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Domination, independence and irredundance with respect to additive induced-hereditary properties

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Abstract

For a given graph G a subset X of vertices of G is called a dominating (irredundant) set with respect to additive induced-hereditary property $\mathscr{P}$, if the subgraph induced by X has the property $\mathscr{P}$ and X is a dominating (an irredundant) set. A set S is independent with respect to $\mathscr{P}$, if $[S] \in \mathscr{P}$.

We give some properties of dominating, irredundant and independent sets with respect to $\mathscr{P}$ and some relations between corresponding graph invariants. This concept of domination and irredundance generalizes acyclic domination and acyclic irredundance given by Hedetniemi et al. (Discrete Math. 222 (2000) 151).

MSC: 05C69

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All graphs considered in this paper are finite and simple.

A graph property $\mathscr{P}$ is any nonempty class of graphs closed under isomorphism. The class of all graphs is denoted by $\mathscr{I}$. We also say that a graph has the property $\mathscr{P}$, if $G \in \mathscr{P}$. By saying that H is a subgraph (induced subgraph) of G, we mean that H is isomorphic to a subgraph (induced subgraph) of G.

A property $\mathscr{P}$ is called *hereditary* (*induced-hereditary*) if it is closed under taking subgraphs (induced subgraphs), respectively. Hereditary properties are induced-hereditary properties too. On the other hand, many well-known induced-hereditary classes of graphs e.g., complete graphs, line graphs, claw-free graphs, interval graphs, perfect graphs, etc. are not hereditary. For a survey see [3].

A property $\mathscr{P}$ is called *additive* if for each graph G all of whose components have the property $\mathscr{P}$ it follows that $G \in \mathscr{P}$, too. In this paper we shall consider only additive induced-hereditary properties.

Any induced-hereditary property $\mathscr{P}$ of graphs is uniquely determined by the set of all minimal forbidden induced subgraphs $C(\mathscr{P})$ and $C(\mathscr{P}) = \{H \in \mathscr{I} : H \notin \mathscr{P}$ but $(H - v) \in \mathscr{P}$ for any $v \in V(H)\}$.

Note that, if $\mathscr{P}$ is additive, then $C(\mathscr{P})$ contains only connected graphs.

For example, we list some important induced-hereditary properties and their sets of forbidden subgraphs.

$\mathscr{O} = \{G \in \mathscr{I} : G$ is totally disconnected$\}$, $C(\mathscr{O}) = \{K_2\}$,
$\mathscr{S}_k = \{G \in \mathscr{I} : \Delta \leqslant k\}$, $C(\mathscr{S}_k) = \{H \in \mathscr{I} : |V(H)| = k+2 = \Delta(H)+1\}$,
$\mathscr{I}_k = \{G \in \mathscr{I} : G$ does not contain $K_{k+2}\}$, $C(\mathscr{I}_k) = \{K_{k+2}\}$.
$\mathscr{D}_k = \{G \in \mathscr{I} : \delta(H) \leqslant k$ for each induced subgraph H of $G\}$,

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$C(\mathscr{D}_1) = \{C_3, C_4, ..\}$.
$\mathscr{O}_k = \{G \in \mathscr{I}$: each component of G has at most $k+1$ vertices$\}$,
$C(\mathscr{O}_k) = \{H \in \mathscr{I} : |V(H)| = k+2,\ H$ is connected$\}$.
$\mathscr{W}_k = \{G \in \mathscr{I}$: length of the longest path in G is at most $k\}$.

For an induced hereditary property $\mathscr{P}$, a nonnegative integer k such that $K_{k+1} \in \mathscr{P}$ but $K_{k+2} \notin \mathscr{P}$ is called the *completeness of* $\mathscr{P}$ and it is denoted $c(\mathscr{P})$.

If such integer does not exist for an induced hereditary property $\mathscr{P}$, then $c(\mathscr{P}) = \infty$.

For example $c(\mathscr{S}_k) = c(\mathscr{O}_k) = c(\mathscr{D}_k) = c(\mathscr{I}_k) = k$ and for additive property $\mathscr{P}$, $c(\mathscr{P}) = 0$ if and only if $\mathscr{P} = \mathscr{O}$.

