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On the Kurosh theorem and separability properties

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Abstract

We prove a Kurosh-type subgroup theorem for free products of LERF groups. This theorem permits a better understanding of how finitely generated subgroups are embedded in finite index subgroups. Consequences include the double coset separability of free products of negatively curved surface groups. Other properties of finitely generated subgroups of such free products are studied as well.

MSC: 20E26; 20E06; 20E05

1. Introduction

A group G is said to be LERF (*locally extended residually finite*) or, by some authors, *subgroup separable* if given a finitely generated subgroup H of G (which shall be abbreviated throughout as $H \leqslant_{\text{f.g.}} G$) and $g \notin H$, there exists a subgroup K of finite index (which we shall write $K \leqslant_{\text{f.i.}} G$) with $H \leqslant K$ and $g \notin K$. If one places a topology on G (called the *profinite topology* [11]) by taking the collection of finite index subgroups as a neighborhood basis of 1, then G is LERF if and only if all its finitely generated subgroups are closed. It is easy to show that finitely presented LERF

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groups have decidable generalized word problem. LERF was introduced by Hall in [10] where he proved that free groups are LERF and, in fact, that one can choose K (in the above notation) so that H is a free factor of K.

In [8], see also [13], Gitik and Rips defined a group G to be *double coset separable* if the setwise product of two finitely generated subgroups of G is closed in the profinite topology. They proved therein that free groups are double coset separable (an independent proof of a more general result can be found in [14]), while Niblo proved [12] that surface groups are double coset separable. A topological characterization of double coset separable groups can be found in [8].

Our main theorem, Theorem 4.1, describes the structure of finitely generated subgroups of free products of LERF groups, combining elements of the topological proof of the Kurosh Theorem [19] with those of Stallings' proof of Hall's theorem [18]. Straightforward corollaries of Theorem 4.1 include Hall's Theorem [10], and the theorem of Burns and Romanovskii [3,15].

To state some more interesting corollaries of this result, we need to introduce a new notion motivated by Hall's Theorem [10]. Recall that a subgroup $H \leqslant G$ is called *malnormal* if

$$g \notin H \Rightarrow gHg^{-1} \cap H = 1.$$

In this case, we write $H \leqslant_{\mathrm{mal}} G$. For example, free factors are malnormal. We say that $H \leqslant G$ is *virtually malnormal* in G if there exists $K \leqslant_{\mathrm{f.i.}} G$ such that $H \leqslant_{\mathrm{mal}} K$. If every finitely generated subgroup of G is virtually malnormal in G, we say that G is LVM (*locally virtually malnormal*). It follows from Hall's theorem that free groups are LVM.

One then has the following immediate corollary of [7, Theorem 1].

Theorem 1.1. *Suppose that H and K are quasiconvex subgroups of a negatively curved, LVM, LERF group G. Then the double coset HK is closed in the profinite topology on G.*

Corollary 1.2. *Let G be a locally quasiconvex, negatively curved, LVM, LERF group. Then G is double coset separable.*

In particular, free groups satisfy the above hypotheses; we shall see shortly that negatively curved surface groups and, in fact, a more general class of amalgamations of free groups do as well.

The primary application of Theorem 4.1 is then the following theorem.

Theorem 1.3. *The free product of a collection of LVM, LERF groups is again an LVM, LERF group.*

Since the adjectives negatively curved and locally quasiconvex all pass through free products [9,6] we have

Corollary 1.4. *If G is a free product of negatively curved, LVM, LERF groups and $H,K \leqslant G$ are quasiconvex in G, then the double coset HK is closed in the profinite topology.*

In particular, if the groups in question are also locally quasiconvex, then G is double coset separable.

