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Journal of Algebra 298 (2006) 120–133

JOURNAL OF Algebra

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Normality of orbit closures for directing modules over tame algebras

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Received 1 February 2005

Available online 9 August 2005

Communicated by Peter Littelmann

Abstract

We show that the orbit closures for directing modules over tame algebras are normal and Cohen–Macaulay. The proof is based on degenerations to normal toric varieties.

Keywords: Module variety; Directing module; Tame algebra

1. Introduction and the main results

Throughout the paper k denotes a fixed algebraically closed field. By an algebra we mean an associative k-algebra with identity, and by a module a finite-dimensional left module. Furthermore, for an algebra A, mod A stands for the category of finite-dimensional left A-modules. By $\mathbb{N}$ and $\mathbb{Z}$ we denote the sets of nonnegative integers and integers, respectively. Finally, if i and j are integers, then by $[i, j]$ we denote the set of all integers k such that $i \leqslant k \leqslant j$.

Let d be a positive integer and denote by $\mathbb{M}(d)$ the algebra of $(d \times d)$-matrices with coefficients in k. For an algebra A the set $\operatorname{mod}_A(d)$ of the A-module structures on the

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.06.023

vector space k^d has a natural structure of an affine variety. Indeed, if $A \simeq k\langle X_1, \dots, X_t\rangle/I$ for $t > 0$ and a two-sided ideal I, then $\operatorname{mod}_A(d)$ can be identified with the closed subset of $(\mathbb{M}(d))^t$ given by the vanishing of the entries of all matrices $\rho(X_1, \dots, X_t)$ for $\rho \in I$. Moreover, the general linear group $\mathrm{GL}(d)$ acts on $\operatorname{mod}_A(d)$ by conjugations and the $\mathrm{GL}(d)$-orbits in $\operatorname{mod}_A(d)$ correspond bijectively to the isomorphism classes of d-dimensional left A-modules. We shall denote by $\mathcal{O}_M$ the $\mathrm{GL}(d)$-orbit in $\operatorname{mod}_A(d)$ corresponding to (the isomorphism class of) a d-dimensional module M in $\operatorname{mod} A$. It is an interesting task to study geometric properties of the Zariski closure $\overline{\mathcal{O}}_M$ of $\mathcal{O}_M$.

The above problem can also be formulated in terms of representations of finite quivers instead of modules over algebras. Here, by a finite quiver Σ we mean a finite set Σ_0 of vertices and a finite set Σ_1 of arrows together with two maps $s, t : \Sigma_1 \to \Sigma_0$, which assign to an arrow its starting and terminating vertex, respectively. Let $\mathbf{d} = (d_x)_{x\in\Sigma_0} \in \mathbb{N}^{\Sigma_0}$ be a dimension vector and let $\mathbb{M}(m,n)$ denote the space of $(m \times n)$-matrices with coefficients in k. The affine space

$$\operatorname{rep}_\Sigma(\mathbf{d}) = \prod_{\alpha\in\Sigma_1} \mathbb{M}(d_{t\alpha}, d_{s\alpha})$$

is called a variety of representations of Σ. The product $\mathrm{GL}(\mathbf{d}) = \prod_{x\in\Sigma_0} \mathrm{GL}(d_x)$ of general linear groups acts on $\operatorname{rep}_\Sigma(\mathbf{d})$ by conjugations:

$$g \cdot V = \left(g_{t\alpha} V_\alpha g_{s\alpha}^{-1}\right)_{\alpha\in\Sigma_1}$$

for $g = (g_x)_{x\in\Sigma_0} \in \mathrm{GL}(\mathbf{d})$ and $V = (V_\alpha)_{\alpha\in\Sigma_1} \in \operatorname{rep}_\Sigma(\mathbf{d})$. The orbit of $V \in \operatorname{rep}_\Sigma(\mathbf{d})$ with respect to this action is denoted by $\mathcal{O}_V$, and its closure by $\overline{\mathcal{O}}_V$. In fact, the module varieties and varieties of representations of quivers are closely related to each other (see [7] for details). In particular, for any algebra A there is a uniquely determined quiver Σ (called the Gabriel quiver of A) such that for each $d \geqslant 1$ and $M \in \operatorname{mod}_A(d)$ there are a dimension vector $\mathbf{d} \in \mathbb{N}^{\Sigma_0}$ and $V \in \operatorname{rep}_\Sigma(\mathbf{d})$ such that $\overline{\mathcal{O}}_M$ is isomorphic to the associated fibre bundle $\mathrm{GL}(d) \times_{\mathrm{GL}(\mathbf{d})} \overline{\mathcal{O}}_V$. Hence $\overline{\mathcal{O}}_M$ is normal, Cohen–Macaulay, unibranch or regular in some codimension if and only if $\overline{\mathcal{O}}_V$ is.

The orbit closures are normal and Cohen–Macaulay varieties (with rational singularities in characteristic zero) provided Σ is a Dynkin quiver of type $\mathbb{A}_n$ or $\mathbb{D}_n$ [5,6], or A is a Brauer tree algebra [13]. Moreover, they are regular in codimension one if Σ is the Kronecker quiver [1], or A is a representation finite algebra [17], i.e., a set $\operatorname{ind} A$ of chosen representatives of isomorphism classes of indecomposable A-modules is finite. Another result states that the variety $\overline{\mathcal{O}}_M$ is unibranch if there are only finitely many modules U in $\operatorname{ind} A$ such that there is a monomorphism from U to M^i for some $i > 0$ [15]. On the other hand, there exists an orbit closure in $\operatorname{rep}_\Sigma((3,3))$, where Σ is the Kronecker quiver, which is neither unibranch nor Cohen–Macaulay (see [16]).

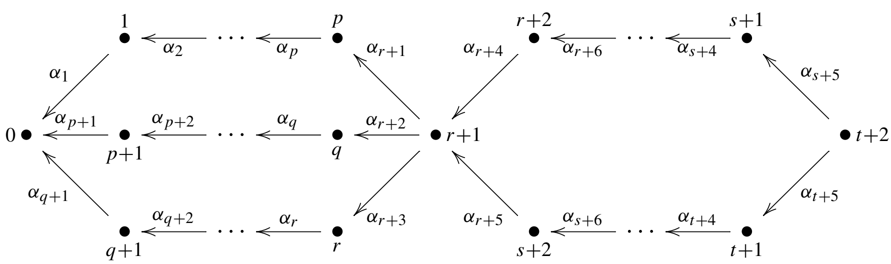
We say that an algebra A is tame if we can chose $\operatorname{ind} A$ in such a way that for every $d > 0$ all d-dimensional modules in $\operatorname{ind} A$ can be described by finitely many one-parameter families. According to Drozd's Tame and Wild Theorem ([11], see also [10]) there is a chance to classify modules only for tame algebras. An indecomposable module M in $\operatorname{mod} A$ is called directing if there exists no sequence

$$M = M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{m-1} \xrightarrow{f_m} M_m = M$$

in $\operatorname{mod} A$, where $m > 0$, $M_1, \ldots, M_{m-1}$ belong to $\operatorname{ind} A$ and $f_1, \ldots, f_m$ are nonzero non-isomorphisms. Bongartz investigated from the geometric point of view a special class of directing modules, so called preprojective ones (see [8, Proposition 6]). Further results in this direction were obtained by Skowroński and the first named author in [3] (see also [2] for the case of decomposable directing modules). The following main theorem of the paper completes the results of [3] to the general case.

