JOURNAL OF ALGEBRA 97, 438–460 (1985)

# Zero Cycles on del Pezzo Surfaces over Local Fields

KEVIN R. COOMBES*

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

AND

DOUGLAS J. MUDER

*Department of Mathematics, University of Chicago, Chicago, Illinois 60637*

*Communicated by I. N. Herstein*

Received March 27, 1984

## 1. INTRODUCTION

Let $X$ be a smooth, projective surface over a field $k$. Say that $X$ is *rational* if $X_E$ is birational with $\mathbb{P}^2_E$ over some extension field $E$. Say that $E$ *splits* $X$ if this birational isomorphism can be realized by a sequence of monoidal transformations centered at $E$-points.

Write $CH_0(X)$ for the Chow group of zero cycles on $X$ modulo rational equivalence, and $A_0(X)$ for the kernel of the degree map $CH_0(X) \to \mathbb{Z}$. Let $E/k$ split $X$, $G = \mathrm{Gal}(E/k)$. Bloch [1] and Colliot-Thélène and Sansuc [5, 6] have studied a map

$$\Phi\colon A_0(X) \to H^1(G, \mathrm{Pic}(X_E) \otimes E^*).$$

Using the work of Mercurjev and Suslin on Brauer groups, Colliot-Thélène [3] succeeded in showing that if $X$ is a smooth, rational surface over a local or global field, then $\Phi$ is injective. It follows by results of Bloch [2] that if $X$ has good reduction over a local field, then $A_0(X) = 0$.

For technical reasons, most of the early results and calculations dealt with conic bundle surfaces. In view of Iskovskih's work [8] classifying rational surfaces into two groups, we were led to ask: How do the singularities of reductions of del Pezzo surfaces over local fields contribute to the Chow group of zero cycles? Our first result is

* Partially supported by the NSF.



0021-8693/85 $3.00

THEOREM 1. *Let $X$ be a del Pezzo surface of degree $d \geqslant 5$, defined over a local field $k$. Then $A_0(X)=0$.*

This happens because the cohomology groups vanish. The first interesting examples thus occur when $d=4$. In this case, Manin [9] has enumerated the possible values of $H^1(G, \mathrm{Pic}(X_E))$, based on an analysis of the orbits of the galois action on the lines of the del Pezzo surface. So, we decided to study degree 4 del Pezzo surfaces in depth.

We will say an $E$-split, $E$-rational del Pezzo surface of degree 4 is *marked* if the set of lines has been named via an intersection preserving correspondence with a fixed abstract configuration of lines.

THEOREM 2. *A fine moduli space for the set of marked split nonsingular del Pezzo surfaces is $M=\mathbb{P}^2 - C$, where $C$ is the union of lines through four fixed points. To a point $(a\colon b\colon c) \in M$ we associate the surface*

$$bsv = ar(u-t) + (c-b)\,su$$

$$btv = cu(r-s) + (a-b)\,rt.$$

As in [12], this family can be extended, over $\mathbb{P}^2$ with the four points blown up, in several ways. The nature of the bad fibres depends somewhat mysteriously on which actions of the Weyl group $W(D_5)$ (of automorphisms of the lines on a degree 4 del Pezzo surface) can be extended to the family.

Armed with these surfaces and with the knowledge that we could write down everything interesting about them, we proceeded to construct examples. In this paper, we restrict our attention to surfaces of type IV in Manin's list. This is the first case where the cohomology of the Picard group is nontrivial. The splitting field is a quadratic extension $E=k(\sqrt{d})$. Each orbit of the galois action on the lines consists of a pair of intersecting lines and $H^1(G, \mathrm{Pic}(X_E)) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Using descent theory, we construct all the type IV surfaces.

Suppose $E/k$ is an unramified extension of local fields. Bad reduction can only occur if the point $P=(a\colon b\colon c)$ specializes onto one of the lines of $\mathbb{P}^2$ with four points blown up. It turns out that the type IV action is invariant under arbitrary permutations of these four points. We find then three kinds of bad reduction

(e) $P$ specializes onto one of the $e_i$ coming from blown up points;

(f) $P$ specializes onto one of the lines $f_{ij}$ connecting two points;

(p) $P$ specializes onto one of the points $e_i \cap f_{ij}$.

