytopes \( C(n, k) \) are well defined up to combinatorial equivalence, and the face lattice of \( C(n, k) \) is (isomorphic to) the face lattice of the cyclic chirotope \( \chi_{n,k+1} \). This justifies our approach to study the cyclic chirotopes rather than their particular realizations as cyclic polytopes.

Given any \( d \)-chirotope \( \chi \) with \( n \) vertices, a subset \( A \in \{1, \ldots, n\} \) is called a halfspace if \( A = X^+ \) for a cocircuit \( X \) of \( \chi \), and it is called a missing face if \( A = X^- \) for a circuit \( X \) of \( \chi \). The following characterization of the faces of a simplicial chirotope in terms of the missing faces is an immediate consequence of the Farkas' Lemma for oriented matroids [2, Theorem 8], [4], [11].

**Lemma 2.2.** A subset \( F \subset \{1, \ldots, n\} \) is a face of the simplicial chirotope \( \chi \) if and only if \( F \) contains no missing face.

The cyclic chirotopes were called alternating matroids by Bland and Las Vergnas [4] due to the following structure of their signed circuits.

**Remark 2.3.** (i) A signed vector \( X \subset \{-1, 0, +1\}^n \) is a circuit of \( \chi_{nd} \) if and only if there exist \( \mu \in \Lambda(n, d + 1) \) and \( \sigma \in \{0, 1\} \) such that
\[
X_i = \begin{cases}
(-1)^{\sigma+1} & \text{if } i = \mu_j; \\
0 & \text{else}.
\end{cases}
\]

(ii) Every missing face of \( \chi_{nd} \) contains at least \( [(d + 1)/2] \) vertices.

Part (ii) follows from part (i), which is an immediate consequence of the definitions of the circuits of \( \chi \). For the reader who is less familiar with the language of oriented matroids let us remark that for realizable affine chirotopes the circuits \( X = (X^+, X^-) \) are exactly the minimal Radon partitions. Observe that in that case our definition differs slightly from Shemer’s definition [14, Definition 2.2] of ‘missing faces’. His ‘missing faces’ correspond to minimal missing faces in our sense.
Let us give straightforward chirotope proofs for two basic properties of cyclic polytopes.

**Corollary 2.4** [10, Theorem 4.7.1]. (i) The face lattice of the cyclic chirotope $\chi^{n,d}$ is simplicial. (ii) If $2l < d$, then any $l$ vertices of $\chi^{n,d}$ form a face.

**Proof.** (i) $\chi^{n,d}$ is a simplicial chirotope and hence its face lattice is simplicial, too. (ii) This follows directly from Lemma 2.2 and Remark 2.3 (ii).

Let $\mu \in \Lambda(n, d - 1)$. The hyperplane $\mu$ is a facet of $\chi^{n,d}$ if and only if the assigned cocircuit, see e.g. [6, Section 2], is positive or negative. In other words: $\mu$ being a facet is equivalent to

$$\chi^{n,d}(\mu, i) \cdot \chi^{n,d}(\mu, j) = \text{sign}(\mu, i) \cdot \text{sign}(\mu, j) \geq 0$$

for all $i, j \in \{1, \ldots, n\}$. Obviously, this is the case if and only if for any two elements $i, j$ in $\mu^c := \{1, \ldots, n\} \setminus \mu$ the cardinality of $\{\mu_k | i < \mu_k < j\}$ is even.

**Example 2.5** ($n = 8, d = 5$). Let us consider the cyclic polytope $C(8, 4)$ or its chirotope $\chi^{8,5}$.

(i) $\mu := (2, 3, 5, 6) \in \Lambda(8, 4)$ is a facet of $\chi^{8,5}$. In the sequence $(1, 2, 3, 4, 5, 6, 7, 8)$ any two elements from $\{1, 4, 7, 8\}$ are separated by an even number of elements from $\mu$.

