ϕ^4 model on a circle

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Abstract

The four-dimensional critical scalar theory at equilibrium with a thermal bath at temperature T is considered. The thermal equilibrium state is labeled by n the winding number of the vacua around the compact imaginary-time direction which compactification radius is $1/T$. The effective action for zero modes is a three-dimensional ϕ^4 scalar theory in which the mass of the scalar field is proportional to n/T resembling the Kaluza–Klein dimensional reduction. Similar results are obtained for the theory at zero temperature but in a one-dimensional potential well. Since parity is violated by the vacua with odd vacuum number n, in such cases there is also a cubic term in the effective potential. The ϕ^3 term contribution to the vacuum shift at one-loop is of the same order of the contribution from the ϕ^4 term in terms of the coupling constant of the four-dimensional theory but becomes negligible as n tends to infinity. Finally, the relation between the scalar classical vacua and the corresponding $SU(2)$ instantons on $S^1 \times \mathbb{R}^3$ in the 't Hooft ansatz is studied.
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1. Introduction

In four dimensions, massless ϕ^4 model with nonpositive potential is equivalent to $SU(2)$ Yang–Mills theory in the 't Hooft ansatz [1–3]. Scalar theories with nonpositive potentials are also familiar in gravity theories [4] while they may be problematic in quantum field theory.

In this Letter we obtain solutions of massless ϕ^4 model with nonpositive potential which are periodic in one direction. We show that this periodic solution can be used for three purposes. The first one is to study the critical ϕ^4 theory at finite temperature. The second one is to study the theory in a one-dimensional potential well by imposing the Dirichlet boundary condition on ϕ to be vanishing at the boundaries. The third application of this solution is to obtain $SU(2)$ instantons on $S^1 \times \mathbb{R}^3$ using the 't Hooft ansatz.

The organization of the Letter is as follows. In Section 2 after a brief review of scalar field theory at finite temperature, we consider the massless ϕ^4 model in one dimension where we

study the periodic solution of the equation of motion to be used in the following sections. In Section 3 we use the result of Section 2 to study the four-dimensional model at finite temperature. The resulting three-dimensional effective action for zero modes is shown to be a massive ϕ^4 theory. The mass of the scalar field is proportional to the winding number of the classical vacua, considered as the thermal equilibrium state, around the compact imaginary-time direction. In Section 4 we study the massless ϕ^4 theory at zero temperature in a potential well. In this case the effective theory is realized in two different sectors corresponding to even and odd vacuum numbers. For even vacuum numbers, parity is conserved and the interaction is given by a ϕ^4 term. In the case of odd vacuum number, parity is violated and a ϕ^3 interaction is added. We study and compare the contribution form both interaction terms to the vacuum shift at one-loop and verify that for large vacuum numbers, the ϕ^3 term contribution is negligible in comparison with the contribution from the ϕ^4 term. In Section 5 we briefly discuss the $SU(2)$ instantons corresponding to the periodic solutions in the 't Hooft ansatz. There we show that indeed the Yang–Mills field equation corresponds to the critical theory with potential $V(\phi) \sim -e^2\phi^4$ where e is the gauge field coupling constant. Section 6 is devoted to conclu-

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sion and is closed by discussing the dependence of the entropy of the thermal vacua on the corresponding winding number.

2. $D = 1$ ϕ^4-model at thermal equilibrium

The static properties of finite temperature QFT can be derived from the partition function $\mathcal{Z} = \mathrm{tr}\, e^{-H/T}$ where H is the Hamiltonian of the quantum field theory and T is the temperature. For a simple theory with boson fields ϕ and Euclidean action $S(\phi)$ the partition function is given by the functional integral

$$\mathcal{Z} = \int [d\phi] \exp\left[-S(\phi)\right], \tag{1}$$

where $S(\phi)$ is the integral of the Lagrangian density $\mathcal{L}(\phi)$,

$$S(\phi) = \int_0^{1/T} dt \int d^d x\, \mathcal{L}(\phi), \tag{2}$$

and the field ϕ satisfies periodic boundary conditions in the imaginary-time direction

$$\phi(t = 0, x) = \phi(t = 1/T, x). \tag{3}$$

The equation of motion of the ϕ^4 model in one dimension, defined by the Euclidean action