Since free groups, finite groups, and negatively curved surface groups are negatively curved, locally quasiconvex, LVM, and LERF, the above corollary applies to free products of such groups. Rips' Theorem on free actions of finitely generated groups on $\mathbb{R}$-trees implies that (finite) free products of free groups and negatively curved surface groups are precisely those finitely generated, negatively curved groups which can act freely on $\mathbb{R}$-trees. Our result thus shows that all such groups are double coset separable. It has been announced by Coulbois [4] that double coset separability (and an, in fact, slightly more general notion) is always preserved by free products. His proof is model-theoretic and does not cover the above result about products of quasiconvex subgroups.

Our results can also be made to apply to various other local properties of LERF groups. Let P be a property of subgroups, e.g. malnormality, being a free factor, being a retract. We write $H \leqslant_P G$ if a subgroup H of G has property P. A group G is said to be LVP (*locally virtually P*) if, for every $H \leqslant_{\text{f.g.}} G$, there exists $K \leqslant_{\text{f.i.}} G$ such that $H \leqslant_P K$. For example, if P is malnormality, then LVP is LVM. Suppose P satisfies the following properties:

A1. $H \leqslant_P G$ and $H \leqslant K \leqslant G$ implies $H \leqslant_P K$;
A2. $H_1 \leqslant_P G_1$ and $H_2 \leqslant_P G_2$ implies $H_1 * H_2 \leqslant_P G_1 * G_2$;
A3. $1 \leqslant_P G$, $G \leqslant_P G$.

Then arguing exactly as we do below for LVM, LERF one can show

Theorem 1.5. *A free product of LVP, LERF groups is again an LVP, LERF group.*

That malnormality satisfies the above axioms will be shown in the sequel. The property of being a retract clearly satisfies the above axioms, as does the property of being a free factor (A1 follows from considering an action of G on a tree with trivial edge stabilizers and with H as a vertex stabilizer). Hence we get the following result, also due to Burns and Romanovskii [3,15].

Theorem 1.6. *Let $\mathscr{H}$ be the class of LERF groups G such that each finitely generated subgroup of G is a free factor in a finite index subgroup of G. Then $\mathscr{H}$ is closed under free products.*

A more general property than being a free factor which satisfies A1–A3 is the property: $H \leqslant_P G$ if H is a vertex group in a graph of groups decomposition of G (this property was suggested as being of interest by D. Cohen in a private communication); A1 is argued as per the case of a free factor; A2 follows by taking the wedge product of the graphs of groups corresponding to G_1 and to G_2 at the vertices corresponding to H_1 and H_2; at this wedged vertex, place the group $H_1 * H_2$.

The argument in the following section proving that certain amalgamations of free groups over cyclic groups are LVM, also implies that the amalgamated product of two free groups over a cyclic subgroup is LVP for the property P above. Free groups are also LVP for this property by Hall's Theorem [10] whence we can conclude that finitely generated groups acting freely on $\mathbb{R}$-trees are as well.

2. Locally virtually malnormal groups

In this section, we present several basic properties of LVM groups. We begin with some examples of malnormal subgroups. It is easy to see that a non-trivial cyclic subgroup $\langle g \rangle$ of a free group is malnormal if and only if g is not a proper power. Also, as alluded to earlier, free factors are always malnormal; that is, if $G = H * L$, then $H \leqslant_{\text{mal}} G$. Thus, Hall's theorem [10] implies that free groups are LVM as previously stated. We shall see shortly that Hall's result also holds for free products of free groups and finite groups, but fails in general for virtually free groups.

We now show that LVM groups enjoy many of the properties of free groups.

Proposition 2.1. *If G is LVM, then the following hold*:

(1) *If $H \leqslant_{\text{f.g.}} G$ contains a non-trivial subgroup normal in G, then $H \leqslant_{\text{f.i.}} G$*;
(2) *If $Z(G) \neq 1$, then G is virtually cyclic*;
(3) *If $H \leqslant_{\text{f.g.}} G$, $g \notin H$, and $gHg^{-1} \cap H \neq 1$, then there exists $K \leqslant_{\text{f.i.}} G$ such that $H \leqslant K$ and $g \notin K$*;
(4) *If G is torsion-free, then G is residually finite.*