Theorem 1.1. *Let M be an indecomposable directing module over a tame algebra. Then the variety $\overline{\mathcal{O}}_M$ is normal and Cohen–Macaulay.*

Using [3, Theorem 2] (see [4, Proposition 2.4] for the correct list of algebras) and the geometric equivalence described in [7] we get that $\overline{\mathcal{O}}_M$ is isomorphic to the associated fibre bundle $\mathrm{GL}(d) \times_{\mathrm{GL}(\mathbf{d})} \overline{\mathcal{O}}_P$, where either $\overline{\mathcal{O}}_P$ is a normal complete intersection, or up to duality, P is defined as follows. Let $0 \leqslant p \leqslant q \leqslant r \leqslant s \leqslant t$, let Δ be the quiver



(if some of the inequalities between 0, p, q, r, s and t are equalities, then we obtain the obvious degenerated version of the above quiver; see also a more detailed discussion about the definition of the quiver $Q(p, q, r, s, t)$ after Proposition 2.3 in Section 2) and let $\mathbf{d}$ be the dimension vector in $\mathbb{N}^{\Delta_0}$, whose $(r + 1)$th coordinate equals 2 and the remaining coordinates are 1. Then $P = P(p, q, r, s, t)$ is the point $(P_\alpha)_{\alpha \in \Delta_1} \in \operatorname{rep}_\Delta(\mathbf{d})$ such that

$$P_{\alpha_{r+1}} = [1 \quad 0], \qquad P_{\alpha_{r+2}} = [-1 \quad -1], \qquad P_{\alpha_{r+3}} = [0 \quad 1],$$
$$P_{\alpha_{r+4}} = [0 \quad 1]^{\mathrm{tr}}, \qquad P_{\alpha_{r+5}} = [1 \quad 0]^{\mathrm{tr}},$$

and the remaining matrices P_α are equal to [1]. Hence Theorem 1.1 is a consequence of the following result.

Theorem 1.2. *Let $P = P(p, q, r, s, t)$ for some integers $0 \leqslant p \leqslant q \leqslant r \leqslant s \leqslant t$. Then the variety $\overline{\mathcal{O}}_P$ is normal, Cohen–Macaulay, and has rational singularities in characteristic zero.*

The idea of the proof is to degenerate such varieties to toric normal varieties using the so-called Sagbi-bases (see [9,12]). These normal toric varieties appear in the following theorem.

Theorem 1.3. *Let Q be a finite quiver without oriented cycles, let* $\mathbf{d}$ *be the dimension vector in* $\mathbb{N}^{Q_0}$ *with the coordinates equal to* 1 *and let V be the point of* $\operatorname{rep}_Q(\mathbf{d})$ *given by the matrices equal to* [1]. *Then* $\overline{\mathcal{O}}_V$ *is a normal toric variety.*

The paper is organized as follows. In Section 2 we prove Theorem 1.3 and investigate the equations defining the toric varieties described in the theorem. Section 3 is devoted to the proof of Theorem 1.2.

2. Toric varieties

Let Q be a finite quiver without oriented cycles and let $\mathbf{d} = (d_i)_{i\in Q_0}$ be the dimension vector in $\mathbb{N}^{Q_0}$ with all d_i equal to 1. Then the algebraic group $\mathrm{GL}(\mathbf{d}) = \prod_{i\in Q_0} k^*$ is a torus and the orbit closures in $\operatorname{rep}_Q(\mathbf{d})$ are affine toric varieties (here we do not assume that toric varieties are normal). In particular, this holds for the orbit closure $\overline{\mathcal{O}}_V$, where $V = (V_\alpha)_{\alpha\in Q_1}$ is the point of $\operatorname{rep}_Q(\mathbf{d})$ with $V_\alpha = [1]$ for any arrow $\alpha \in Q_1$. Let $\mathbf{e}_\alpha = \mathbf{e}_{t\alpha} - \mathbf{e}_{s\alpha}$ for $\alpha \in Q_1$, where $(\mathbf{e}_i)_{i\in Q_0}$ is the standard basis of $\mathbb{Z}^{Q_0}$. It follows from the definition of the action of $\mathrm{GL}(\mathbf{d})$ on $\operatorname{rep}_Q(\mathbf{d})$ that $\overline{\mathcal{O}}_V$ corresponds to the semigroup

$$\mathcal{C}_Q = \sum_{\alpha\in Q_1} \mathbb{N}\cdot \mathbf{e}_\alpha \subset \mathbb{Z}^{Q_0},$$

which means that the algebra $k[\overline{\mathcal{O}}_V]$ of regular functions on $\overline{\mathcal{O}}_V$ may be identified with the subalgebra of $k[T_i, T_i^{-1}]_{i\in Q_0}$ generated by $T^{\mathbf{e}_\alpha}$, $\alpha \in Q_1$, where for $\mathbf{x} = (x_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}$ we put $T^{\mathbf{x}} = \prod_{i\in Q_0} T_i^{x_i}$. According to this identification, $k[\overline{\mathcal{O}}_V]$ as a vector space has a basis formed by $T^{\mathbf{x}}$, $\mathbf{x} \in \mathcal{C}_Q$. It is well known that an affine toric variety is normal if and only if the corresponding semigroup $\mathcal{C}$ is saturated, i.e., if a lattice point $\mathbf{x}$ belongs to the subgroup of $\mathbb{Z}^n$ generated by $\mathcal{C}$ and $\lambda\mathbf{x} \in \mathcal{C}$ for some $\lambda \in \mathbb{N} \setminus \{0\}$, then $\mathbf{x} \in \mathcal{C}$. It is known that $\mathcal{C}_Q$ is a saturated semigroup (see [14, Example 3.7]), but for completeness we include a short proof below.

For a vector $\mathbf{x} = (x_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}$ and a subset F of Q_0 we abbreviate by $\mathbf{x}_F$ the sum $\sum_{i\in F} x_i$. A subset F of Q_0 is called a filter in Q if

$$s\alpha \in F \quad\Rightarrow\quad t\alpha \in F$$

for any arrow $\alpha \in Q_1$. Let X_Q be the subset of all $\mathbf{x} \in \mathbb{Z}^{Q_0}$ such that $\mathbf{x}_{Q_0} = 0$ and $\mathbf{x}_F \geqslant 0$ for any filter F in Q. Obviously X_Q is a saturated semigroup. Hence Theorem 1.3 is a consequence of the following fact.

Proposition 2.1. $\mathcal{C}_Q = X_Q$.

Proof. Obviously $\mathcal{C}_Q \subseteq X_Q$. Let $\mathbf{x} = (x_i)_{i \in Q_0} \in X_Q$. In order to prove that $\mathbf{x} \in \mathcal{C}_Q$ we proceed by a double induction, first: on the cardinality of Q_0, and second: on the integer $\sum_{F \in \mathcal{F}} \mathbf{x}_F \geqslant 0$, where $\mathcal{F}$ is the set of all filters in Q.