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, we denote by $N(v)$ the set of vertices of G adjacent to v (neighbours of v) and for $A \subseteq V$, by $N(A)$ the set of neighbours of vertices of A. By $N[v]$, we denote $N(v) \cup \{v\}$ and $N[A] = N(A) \cup A$.

A nonempty set $D \subseteq V$ is called *dominating in G* if every vertex v in $V - D$ is adjacent to a vertex of D, i.e. $N(v) \cap D \neq \emptyset$.

The minimum (maximum) cardinality of a minimal dominating set in G is called the *lower* (*upper*) *domination number* and it is denoted by $\gamma(G)(\Gamma(G))$, respectively.

A set $I \subseteq V$ is said to be *irredundant in G* if for each vertex $v \in I$, $N[v] - N[I - \{v\}] \neq \emptyset$.

The minimum (maximum) cardinality of a maximal irredundant set in G is called the *lower* (*upper*) *irredundant number of G* and it is denoted by $ir(G)(IR(G))$, respectively.

We use the notation $G[A]$ (shortly $[A]$) for a subgraph induced by the set $A \subseteq V(G)$.

A set $S \subseteq V$ is said to be *independent* if $G[S] \in \mathscr{O}$, ($N(v) \cap S = \emptyset$ for each vertex $v \in S$). If S is a maximal independent set of G and v is not in S, then $G[S \cup \{v\}]$ contains as a subgraph K_2, i.e. the subgraph which is forbidden for the property $\mathscr{O}$.

The minimum (maximum) cardinality of a maximal independent set in G is called the *lower* (*upper*) *independence number* and it is denoted by $i(G)(\alpha(G))$, respectively.

In 1978 Cockayne et al. [5] first defined what now is a well-known inequality chain of domination related parameters of a graph G

$$ir(G) \leqslant \gamma(G) \leqslant i(G) \leqslant \alpha(G) \leqslant \Gamma(G) \leqslant IR(G). \tag{1}$$

Since then more than 100 research papers have been published in which this inequality chain is the focus of study. Researchers have considered conditions under which two or more of these parameters are equal or parameters whose values lie between two consecutive parameters in (1), extensions of the inequality chain (1) in either direction, inequality chain similar to (1) for other parameters, etc.

In this paper we introduce a new type of domination and irredundance motivated by the concept of acyclic domination and acyclic irredundance introduced by Hedetniemi et al. [6].

Let G be a graph and $\mathscr{P}$ be an additive induced-hereditary property.

A set $X \subseteq V$ is called a *dominating* (*irredundant*) *set with respect to the additive induced-hereditary property* $\mathscr{P}$ (shortly with respect to $\mathscr{P}$) if $G[X] \in \mathscr{P}$ and X is dominating (irredundant), respectively.

Dominating (irredundant) sets with respect to $\mathscr{D}_1$ are acyclic dominating, (acyclic irredundant) sets [6].

Dominating (irredundant) sets with respect to $\mathscr{I}$ are dominating (irredundant) sets in G in the ordinary sense.

Dominating (irredundant) sets with respect to $\mathscr{O}$ are independent dominating (irredundant) sets in G.

By $\gamma^{\mathscr{P}}(G)$, $\Gamma^{\mathscr{P}}(G)$, ($ir^{\mathscr{P}}(G)$, $IR^{\mathscr{P}}(G)$) we denote the *lower*, *upper dominating* (*irredundant*) *number of G with respect to* $\mathscr{P}$, as the minimum, maximum cardinality of a minimal dominating (a maximal irredundant) set with respect to $\mathscr{P}$ in G.

Let us recall the definition of $\mathscr{P}$-independent set [4].

A set $S \subseteq V$ is called $\mathscr{P}$*-independent* (shortly $\mathscr{P}$*-set*) if $G[S]$ has the property $\mathscr{P}$.

By $i^{\mathscr{P}}(G)$, $\alpha^{\mathscr{P}}(G)$ the minimum, maximum of the cardinalities of a maximal $\mathscr{P}$-set in G are denoted.