Proof. For (1), let $H \leqslant_{\text{mal}} K \leqslant_{\text{f.i.}} G$; since $gHg^{-1} \cap H \neq 1$ for all $g \in G$, it must be that $H = K$. Since any cyclic subgroup of the center of G is normal, (2) follows from (1). For (3), since G is LVM, we can find a subgroup K such that $H \leqslant_{\text{mal}} K \leqslant_{\text{f.i.}} G$. By the definition of malnormality, it follows that $g \notin K$. As for (4), if G is torsion-free, then for all $g \in G$, $g \notin \langle g^2 \rangle$ and so an application of (3) with $H = \langle g^2 \rangle$ shows that we can separate g from 1 by a finite index subgroup. □

We do not know whether LVM groups in general are residually finite.

The following lemma about malnormal subgroups is straightforward and we leave the proof to the reader.

Lemma 2.2. *If $H \leqslant_{\text{mal}} K$ and $K \leqslant_{\text{mal}} G$, then $H \leqslant_{\text{mal}} G$. If $H \leqslant_{\text{mal}} G$ and $H \leqslant K \leqslant G$, then $H \leqslant_{\text{mal}} K$. In particular, if $H \leqslant_{\text{mal}} K$ and $G = K * L$, then $H \leqslant_{\text{mal}} G$.*

The above lemma implies, in particular, that a subgroup of an LVM group is LVM. We observe, however, that LVM is not a virtual property. For instance, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is virtually cyclic, but not LVM since $0 \times \mathbb{Z}/2\mathbb{Z}$ is a finitely generated normal subgroup of infinite index, contradicting Proposition 2.1 (1). More generally, extensions of finitely generated groups by infinite groups cannot be LVM.

The first author observed in [7] that Scott's result [16] implies that negatively curved surface groups are LVM. In fact, the following more general result is true. Suppose G is a free product of free groups with cyclic amalgamation such that the amalgamating subgroup is malnormal in each factor. Then, given $H \leqslant_{\text{f.g.}} G$, the proofs of [5, Lemma 4.5] and [6, Theorem 3.4] construct $K \leqslant_{\text{f.i.}} G$ with graph of groups decomposition (c.f. [17]) such that H is a malnormal subgroup of a vertex group, and the edge groups are cyclic with the following property: if E is an edge group contained in a vertex group V, and $g \in V \setminus E$, then gEg^{-1} intersects the images in V of the edge groups trivially. The vertex group containing H comes from the precover of [5, Lemma 4.5] (and H is, in fact, a free factor in this vertex group), while the remainder of the graph of groups structure in constructed in [6, Theorem 3.4]. It is a straightforward exercise, not dissimilar to Lemma 4.3 below, to show that this data implies $H \leqslant_{\text{mal}} K$. Thus such an amalgamation is LVM. Such amalgamations are also LERF [2,5] (see also the thesis of D. Wise), negatively curved [1], and locally quasiconvex [6, Corollary 3.8]. It is easy to see that all negatively curved surface groups arise as amalgamations of this sort.

We end this section by giving an algebraic characterization of LVM, LERF groups.

Proposition 2.3. *A group G is LVM, LERF if and only if given $H \leqslant_{\text{f.g.}} G$ and $A \subseteq G$ finite with $A \cap H = \emptyset$, there exists $K \leqslant_{\text{f.i.}} G$ such that $H \leqslant_{\text{mal}} K$ and $K \cap A = \emptyset$.*

Proof. Clearly the condition stated is sufficient for G to be LVM, LERF. As for necessity, suppose H and A are as above. By LERF, there exists $K_1 \leqslant_{\text{f.i.}} G$ such that $H \leqslant K_1$ and $K_1 \cap A = \emptyset$; by LVM, there exists $K_2 \leqslant_{\text{f.i.}} G$ such that $H \leqslant_{\text{mal}} K_2$. By Lemma 2.2, $H \leqslant_{\text{mal}} K = K_1 \cap K_2$ while, clearly, $K \cap A = \emptyset$. □

3. Geometry, LERF, and LVM

This section gives a topological condition for the fundamental group of a 2-complex to be LERF. All 2-complexes and their morphisms will be assumed to be combinatorial. First, we give a combinatorial version of a topological result due to Scott [16]. If L is a 2-complex, a *partial automorphism* of L is an isomorphism between (possibly empty) subcomplexes of L. It follows easily that a finite 2-complex has only finitely many partial automorphisms.