Assume that char $(k)=0$ and $k$ has residue characteristic $p>2$. Our main result is:

THEOREM 3. *Let $E/k$ be an unramified quadratic extension of local fields. Let $X$ be a degree 4 del Pezzo $k$-surface split by $E$. Then either $A_0(K)=0$ or $X$ is a type IV surface with bad reduction. The type of bad reduction determines*

$$\begin{aligned} A_0(X) &= \mathbb{Z}/2 && (f) \\ &= 0 && (e) \text{ even} \\ &= \mathbb{Z}/2 \oplus \mathbb{Z}/2 && (e) \text{ odd} \\ &= \mathbb{Z}/2 && (p) \text{ even} \\ &= \mathbb{Z}/2 \oplus \mathbb{Z}/2 && (p) \text{ odd.} \end{aligned}$$

Here the distinction between even and odd cases is defined using the parameters $(a: b: c)$. We assume $a$, $b$, $c$ are relatively prime integers. Bad reduction occurs because one of

$$a, \qquad b, \qquad c, \qquad b-a, \qquad b-c, \qquad b-a-c$$

has positive valuation. We say the reduction is even or odd according as the smallest nonzero valuation among the above six quantities is even or odd.

Bloch has suggested, for reasons related to the study of $L$-functions of rational surfaces, that the structure of the group $A_0(X)$ should be calculable from a knowledge of the components of the closed fibre of a minimal regular model. While we do not consider this question directly, our theorem does shed some light on potential difficulties. In particular, the Neron model must reflect the difference between the even and odd cases of Theorem 3.

## 2. DEL PEZZO SURFACES

Let $X$ be a smooth rational surface over a field $k$. We propose to study the group of zero cycles of degree zero on $X$ by means of a homomorphism

$$\Phi: A_0(X) \to H^1(G, \operatorname{Pic}(X_E) \otimes E^*)$$

where $G$ is the galois group of a splitting extension $E/k$. This map was introduced by Bloch [1] and Colliot-Thélène and Sansuc [6]. It can briefly be defined as follows.

Since $X_E$ is $E$-birational to $\mathbb{P}^2_E$, the cohomology of the Gersten complex resolving $\mathcal{K}_{2,X_E}$:

$$K_2 E(X) \to \coprod_{x \in X^1_E} E(x)^* \to \coprod_{x \in X^2_E} \mathbb{Z}$$

is isomorphic to

$$K_2E \quad \mathrm{Pic}(X_E)\otimes E^* \quad \mathbb{Z}.$$

So, there are exact sequences

$$0 \to K_2E(X)/K_2E \to \quad A \quad \to \mathrm{Pic}(X_E)\otimes E^* \to 0$$

$$0 \to \quad A \quad \to \coprod_{X_E^1} E(x)^* \to \coprod_{X_E^2}{}^0 \mathbb{Z} \quad \to 0.$$

Here we write $\amalg^0$ for the degree zero part of the direct sum. The second exact sequence tells us that $H^1(G, A) = A_0(X)$. So, the cohomology of the first exact sequence is

$$\cdots \to H^1(G, K_2E(X)/K_2E) \to A_0(X) \xrightarrow{\Phi} H^1(G, \mathrm{Pic}(X_E)\otimes E^*) \to \cdots.$$

Colliot-Thélène and Coray [4], Bloch [1], Colliot-Thélène and Sansuc [5], and Colliot-Thélène [3] show that if $X$ is smooth and rational over a local or global field, then $\Phi$ is injective and $A_0(X)$ is finite. For arithmetic purposes, the calculation of $A_0(X)$ thus reduces to a question of which cohomology classes are represented by points.

We wish to recall the following facts.

(2.1) If $X$ is a smooth rational surface with good reduction over a local field, then $A_0(X) = 0$ [3].

(2.2) Let $q \in \mathbb{Z}$ be prime. For any abelian group $A$, let $A(q)$ denote the $q$-primary part. For $E/k$ the splitting field of $X$, let $E_q$ be the fixed field of a $q$-Sylow subgroup $G_q$. Then $A_0(X)(q) \hookrightarrow A_0(X_{E_q})(q)$ and $H^1(G, \mathrm{Pic}(X_E)\otimes E^*) \hookrightarrow H^1(G_q, \mathrm{Pic}(X_E)\otimes E^*)$ [2].

(2.3) $A_0(X)$ and $H^1(G, \mathrm{Pic}(X_E)\otimes E^*)$ are $k$-birational invariants of $X$ [4, 10].