$$S = \int dt \left(\frac{1}{2}\phi'^2 - \frac{g}{4}\phi^4\right), \tag{4}$$

where a $'$ denotes one time derivation with respect to t, is the following nonlinear Laplace equation

$$\phi'' + g\phi^3 = 0. \tag{5}$$

This equation can be easily integrated once to obtain

$$\frac{1}{2}\phi'^2 + \frac{g}{4}\phi^4 = c. \tag{6}$$

For $c = 0$ the solution $\phi \sim t^{-1}$ is singular at $t = 0$. For $c > 0$, defining $c = L^{-4}$ one obtains[1]

$$\phi = \frac{1}{L}\left(\frac{4}{g}\right)^{1/4} \mathrm{sn}\left(g^{1/4}\frac{t}{L}\middle|-1\right), \tag{7}$$

in which $\mathrm{sn}(u|m) = \sin(\varphi)$ is the Jacobi elliptic function in which $\varphi = \mathrm{am}(u|m)$ is the inverse of Jacobi elliptic function of the first kind, $F(\varphi|m)$ defined by the relation[2]

$$F(\varphi|m) = \int_0^\varphi \left(1 - m \sin^2\theta\right)^{-1/2} d\theta. \tag{8}$$

Defining

$$u(\varphi) = F(\varphi|-1) = \int_0^\varphi d\theta\, \frac{1}{\sqrt{1 + \sin^2\theta}}, \tag{9}$$

one can easily verify that $u(\varphi)$ is a periodic function

$$u(\varphi + 2n\pi) = u(\varphi) + 4nK(-1), \quad n \in \mathbb{N}, \tag{10}$$

in which $K(m) = F(\frac{\pi}{2}|m)$ denotes the complete elliptic integral of the first kind. Consequently the solution (7) is periodic,

$$\phi(t) = \phi\left(t + n\frac{4LK(-1)}{g^{1/4}}\right), \quad n \in \mathbb{N}. \tag{11}$$

By identifying the period with $1/T$, one can determine L in terms of T, as follows:

$$L_n = \frac{g^{1/4}}{4K(-1)}\frac{T^{-1}}{n}, \quad n \in \mathbb{N}. \tag{12}$$

Thus, there exist a set of *classical vacua* ϕ_n with winding number $n \in \mathbb{N}$ around the compact imaginary-time direction given by

$$\phi_n(t) = \sqrt{\frac{2}{g}}\,\omega_n\, \mathrm{sn}(\omega_n t | -1), \quad \omega_n = n\omega_1, \tag{13}$$

where $\omega_1 = 4K(-1)T$. It is straightforward to calculate the action corresponding to ϕ_n, which is given by

$$S(\phi_n) = n^4\frac{(4K(-1))^4}{3g}T^3. \tag{14}$$

Consequently there is *an action barrier*,

$$\Delta S \sim \frac{(4K(-1))^4}{3g}T^3, \tag{15}$$

separating different vacuum states. Given the vacua ϕ_n, it is natural to search for the corresponding kink solutions interpolating between different vacua. We leave this question as an open problem. In Section 5 we obtain $SU(2)$ instantons on $S^1 \times \mathbb{R}^3$ in the 't Hooft ansatz corresponding to ϕ_n.

3. $D = 4$ ϕ^4 model at thermal equilibrium

In this section we study the $D = 4$ ϕ^4 model at thermal equilibrium corresponding to the vacua ϕ_n given by Eq. (13). The idea is similar to the Kaluza–Klein dimensional reduction (see e.g. [6]). Considering a $(d + 1)$-dimensional spacetime with one compactified dimension $\mathcal{M} = \mathbb{R}^{1,d-1} \times S^1$, a general (scalar) field $\sigma(x^\mu, t)$ in which x^μ's are coordinates of $\mathbb{R}^{1,d-1}$ and t is the coordinate on S^1, can be decomposed via a Fourier transformation into its zero mode $\sigma_0(x^\mu)$ and the Kaluza–Klein modes $\sigma_i(x^\mu)$ which correspond to the ith winding state along S^1. In the original Kaluza–Klein method, the classical vacuum state is unique and corresponds to $\sigma = 0$. If the there are different local vacuum states in the theory, it is natural to anticipate the emergence of different effective theories for zero-modes characterizing the corresponding vacuum state. This is the case that one encounters when zero-modes of the $D = 4$ ϕ^4 model at thermal equilibrium corresponding to the vacua ϕ_n is considered. In this theory, at any local vacua ϕ_n, the emergence of a Kaluza–Klein mode gives a transition to a different vacua by classically penetrating through the barrier (15). The classical transition rates can be obtained by calculating the interaction