We shall use notation γ*-set*, $\gamma^{\mathscr{P}}$*-set*, *i-set*, $i^{\mathscr{P}}$*-set*, etc., for a minimal dominating set of cardinality $|\gamma(G)|$, for a minimal dominating set with respect to $\mathscr{P}$ of cardinality $|\gamma^{\mathscr{P}}(G)|$, etc.

Lemma 1. *If X is a minimal dominating set with respect to* $\mathscr{P}$ *in G*, *X is a minimal dominating set in G.*

Proof. Since $[X - \{v\}] \in \mathscr{P}$ for each vertex $v \in X$, then the set X has to be a minimal dominating set in G. □

By Lemma 1 we obtain

$$\gamma(G) \leqslant \gamma^{\mathscr{P}}(G) \leqslant \Gamma^{\mathscr{P}}(G) \leqslant \Gamma(G). \tag{2}$$

There are certain classes of graphs with the numbers $\gamma(G)$ and $\gamma^{\mathcal{P}}(G)$ equal. It is very easy to see that:

1.1. If $G \in \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{P}$, then $\gamma(G) = \gamma^{\mathcal{P}}(G)$.
1.2. Let $\mathcal{P}$ be an induced hereditary property with $0 \leqslant c(P) < \infty$ and $\gamma(G) \leqslant c(\mathcal{P}) + 1$, then $\gamma(G) = \gamma^{\mathcal{P}}(G)$.
1.3. If $\gamma^{\mathcal{D}_1}(G) = \gamma(G)$ [6], then for any additive hereditary property $\mathcal{P}$ such that $\mathcal{D}_1 \subseteq \mathcal{P}$, $\gamma(G) = \gamma^{\mathcal{P}}(G)$.

Lemma 2. *If X is a maximal $\mathcal{P}$-set in G, then X is a dominating set with respect to $\mathcal{P}$ in G.*

Proof. For any $v \in V - X$, the graph $G[X \cup \{v\}]$ contains a subgraph $H \in C(\mathcal{P})$. $\mathcal{P}$ is an additive property, so H is connected. It implies that the vertex v is adjacent to some vertex of X. Hence X is a dominating set with respect to $\mathcal{P}$ (X is not necessarily minimal). □

From Lemma 2 we obtain

$$\gamma^{\mathcal{P}}(G) \leqslant i^{\mathcal{P}}(G). \tag{3}$$

Lemma 3. *If X is a maximal independent set in G, then X is a minimal dominating set with respect to $\mathcal{P}$.*

Proof. Let X be an *i-set* in G. Since $\mathcal{O} \subseteq \mathcal{P}$, for any additive induced-hereditary property $\mathcal{P}$, then $[X] \in \mathcal{P}$. It is not difficult to see that X is a minimal dominating set in G (see [5]), hence X is a minimal dominating with respect to $\mathcal{P}$. □

From Lemma 3 we obtain

$$\gamma^{\mathcal{P}}(G) \leqslant i(G) \leqslant \alpha(G) \leqslant \Gamma^{\mathcal{P}}(G). \tag{4}$$

From (1), (2) and (4) we have:

$$ir(G) \leqslant \gamma(G) \leqslant \gamma^{\mathcal{P}}(G) \leqslant i(G) \leqslant \alpha(G) \leqslant \Gamma^{\mathcal{P}}(G) \leqslant \Gamma(G) \leqslant IR(G). \tag{5}$$

We can also obtain a subchain of (5) in the following way:

I. If $\mathcal{P} \subseteq \mathcal{Q}$, then $\gamma^{\mathcal{Q}} \leqslant \gamma^{\mathcal{P}}$.
II. $(\mathcal{O} \subseteq P \subseteq \mathcal{I}) \Rightarrow (\gamma(G) = \gamma^{\mathcal{I}}(G) \leqslant \gamma^{\mathcal{P}}(G) \leqslant \gamma^{\mathcal{O}}(G) = i(G))$.