PROPOSITION 2.4. *Let $X$ be a smooth rational surface over a local field. Then Brauer equivalence of points on $X$ is the same as rational equivalence.*

*Proof.* Recall (Manin [10]) the definition of Brauer equivalence: We identify $H^1(G, \mathrm{Pic}\, X_E)$ with the group $\mathrm{Br}(X; E)/\mathrm{Br}(k; E)$ of Azumaya algebras on $X$, split by $E$, modulo constant algebras. Given an Azumaya algebra $A$ and a point $x$, we get an element $\mathrm{Cor}_{k(x)/k}\, A(x) \in \mathrm{Br}(k)$. This extends by linearity and defines a pairing

$$A_0(X) \otimes H^1(G, \mathrm{Pic}\, X_E) \to \mathrm{Br}(k).$$

We say that points are Brauer equivalent if they have the same image in $\mathrm{Hom}(H^1(G, \mathrm{Pic}\, X_E), \mathrm{Br}(k))$.

When $k$ is a local field, $\mathrm{Br}(k)=\mathbb{Q}/\mathbb{Z}$ and local class field theory identifies this Hom-group with $H^1(G, \mathrm{Pic}(X_E)\otimes E^*)$. Bloch [2] has checked that $\Phi\colon A_0(X)\to H^1(G, \mathrm{Pic}(X_E)\otimes E^*)$ agrees with the map induced by this pairing. The result follows from the injectivity of $\Phi$. (See also [3].)

We now restrict ourselves to the case of del Pezzo surfaces. A nonsingular rational surface is called a del Pezzo surface if its anticanonical sheaf is ample. The chief invariant of a del Pezzo surface is its degree $d=(-K)^2$, the self-intersection of the anticanonical class. The degree can only take on the values $1\leqslant d\leqslant 9$; when $3\leqslant d\leqslant 9$, the anticanonical class gives an embedding as a surface of degree $d$ in $\mathbb{P}^d$, isomorphic over an algebraically closed field either to $\mathbb{P}^2$ with $9-d$ points blown up or to the embedding of $\mathbb{P}^1\times\mathbb{P}^1$ in $\mathbb{P}^8$. For more detailed information, see Manin [10] or Nagata [11].

THEOREM 2.5. *Let $X$ be a del Pezzo surface of degree $d\geqslant 5$ over a field $k$. Write $B_0(X)$ for the group of degree zero, zero cycles on $X$ modulo Brauer equivalence. Then $B_0(X)=0$.*

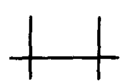
COROLLARY. *If $X$ is del Pezzo of degree $d\geqslant 5$ over a local field, then $A_0(X)=0$.*

*Proof.* Philosophically, this result holds because $PGL(3)$ allows us to choose $4\geqslant 9-d$ points on $\mathbb{P}^2$. There simply is not enough freedom to get interesting phenomena. We will prove the theorem by showing that either $X$ is already $k$-birational to $\mathbb{P}^2$ or $H^1(G, \mathrm{Pic}(X_E))=0$.

*degree* 9. These surfaces are all Severi–Brauer twisted forms of $\mathbb{P}^2$. So, $\mathrm{Pic}\, X_E=\mathbb{Z}$ with trivial galois action and $H^1(G, \mathrm{Pic}\, X_E)=0$.

*degree* 8. Here there are two possibilities. In the first case, $X$ is isomorphic to the blowup of $\mathbb{P}^2$ at a $k$-point, so is already birational to $\mathbb{P}^2$. In the second case, $X$ is a twisted form of $\mathbb{P}^1\times\mathbb{P}^1$ so that $\mathrm{Pic}\, X_E=\mathbb{Z}\oplus\mathbb{Z}$ is a permutation module, and $H^1(G, \mathrm{Pic}\, X_E)=0$.

*degree* 7. Manin has shown that these surfaces are all $k$-birationally trivial. For, the configuration of lines is



and the pair of nonintersecting lines can be blown down as a set over $k$.

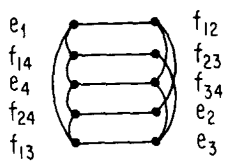
*degree* 6. Here we use the fact (2.2) that it is enough to consider $p$-torsion independently, via the $p$-Sylow subgroup. The configuration of lines forms a hexagon, with group of symmetries $D_6$. The 2-Sylow subgroup is $\mathbb{Z}/2\oplus\mathbb{Z}/2$ and one of the orbits consists of a pair of skew lines. These lines can thus be blown down over $k$, making the surface $k$-birational to a degree 8 surface. The 3-Sylow subgroup $\mathbb{Z}/3$ has an orbit consisting of three skew lines. Blowing this set down gives a $k$-birational map to a Severi–Brauer surface. In both cases, we have $X$ birational to something for which $B_0(X)=0$.