Assume first that there is no arrow in Q_1 (for example, this holds if Q_0 has only one element). Then for any $i \in Q_0$, $\{i\}$ is a filter in Q and thus $x_i \geqslant 0$. On the other hand, $\sum_{i \in Q_0} x_i = 0$, which gives $\mathbf{x} = 0 \in \mathcal{C}_Q$.

Assume now that there is a proper nonempty filter F in Q such that $\mathbf{x}_F = 0$. Let Q' and Q'' be the full subquivers of Q such that $Q'_0 = F$ and $Q''_0 = Q_0 \setminus F$. Then $\mathbf{x} = \mathbf{x}' + \mathbf{x}''$ according to the canonical isomorphism

$$\mathbb{Z}^{Q_0} \simeq \mathbb{Z}^{Q'_0} \oplus \mathbb{Z}^{Q''_0}.$$

Observe that $\mathbf{x}' \in X_{Q'}$ and $\mathbf{x}'' \in X_{Q''}$. By the inductive assumption, $\mathbf{x}' \in \mathcal{C}_{Q'}$ and $\mathbf{x}'' \in \mathcal{C}_{Q''}$. Consequently, $\mathbf{x} \in \mathcal{C}_{Q'} \oplus \mathcal{C}_{Q''} \subseteq \mathcal{C}_Q$.

Hence we may assume that Q_1 is nonempty and that $\mathbf{x}_F > 0$ for any nonempty proper filter F in Q. Choose $\alpha \in Q_1$ and let $\mathbf{y} = \mathbf{x} - \mathbf{e}_\alpha$. Obviously $\mathbf{y}_{Q_0} = 0$. Since there are no oriented cycles in Q, there is a filter F in Q with $t\alpha \in F$ and $s\alpha \notin F$. For any such filter $\mathbf{y}_F = \mathbf{x}_F - 1 \geqslant 0$, while for the remaining ones $\mathbf{y}_F = \mathbf{x}_F \geqslant 0$. Hence $\mathbf{y} \in X_Q$ and $\sum_{F \in \mathcal{F}} \mathbf{y}_F < \sum_{F \in \mathcal{F}} \mathbf{x}_F$. By our inductive assumption $\mathbf{y} \in \mathcal{C}_Q$, which gives $\mathbf{x} = \mathbf{y} + \mathbf{e}_\alpha \in \mathcal{C}_Q$. □

Now we consider the problem of finding equations defining $\overline{\mathcal{O}}_V$. More precisely, we want to describe generators of the ideal $I_{\mathcal{C}_Q}$, which is the kernel of the algebra homomorphism

$$k[S_\alpha]_{\alpha \in Q_1} \to k\big[T_i, T_i^{-1}\big]_{i \in Q_0}, \quad S_\alpha \mapsto T^{\mathbf{e}_\alpha}.$$

For $\mathbf{w} = (w_\alpha)_{\alpha \in Q_1} \in \mathbb{Z}^{Q_1}$ we define $\mathbf{w}^+ = (w_\alpha^+)_{\alpha \in Q_1}$, $\mathbf{w}^- = (w_\alpha^-)_{\alpha \in Q_1} \in \mathbb{Z}^{Q_1}$ by

$$w_\alpha^+ = \max\{w_\alpha, 0\} \quad \text{and} \quad w_\alpha^- = \max\{-w_\alpha, 0\} \quad \text{for } \alpha \in Q_1.$$

Let $\mathcal{U} : \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}$ be the group homomorphism such that $\mathcal{U}(\mathbf{f}_\alpha) = \mathbf{e}_\alpha$ for $\alpha \in Q_1$, where $(\mathbf{f}_\alpha)_{\alpha \in Q_1}$ is the standard basis of $\mathbb{Z}^{Q_1}$. Then $I_{\mathcal{C}_Q}$ is generated by the binomials

$$S^{\mathbf{w}^+} - S^{\mathbf{w}^-} \quad \text{with } \mathbf{w} \in \operatorname{Ker}(\mathcal{U}),$$

where

$$S^{\mathbf{w}} = \prod_{i \in Q_1} S_\alpha^{w_\alpha} \quad \text{for } \mathbf{w} = (w_\alpha)_{\alpha \in Q_1} \in \mathbb{N}^{Q_1}$$

(see [14, Lemma 1.1]). Note that $\operatorname{Ker}(\mathcal{U})$ consists of the vectors $\mathbf{w} = (w_\alpha)_{\alpha \in Q_1} \in \mathbb{Z}^{Q_1}$ such that

$$\sum_{s\alpha = i} w_\alpha = \sum_{t\alpha = i} w_\alpha \quad \text{for all } i \in Q_0. \tag{1}$$

In the case of toric varieties occurring in Theorem 1.3 we shall indicate a special finite subsets of $\mathrm{Ker}(\mathcal{U})$ for which the corresponding binomials generate the ideal $I_{\mathcal{C}_Q}$.

Let Q^* be the double quiver of Q, i.e., the quiver with the same set of vertices as Q and the set of arrows $Q_1 \cup Q_1^-$, where $Q_1^- = \{\alpha^- \mid \alpha \in Q_1\}$ is the set of the formal inverses α^- of arrows α in Q with $s\alpha^- = t\alpha$ and $t\alpha^- = s\alpha$. By a nonoriented path in Q we mean an oriented path in Q^* which does not contain neither $\alpha\alpha^-$ nor $\alpha^-\alpha$ for $\alpha \in Q_1$ as a subpath. By a nonoriented cycle in Q we mean a nontrivial nonoriented path in Q which starts and terminates at the same vertex. A nonoriented cycle is called primitive if it does not contain a proper subpath which is a nonoriented cycle.

With a primitive nonoriented cycle $\beta_1 \cdots \beta_l$ in Q we may associate a vector $\mathbf{u} = (u_\alpha)_{\alpha \in Q_1} \in \mathbb{Z}^{Q_1}$ in the following way:

$$u_\alpha = \begin{cases} 1, & \alpha = \beta_i \text{ for some } i \in [1, l], \\ -1, & \alpha^- = \beta_i \text{ for some } i \in [1, l], \\ 0, & \text{otherwise}, \end{cases} \qquad \alpha \in Q_1.$$

Note that $\mathbf{u} \in \mathrm{Ker}(\mathcal{U})$. Let $\mathcal{Z}$ be the set of all vectors obtained from primitive nonoriented cycles in Q in the way described above. Observe that $\mathcal{Z} = -\mathcal{Z}$, which means that $-\mathbf{u} \in \mathcal{Z}$ for any $\mathbf{u} \in \mathcal{Z}$. Thus we can choose a subset $\mathcal{Z}'$ of $\mathcal{Z}$ such that $\mathcal{Z} = \mathcal{Z}' \cup (-\mathcal{Z}')$ and $\mathcal{Z}' \cap (-\mathcal{Z}') = \emptyset$. Note that the elements of $\mathcal{Z}'$ correspond bijectively to the equivalence classes of primitive nonoriented cycles in Q under the relation which identify a cycle with all its rotations and all rotations of its inversion (since these notions seem to be self-explained we will not give precise definitions here). Our next aim is to show that the binomials corresponding to the elements of $\mathcal{Z}'$ (hence to the equivalence classes of primitive nonoriented cycles in Q) generate $\mathrm{Ker}(\mathcal{U})$. We start with the following auxiliary observation.