Proposition 3.1. *Suppose G is a LERF group acting without fixed points on a 2-complex $\hat{X}$. Then given $H \leqslant_{\text{f.g.}} G$ and a finite subcomplex C of $\hat{X}/H$, there exists $K \leqslant_{\text{f.i.}} G$ with $H \leqslant K$ such that the restriction of the projection map $\varphi : \hat{X}/H \to \hat{X}/K$ to C is an embedding.*

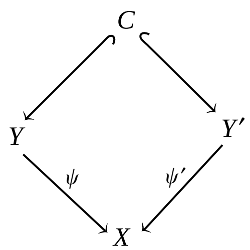
Proof. Since C is finite, we can find a finite lift D of C to $\hat{X}$ by choosing a lift of each closed n-cell. Let $G_D = \{g \in G \mid gD \cap D \neq \emptyset\}$. We claim G_D is finite. Indeed, by restricting the action of G to D, each element of G_D gives rise to a partial automorphism of D

with non-empty domain. Furthermore, since G acts without fixed points, each element of G_D induces a distinct partial automorphism. It follows, by the remark proceeding the proposition, that G_D is finite. Since G is LERF, we can find $K \leqslant_{\text{f.i.}} G$ with $H \leqslant K$ and $(G_D \setminus H) \cap K = \emptyset$.

We show φ restricts to an embedding on C. Let e_1, e_2 be n-cells of C and f_1, f_2 be lifts of e_1, e_2, respectively, to D. Then $\varphi(e_1) = \varphi(e_2)$ if and only if there exists $k \in K$ with $kf_1 = f_2$, whence $k \in G_D \cap K \subseteq H$. It follows that $e_1 = e_2$ and so φ is an embedding as desired. □

The following result is a variant on [5, Theorem 1.1] in the case of the standard 2-complex of a group, though the proof here is different.

Theorem 3.2. *Let X be a connected 2-complex. Then $\pi_1(X)$ is LERF if and only if given a connected covering $\psi : Y \to X$ and a finite connected subgraph $C \subseteq Y$, there exists a finite connected covering $\psi' : Y' \to X$ such that $C \subseteq Y'$, and the diagram*



commutes.

Proof. First we show sufficiency. Fix a vertex v_0 of X. Let H be a finitely generated subgroup of $G = \pi_1(X, v_0)$ and suppose $g \in G \setminus H$. Let $\psi : (Y, \tilde{v}_0) \to (X, v_0)$ be the covering with $\pi_1(Y, \tilde{v}_0) = H$, let $\{h_1, \ldots, h_n\}$ be a generating set for H, and let C be the subgraph of Y consisting of the edges and vertices of the lifts of $h_1, \ldots, h_n$ and g starting at $\tilde{v}_0$. Then C is finite and connected. So, by assumption, there is a finite connected covering $\psi' : (Y', \tilde{v}_0') \to (X, v_0)$ such that $\psi|_C$ factors through an embedding followed by ψ'. But then $\pi_1(Y', \tilde{v}_0')$ is a finite index subgroup of G containing H, but not g. We can conclude G is LERF.

For necessity, we observe that given a cover Y and a finite subgraph C of Y, it suffices to prove the statement for the cover Y_1 associated to $H = \psi(\pi_1(C)) \leqslant_{\text{f.g.}} \pi_1(X)$. Indeed, from $H \leqslant \pi_1(Y)$ we can conclude that the covering $Y_1 \to X$ factors through $\psi : Y \to X$ and that the map $\psi|_C : C \to X$ lifts to Y_1. It follows directly that the lift of $\psi|_C$ to Y_1 is, in fact, an embedding.