(ii) $\tilde{\mu} := (2, 3, 5, 7)$ is not a facet of $\chi^{8,5}$. In the sequence $(1, 2, 3, 4, 5, 6, 7, 8)$ the numbers 4 and 6 are separated by an odd number of elements of $\tilde{\mu}$. This means geometrically that in the chirotope $\chi^{8,5}$ the vertices 4 and 6 are located on different sides of the hyperplane 2357.

We see that the facial structure of the cyclic chirotopes is described by the following well-known rule.

**Theorem 2.6** (Gale's Evenness Condition, [10], [13]). A hyperplane $\mu \in \Lambda(n, d - 1)$ is a facet of $\chi^{n,d}$ if and only if in the sequence $(1, 2, 3, \ldots, n)$ any two elements from $\mu^c$ are separated by an even number of elements from $\mu$.

To further illustrate this condition let us discuss the affine Gale diagram [17] of $\chi^{8,5}$ given in Fig. 1. By definition, an affine Gale diagram of a chirotope $\chi$ is a realization of an affine reorientation of the dual chirotope $\chi^*$. In our diagrams we indicate the reoriented points with a '−'.

The Radon partition $(17, 48)$ in our affine diagram agrees with the partition of $\{1, 4, 7, 8\}$ into negatively and positively signed points. Therefore 1478 is a cofacet of $\chi^{8,5}$, whence 2356

![Figure 1. Affine Gale diagram of the cyclic chirotope $\chi^{8,5}$.](image-url)
is a facet. Likewise, 2357 is not a facet because the negatively signed point 1 is not in the convex hull of the positively signed triangle 468 in Fig. 1.

Observe that the chirotope of our affine Gale diagram is again cyclic. Using the standard notation $\mathcal{X}$ for the reorientation of a subset $A \subset \{1, \ldots, n\}$ of the $n$ vertices of a $d$-chirotope $\chi$, we have the relation

$$(\chi^{8.3})^* = \chi^{8.3}. $$

This self-duality is a general property of all cyclic chirotopes. Cordovil and Duchet proved the corresponding result for alternating matroids; here we provide a different proof based on chirotope duality. Notice the analogy to certain duality arguments in the Grassmann algebra. For any $n \in \mathbb{N}$ let $U_\mu := \{i \in \{1, \ldots, n\} | i \text{ odd}\}$.

**Theorem 2.7.** For all $n \geq d$ we have the self-duality relation

$$(\chi^{n,d})^* = v_\chi^n \chi^{n-d}$$

for the reorientation classes of cyclic chirotopes.

**Proof.** Let $\mu \in \Lambda(n, n - d)$.

$$v_\chi^n \chi^{n-d}(\mu) = \chi^n, n - d(\mu) \cdot (-1)^{\mu \cap U_\mu}$$

$$= (-1)^{\mu \cap U_\mu}$$

$$= (-1)^{\sum_{i \in U_\mu} i}$$

$$= \chi^{n, d}(\mu^C) \cdot (-1)^{\sum_{i \in U_\mu} i}$$

$$= (\chi^{n,d})^*(\mu).$$

3. A Characterization of Neighborly Chirotopes

Given $l \in \mathbb{N}$, a chirotope $\chi$ is called $l$-neighborly if every $l$-element subset of $\{1, \ldots, n\}$ spans a face of $\chi$. A $d$-chirotope $\chi$ is called neighborly if it is $[(d - 1)/2]$-neighborly. This definition generalizes the usual definition for polytopes.

**Remark 3.1.** (i) A chirotope $\chi$ is 1-neighborly if and only if every vertex of $\chi$ is extreme, i.e. every contraction is affine.

(ii) Every minor by deletion of a neighborly chirotope is neighborly.

Clearly, there are no non-trivial 2-neighborly $d$-chirotopes for $d \leq 4$. We have seen in Corollary 2.4 that every cyclic chirotope $\chi^{n,d}$ is neighborly. Hence for every $d > 4$ and all $n \geq d$ there exists a realizable 2-neighborly $d$-chirotope with $n$ vertices, i.e. the segment between any two vertices of the corresponding polytope forms an edge. This fact, apart from being a surprise for the three-dimensional intuition of the novice, is of considerable interest for linear programming.