Lemma 2.2. *If* $\mathbf{w} \in \mathrm{Ker}(\mathcal{U})$ *is nonzero, then there exists* $\mathbf{u} \in \mathcal{Z}$ *such that* $\mathbf{u}^+ \leqslant \mathbf{w}^+$ *and* $\mathbf{u}^- \leqslant \mathbf{w}^-$.

Proof. Let $\mathbf{w} = (w_\alpha)_{\alpha \in Q_1}$ be a nonzero element of $\mathrm{Ker}(\mathcal{U})$. We construct inductively an infinite nonoriented path $\omega = \beta_1\beta_2\beta_3\cdots$ in Q, such that for each $j \geqslant 1$ either $\beta_j = \alpha$ for an arrow $\alpha \in Q_1$ with $w_\alpha > 0$, or $\beta_j = \alpha^-$ for an arrow $\alpha \in Q_1$ with $w_\alpha < 0$. We take an arbitrary arrow $\alpha \in Q_1$ with $w_\alpha \neq 0$ in order to define β_1. Assume now that β_n is defined. If $\beta_n = \alpha$ for $\alpha \in Q_1$, then it follows from the equality (1) for $i = t\alpha_n$ that there is an arrow $\alpha' \neq \alpha$ such that either $s\alpha' = t\alpha$ and $w_{\alpha'} > 0$, or $t\alpha' = t\alpha$ and $w_{\alpha'} < 0$. In the former case we put $\beta_{n+1} = \alpha'$, and in the latter $\beta_{n+1} = \alpha'^-$. If $\beta_n = \alpha^-$ for $\alpha \in Q_1$, then we consider the equality (1) for $i = s\alpha$ and we define β_{n+1} in a similar way as above. Since the quiver Q is finite, there exists a primitive nonoriented cycle which is a subpath of ω. The vector corresponding to this cycle satisfies the claim. $\square$

Now we can prove the announced result.

Proposition 2.3. *Let Q be a finite quiver without oriented cycles and assume the above notation. Then the ideal $I_{\mathcal{C}_Q}$ is generated by the binomials*

$$S^{\mathbf{u}^+} - S^{\mathbf{u}^-}, \quad \mathbf{u} \in \mathcal{Z}'.$$

Proof. Since

$$S^{\mathbf{v}^+} - S^{\mathbf{v}^-} = -\left(S^{\mathbf{u}^+} - S^{\mathbf{u}^-}\right)$$

if $\mathbf{v} = -\mathbf{u}$ and $\mathbf{u} \in \mathbb{Z}^{Q_1}$, it suffices to prove that if $\mathbf{w} = (w_\alpha)_{\alpha \in Q_1}$ belongs to $\operatorname{Ker}(\mathcal{U})$, then $S^{\mathbf{w}^+} - S^{\mathbf{w}^-}$ belongs to the ideal generated by the binomials

$$S^{\mathbf{u}^+} - S^{\mathbf{u}^-}, \quad \mathbf{u} \in \mathcal{Z}.$$

We proceed by induction on $|\mathbf{w}| = \sum_{\alpha \in Q_1} |w_\alpha| \geqslant 0$. If $|\mathbf{w}| = 0$, then $\mathbf{w} = 0$ and we are done. Otherwise by the previous lemma, there is a vector $\mathbf{u} \in \mathcal{Z}$ such that $\mathbf{u}^+ \leqslant \mathbf{w}^+$ and $\mathbf{u}^- \leqslant \mathbf{w}^-$. Then

$$\mathbf{w}^+ = \mathbf{u}^+ + \mathbf{v}^+ \quad \text{and} \quad \mathbf{w}^- = \mathbf{u}^- + \mathbf{v}^- \quad \text{for } \mathbf{v} = \mathbf{w} - \mathbf{u}.$$

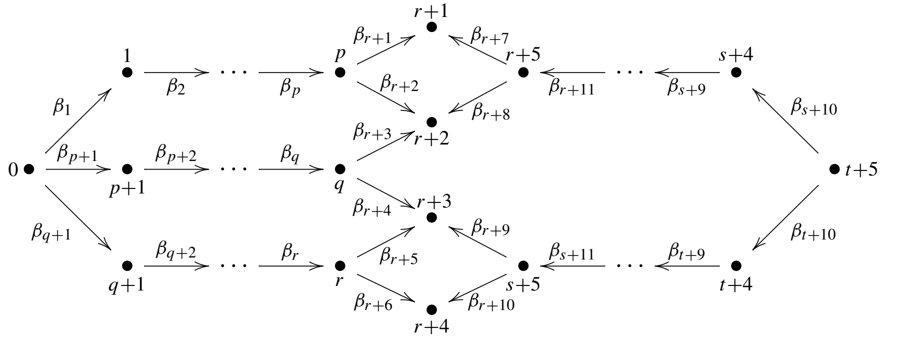
Moreover, $\mathbf{v} \in \operatorname{Ker}(\mathcal{U})$ and $|\mathbf{v}| = |\mathbf{w}| - |\mathbf{u}| < |\mathbf{w}|$. Since

$$S^{\mathbf{w}^+} - S^{\mathbf{w}^-} = S^{\mathbf{v}^+}\left(S^{\mathbf{u}^+} - S^{\mathbf{u}^-}\right) + S^{\mathbf{u}^-}\left(S^{\mathbf{v}^+} - S^{\mathbf{v}^-}\right),$$

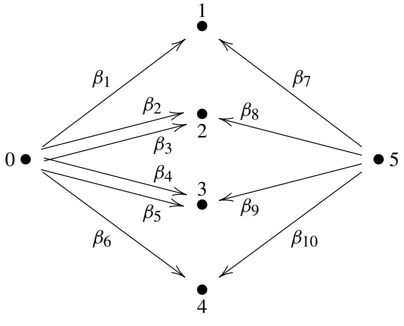
the claim follows by the inductive assumption. □

The above proposition gives us a finite set of generators of $I_{\mathcal{C}_Q}$. As we shall see below, this set usually is not minimal.

We restrict now our findings to a quiver Q of a special form. Let $0 \leqslant p \leqslant q \leqslant r \leqslant s \leqslant t$. We define a quiver $Q = Q(p, q, r, s, t)$ in the following way. If $0 < p < q < r < s < t$, then Q is the quiver



If $0 = p$ ($p = q$, $q = r$, $r = s$ or $s = t$, respectively) then we cancel appropriate arrows and identify vertices 0 and p (0 and q, 0 and r, $r + 5$ and $t + 5$, or $s + 5$ and $t + 5$, respectively). Thus in the most extremal case $0 = p = q = r = s = t$ we get the quiver



with 6 vertices and 10 arrows.