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[1] In [5] similar wave functions are obtained for a massive harmonic oscillator.
[2] For definition and calculations on elliptic functions one can use the software Mathematica5, Wolfram Research, Inc.

term $g \int \phi^4$ in which ϕ is replaced with the corresponding Kaluza–Klein mode expansion. These terms are negligible at high temperature/weak coupling limit as can be verified from Eq. (15). Of course, similar to the Kaluza–Klein dimensional reduction method, in order to reconstruct the original theory in the decompactification limit $T \to 0$, all of these terms should be considered. In the following, we do not consider the Kaluza–Klein excitations and restrict ourselves to the effective action for zero-modes in a local vacua labeled by ϕ_n.

Assuming that

$$\phi(\vec{x}, t) = \phi_n(t) + T^{1/2} \bar{\phi}(\vec{x}), \tag{16}$$

we obtain the three-dimensional critical action for zero-mode $\phi(\vec{x})$. The coefficient $T^{1/2}$ is considered in the definition of $\phi(\vec{x})$ to insure that its classical mass-dimension is equal to $1/2$, the mass dimension of free scalar fields in three dimensions. One should note that since we are considering the critical scalar theory at finite temperature, the only mass scale at hand is the background temperature T to adjust the mass dimension in the dimensional reduction procedure.

The effective action can be obtained simply by inserting Eq. (16) into the action

$$S[\phi] = \int_0^{1/T} dt \int d^3x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{g}{4} \phi^4 \right). \tag{17}$$

In the formal expansion of $S[\phi] = S[\phi_n + T^{1/2}\bar{\phi}]$,

$$S[\phi] = S[\phi_n] + T^{1/2} \int \bar{\phi} \left(\frac{\delta S}{\delta \phi} \right)_{\phi_n} + S_n[\bar{\phi}], \tag{18}$$

the linear term in $\bar{\phi}$ is vanishing since ϕ_n is a solution of the equation of motion. Using Eq. (14) one verifies that here

$$\frac{S[\phi_n]}{V} = n^4 \frac{(2K(-1))^4}{3g} T^3, \tag{19}$$

where $V = \int d^3x$ is the volume of the *room*. The effective action is given by $S[\bar{\phi}]$,

$$S_n[\bar{\phi}] = \int d^3x \left(\frac{1}{2} \sum_{i=1}^3 (\partial_i \bar{\phi})^2 - \frac{1}{2} m_n^2 \bar{\phi}^2 - g_3 \bar{\phi}^3 - \frac{g_4}{4} \bar{\phi}^4 \right), \tag{20}$$

in which $g_4 = Tg$ and,

$$m_n^2 = 3gT \int_0^{1/T} \phi_n^2 = 6n\omega_n \left(4E(-1) - 4K(-1) \right), \tag{21}$$

$$g_3 = gT^{3/2} \int_0^{1/T} \phi_n = 0, \tag{22}$$

where $E(m)$ gives the complete elliptic integral,

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} \, d\theta. \tag{23}$$

$K(-1) \simeq 1.31$ and $E(-1) \simeq 1.91$. Consequently, the effective action for zero modes is a three-dimensional ϕ^4 scalar theory in which the mass of the scalar field are given by

$$m_n = nm_1, \quad n \in \mathbb{N}, \tag{24}$$

resembling the Kaluza–Klein reduction, where

$$m_1 = \left\{ 6 \left(4K(-1) \right) \left[4E(-1) - 4K(-1) \right] \right\}^{1/2} T \simeq 8.6T. \tag{25}$$

4. Dirichlet boundary condition

In this section we study the $D = 4 \, \phi^4$ theory at zero temperature with Dirichlet boundary condition

$$\phi(0, \vec{x}) = \phi(\ell, \vec{x}) = 0, \tag{26}$$

instead of imposing the periodicity condition $\phi(t) = \phi(t + 1/T)$.