Corollary 4. *If $\gamma(G) = i(G)$, then $\gamma(G) = \gamma^{\mathcal{P}}(G)$ for any induced hereditary property $\mathcal{P}$.*

Lemma 5. *If X is a minimal dominating set with respect to $\mathcal{P}$ in G, then X is a maximal irredundant set with respect to $\mathcal{P}$ in G.*

Proof. Let X be a minimal dominating set with respect to $\mathcal{P}$ in G. By Lemma 1, X is a minimal dominating set in G and by [5], X is a maximal irredundant set in G. Since $[X] \in \mathcal{P}$, then X is a maximal irredundant set with respect to $\mathcal{P}$. □

From Lemma 5 we obtain

$$ir^{\mathcal{P}}(G) \leqslant \gamma^{\mathcal{P}}(G) \leqslant \Gamma^{\mathcal{P}}(G) \leqslant IR^{\mathcal{P}}(G). \tag{6}$$

By (6) and (3) it follows:

$$ir^{\mathcal{P}}(G) \leqslant \gamma^{\mathcal{P}}(G) \leqslant i^{\mathcal{P}}(G). \tag{7}$$

Theorem 6. *For any graph G*

(1) $\alpha^{\mathcal{P}}(G) \geqslant \alpha(G)$ *for any additive induced-hereditary property $\mathcal{P}$.*
(2) *Let $|V(G)| \geqslant \alpha(G) + k$. Then $\alpha^{\mathcal{P}}(G) \geqslant \alpha(G) + k$ for the property $\mathcal{D}_k$ and $\mathcal{I}_k$, $k \geqslant 1$.*
(3) $\mathrm{IR}^{\mathcal{P}}(G) \leqslant \alpha^{\mathcal{P}}(G)$ *for any additive induced-hereditary property $\mathcal{P}$.*

Proof. (1) If $\mathscr{P} \subseteq \mathscr{Q}$, then each maximal $\mathscr{P}$-set is a $\mathscr{Q}$-set, thus $\alpha^{\mathscr{Q}} \geqslant \alpha^{\mathscr{P}}$. By the fact that $\mathscr{O} \subseteq \mathscr{P}$ we have (1).

(2) Let X be an α-set in G and let X' be a set of $k \geqslant 1$ vertices and $X \cap X' = \emptyset$. It is easy, to see that $[X \cup X'] \in \mathscr{I}_k$, since it does not contain the subgraph K_{k+2}.

Let H be an induced subgraph of $[X \cup X']$. If $v \in V(H)$ and $v \in X$, then $\deg_H(v) \leqslant k$. If $V(H) \subseteq X'$, then $\deg_H(v) \leqslant k-1$ for each $v \in V(H)$. Finally, $\delta(H) \leqslant k$ for each induced subgraph H of $[X \cup X']$ i.e., $[X \cup X'] \in \mathscr{D}_k$. It means that $\alpha^{\mathscr{P}}(G) \geqslant |X \cup X'| = \alpha(G) + k$.

(3) Let X be an $IR^{\mathscr{P}}$-set in G. Since $[X] \in \mathscr{P}$, (3) is proved. □

Remark 6.1. The inequality (2) is not true for the property $\mathscr{O}_k$, $\mathscr{S}_k$, $\mathscr{W}_k$.

From Theorem 6 and (6) we obtain

$$\Gamma^{\mathscr{P}}(G) \leqslant IR^{\mathscr{P}} \leqslant \alpha^{\mathscr{P}}(G). \tag{8}$$

Theorem 7. *If $\mathscr{P} = \mathscr{D}_k$ or $\mathscr{P} = \mathscr{I}_k$, then there exists a graph G such that*

(1) *$i(G) - \gamma^{\mathscr{P}}(G)$ is arbitrarily large,*
(2) *$i^{\mathscr{P}}(G) - \gamma^{\mathscr{P}}(G)$ is arbitrarily large,*
(3) *$\gamma^{\mathscr{P}}(G) - \gamma(G) > r$ for any positive integer r.*

Proof. Let $H \in C(\mathscr{P})$. Let G be the graph formed by attaching m, $m \geqslant 2$ independent vertices to each vertex of H (i.e., G is the corona $H \circ mK_1$, $m \geqslant 2$). G has the property that it contains an induced subgraph H and G does not contain any other forbidden subgraphs of $C(\mathscr{P})$. Let $n = |V(H)|$.