Recall that $\mathbf{f}_{\beta_1}, \ldots, \mathbf{f}_{\beta_{t+10}}$ is the standard basis of $\mathbb{Z}^{Q_1}$. Let

$$\mathbf{u}_i = \mathbf{f}_{\beta_{r+i}} \quad \text{for } i \in [1, 10] \quad \text{and}$$
$$\mathbf{u}_{11} = \mathbf{f}_{[1,p]}, \qquad \mathbf{u}_{12} = \mathbf{f}_{[p+1,q]}, \qquad \mathbf{u}_{13} = \mathbf{f}_{[q+1,r]},$$
$$\mathbf{u}_{14} = \mathbf{f}_{[r+11,s+10]}, \qquad \mathbf{u}_{15} = \mathbf{f}_{[s+11,t+10]},$$

where $\mathbf{f}_{[i,j]} = \sum_{l \in [i,j]} \mathbf{f}_{\beta_l}$ for $i, j \in [1, t + 10]$. Observe that it may happen that $\mathbf{u}_i = 0$ for some $i \in [11, 15]$. With the above notation $\mathcal{Z}'$ consists, up to sign, of the following vectors:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_2 + \mathbf{u}_{11} - \mathbf{u}_3 - \mathbf{u}_{12}, \\
\mathbf{v}_2 &= \mathbf{u}_4 + \mathbf{u}_{12} - \mathbf{u}_5 - \mathbf{u}_{13}, \\
\mathbf{v}_3 &= \mathbf{u}_1 + \mathbf{u}_8 - \mathbf{u}_2 - \mathbf{u}_7, \\
\mathbf{v}_4 &= \mathbf{u}_5 + \mathbf{u}_{10} - \mathbf{u}_6 - \mathbf{u}_9, \\
\mathbf{v}_5 &= \mathbf{u}_3 + \mathbf{u}_9 + \mathbf{u}_{15} - \mathbf{u}_4 - \mathbf{u}_8 - \mathbf{u}_{14}, \\
\mathbf{v}_6 &= \mathbf{u}_1 + \mathbf{u}_9 + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_4 - \mathbf{u}_7 - \mathbf{u}_{12} - \mathbf{u}_{14}, \\
\mathbf{v}_7 &= \mathbf{u}_3 + \mathbf{u}_{10} + \mathbf{u}_{12} + \mathbf{u}_{15} - \mathbf{u}_6 - \mathbf{u}_8 - \mathbf{u}_{13} - \mathbf{u}_{14}, \\
\mathbf{v}_8 &= \mathbf{u}_1 + \mathbf{u}_{10} + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_6 - \mathbf{u}_7 - \mathbf{u}_{13} - \mathbf{u}_{14}, \\
\mathbf{v}_9 &= \mathbf{u}_1 + \mathbf{u}_8 + \mathbf{u}_{11} - \mathbf{u}_3 - \mathbf{u}_7 - \mathbf{u}_{12}, \\
\mathbf{v}_{10} &= \mathbf{u}_4 + \mathbf{u}_{10} + \mathbf{u}_{12} - \mathbf{u}_6 - \mathbf{u}_9 - \mathbf{u}_{13}, \\
\mathbf{v}_{11} &= \mathbf{u}_2 + \mathbf{u}_4 + \mathbf{u}_{11} - \mathbf{u}_3 - \mathbf{u}_5 - \mathbf{u}_{13}, \\
\mathbf{v}_{12} &= \mathbf{u}_1 + \mathbf{u}_3 + \mathbf{u}_9 + \mathbf{u}_{15} - \mathbf{u}_2 - \mathbf{u}_4 - \mathbf{u}_7 - \mathbf{u}_{14}, \\
\mathbf{v}_{13} &= \mathbf{u}_3 + \mathbf{u}_5 + \mathbf{u}_{10} + \mathbf{u}_{15} - \mathbf{u}_4 - \mathbf{u}_6 - \mathbf{u}_8 - \mathbf{u}_{14},
\end{aligned}$$

$$\mathbf{v}_{14} = \mathbf{u}_2 + \mathbf{u}_9 + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_4 - \mathbf{u}_8 - \mathbf{u}_{12} - \mathbf{u}_{14},$$
$$\mathbf{v}_{15} = \mathbf{u}_3 + \mathbf{u}_9 + \mathbf{u}_{12} + \mathbf{u}_{15} - \mathbf{u}_5 - \mathbf{u}_8 - \mathbf{u}_{13} - \mathbf{u}_{14},$$
$$\mathbf{v}_{16} = \mathbf{u}_1 + \mathbf{u}_4 + \mathbf{u}_8 + \mathbf{u}_{11} - \mathbf{u}_3 - \mathbf{u}_5 - \mathbf{u}_7 - \mathbf{u}_{13},$$
$$\mathbf{v}_{17} = \mathbf{u}_2 + \mathbf{u}_4 + \mathbf{u}_{10} + \mathbf{u}_{11} - \mathbf{u}_3 - \mathbf{u}_6 - \mathbf{u}_9 - \mathbf{u}_{13},$$
$$\mathbf{v}_{18} = \mathbf{u}_1 + \mathbf{u}_3 + \mathbf{u}_9 + \mathbf{u}_{12} + \mathbf{u}_{15} - \mathbf{u}_2 - \mathbf{u}_5 - \mathbf{u}_7 - \mathbf{u}_{13} - \mathbf{u}_{14},$$
$$\mathbf{v}_{19} = \mathbf{u}_2 + \mathbf{u}_5 + \mathbf{u}_{10} + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_4 - \mathbf{u}_6 - \mathbf{u}_8 - \mathbf{u}_{12} - \mathbf{u}_{14},$$
$$\mathbf{v}_{20} = \mathbf{u}_1 + \mathbf{u}_3 + \mathbf{u}_5 + \mathbf{u}_{10} + \mathbf{u}_{15} - \mathbf{u}_2 - \mathbf{u}_4 - \mathbf{u}_6 - \mathbf{u}_7 - \mathbf{u}_{14},$$
$$\mathbf{v}_{21} = \mathbf{u}_2 + \mathbf{u}_9 + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_5 - \mathbf{u}_8 - \mathbf{u}_{13} - \mathbf{u}_{14},$$
$$\mathbf{v}_{22} = \mathbf{u}_1 + \mathbf{u}_4 + \mathbf{u}_8 + \mathbf{u}_{10} + \mathbf{u}_{11} - \mathbf{u}_3 - \mathbf{u}_6 - \mathbf{u}_7 - \mathbf{u}_9 - \mathbf{u}_{13},$$
$$\mathbf{v}_{23} = \mathbf{u}_1 + \mathbf{u}_9 + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_5 - \mathbf{u}_7 - \mathbf{u}_{13} - \mathbf{u}_{14},$$
$$\mathbf{v}_{24} = \mathbf{u}_2 + \mathbf{u}_{10} + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_6 - \mathbf{u}_8 - \mathbf{u}_{13} - \mathbf{u}_{14},$$
$$\mathbf{v}_{25} = \mathbf{u}_1 + \mathbf{u}_5 + \mathbf{u}_{10} + \mathbf{u}_{11} + \mathbf{u}_{15} - \mathbf{u}_4 - \mathbf{u}_6 - \mathbf{u}_7 - \mathbf{u}_{12} - \mathbf{u}_{14},$$
$$\mathbf{v}_{26} = \mathbf{u}_1 + \mathbf{u}_3 + \mathbf{u}_{10} + \mathbf{u}_{12} + \mathbf{u}_{15} - \mathbf{u}_2 - \mathbf{u}_6 - \mathbf{u}_7 - \mathbf{u}_{13} - \mathbf{u}_{14}.$$

Indeed, recall that the elements of $\mathcal{Z}'$ correspond to the equivalence classes of the primitive nonoriented cycles in Q. Note that each such equivalence class is determined by a nonempty subset of the set consisting of the five inner polygons visible on the picture of the quiver Q. There are $2^5 - 1 = 31$ such nonempty subsets, 26 of them lead to our vectors $\mathbf{v}_i$, $i \in [1, 26]$, and none of the remaining five subsets corresponds to the equivalence class of a primitive nonoriented cycle in Q (they may be seen as corresponding to equivalence classes of two disjoint primitive cycles).