Rewriting the identity (10) as

$$u(\varphi + n\pi) = u(\varphi) + 2nK(-1), \quad n \in \mathbb{N}, \tag{27}$$

and noting that $\mathrm{sn}(0) = 0$, one easily verifies that the Dirichlet boundary condition $\phi(\ell) = 0$ is satisfied by assuming that

$$\ell = \frac{2nLK(-1)}{g^{1/4}}, \quad n \in \mathbb{N}, \tag{28}$$

from which one instead of Eq. (12) obtains

$$L_n = \frac{g^{1/4}}{2K(-1)} \frac{\ell}{n}, \quad n \in \mathbb{N}. \tag{29}$$

Thus, here the set of *classical vacua* ϕ_n is given by Eq. (13) in which one should assume that

$$\omega_1 = 2K(-1)T. \tag{30}$$

All the results of Section 3 are still valid after replacing $1/T \to \ell$ and $\omega_n \to \omega_n/2$ or replacing $4K(-1)$ and $4E(-1)$ by $2K(-1)$ and $2E(-1)$ respectively.

Since parity is violated in the four-dimensional theory by the vacua ϕ_n with odd vacuum number n, it is natural that the parity violating term $g_3 \bar{\phi}^3$ now appear in the expansion (18) in this case,

$$g_3 = \frac{g}{\ell^{3/2}} \int_0^\ell \phi_n^2 = \frac{\pi}{\ell^{3/2}} \sqrt{\frac{g}{2}} \Delta(n), \tag{31}$$

where

$$\Delta(n) = \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \tag{32}$$

The general form of the effective potential for equilibrium states given by vacua with even and odd vacuum number n is plotted in Figs. 1 and 2 respectively. From these figures one verifies that the ϕ^3 term do not affect the theory dramatically for example by generating a local minima. In the following we compute the one-loop contribution to the scalar field self-energy for the even and odd cases and show that for large n, the vacuum shift in two sectors are similar. To this aim, we assume the validity of perturbation around the zero of the potential at $\phi = 0$. As we
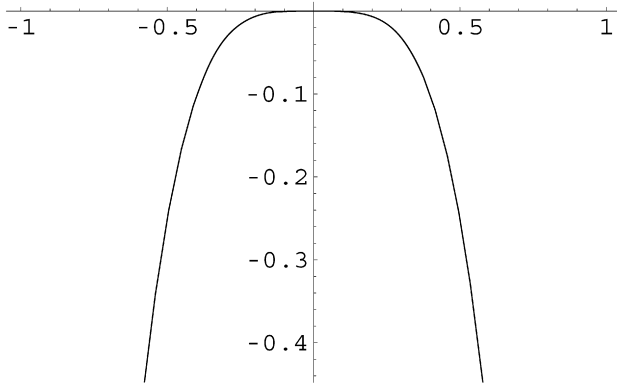
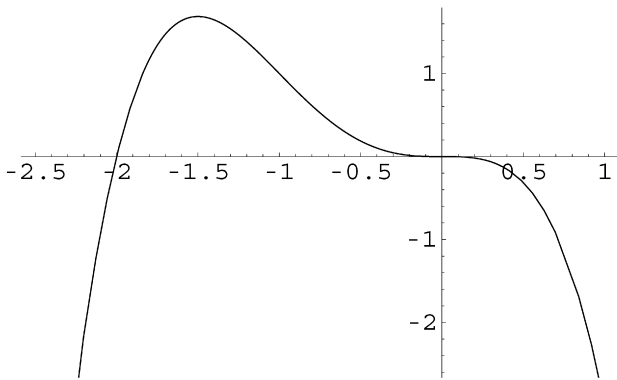Fig. 1. The effective potential: the even-vacua.

Fig. 2. The effective potential: the odd-vacua.

will show at the end of this section even in the case of the odd-vacua, where the potential is of the form plotted in Fig. 2, one should do perturbation around $\phi = 0$.