So assume now that $H = \pi_1(Y)$ (that is, $Y = Y_1$), and let $\varphi : \tilde{X} \to X$ be the universal cover. Then $\pi_1(X)$ is a LERF group acting without fixed points on the 2-complex $\tilde{X}$, and $Y = \tilde{X}/H$. The result now follows from Proposition 3.1 taking $Y' = \tilde{X}/K$ (where K is as in the conclusion of that proposition). □

We can now interpret the combination of LERF and LVM topologically.

Corollary 3.3. *In Theorem* 3.2, *if* $\pi_1(X)$ *is LVM, LERF then* Y' *can be chosen so that* $\psi(\pi_1(C)) \leqslant_{\mathrm{mal}} \pi_1(Y')$.

Proof. Let $G = \pi_1(X)$ and suppose that C is a finite graph as in the statement of Theorem 3.2. Then $H=\psi(\pi_1(C)) \leqslant_{\mathrm{f.g.}} G$ so we can find $K_1 \leqslant_{\mathrm{f.i.}} G$ such that $H \leqslant_{\mathrm{mal}} K_1$. By Theorem 3.2, there is a finite cover $\varphi : X' \to X$ such that $\psi|_C$ factors through an embedding followed by φ. Let $K_2 = \pi_1(X') \cap K_1$. Then $H \leqslant_{\mathrm{mal}} K_2$ by Lemma 2.2, and $K_2 \leqslant_{\mathrm{f.i.}} G$. Furthermore, if $\psi' : Y' \to X$ is the finite connected covering with $\pi_1(Y')=K_2$, then $\psi|_C$ factors through an embedding followed by ψ' as desired. □

4. A Kurosh-type theorem

Let $G_v\,(v \in V)$ be a collection of groups. For each $v \in V$, choose a connected 2-complex Γ_v with a single vertex, denoted v by abuse of notation, such that $\pi_1(\Gamma_v, v)= G_v$. Such a complex will be called a *vertex complex*. We construct a 2-complex Γ as follows: we take the disjoint union of the $\Gamma_v\,(v \in V)$ and a new vertex v_0; v_0 is then connected to v by an edge $e_v\,(v \in V)$. It is straightforward to see that $\pi_1(\Gamma, v_0) = *_{v\in V} G_v$.

Following the usual convention, we write $H^g = g^{-1}Hg$ for $H \leqslant G$ and $g \in G$.

Theorem 4.1. *Let* G_v, $v \in V$, *be a collection of LERF groups and* G *be the free product of the* G_v, $v \in V$. *Suppose that* $H \leqslant_{\mathrm{f.g.}} G$ *and* $A \subseteq G$ *is finite with* $H \cap A = \emptyset$. *Then there exist*: $v_1,\dots,v_r \in V$ (*not necessarily distinct*); $H_i \leqslant_{\mathrm{f.g.}} K_i \leqslant_{\mathrm{f.i.}} G_{v_i}$, $i = 1,\dots,r$; $g_1,\dots,g_r \in G$; *and subgroups* $F_1, F_2, K_0 \subseteq G$ *such that*:

$$H = H_1^{g_1} * \cdots * H_r^{g_r} * F_1;\;\; K = K_1^{g_1} * \cdots * K_r^{g_r} * F_1 * F_2 * K_0 \leqslant_{\mathrm{f.i.}} G, \tag{4.1}$$

F_1, F_2 *are free*; K_0 *is a finite free product of conjugates of some of the* G_v; *and* $K \cap A = \emptyset$.

Furthermore, if G_{v_i} *is LVM*, $i \in \{1,\dots,r\}$, *we may take* $H_i \leqslant_{\mathrm{mal}} K_i$ *while if* G_{v_i} *is finite, we may take* $H_i = K_i$.