Lemma 2.4. *Let $Q = Q(p,q,r,s,t)$ for $0 \leqslant p \leqslant q \leqslant r \leqslant s \leqslant t$. Then the ideal $I_{\mathcal{C}_Q}$ is generated by the binomials*

$$S^{\mathbf{v}_i^+} - S^{\mathbf{v}_i^-}, \quad i \in [1, 8].$$

Proof. By Proposition 2.3, it suffices to show that the above binomials generate the remaining binomials

$$S^{\mathbf{v}_i^+} - S^{\mathbf{v}_i^-}, \quad i \in [9, 26].$$

This is a quite easy, but tedious verification. Hence we prove the claim only for $i = 9$ and $i = 21$, leaving the other cases to the reader:

$$\begin{aligned} S^{\mathbf{v}_9^+} - S^{\mathbf{v}_9^-} &= S^{\mathbf{u}_1} S^{\mathbf{u}_8} S^{\mathbf{u}_{11}} - S^{\mathbf{u}_3} S^{\mathbf{u}_7} S^{\mathbf{u}_{12}} \\ &= S^{\mathbf{u}_{11}} \big(S^{\mathbf{u}_1} S^{\mathbf{u}_8} - S^{\mathbf{u}_2} S^{\mathbf{u}_7} \big) + S^{\mathbf{u}_7} \big(S^{\mathbf{u}_2} S^{\mathbf{u}_{11}} - S^{\mathbf{u}_3} S^{\mathbf{u}_{12}} \big) \\ &= S^{\mathbf{u}_{11}} \big(S^{\mathbf{v}_3^+} - S^{\mathbf{v}_3^-} \big) + S^{\mathbf{u}_7} \big(S^{\mathbf{v}_1^+} - S^{\mathbf{v}_1^-} \big), \end{aligned}$$

$$\begin{aligned} S^{\mathbf{v}_{21}^+} - S^{\mathbf{v}_{21}^-} &= S^{\mathbf{u}_2} S^{\mathbf{u}_9} S^{\mathbf{u}_{11}} S^{\mathbf{u}_{15}} - S^{\mathbf{u}_5} S^{\mathbf{u}_8} S^{\mathbf{u}_{13}} S^{\mathbf{u}_{14}} \\ &= S^{\mathbf{u}_9} S^{\mathbf{u}_{15}} \big(S^{\mathbf{u}_2} S^{\mathbf{u}_{11}} - S^{\mathbf{u}_3} S^{\mathbf{u}_{12}} \big) + S^{\mathbf{u}_{12}} \big(S^{\mathbf{u}_3} S^{\mathbf{u}_9} S^{\mathbf{u}_{15}} - S^{\mathbf{u}_4} S^{\mathbf{u}_8} S^{\mathbf{u}_{14}} \big) \\ &\quad + S^{\mathbf{u}_8} S^{\mathbf{u}_{14}} \big(S^{\mathbf{u}_4} S^{\mathbf{u}_{12}} - S^{\mathbf{u}_5} S^{\mathbf{u}_{13}} \big) \\ &= S^{\mathbf{u}_9} S^{\mathbf{u}_{15}} \big(S^{\mathbf{v}_1^+} - S^{\mathbf{v}_1^-} \big) + S^{\mathbf{u}_{12}} \big(S^{\mathbf{v}_5^+} - S^{\mathbf{v}_5^-} \big) \\ &\quad + S^{\mathbf{u}_8} S^{\mathbf{u}_{14}} \big(S^{\mathbf{v}_2^+} - S^{\mathbf{v}_2^-} \big). \qquad \square \end{aligned}$$

3. Degenerations to toric varieties

Let Δ, $\mathbf{d}$ and P be as in Theorem 1.2. As usual $\mathbf{e}_1, \ldots, \mathbf{e}_{t+5}$ denote the standard basis of $\mathbb{Z}^{t+5}$. For $i, j \in [1, t+5]$, $\mathbf{e}_{[i,j]} = \sum_{l \in [i,j]} \mathbf{e}_l$. If $\mathbf{x} = (x_i)_{i \in [1,t+5]} \in k^{t+5}$ and $\mathbf{w} = (w_i)_{i \in [1,t+5]} \in \mathbb{N}^{t+5}$, then $\mathbf{x}^{\mathbf{w}} = \prod_{i \in [1,t+5]} x_i^{w_i}$.

Our aim in this section is to prove Theorem 1.2. As the first step we describe the coordinate ring of $\overline{\mathcal{O}}_P$. Note that $\dim \overline{\mathcal{O}}_P = t + 5$. Indeed, $\dim \overline{\mathcal{O}}_P = \dim \mathrm{GL}(\mathbf{d}) - \dim \mathrm{Stab}_{\mathrm{GL}(\mathbf{d})}(P)$, where $\mathrm{Stab}_{\mathrm{GL}(\mathbf{d})}$ denotes the subgroup of all $g \in \mathrm{GL}(\mathbf{d})$ such that $g \cdot P = P$. Easy calculations show $\dim \mathrm{GL}(\mathbf{d}) = t + 6$ and $\mathrm{Stab}_{\mathrm{GL}(\mathbf{d})}(P) \simeq k^*$, thus the formula follows.

Let $\Phi : k^{t+5} \to \mathrm{rep}_\Delta(\mathbf{d})$ be given by

$$\begin{aligned} \Phi(\mathbf{x})_{\alpha_i} &= [x_i], \quad i \in [1, r] \cup [r+6, t+5], \\ \Phi(\mathbf{x})_{\alpha_{r+1}} &= \mathbf{x}^{\mathbf{e}_{[p+1,r]}} [\, x_{r+1} \quad x_{r+3} \,], \\ \Phi(\mathbf{x})_{\alpha_{r+2}} &= \mathbf{x}^{\mathbf{e}_{[1,p]}} \mathbf{x}^{\mathbf{e}_{[q+1,r]}} [\, -x_{r+1} - x_{r+4} \quad -x_{r+2} - x_{r+3} \,], \\ \Phi(\mathbf{x})_{\alpha_{r+3}} &= \mathbf{x}^{\mathbf{e}_{[1,q]}} [\, x_{r+4} \quad x_{r+2} \,], \\ \Phi(\mathbf{x})_{\alpha_{r+4}} &= [\, -x_{r+3} \quad x_{r+1} \,]^{\mathrm{tr}} x_{r+5} \mathbf{x}^{\mathbf{e}_{[s+6,t+5]}}, \\ \Phi(\mathbf{x})_{\alpha_{r+5}} &= [\, x_{r+2} \quad -x_{r+4} \,]^{\mathrm{tr}} x_{r+5} \mathbf{x}^{\mathbf{e}_{[r+6,s+5]}}, \end{aligned}$$

for $\mathbf{x} = (x_i)_{i \in [1,t+5]} \in k^{t+5}$. The next observation is the following.

Lemma 3.1. $\overline{\Phi(k^{t+5})} = \overline{\mathcal{O}}_P$.