It is easy to check that

$i(G) = \alpha(H) + (n - \alpha(H))m$,
$i^{\mathscr{P}}(G) = n - 1 + nm = n(m+1) - 1$,
$\gamma(G) = n$,
$\gamma^{\mathscr{P}}(G) = n - 1 + m$.

Finally $i(G) - \gamma^{\mathscr{P}}(G) = (m-1)(n-1-\alpha(H))$,

$i^{\mathscr{P}}(G) - \gamma^{\mathscr{P}}(G) = m(n-1)$,
$\gamma^{\mathscr{P}}(G) - \gamma(G) = m - 1$. □

Theorem 8. *For any additive induced-hereditary property $\mathscr{P}$ with $1 \leqslant c(\mathscr{P}) < \infty$, for every positive integer r there exists a pair of graphs G_1, G_2 such that $\Gamma(G_1) - \Gamma^{\mathscr{P}}(G_1) > r$ and $\Gamma^{\mathscr{P}}(G_2) - \alpha(G_2) > r$.*

Proof. Let G_1 be the Cartesian product $K_m \times K_2$, where $m \geqslant c(\mathscr{P}) + 2$. It is easy to see that $\Gamma(G_1) = m$ and $\Gamma^{\mathscr{P}}(G_1) = 2$, thus $\Gamma(G_1) - \Gamma^{\mathscr{P}}(G_1) = m - 2$. Let $H = K_{k+3} \times (K_{k+2} - e)$ where $k = c(\mathscr{P})$. We denote vertices of two nonadjacent copies of K_{k+3} by $x_1, x_2, \ldots, x_{k+3}$ and $y_1, y_2, \ldots, y_{k+3}$, respectively. Taking vertices x_1, x_2 and $y_3, y_4, \ldots, y_{k+3}$ we obtain a minimal dominating set with respect to $\mathscr{P}$. Hence $\Gamma^{\mathscr{P}}(H) \geqslant k+3$ and it is possible to verify that there is no minimal dominating set with respect to $\mathscr{P}$ of cardinality greater than $k+3$. Thus $\Gamma^{\mathscr{P}}(H) = k+3$. On the other hand, H is covered by $k+2$ cliques so $\alpha(H) = k+2$. For any positive integer r, let G_2 be the graph formed by identifying the vertex x from r copies of H. Then $\Gamma^{\mathscr{P}}(G_2) = r(k+3)$, $\alpha(G_2) = r(k+2)$ and $\Gamma^{\mathscr{P}}(G_2) - \alpha(G_2) = r$. □

Allan and Laskar [1], Bollobás and Cockayne [2] independently established relationship between $\gamma(G)$ and $ir(G)$. Similar relations were considered in [6]. We generalize these results for an arbitrary additive induced-hereditary property $\mathscr{P}$.

Theorem 9. *For any graph G and for any additive induced-hereditary property $\mathscr{P}$*

$$\gamma(G) \leqslant 2ir^{\mathscr{P}}(G). \tag{9}$$

Proof. Let $I = \{x_1, x_2, \ldots, x_k\}$ be an $ir^{\mathscr{P}}$-set. For every $v \in V - I$ the set $I \cup \{v\}$ is not irredundant or it does not have property $\mathscr{P}$. By the definition of irredundancy for each vertex $x_i \in I$, there exists $y_i \in (N[x_i] - N[I - \{x_i\}])$, where y_i is a private neighbour of x_i (the vertex x_i can be its own neighbour). Let $A = I \cup \{y_1, y_2, \ldots, y_k\}$. We claim, that $N[A] = V$, for the contrary, suppose there exists vertex $v \notin N[A]$. Let $B = I \cup \{v\}$, by the above assumption, $v \notin N[B - \{v\}]$, so B is an irredundant set and by the additivity of $\mathscr{P}$ the set B has the property $\mathscr{P}$, contradiction with maximality of I. □