Proof. Let Γ be constructed as above and Δ be the graph consisting of the closure of the edges $e_v(v \in V)$. We proceed as in the proof of Theorem 3.2 and the Kurosh Subgroup Theorem [19, Section 4.3.9].

Let $\varphi : (\bar{\Gamma}, \overline{v_0}) \to (\Gamma, v_0)$ be the cover corresponding to H. It is easy to see that, for $v \in V$, each component of $\varphi^{-1}(\Gamma_v)$ is a cover of Γ_v. Let Γ' be the subgraph of $\bar{\Gamma}$ obtained by lifting A and a finite set of generators of H to paths starting at $\overline{v_0}$. We adjoin to Γ' (without changing its name) a finite number of extra edges of $\bar{\Gamma}$ so that, for each $v \in V$ and each component C of $\varphi^{-1}(\Gamma_v)$, $\Gamma' \cap C$ is connected. If C is finite, we add all the edges of C to Γ'. Let $C_1,\dots,C_r$ be the complete set of subgraphs of Γ' which are components of $\Gamma' \cap \varphi^{-1}(\Gamma_{v_i})$ for some $v_i \in V$ $(i = 1,\dots,r)$.

By construction of Γ', since a path from v_0 can only enter Γ_{v_i} passing through e_{v_i}, we can choose $(i = 1,\dots,r)$ a preimage $\overline{v_i} \in C_i$ of v_i such that the edge e_{v_i} lifts in Γ' to an edge $\overline{e_{v_i}}$ ending at $\overline{v_i}$. Let

$$H_i = \varphi(\pi_1(C_i, \overline{v_i})) \leqslant_{\mathrm{f.g.}} G_{v_i} \;\; (1 = 1,\dots,r);$$

these subgroups are finitely generated since the C_i are all finite. Choose maximal trees for each C_i and extend to a maximal tree T of Γ'; then extend T to a maximal tree $\bar{T}$ of $\bar{\Gamma}$, first by extending to the preimages of the vertex complexes, and then to the preimage of Δ. Let $g_i \in G$ be the image of the path in T from the initial vertex of $\overline{e_{v_i}}$ to $\overline{v_0}$. Using the usual procedure for computing the fundamental group of a graph, it is easy to see that $H = \varphi(\pi_1(\Gamma', \overline{v_0}))$ is generated by the $H_i^{g_i}$ $(i = 1, \dots, r)$ under the natural identification of G_v as a subgroup of G and by the subgroup F_1 generated by the loops corresponding to edges of $\Gamma' \cap \varphi^{-1}(\Delta) \setminus T$. But calculating H using $\bar{T}$ and $\bar{\Gamma}$, we see, since all the 2-cells of $\bar{\Gamma}$ lie in components of the $\varphi^{-1}(\Gamma_v)(v \in Ver(\Gamma))$, that

$$H = H_1^{g_1} * \cdots * H_r^{g_r} * F_1$$

and F_1 is free on the aforementioned set. An easier way to think of this decomposition is to consider the usual Kurosh factorization of H, as per [19, Section 4.3.9]; then each $H_i^{g_i}$ is a subgroup of one of the usual free factors, while F_1 is a subgroup of the free part.

Now, using Theorem 3.2, we can embed C_i in a finite cover D_i of Γ_{v_i} $(i = 1, \dots, r)$; if G_{v_i} is also LVM we can, by Corollary 3.3, choose D_i so that $H_i \leqslant_{\text{mal}} \pi_1(D_i)$; if G_{v_i} is finite, then, by construction C_i is already a finite cover. Let Γ'' be the union of Γ' and the D_i $(i = 1, \dots, r)$; see Fig. 1.

Let n be the largest cardinality of a preimage in Γ'' of a vertex of Γ.