*degree* 5. The group of automorphisms of the lines on $X$ has order $2^3 \cdot 3 \cdot 5$. We consider 2-torsion, 3-torsion, and 5-torsion separately.

The 2-Sylow subgroup is isomorphic to $D_4$ acting in the obvious way on the four blown up points of $\mathbb{P}^2$. So, one of the orbits consists of four skew lines, which may be blown down again to get $X$ birational to a Severi-Brauer surface.

The 3-Sylow subgroup $\mathbb{Z}/3$ can be taken to have generator $(e_1 e_2 e_3)$ where the exceptional curves $e_i$ arise from the blown up points. So, the set $\{e_1, e_2, e_3\}$ can be blown down over $k$ to find $X$ birational to a degree 8 surface.

Finally, we come to a case where we must actually calculate a cohomology group. We introduce the following graph to represent the lines on $X_E$:




Each vertex represents a line; two vertices are connected if and only if the corresponding lines intersect. The generator $\sigma$ of the 5-Sylow subgroup cycles the entire graph up one step.

The group $\mathrm{Pic}(X_E) \approx \mathbb{Z}^5$ with generators $l, e_1, e_2, e_3, e_4$. The remaining lines are defined by $f_{ij} = l - e_i - e_j$ in the Picard group. The first cohomology group, since $G$ is cyclic, is defined as $\mathrm{Ker}(N)/\mathrm{Im}(1-\sigma)$. One calculates that

$$N(al + b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4)$$
$$= (3a + b_1 + b_2 + b_3 + b_4)(-K)$$

where $-K = 3l - e_1 - e_2 - e_3 - e_4$ is the anticanonical class, and

$$\begin{aligned}
(1-\sigma)(e_1 - l) &= e_1 - e_4 \\
(1-\sigma)(l - e_2) &= e_1 - e_2 \\
(1-\sigma)(e_3) &= e_3 - e_2 \\
(1-\sigma)(e_4 - e_1 + l) &= 3e_4 - l
\end{aligned}$$

so that $1-\sigma$ hits the generators of $\mathrm{Ker}(N)$. So, $H^1(G, \mathrm{Pic}\, X_E) = 0$.

*Remark.* The discussion of surfaces of degree 5 could be more quickly reduced to the case of 5-torsion by noting that Manin shows that the existence of a point on $X$ defined over an extension of degree prime to five forces $X$ to be $k$-birationally trivial. Swinnerton-Dyer [13] shows that every degree 5 del Pezzo surface actually has a rational point.

## 3. Moduli of del Pezzo Surfaces of Degree 4

In this section, we begin the construction of a class of rational surfaces for which $A_0(X)$ is nontrivial. We first construct a good family of del Pezzo surfaces of degree 4 (written $\nabla^4$) with all rational lines. The most useful property of this family is that an isomorphism of fibres extends uniquely to an action of the Weyl group $W(D_5)$ on the entire family. The construction of $\nabla^4$ surfaces with specified galois action then proceeds by standard descent techniques.

We recall the geometry of nonsingular split $\nabla^4$ surfaces. Every $\nabla^4$ contains exactly 16 lines, subject to certain incidence relations derived from the fact that

5 are the blowups of points of $\mathbb{P}^2 (e_i)$

10 are the transforms of lines through 2 points $(f_{ij})$

1 is the transform of the conic through all 5 points $(g)$.

Let $S$ be an abstract set of lines with these incidence relations. We say that a $\nabla^4$ is marked by giving an isomorphism from $S$ to the set of lines on the surface. A marking thus corresponds to choosing, in order, a quintuple of skew lines to be viewed as the inverse image of points of $\mathbb{P}^2$. With the obvious meaning, let $M$ be the set of isomorphism classes of marked, nonsingular, split $\nabla^4$ surfaces. So, $M$ is a set-theoretic fine moduli space for such surfaces; we will give a geometric description.

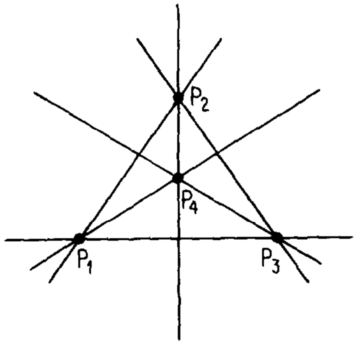
Write $F_m$ for the surface associated to $m \in M$. An isomorphism $F_m \to F_{m'}$ must arise by renaming the lines. Since the group of automorphisms of the configuration of lines is $W(D_5)$, it is clear that an isomorphism of fibres extends uniquely to an action on all the surfaces by an element of $W(D_5)$.