Proof. Let

$$U = \big\{ \mathbf{x} = (x_i)_{i \in [1,t+5]} \in k^{t+5} \mid x_i \neq 0,\ i \in [1, r] \cup [r+5, t+5], \\ x_{r+1} x_{r+2} \neq x_{r+3} x_{r+4} \big\}.$$

Then U is an open subset of k^{t+5} and $\Phi|_U$ is injective, thus $\dim \overline{\Phi(k^{t+5})} = t + 5 = \dim \overline{\mathcal{O}}_P$. Since $\overline{\mathcal{O}}_P$ is irreducible, it is enough to show that $\Phi(U) \subset \mathcal{O}_P$. Let $\mathbf{x} = (x_i)_{i \in [1,t+5]} \in U$ and $X = \left[\begin{smallmatrix} x_{r+1} & x_{r+3} \\ x_{r+4} & x_{r+2} \end{smallmatrix}\right]$. Then $g = (g_i)_{i \in [1,t+2]}$ given by

$$\begin{aligned}
g_i &= \mathbf{x}^{\mathbf{e}[1,i]}, \quad i \in [0, p],\\
g_i &= \mathbf{x}^{\mathbf{e}[p+1,i]}, \quad i \in [p+1, q],\\
g_i &= \mathbf{x}^{\mathbf{e}[q+1,i]}, \quad i \in [q+1, r],\\
g_{r+1} &= \mathbf{x}^{\mathbf{e}[1,r]} X,\\
g_i &= \mathbf{x}^{\mathbf{e}[1,r]} \det X x_{r+5} \mathbf{x}^{\mathbf{e}[r+6,i+3]} \mathbf{x}^{\mathbf{e}[s+6,t+5]}, \quad i \in [r+2, s+1],\\
g_i &= \mathbf{x}^{\mathbf{e}[1,r]} \det X x_{r+5} \mathbf{x}^{\mathbf{e}[r+6,s+5]} \mathbf{x}^{\mathbf{e}[s+6,i+3]}, \quad i \in [s+2, t+2],
\end{aligned}$$

belongs to $\mathrm{GL}(\mathbf{d})$ and $g \cdot \Phi(\mathbf{x}) = P$.

Obviously, the above lemma implies that $k[\overline{\mathcal{O}}_P] = k[a_1, \ldots, a_{t+10}]$, where $a_1, \ldots, a_{t+10}$ are polynomials in $k[T_1, \ldots, T_{t+5}]$ defined by

$$\begin{aligned}
a_i &= T_i, \quad i \in [1, r],\\
a_{r+1} &= T^{\mathbf{e}[p+1,r]} T_{r+1},\\
a_{r+2} &= T^{\mathbf{e}[p+1,r]} T_{r+3},\\
a_{r+3} &= T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_{r+2} + T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_{r+3},\\
a_{r+4} &= T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_{r+1} + T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_{r+4},\\
a_{r+5} &= T^{\mathbf{e}[1,q]} T_{r+4},\\
a_{r+6} &= T^{\mathbf{e}[1,q]} T_{r+2},\\
a_{r+7} &= T_{r+1} T_{r+5} T^{\mathbf{e}[s+6,t+5]},\\
a_{r+8} &= T_{r+3} T_{r+5} T^{\mathbf{e}[s+6,t+5]},\\
a_{r+9} &= T_{r+4} T_{r+5} T^{\mathbf{e}[r+6,s+5]},\\
a_{r+10} &= T_{r+2} T_{r+5} T^{\mathbf{e}[r+6,s+5]},\\
a_i &= T_{i-5}, \quad i \in [r+11, t+10].
\end{aligned}$$

As before, $T^{\mathbf{w}} = \prod_{i\in[1,t+10]} T_i^{w_i}$ for $\mathbf{w} = (w_i)_{i\in[1,t+10]} \in \mathbb{N}^{t+10}$.

We order the elements of $\mathbb{N}^{t+5}$ by the reversed lexicographic order, i.e., we say that $\mathbf{u} = (u_i)_{i\in[1,t+5]}$ is smaller than $\mathbf{v} = (v_i)_{i\in[1,t+5]}$ if there exists $i \in [1, t+5]$ such that $u_i < v_i$ and $u_j = v_j$ for all $j \in [i+1, t+5]$. The induced order of the monomials in $k[T_1, \ldots, T_{t+5}]$ is a term order in the sense of [12, 1.3].

For $a = \sum_{\mathbf{v}\in\mathbb{N}^{t+5}} \lambda_{\mathbf{v}} T^{\mathbf{v}} \in k[T_1, \ldots, T_{t+5}]$, $a \neq 0$, we define the initial monomial $\mathrm{in}(a)$ as $T^{\mathbf{u}}$, where $\mathbf{u} = \max\{\mathbf{v} \in \mathbb{N}^{t+5} \mid \lambda_{\mathbf{v}} \neq 0\}$. If A is a subalgebra of $k[T_1, \ldots, T_{t+5}]$, then by the initial algebra $\mathrm{in}(A)$ of A we mean the subalgebra of A generated by $\{\mathrm{in}(a) \mid a \in A\}$. According to [9, Corollary 2.3(b)] in order to prove Theorem 1.2 it is enough to show that $\mathrm{in}(k[a_1, \ldots, a_{t+10}])$ is finitely generated and normal. Using Theorem 1.3 it will follow if we show isomorphisms $\mathrm{in}(k[a_1, \ldots, a_{t+10}]) \simeq k[\mathrm{in}(a_1), \ldots, \mathrm{in}(a_{t+10})] \simeq k[\overline{\mathcal{O}}_V]$, where V is the point of $\mathrm{rep}_Q((1)_{i\in[1,t+5]})$ with all matrices equal to $[1]$. Here $Q = Q(p, q, r, s, t)$ is the quiver defined in Section 2.

We first show the latter isomorphism, or in other words, we describe $k[\overline{\mathcal{O}}_V]$. The method is analogous to the one applied above in order to describe $k[\overline{\mathcal{O}}_P]$. Let $\Psi : k^{t+5} \to \operatorname{rep}_Q((1)_{i\in[1,t+5]})$ be defined by

$$\begin{aligned}
\Phi(\mathbf{x})_{\beta_i} &= x_i, \quad i \in [1,r],\\
\Phi(\mathbf{x})_{\beta_{r+1}} &= \mathbf{x}^{\mathbf{e}[p+1,r]} x_{r+1},\\
\Phi(\mathbf{x})_{\beta_{r+2}} &= \mathbf{x}^{\mathbf{e}[p+1,r]} x_{r+3},\\
\Phi(\mathbf{x})_{\beta_i} &= \mathbf{x}^{\mathbf{e}[1,p]} \mathbf{x}^{\mathbf{e}[q+1,r]} x_i, \quad i \in [r+3, r+4],\\
\Phi(\mathbf{x})_{\beta_{r+5}} &= \mathbf{x}^{\mathbf{e}[1,q]} x_{r+4},\\
\Phi(\mathbf{x})_{\beta_{r+6}} &= \mathbf{x}^{\mathbf{e}[1,q]} x_{r+2},\\
\Phi(\mathbf{x})_{\beta_{r+7}} &= x_{r+1} x_{r+5} \mathbf{x}^{\mathbf{e}[s+6,t+5]},\\
\Phi(\mathbf{x})_{\beta_{r+8}} &= x_{r+3} x_{r+5} \mathbf{x}^{\mathbf{e}[s+6,t+5]},\\
\Phi(\mathbf{x})_{\beta_{r+9}} &= x_{r+4} x_{r+5} \mathbf{x}^{\mathbf{e}[r+6,s+5]},\\
\Phi(\mathbf{x})_{\beta_{r+10}} &= x_{r+2} x_{r+5} \mathbf{x}^{\mathbf{e}[r+6,s+5]},\\
\Phi(\mathbf{x})_{\beta_i} &= x_{i-5}, \quad i \in [r+11, t+10],
\end{aligned}$$