It is not difficult to see that for $l > [(d - 1)/2]$ the simplex is the only $l$-neighborly $d$-chirotope. For, every $d$-chirotope $\chi$ with $n \geq d + 1$ vertices has a circuit $X$ with at most $d + 1$ elements. Both $X^+$ and $X^-$ are missing faces of $\chi$ and hence not faces. Since either $X^+$ or $X^-$ contains at most $[(d + 1)/2]$ elements, $\chi$ cannot be $l$-neighborly for any $l > [(d - 1)/2]$.

The following characterization of neighborliness for chirotopes in terms of their circuits follows immediately from Lemma 2.2.
PROPOSITION 3.2. A $d$-chirotope $\chi$ with $n$ vertices is neighborly if and only if every missing face of $\chi$ contains at least $[(d + 1)/2]$ vertices.

As is customary in the theory of neighborly polytopes, we restrict ourselves to even-dimensional neighborly polytopes, that is, more generally, odd-dimensional neighborly chirotopes.

PROPOSITION 3.3. A $(2k + 1)$-chirotope $\chi$ is neighborly if and only if $\chi$ is simplicial and $|X^+| = |X^-|$ for all circuits $X$ of $\chi$.

PROOF. If $\chi$ is not simplicial, then $\chi$ has a circuit $X$ with $\leq 2k + 1$ elements, so either the missing face $X^+$ or the missing face $X^-$ have $\leq k$ elements. Hence $\chi$ is not $(k + 1)$-neighborly. But for a simplicial $(2k + 1)$-chirotope every circuit has exactly $2k + 2$ elements; in this case the condition $|X^+| = |X^-|$ for all circuits $X$ of $\chi$ is equivalent to every missing face containing at least $k + 1$ vertices.

Let us note another reformulation of the above proposition.

COROLLARY 3.4. A $(2k + 1)$-chirotope $\chi$ is neighborly if and only if every missing face of $\chi$ contains exactly $k + 1$ elements.

Given two $d$-chirotopes $\chi$, $\tilde{\chi}$, we call $\tilde{\chi}$ a mutant of $\chi$ if the two chirotopes differ in exactly one simplex orientation. If we switch the orientation of exactly one simplex $\lambda \in \Lambda(n, d)$ from $+1$ to $-1$ or vice versa, then only circuits $X \subseteq \lambda \cup \{i\}$ are affected: the sign of $X$ changes. Hence Proposition 3.3 implies:

REMARK 3.5. A neighborly $(2k + 1)$-chirotope has no neighborly mutants, i.e. if the orientation of exactly one basis is switched, a neighborly $(2k + 1)$-chirotope loses its neighborliness.

Let us give an elementary proof in the chirotope language for the well-known fact that every neighborly $2k$-polytopes with up to $2k + 3$ vertices is cyclic while this is not true any longer if the number of vertices exceeds $2k + 3$. The symmetric group $S_n$ acts on the set of $d$-chirotopes with $n$ vertices by relabelling the vertices. The orbits of this action define a natural concept of isomorphism among the $d$-chirotopes with $n$ vertices. In addition, we call the two chirotopes $\chi$ and $-\chi$ isomorphic.

Following Cordovil and Duchet [8], a pair of vertices $(i, j)$ of a $d$-chirotope $\chi$ with $n$ vertices is sign-invariant if $i$ and $j$ are separated by either no or all hyperplanes spanned in $\{1, \ldots, n\}\{i, j\}$. Clearly, two isomorphic chirotopes $\chi$ and $\chi'$ have isomorphic graphs $G(\chi)$ and $G(\chi')$ of sign-invariant pairs.

THEOREM 3.6. (i) Let $\chi$ be a neighborly $(2k + 1)$-chirotope $\Pi$ with $n \leq 2k + 3$ vertices. Then $\Pi$ is isomorphic to $\chi^{n, 2k + 1}$.

(ii) For all $k \in \mathbb{N}$ there is a realizable neighborly $(2k + 1)$-chirotope with $2k + 4$ vertices which is not isomorphic to the cyclic chirotope $\chi^{2k + 4, 2k + 1}$.