Start with the fact that marked, nonsingular, split $\nabla^4$ surfaces are each isomorphic to the blowup of $\mathbb{P}^2$ at 5 points in general position. We may eliminate the action of $\mathrm{Aut}(\mathbb{P}^2) = PGL(3)$ by fixing the first four points. So, every $F_m$ is isomorphic to one of the surfaces obtained by blowing up

$$\begin{array}{ll} P_1 & (1:0:0) \\ P_2 & (0:0:1) \\ P_3 & (1:1:0) \\ P_4 & (0:1:1) \\ P_5 & (a:b:c). \end{array}$$

Being in general position means that the arbitrary point $P_5$ should not lie on any of the lines of

$C$:

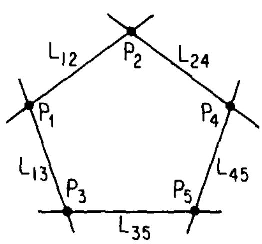



PROPOSITION 3.1. *A fine moduli space for marked, split, nonsingular* $\nabla^4$ *surfaces is* $M = \mathbb{P}^2 - C$. *To a point* $(a : b : c) \in M$ *we associate the surface*

$$bsv = ar(u - t) + (c - b)\, su$$

$$btv = cu(r - s) + (a - b)\, rt.$$

*Proof.* It is clear from the above discussion that $\mathbb{P}^2 - C$ is a fine moduli space. We show how to get the equations of the surface given by blowing up the points $P_1, \ldots, P_5$. Consider the configuration




A basis for the space of cubic curves through the $P_i$ is given by the degenerate curves

$$r = L_{12} L_{45} L_{13} = byz(a(y - z) + (c - b)\, x)$$

$$s = L_{45} L_{13} L_{24} = bxz(a(y - z) + (c - b)\, x)$$

$$t = L_{13} L_{24} L_{35} = bxz(c(y - x) + (a - b)\, z)$$

$$u = L_{24} L_{35} L_{12} = byx(c(y - x) + (a - b)\, z)$$

$$v = L_{35} L_{12} L_{45} = y(a(y - z) + (c - b)\, x)(c(y - x) + (a - b)\, z).$$

These satisfy the following relations

$$\frac{s}{r}=\frac{x}{y}$$

$$\frac{t}{u}=\frac{z}{y}$$

$$\frac{v}{u}=\frac{a(y-z)+(c-b)\,x}{bx}=\frac{a(1-t/u)+(c-b)\,s/r}{bs/r}$$

$$\frac{v}{r}=\frac{c(y-x)+(a-b)z}{bz}=\frac{c(1-s/r)+(a-b)\,t/u}{bt/u}.$$

Clearing denominators in the last two expressions gives the above equations.

In the tables at the end of this section we give a complete enumeration of

I: Equations of the lines on $F_{(a:b:c)}$.

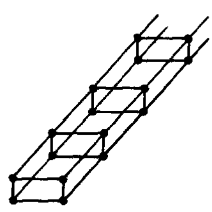
II: Equations of the quadritangents; i.e., those hyperplane sections consisting of four lines.

III: Coordinates of all the points of intersection of the lines.

The most important geometric invariants of a $\nabla^4$ surface are its cross ratios. Cayley first realized their importance for cubic surfaces; they were used by Naruki [12] to construct a moduli space for cubic surfaces. We define the cross ratios as follows: Fix a pair $L$ of intersecting lines on $X$. This determines a pencil of hyperplanes in $\mathbb{P}^4$ containing $L$. Exactly four members of this pencil are quadritangents. Each ordering $t_1,..., t_4$ of these quadritangents determines a cross ratio

$$r(t_1,..., t_4; X)=\frac{(t_1-t_4)(t_2-t_3)}{(t_1-t_3)(t_2-t_4)}.$$

One would expect many cross ratios: 6 for each of the 40 pairs of intersecting liens. Some reflection shows that the lines are more rigid than that. Using Manin's system of graphing the lines (vertices for lines, simplices for intersections), we get the graph:



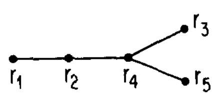
We have left out some simplices (the back face connects to the front and each face cross-connects to the face two steps back). However, each face is a quadritangent, and the cross ratios are calculated on one of the lines where four quadritangents hit.