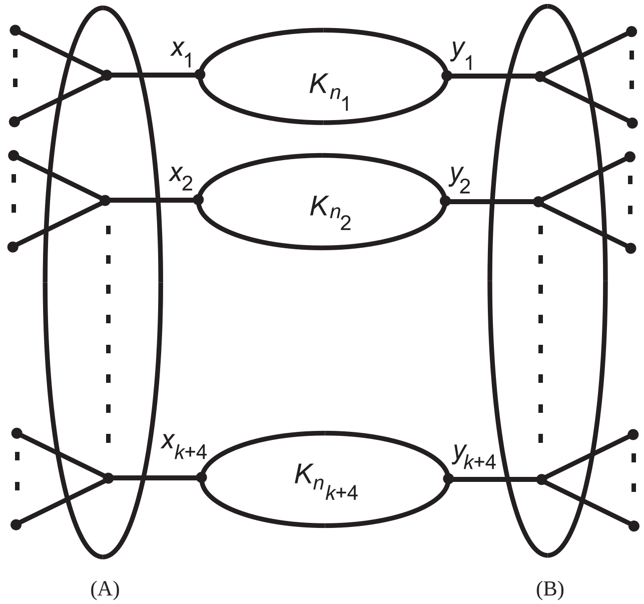

Fig. 1.

The inequality $\gamma^{\mathscr{P}}(G) \leqslant 2ir^{\mathscr{P}}(G)$ is not true for $\mathscr{P} = \mathscr{D}_k$ and $\mathscr{I}_k$, $k \geqslant 1$. The graph presented in Fig. 1 has $\gamma^{\mathscr{P}} > 2ir^{\mathscr{P}}$. The sets A and B each induce the complete graph on $k+4$ vertices and each vertex of $A \cup B$ is adjacent to $t \geqslant 2$ vertices of degree 1. For each $1 \leqslant i \leqslant k+4$, K_{n_i} is a complete graph on n_i vertices, $n_i \geqslant 3$. It is easy to check that $X = A' \cup B' \cup A'' \cup B'' \cup C$ is dominating, where A' is a set of $k+1$ vertices in A, B' is the set of $k+1$ in B and A'' is the set of $3t$ vertices of degree 1 adjacent to vertices of $A - A'$, B'' is the set of $3t$ vertices of degree 1 adjacent to $B - B'$, $C = \{x_1, x_2, \ldots, x_{k+4}\}$. If $\mathscr{P} = \mathscr{I}_k$ then it is obvious that $[X]$ does not contain K_{k+2} as induced subgraph. If $\mathscr{P} = \mathscr{D}_k$ it is not difficult to verify, that any induced subgraph of $[X]$ has a vertex of degree at most k. Hence in both cases $[X] \in \mathscr{P}$. Because any other $\mathscr{P}$-set which is dominating has more than $|X|$ vertices, we have $\gamma^{\mathscr{P}}(G) = 2(k+1) + 6t + k + 4$. The set $I = \{x_1, x_2, \ldots, x_{k+4}, y_1, y_2, \ldots, y_{k+4}\}$ is a minimum maximal irredundant set and $[I]$ has the property $\mathscr{P}$, so $ir^{\mathscr{P}}(G) = 2(k+4)$. For $t \geqslant 2 : \gamma^{\mathscr{P}}(G) > 2ir^{\mathscr{P}}(G)$.

Theorem 10. *If $c(\mathscr{P}) = k$, $1 \leqslant k < \infty$, then $\gamma(G)$ and $ir^{\mathscr{P}}(G)$ are incomparable as are $ir(G)$ and $ir^{\mathscr{P}}(G)$.*

Proof. For example, consider the graph $G = K_{k+2} \circ mK_1$, $m \geqslant 2$.

Then $ir(G) = k+2$, and $ir^{\mathscr{P}}(G) = k+1+m$, $\gamma(G) = k+2$.

In this case (1) $ir^{\mathscr{P}}(G) > ir(G)$ and (2) $ir^{\mathscr{P}}(G) > \gamma(G)$.

To show the converse inequality to (1) and (2) we construct a graph G (Fig. 2).