PROOF. For $n = 2k + 1$ there is, up to isomorphism, only one chirotope: the simplex. For $n = 2k + 2$ the chirotope $\chi$ has only one circuit $X$, and $\sigma$ is determined by $X$. But the circuit $X$ is determined, up to isomorphism, by Theorem 3.3.

Next consider the case $n = 2k + 3$, and let $(\bar{V}, v)$ be an affine Gale diagram [17] of $\chi$. The set $\bar{V} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{2k+3}\}$ is a subset of the real line. Since $\chi$ is simplicial, the affine
Figure 2. Affine Gale diagrams of $\chi^{2k+1,3k+1}$ (a), $\chi^{2k+3,2k+1}$ (b) and a non-cyclic neighborly $(2k+1)$-chirotope with $2k + 4$ vertices (c).

Gale diagram of $V$ is simplicial as well, i.e. the $v_i$ are distinct. Without loss of generality we can assume that $v_1 < v_2 < \cdots < v_{2k+3}$. We have to show that $\tilde{P}$ equals the affine Gale diagram in Fig. 2(a) which represents the cyclic chirotope $\chi^{2k+3,2k+1}$.

By Proposition 3.3, every half-space of $\chi^*$ consists of $k + 1$ elements. Hence for any $r \leq 2k + 2$

$$|\{i < r\} \cap v| + |\{i > r\} \cap v| = k + 1,$$

$$|\{i < r + 1\} \cap v| + |\{i > r + 1\} \cap v| = k + 1.$$ 

Subtracting these two equations we obtain

$$|\{r\} \cap v| - |\{r + 1\} \cap v| = 0.$$ 

In other words, for all $r \leq 2k + 2$ either $r \in v$ or $r + 1 \in v$. This shows that $v$ consists either of all odd numbers or of all even numbers $\leq 2k + 3$, which proves the claim.

Figure 2(c) shows the affine Gale diagram of a neighborly $(2k + 1)$-chirotope $\Pi$ with $2k + 4$ vertices. It is easy to see that $\Pi$ is not isomorphic to $\chi^{2k+3,2k+1}$ whose affine Gale diagram is given in Fig. 2(b). For, the graph $G(\chi^{2k+3,2k+1})$ of sign invariant pairs of the cyclic chirotope $\chi^{2k+3,2k+1}$ is a $(2k + 4)$-cycle while the graph $G(\chi)$ of the chirotope $\chi$ in Fig. 2(c) has two connected components $\{1, 2, 3, 4\}$ and $\{5, 6, \ldots, 2k + 2, 2k + 3\}$. \hfill \Box

4. RIGIDITY OF NEIGHBORLY CHIROTOPES

In the last two sections we consider only neighborly chirotopes of odd rank. Shemer proved that the corresponding even-dimensional polytopes have a remarkable property: The face lattice of every subpolytope of an even-dimensional neighborly polytope $P$ is uniquely determined by the face lattice of $P$ [14, Theorem 2.12]. This implies in particular that the vertices of such a polytope are necessarily in general position, a property that is not shared by any non-trivial simplicial 3-polytope.

In this section we generalize Shemer's theorem to the combinatorial setting of oriented matroids or chirotopes, thereby deriving a new and short proof for the realizable case. It is a non-trivial fact that non-realizable neighborly $(2k + 1)$-chirotopes and consequently non-polytopal neighborly matroid spheres do exist, and so far only one (minimally) example is known. In [5] J. Bokowski and K. Garms give a non-realizability proof for the neighborly 5-chirotope with 10 vertices of the 3-sphere $M_{425}$ from Altshuler's list [1].