for $\mathbf{x} = (x_i)_{i\in[1,t+5]} \in k^{t+5}$. With arguments similar to those used in the proof of Lemma 3.1, one shows that

$$\overline{\Phi\big(k^{t+5}\big)} = \overline{\mathcal{O}}_V,$$

hence $k[\overline{\mathcal{O}}_V]$ may be identified with the subalgebra of $k[T_1, \dots, T_{t+5}]$ generated by polynomials $b_1, \dots, b_{t+10}$, where

$$\begin{aligned}
b_i &= T_i, \quad i \in [1,r],\\
b_{r+1} &= T^{\mathbf{e}[p+1,r]} T_{r+1},\\
b_{r+2} &= T^{\mathbf{e}[p+1,r]} T_{r+3},\\
b_i &= T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_i, \quad i \in [r+3, r+4],\\
b_{r+5} &= T^{\mathbf{e}[1,q]} T_{r+4},\\
b_{r+6} &= T^{\mathbf{e}[1,q]} T_{r+2},\\
b_{r+7} &= T_{r+1} T_{r+5} T^{\mathbf{e}[s+6,t+5]},\\
b_{r+8} &= T_{r+3} T_{r+5} T^{\mathbf{e}[s+6,t+5]},\\
b_{r+9} &= T_{r+4} T_{r+5} T^{\mathbf{e}[r+6,s+5]},\\
b_{r+10} &= T_{r+2} T_{r+5} T^{\mathbf{e}[r+6,s+5]},\\
b_i &= T_{i-5}, \quad i \in [r+11, t+10].
\end{aligned}$$

It is an obvious observation that $b_i = \mathrm{in}(a_i)$ for all $i \in [1, t+10]$, which shows that $k[\mathrm{in}(a_1), \dots, \mathrm{in}(a_{t+10})] \simeq k[\overline{\mathcal{O}}_V]$.

Observe that the kernel I of the algebra homomorphism

$$k[S_{\beta_1}, \dots, S_{\beta_{t+10}}] \to k[T_1, \dots, T_{t+5}], \quad S_{\beta_i} \mapsto b_i,$$

equals the ideal I_{C_Q} defined in Section 2, as both of them are the ideals of $\overline{\mathcal{O}}_V$ in $\mathrm{rep}_Q((1)_{i\in[1,t+5]})$. By Lemma 2.4, I is generated by the binomials

$$\xi_i = S^{\mathbf{v}_i^+} - S^{\mathbf{v}_i^-}, \quad i \in [1, 8],$$

where $\mathbf{v}_1, \dots, \mathbf{v}_8$ are as in Section 2.

As the final step we show that $\mathrm{in}(k[a_1, \dots, a_{t+10}]) \simeq k[b_1, \dots, b_{t+10}]$ (if this condition holds, then one says that $a = (a_1, \dots, a_{t+10})$ is a Sagbi basis of the algebra $k[a_1, \dots, a_{t+10}]$). According to [9, Proposition 1.1] it is enough to show that there exist $\lambda_{i,\mathbf{u}} \in k$, $i \in [1, 8]$, $\mathbf{u} \in I_i = \{\mathbf{v} \in \mathbb{N}^{t+10} \mid \mathrm{in}(a^{\mathbf{v}}) \leqslant \mathrm{in}(\xi_i(a))\}$, such that

$$\xi_i(a) = \sum_{\mathbf{u}\in I_i} \lambda_{i,\mathbf{u}} a^{\mathbf{u}}.$$

Here, $a^{\mathbf{u}} = a_1^{u_{\beta_1}} \cdots a_{t+10}^{u_{\beta_{t+10}}}$ for $\mathbf{u} = (u_{\beta_i})_{i\in[1,t+10]} \in \mathbb{N}^{Q_1}$ and $\xi(a)$ denotes the image of $\xi \in k[S_{\beta_1}, \dots, S_{\beta_{t+10}}]$ via the map

$$k[S_{\beta_1}, \dots, S_{\beta_{t+10}}] \to k[T_1, \dots, T_{t+5}], \quad S_{\beta_i} \mapsto a_i.$$

But

$$\begin{aligned}
\xi_i(a) &= 0, \quad i \in \{3, 4, 8\},\\
\xi_1(a) &= -T^{\mathbf{e}[1,r]} T_{r+2} = -a^{\mathbf{e}[q+1,r]} a_{r+6},\\
\xi_2(a) &= T^{\mathbf{e}[1,r]} T_{r+1} = a^{\mathbf{e}[1,p]} a_{r+1},\\
\xi_6(a) &= -T^{\mathbf{e}[1,r]} T_{r+1} T_{r+1} T_{r+5} T^{\mathbf{e}[r+6,t+5]}\\
&= -a^{\mathbf{e}[1,p]} a_{r+1} a_{r+7} a^{\mathbf{e}[r+11,s+10]},\\
\xi_7(a) &= T^{\mathbf{e}[1,r]} T_{r+2} T_{r+2} T_{r+5} T^{\mathbf{e}[r+6,t+5]}\\
&= a^{\mathbf{e}[q+1,r]} a_{r+6} a_{r+10} a^{\mathbf{e}[s+11,t+10]},\\
\xi_5(a) &= T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_{r+2} T_{r+4} T_{r+5} T^{\mathbf{e}[r+6,t+5]}\\
&\quad - T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_{r+1} T_{r+3} T_{r+5} T^{\mathbf{e}[r+6,t+5]}\\
&= a_{r+4} a_{r+10} a^{\mathbf{e}[s+11,t+10]} - a_{r+3} a_{r+7} a^{\mathbf{e}[r+11,s+10]},
\end{aligned}$$

and the initial monomial

$$\mathrm{in}\big(a_{r+3} a_{r+7} a^{\mathbf{e}[r+11,s+10]}\big) = T^{\mathbf{e}[1,p]} T^{\mathbf{e}[q+1,r]} T_{r+1} T_{r+3} T_{r+5} T^{\mathbf{e}[r+6,t+5]}$$

is smaller than

$$\mathrm{in}\big(a_{r+4}a_{r+10}a^{\mathbf{e}[s+11,t+10]}\big) = \mathrm{in}\big(\xi_5(a)\big) = T^{\mathbf{e}[1,p]}T^{\mathbf{e}[q+1,r]}T_{r+2}T_{r+4}T_{r+5}T^{\mathbf{e}[r+6,t+5]},$$

which finishes the proof. □

Acknowledgments

The paper was written during the authors' joint stay at the University of Berne. Authors gratefully acknowledge the support from the Schweizerischer Nationalfonds and the Polish Scientific Grant KBN No. 1 P03A 018 27.

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