Let $\mathscr{P}$ be a property with $k = c(\mathscr{P})$, where $1 \leqslant k < \infty$. Let $G = (K \cup V \cup W \cup Z, E)$. Let $K = \{k_1, \ldots, k_{k+1}\}$ be the set of vertices of a complete graph, $V = \{v_1, \ldots, v_{k+2}\}$, $W = \{w_1, w_2, \ldots, w_{k+2}\}$, $Z = \{z_1, z_2, \ldots, z_{k+1}\}$ be independent sets of vertices. For each vertex v_i, $1 \leqslant i \leqslant k+2$ the vertex v_i is adjacent to each vertex of K, v_i is adjacent to w_i for each $1 \leqslant i \leqslant k+2$, and the vertex k_i with z_i for each $1 \leqslant i \leqslant k+1$, and the vertex w_i with z_i for each $1 \leqslant i \leqslant k+1$ and the vertex w_{k+2} is adjacent to each z_i, $1 \leqslant i \leqslant k+1$. We can show that $K \cup \{v_{k+2}\}$ is a maximal irredundant set and each maximal irredundant set in G has at least $k+2$ vertices. So $ir(G) = k+2$. On the other hand, the set K is a maximal irredundant set with respect to $\mathscr{P}$, so $ir^{\mathscr{P}}(G) \leqslant k+1$. It is not difficult to calculate $\gamma(G)$ and it is equal $k+2$. Hence $ir^{\mathscr{P}}(G) < ir(G)$ and $ir^{\mathscr{P}}(G) < \gamma(G)$. □

Remark 10.1. Hedetniemi et al. in [6] showed that $i(G)$ and $i^{\mathscr{P}}(G)$ are incomparable, as are $i^{\mathscr{P}}(G)$ with $IR(G)$, $IR^{\mathscr{P}}(G)$, $\Gamma(G)$, $\Gamma^{\mathscr{P}}(G)$ and $\alpha^{\mathscr{P}}(G)$, with $IR(G)$ and $\Gamma(G)$, also $\Gamma(G)$ with $IR^{\mathscr{P}}(G)$, for the property $\mathscr{D}_1$. The relationships between all these parameters established in (1)–(9) are presented in the Hasse diagram shown in Fig. 3.

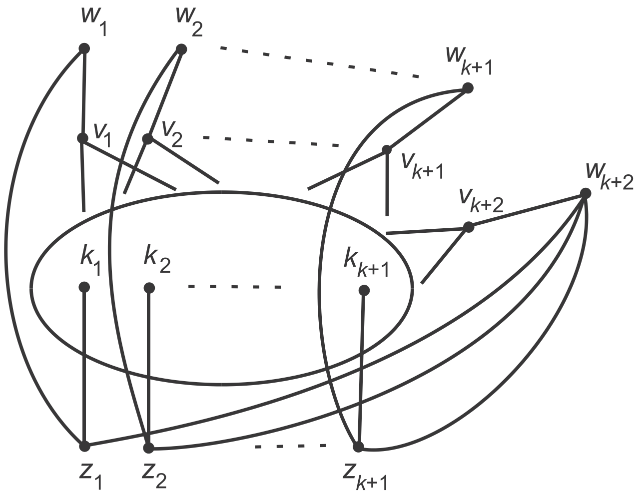

Fig. 2.

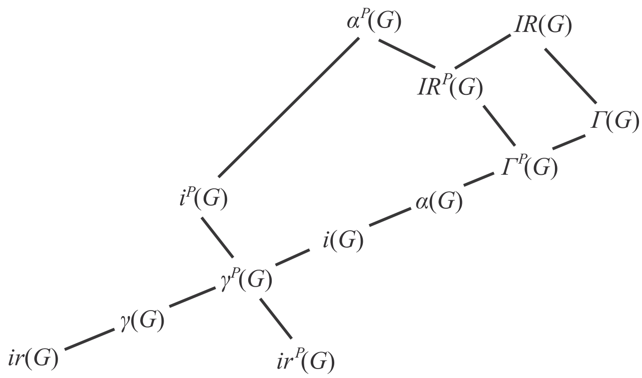

Fig. 3.

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