We call a $d$-chirotope $\chi$ with $n$ vertices rigid if there is no other $d$-chirotope (except $-\chi$) which has the same face lattice as $\chi$. For a discussion of chirotope rigidity see [16, Section 3]. In order to prove our rigidity theorem for neighborly chirotopes we need one lemma.
LEMMA 4.1. Let $\chi$ be a neighborly $(2k + 1)$-chirotope with $n > 2k + 2$ vertices. Then $\alpha \in \Lambda(n - 1, k + 1)$ is a missing face of $\chi \setminus n$ if and only if there is an $i \in \{1, \ldots, k + 1\}$, such that both $\alpha$ and $(\alpha \setminus \{\alpha_i\}) \cup \{n\}$ are missing faces of $\chi$.

PROOF. (if): If $\alpha$ and $(\alpha \setminus \{\alpha_i\}) \cup \{n\}$ are missing faces of $\chi$, then the signed vectors $X, Y$ defined by

$$
X^+ := \alpha, \quad X^- := \{1, \ldots, n\} \setminus \alpha, \\
Y^+ := (\alpha \setminus \{\alpha_i\}) \cup \{n\}, \quad Y^- := \{(1, \ldots, n - 1) \setminus \alpha \} \cup \{\alpha_i\}
$$

define a Radon partition in $\chi$, i.e. they are elements of the signed circuit span of $\chi$. By the classical axiomatization of oriented matroids given in [4], we can 'subtract' $X$ and $Y$, which means that either $X \setminus n$ or $Y \setminus n$ are in the signed span of $\chi$, too. Assume $Y \setminus n$ was a Radon partition in $\chi \setminus n$. The non-face $(Y \setminus n)^+$ contains only $k$ elements, which contradicts the neighborliness of $\chi \setminus n$. Hence, by Remark 3.1(a), the signed vector $X \setminus n$ defines a Radon partition in $\chi \setminus n$. Since $X^+ = (X \setminus n)^+ = A$ has $k + 1$ elements, this means that $A$ is a missing face of $\chi \setminus n$.

(only if): If $\alpha$ is a missing face $\chi \setminus n$, then the signed vector $Z$ defined by

$$
Z^+ := \alpha, \quad Z^- := \{1, \ldots, n - 1\} \setminus \alpha
$$

is in the signed circuit span of $\chi \setminus n$ and hence in the signed circuit span of $\chi$. We can add the new point $n$ to $Z^+$, thereby obtaining another element $Z$ in the circuit span of $\chi$. Writing $Z$ as a conformal union [2] of circuits of $\chi$, we obtain a representation of $(Z)^+ = \alpha \cup n$ as union of $(k + 1)$-element missing faces, one of which is $\alpha$. Hence there exists a missing face of the form $(\alpha \setminus \{\alpha_i\}) \cup n$ of $\chi$.

THEOREM 4.2. Every neighborly $(2k + 1)$-chirotope is rigid.

PROOF. Let $\chi$ be a neighborly $(2k + 1)$-chirotope with $n$ vertices. By Corollary 3.4, the missing faces of $\chi$ are precisely the nonfaces of cardinality $k + 1$, and hence the set of missing faces of $\chi$ is determined uniquely by the face lattice of $\chi$.

We prove by induction on $n$ that $\chi$ is (up to reflection) uniquely determined by its missing faces. If $n = 2k + 2$, then $\chi$ has only one circuit $X$. Therefore the two only missing faces $X^+$ and $X^-$ of $\chi$ determine $\chi$.

Now let $n > 2k + 2$ and assume that the theorem is true for all neighborly $(2k + 1)$-chirotopes with less vertices. It is sufficient to show that for every circuit $X$ of $\chi$ the sets $X^+$ and $X^-$ are determined. Choose an $i \in X^0$. By Lemma 4.1, the missing faces of $\chi \setminus i$ are determined uniquely. Since $X$ is also a circuit of $\chi \setminus i$, it is determined by the induction hypothesis.

5. ON A PROBLEM OF M. A. PERLES

M. A. Perles [private communication] posed the problem whether the even-dimensional neighborly polytopes are characterized among all simplicial polytopes by the rigidity property in Theorem 4.2. Using Gale diagrams it is easily derived that this is the case for $k$-polytopes with $k + 3$ vertices, and we provide a proof of this fact using our terminology. As main result of this section we shall answer Perles’ question by showing that this is not true in general.