If v_0 has n preimages, then, for each $v \in V$, we add a finite number of disjoint copies of Γ_v to Γ'' so that v has exactly n preimages in the resulting (probably disconnected) 2-complex Γ'''. Observe that, for each edge $v_0 \xrightarrow{e_v} v$ of Δ, there is at most one lift of e_v to any preimage of v_0 in Γ''' because all such come from $\bar{\Gamma}$ which is a cover. Since there are exactly n preimages of v_0 and v, we can add lifts until there is exactly one lift of e_v starting from each preimage of v_0. Doing this for all the edges of Δ results in a finite cover $\psi : (\hat{\Gamma}, \overline{v_0}) \to (\Gamma, v_0)$.

If v_0 has less than n preimages in Γ'', then choose $v \in V$ with n preimages. Since v has more preimages than v_0, our previous observation implies that we can keep adding lifts of e_v until each preimage of v_0 in Γ'' has exactly one such lift in the resulting graph Γ_0. Now attach, by the endpoint, a copy of e_v to each preimage of v which is not already the endpoint of a lift of e_v. In this manner, we obtain a connected 2-complex containing Γ'' such that e_v has exactly n lifts and v_0 exactly n preimages. Now we can continue as in the previous case to obtain a finite cover $\psi : (\hat{\Gamma}, \overline{v_0}) \to (\Gamma, v_0)$ containing Γ'.

Let $K = \pi_1(\hat{\Gamma}, \overline{v_0})$; then $K \leqslant_{\text{f.i.}} G$, $H \leqslant K$, and $K \cap A = \emptyset$. Extend T to a maximal tree T' of $\hat{\Gamma}$ by first extending the tree to the preimages of the vertex complexes, and then to the preimage of Δ. Then

$$H_i \leqslant K_i = \pi_1(D_i, \overline{v_i}) \leqslant_{\text{f.i.}} G_{v_i} \quad (i = 1, \dots, r);$$

if G_{v_i} is LVM, $H_i \leqslant_{\text{mal}} K_i$; if G_{v_i} is finite $H_i = K_i$. If we let F be the subgroup generated by the loops corresponding to edges of $\psi^{-1}(\Delta) \setminus T'$, then a calculation similar to that above (or that in [19, Section 4.3.9]) shows that