The one-loop contribution to the self-energy from the ϕ^3 term, depends on p the momenta of the external line,

$$\Sigma_n^{(3)}(p) = \frac{\Delta(n)\pi^2}{2} \frac{g}{\ell^3} \int d^3k \frac{1}{k^2 + m_n^2} \frac{1}{(\vec{k} + \vec{p})^2 + m_n^2}$$
$$= \Delta(n)\pi^4 \frac{g}{\ell^3} \frac{2}{p} \sin^{-1}\left(\frac{1}{\sqrt{1 + 4m_n^2/p^2}}\right). \quad (33)$$

From this result, the one-loop contribution from the ϕ^3 term to the vacuum to vacuum transition (the *unobservable* vacuum shift) can be obtained

$$\Sigma_n^{(3)}(0) = \Delta(n)\pi^4 \frac{g}{\ell^3} \frac{1}{m_n} \sim \frac{\Delta(n)}{n} \frac{g}{\ell^2}. \quad (34)$$

The one-loop contribution to the self-energy from the ϕ^4 term, is divergent. The divergent term is linear in the cut-off Λ and is given by the one-loop diagram for mass-less scalar ($n = 0$)

$$\Sigma_n^{(4)} = -\frac{g}{4\ell} \int d^3k \frac{1}{k^2 + m_n^2} = \Sigma_0^{(4)} + \frac{g}{\ell}\pi^2 m_n, \quad (35)$$

where $\Sigma_0^{(4)}$ is the one-loop vacuum shift for the $n = 0$ case

$$\Sigma_0^{(4)} = -\frac{g}{4\ell} \int d^3k \frac{1}{k^2} = -\frac{g}{\ell}\pi\Lambda. \quad (36)$$

Consequently,

$$\Sigma_n^{(4)} - \Sigma_0^{(4)} = \frac{g}{\ell}\pi^2 m_n \sim n\frac{g}{\ell^2}. \quad (37)$$

The ϕ^4 theory in three dimensions is super renormalizable. After renormalization the vacuum shift is given by

$$\Sigma_n = \Sigma_n^{(3)} + \Sigma_n^{(4)} - \Sigma_0^{(4)} = \frac{g}{\ell}\pi^2 m_n\left(1 + \frac{\Delta(n)\pi^2}{m_n^2\ell^2}\right). \quad (38)$$

Consequently the ϕ^3 term slightly modifies the vacuum shift, but, the relative difference between the vacuum shift for the even and odd cases, decreases as $1/n^2$. This result is consistent with our physical intuition because for large n the global behavior of the potential barrier is important thus Figs. 1 and 2 seem more similar to each other as n increases.

The only thing that we should check is whether the perturbation point $\phi = 0$ we used for both the even and odd cases is the correct perturbation base point in the odd case. This can be verified as follows. The potential term plus the mass term in the odd-case is given by

$$\tilde{V}_n(\phi) = \frac{1}{2}m_n^2\phi^2 + g_3\phi^3 + \frac{g_4}{4}\phi^4. \quad (39)$$

If there is another point $\phi = \phi_0$ around which we could do the perturbation, then $\tilde{V}(\phi - \phi_0)$ should, first of all, be of the form

$$\tilde{V}(\delta\phi) = \text{const} + \frac{1}{2}M_0^2(\delta\phi)^2 + \mathcal{O}(\delta\phi)^3, \quad (40)$$

for some constant M_0^2. This means that ϕ_0 should be a stationary point of $\tilde{V}(\phi)$. To obtain the stationary point, we have to solve the equation

$$\delta\tilde{V}(\phi)/\delta\phi = m_n^2\phi + 3g_3\phi^2 + g_4\phi^3 = 0. \quad (41)$$

This equation in addition to $\phi = 0$ has another solution if

$$9g_3^2 > 4m_n^2 g_4 \quad (42)$$

inserting $g_4 = g/\ell$, $m_n \simeq 4.3n/\ell$ and g_3 from Eq. (31) into Eq. (42) one verifies that a solution like ϕ_0 exists if

$$n^2 < 0.6. \quad (43)$$

Since the ϕ^3 term exists only for odd n, i.e. $n \geqslant 1$, therefore the only solution to Eq. (41) is $\phi = 0$ and consequently the one-loop calculations above are correct.