PROPOSITION 5.1. Let $\chi$ be a rigid, simplicial $d$-chirotope with $n = d + 2$ vertices. Then $d$ is odd and $\chi$ is neighborly.
PROOF. Consider an affine Gale diagram \((\tilde{V}, v)\) of \(\chi\). Since \(\chi\) is simplicial, \(V\) is a set of \(n\) distinct real numbers, and we can write \(\tilde{V} = \{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n\}\) with \(\tilde{v}_1 < \tilde{v}_2 < \cdots < \tilde{v}_n\). By Theorem 3.6(i) we have to show that \(\chi\) is cyclic, i.e. that \(v\) consists of either all odd or all even numbers between 1 and \(n\).

Assume there existed an \(r\) such that \((r \in v \text{ and } r + 1 \notin v)\) or \((r \notin v \text{ and } r + 1 \in v)\). In both cases we can interchange \(v_r\) and \(v_{r+1}\) in the affine Gale diagram \(\tilde{V}\) without altering the face lattice of \(\chi\). For the chirotope \(\chi\) this means that we can switch the orientation of a simplex without altering the face lattice of \(\chi\), i.e. \(\chi\) is not rigid. Hence \(\chi\) is cyclic.

Notice that for even \(d\) the cyclic chirotope \(\chi_{d+2,d}\) is not rigid. We can switch the orientation of the simplex \((2, 3, \ldots, d+1)\), thereby obtaining another chirotope with the same face lattice. Hence \(\chi\) is a neighborly chirotope of odd rank.

THEOREM 5.2. For all \(j \geq 4\) there is a rigid, simplicial, realizable \((3j - 3)\)-chirotope \(\mathcal{C}_j\) with \(3j\) vertices such that \(\mathcal{C}_j\) is not neighborly.

In Fig. 3 we see an affine Gale diagram of the smallest chirotope \(\mathcal{C}_4\) in our family.

PROOF. Consider the cyclic chirotope \(\chi^{3j,3}\) and extend these with another \(j\) vertices by the principal extensions \([12, 13]\)

\[
2j + i := [(2i - 1)\text{--}, 2i\text{--}, (2i + 1)\text{--}] (\text{mod } 2j), \quad i = 1, \ldots, j
\]

to a 3-chirotope \((\tilde{\chi}_j)\) with \(3j\) vertices. Geometrically speaking, we insert a negatively signed point \(2j + i\) inside the circle \(\chi^{3j,3}\) close to the mid-point of the line \(2i - 1, 2i\).

Let us see that its dual \(\mathcal{C}_j := \tilde{\chi}_j^*\) has the desired properties to be rigid but not neighborly:

1. \(\mathcal{C}_j\) is clearly simplicial and realizable, since it is constructed from the realizable simplicial cyclic chirotope by simplicial principal extensions.

2. \(\mathcal{C}_j\) is not neighborly, since the \(j\)-element set \(\{2j + 1, \ldots, 3n\}\) is not a face of \(\mathcal{C}_j\) and \(j < [(3j - 1)/2]\) for all \(j \geq 4\).

3. The following simplices form a reduced system \([16, \text{Definition 3.5}]\) of \(\mathcal{C}_j^* = \tilde{\chi}_j^*\):

\[
\mathcal{R} := \{(1, 2, 3), (2, 3, 4), \ldots, (2j - 1, 2j, 1), (2j, 1, 2)\}
\cup \{(2j + i, k, l) | i = 1, \ldots, j, k \in \{1, \ldots, 2j\}, l \in \{2i - 1, 2i\}\} \subset \Lambda(3j, 3).
\]

But every element of the reduced system \(\mathcal{R}\) is contained in a cofacet of \(\mathcal{C}_j\) of the form \(\{2j + i, 2i - 1, 2i, l\}\). So, \(\mathcal{C}_j\) has a reduced system of outer simplices, which shows, by \([16, \text{Lemma 3.6}]\), that \(\mathcal{C}_j\) is rigid.

\(\square\)
REFERENCES


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