$$K = K_1^{g_1} * \cdots * K_r^{g_r} * F * K_0,$$

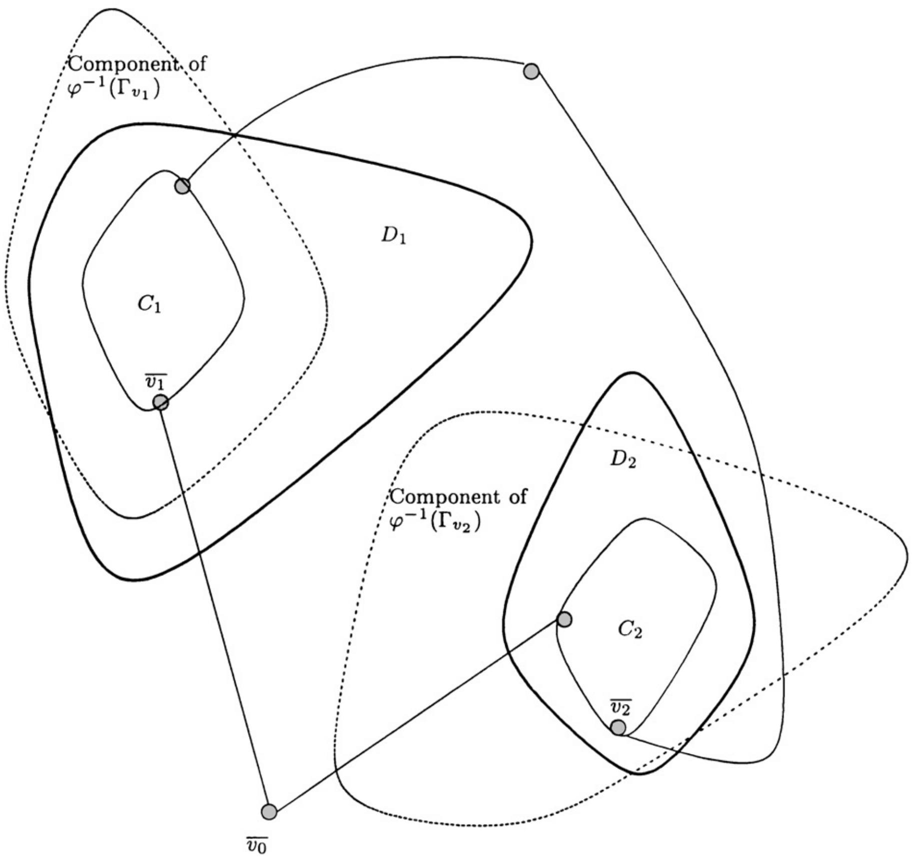


Fig. 1. Constructing Γ''.

where K_0 is a finite free product of conjugates of some of the G_v coming from the various copies of the Γ_v needed to bring the number of preimages of each vertex up to n, and where F is free on the aforementioned set. However, since $T' \cap \psi^{-1}(\Delta)$ is obtained by adding edges to $T \cap \Gamma' \cap \varphi^{-1}(\Delta)$, it follows that there $F = F_1 * F_2$ where F_2 is generated by those loops corresponding to edges of $\psi^{-1}(\Delta) \setminus (\Gamma' \cup T')$. □

Corollary 4.2 (Burns [3], Romanovskii [15]). *Free products of LERF groups are LERF.*

Theorem 1.3, is an immediate consequence of Theorem 4.1, Lemma 2.2 and the following lemma.

Lemma 4.3. *Suppose that $H_i \leqslant_{\mathrm{mal}} G_i$, $i = 1, \ldots, n$. Then*

$$H_1 * \cdots * H_n \leqslant_{\mathrm{mal}} G_1 * \cdots * G_n.$$

Proof. By induction, it suffices to prove that if $H \leqslant_{\text{mal}} K$ and $J \leqslant_{\text{mal}} L$, then $H * J \leqslant_{\text{mal}} K * L = G$. Let $g \in G \setminus (H * J)$ and let

$$g = k_1 l_1 \cdots k_r l_r$$

be a factorization in normal form; that is, $k_i \in K$, $l_i \in L$ and only k_1, l_r are permitted to be 1.

Suppose that $1 \neq m \in H * J$ with normal form

$$m = h_1 j_1 \cdots h_s j_s.$$

We show that $g^{-1}mg \notin H * J$. Without loss of generality, we may assume that $k_1 \neq 1$. By induction, we may assume that $k_1 \notin H$ (otherwise, replace m by $k_1^{-1}mk_1$ and g by $l_1 \cdots k_r l_r$). If $m \notin H$ (that is, $j_1 \neq 1$), then $g^{-1}mg$ has normal form

$$g^{-1}mg = \begin{cases} l_r^{-1}k_r^{-1} \cdots l_1^{-1}(k_1^{-1}h_1)j_1 \cdots h_s j_s k_1 l_1 \cdots k_r l_r, & j_s \neq 1, \\ l_r^{-1}k_r^{-1} \cdots l_1^{-1}(k_1^{-1}h_1)j_1 \cdots (h_s k_1) l_1 \cdots k_r l_r & \text{otherwise.} \end{cases}$$

Here we are using that $k_1 \notin H$ implies that $k_1^{-1}h_1, h_s k_1 \neq 1$. Since $k_1 \notin H$, we can conclude further that $k_1^{-1}h_1 \notin H$ and so, by the normal form theorem for free products, it follows that $g^{-1}mg \notin H * J$.

If $m \in H$, then $g^{-1}mg$ has normal form

$$g^{-1}mg = l_r^{-1}k_r^{-1} \cdots l_1^{-1}(k_1^{-1}mk_1)l_1 \cdots k_r l_r.$$

Since $k_1 \in K \setminus H$ and $H \leqslant_{\text{mal}} K, k_1^{-1}mk_1 \notin H$. Thus, again appealing to the normal form theorem, we see that $g^{-1}mg \notin H * J$. □

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