5. $SU(2)$ instanton on $S^1 \times \mathbb{R}^3$

In this section we discuss $SU(2)$ instantons corresponding to the vacua ϕ_n, in the 't Hooft ansatz [1]. The 't Hooft ansatz for the Yang–Mills potential A_μ^a is given by

$$A_\mu^a = \eta_{\mu\nu}^a \partial^\nu \psi / \psi, \quad (44)$$

where $\eta_{\mu\nu}^a$ are the 't Hooft tensors

$$\eta_{\mu\nu}^a = \epsilon_{0a\mu\nu} + i\eta_{a\mu}\eta_{0\nu} - i\eta_{a\nu}\eta_{0\mu}, \quad (45)$$

in which $\eta_{\mu\nu} = (-, +, +, +)$ is the Minkowski metric. Assuming that $\psi = \psi(x^0)$, the gauge field A_μ^a and the fieldstrength

$F_{\mu\nu}$ are given as follows:

$$A_\mu^a = i\eta_{a\mu}\phi, \tag{46}$$

and

$$F_{ij}^a = -e\epsilon_{aij}\phi^2, \qquad F_{0i}^a = i\delta_{ai}\partial_0\phi, \tag{47}$$

in which $\phi = \partial_t\psi/\psi$ and e is the gauge field coupling constant. The field equation $\partial^\mu F_{\mu\nu}^a + e\epsilon^{abc}A^{b\mu}F_{\mu\nu}^c = 0$, reads

$$\partial_0^2\phi - 2e^2\phi^3 = 0. \tag{48}$$

Thus the field equation for an instanton is given by Eq. (48) after a Wick rotation $x^0 \to ix^0$,

$$\partial_t^2\phi + g\phi^3 = 0, \tag{49}$$

where $g = 2e^2$. Using Eq. (47) the instanton action

$$S_{\text{YM}} = -\frac{1}{4}\int d^4x \, F_{\mu\nu}^a F^{a\mu\nu}, \tag{50}$$

satisfies the identity $S_{\text{YM}} = 3S[\phi]$ where $S[\phi]$ is the action of the critical scalar theory in $D = 4$ given in Eq. (19).

Consequently, the set of solutions ϕ_n for the critical scalar theory gives a set of solution to the $SU(2)$ instanton field equation. Furthermore the 't Hooft ansatz provide a powerful motivation to study the critical scalar theory with potential $V(\phi) \sim -\phi^4$ since as is observed above, the coupling constant $g \sim e^2$ in Eq. (49) cannot be negative.

6. Conclusion

The massless ϕ^4 model in one dimension has a set of periodic solutions,

$$\phi_n(t) = \sqrt{\frac{2}{g}}\omega_n \, \text{sn}(\omega_n t | -1), \quad \omega_n = n\omega_1, \tag{51}$$

if the scalar potential is $V(\phi) \sim -\phi^4$. These solutions can be used to study the theory at finite temperature in higher dimensions or to study such theories in a one-dimensional potential well with Dirichlet boundary condition on the scalar field to be vanishing on the boundaries. The resulting effective action is a scalar theory in one dimension lower. Here we studied dimensional reduction $d = 4 \to d = 3$. The mass of scalar field in the effective theory appeared to be proportional to n, resembling the Kaluza–Klein dimensional reduction. In the thermal theory, n is the winding number of the classical vacua around the compact time direction but in the theory with Dirichlet boundary condition it denotes the vacuum number. In the thermal theory, the scalar interaction in the effective theory is given by a ϕ^4 term. The effective action in the case of ϕ^4 model in one dimensional potential well with Dirichlet boundary condition, should be studied in two different sectors. If the vacuum number is even, the effective interaction is given by a ϕ^4 term. But if the vacuum number is odd, due to parity violation, there is also a cubic term in the effective potential. Both theories of course are super renormalizable.

Finally one can use ϕ_n to construct $SU(2)$ invariant solutions of the $SU(2)$ Yang–Mills field equation on $S^1 \times \mathbb{R}^3$, in the 't Hooft ansatz,

$$A^{(n)a}_{\;\;\;\mu} = i\eta_{a\mu}\phi_n, \quad n \in \mathbb{N}. \tag{52}$$

Considering A_μ^a in Eq. (52) as an instanton the corresponding action can be shown to be equivalent to

$$S_{\text{YM}} = n^4 \frac{(4K(-1))^4}{g\ell^3}, \tag{53}$$

where ℓ is the radius of the S^1.

Our motivation for the present work has been to generalize the classical Fubini's solution [7] of the massless phi-fourth model. The Fubini's solution is invariant under the de Sitter subgroup of the full conformal symmetry group of the classical massless phi-fourth model. His motivation has been to find a natural mass-scale in the physics of hadrons. In [8] we showed that the Fubini's vacua, in the phi-fourth model with nonpositive potential, can be interpreted as an open FRW de Sitter background. Furthermore, semiclassical arguments showed that the entropy associated to the Fubini's vacua is equivalent to the entropy of a de Sitter vacua. This is a good sign for the relevance of nonpositive potentials to physics specially when opposed to the phi-fourth model with a positive potential which is known to be trivial, see e.g. [9]. The connection between the Fubini's solution and the dS vacua is recently studied in the context of M-theory in [10].

The Fubini's solution is invariant under none of the translations of the conformal symmetry group. Thus an interesting question is whether there exist a classical vacuum invariant under some translations if not all [8]. As we saw above, such solutions at least provide new instantons of the $SU(2)$ Yang–Mills theory which are probably useful in a braneworld scenario. In such a scenario and probably in the physics of superconductors, the scalar theory itself is interesting as it is exhibiting a new mass generating mechanism.

We close this section by giving some comments on the dependence of the entropy $\mathcal{S}_n$ of vacua ϕ_n on n, the winding number, in the case of the scalar theory at thermal equilibrium. Classically, one may define an entropy by

$$T\delta\mathcal{S} = \delta S. \tag{54}$$

Since the action S_n is not a linear function of n, this formula will be applicable only for large values of n. In this limit, $\delta n^4 \simeq 4n^3$, thus Eq. (54) can be integrated once to obtain

$$\mathcal{S}_n = n^4 \frac{(4K(-1))^4}{g}T^2, \quad n \gg 1. \tag{55}$$

To obtain the entropy semi-classically, one might determine the quantum state $|\psi_n\rangle$ corresponding to the classical vacua ϕ_n and define the entropy by the relation

$$\langle\psi_n|\psi_n\rangle \sim e^{-\mathcal{S}_n}. \tag{56}$$

To construct the quantum state $|\psi_n\rangle$, one may proceed as follows. One counts the number of plane waves e^{ikx} with a given momenta k superposed to construct ϕ_n and then create the same number of free-particle states from $|0\rangle$, the state of nothing. The resulting spectrum of free-particle states describes the

quantum state $|\psi_n\rangle$ corresponding to the classical vacua ϕ_n. Consequently,

$$|\psi_n\rangle \sim \exp\left(\sum_m \sqrt{A_m^{(n)}}\, a_m^\dagger\right)|0\rangle, \qquad (57)$$

where $A_m^{(n)}$ are given by the Fourier transform of ϕ_n,

$$\phi_n(t) = \sum_m A_m^{(n)} \sin(2\pi m T t). \qquad (58)$$

Thus to obtain $A_m^{(n)}$ one should calculate the following integral

$$A_m^{(n)} = \frac{\omega_n}{\pi}\sqrt{\frac{2}{g}}\int_0^{2\pi} \mathrm{sn}\left(n\frac{K(-1)}{\pi}\theta\,\bigg|-1\right)\sin(m\theta)\,d\theta. \qquad (59)$$

We expect that the entropy obtained in this way become equivalent to the result of Eq. (54) for large n.

Acknowledgement

References

[1] G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8;
G. 't Hooft, Phys. Rev. D 14 (1976) 3432.
[2] A. Actor, Rev. Mod. Phys. 51 (1979) 461.
[3] F. Loran, Phys. Rev. D 71 (2005) 126003, hep-th/0501189.
[4] J. Maldacena, C. Nunez, Int. J. Mod. Phys. A 16 (2001) 822, hep-th/0007018.
[5] C.A.A. de Carvalho, R.M. Cavalcanti, E.S. Fraga, S.E. Joras, Ann. Phys. (N.Y.) 273 (1999) 146;
C.A.A. de Carvalho, R.M. Cavalcanti, E.S. Fraga, S.E. Joras, Phys. Rev. E 61 (2001) 6392.
[6] L. O'Raifeartaigh, The Dawning of Gauge Theory, Princeton Series in Physics, Princeton Univ. Press, Princeton, 1997.
[7] S. Fubini, Nuovo Cimento A 34 (1976) 521.
[8] F. Loran, hep-th/0612089, Mod. Phys. Lett. A, in press.
[9] R. Jackiw, C. Nunez, S.-Y. Pi, Phys. Lett. A 347 (2005) 47.
[10] S. de Haro, A.C. Petkou, hep-th/0606276;
S. de Haro, I. Papadimitriou, A.C. Petkou, hep-th/0611315.