We have calculated the cross ratios and displayed the results in Table IV. It turns out that there are actually only five families of six cross ratios, each corresponding to a choice of the faces above. In Table IV we list only one representative of each family of six cross ratios.

The distinct cross ratios define a rational map $\mathbb{P}^2 \to (\mathbb{P}^1)^5$. This map is undefined only at the points $P_1, P_2, P_3, P_4$ and can be desingularized by blowing them up. Let $N$ be the blow up of $\mathbb{P}^2$ at these four points. The cross-ratios then define a regular embedding of $N \to (\mathbb{P}^1)^5$. In this sense, $N$ may be viewed as the natural completion of the moduli space $M$.

The use of cross ratios thus justifies the naive idea that $N$ should parametrize $\nabla^4$ surfaces: after all, should not "$\mathbb{P}^2$ with five points blown up" be described by first blowing up four points (which can be done in only one way) to get $N$ and then blowing up an arbitrary point of $N$? We can guess what surfaces should complete the family over $N$. Timms [14] thought of obtaining $\nabla^4$ surfaces by projecting from points of $N$. Projecting from a general point gives a nonsingular surface; projecting from a general point on a line of $N$ gives a surface with one conic node; and projecting from the intersection of two lines of $N$ gives a surface with two conic nodes. We do not yet know if a family with these properties exists over $N$.

We now discuss the action of the Weyl group $W(D_5)$. This is the group of automorphisms of the lines. It is associated to the Dynkin diagram

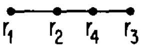



with roots

| | |
|---|---|
| $r_1$ | $(e_1 e_2)$ |
| $r_2$ | $(e_2 e_3)$ |
| $r_3$ | quadratic transformation at $e_1, e_2, e_3$ |
| $r_4$ | $(e_3 e_4)$ |
| $r_5$ | $(e_4 e_5)$. |

It is clear that this acts on $M$ by acting on the $P_i \in \mathbb{P}^2$, and extends to an equivariant action on $F \to M$ by renaming the lines. In Table V we exhibit the action of the simple roots on this family.

It is also clear how the action of $W(D_5)$ extends to $N$. The group of automorphisms of $N$ is the Weyl group $W(A_4)$ associated to




The point is that the roots $r_3$ (corresponding to the degeneration of three points becoming collinear) and $t_5$ (corresponding to two points becoming coincident) are essentially the same. More precisely, they act the same way on $M$, hence give the obvious representation $W(D_5) \to W(A_4)$ (Tables I–V).

## 4. Manin's Type IV Action

We begin by describing how descent theory will be applied in our situation to construct examples of surfaces with specified galois action on the lines. We have a universal family $F \to M$ of marked $E$-split del Pezzo surfaces of degree 4. Suppose now that we are given a $k$-surface $X$ which is split by $E/k$. Let $G = \mathrm{Gal}(E/k)$.

Choose a marking of the lines on $X_E$. This determines a unique point $m \in M(E)$ and a unique $E$-isomorphism $\alpha\colon X_E \to F_m$. Now let $\gamma \in G$ be arbitrary. Then the action of $\gamma$ on $X_E$ defines a new naming of the lines, hence uniquely determines an $E$-isomorphism $\beta$: $X_E \to F_{\bar{m}}$. Since $\beta\alpha^{-1}\colon F_m \to F_{\bar{m}}$ is an $E$-automorphism of fibres, it extends uniquely to an $E$-

TABLE I

Lines on the Degree 4 del Pezzo Surface

| | | | |
|---|---|---|---|
| $e_1$ | $r = 0$ | $bv = (c-b)\,u$ | $cs = (b-c)\,t$ |
| $e_2$ | $u = 0$ | $bv = (a-b)\,r$ | $at = (b-a)\,s$ |
| $e_3$ | $t = 0$ | $r = s$ | $bv = (a+c-b)\,u$ |
| $e_4$ | $s = 0$ | $u = t$ | $bv = (a+c-b)\,r$ |
| $e_5$ | $v = 0$ | $cu = bt$ | $ar = bs$ |
| $g$ | $bv = (a-b)\,r + (c-b)\,u$ | $c(s-r) = (b-c)\,t$ | $a(t-u) = (b-a)\,s$ |
| $f_{12}$ | $r = 0$ | $u = 0$ | $v = 0$ |
| $f_{13}$ | $r = 0$ | $s = 0$ | $t = 0$ |
| $f_{14}$ | $u = t$ | $bv = (a+c-b)\,r - cs$ | $bv = (c-b)\,t$ |
| $f_{15}$ | $cu = bt$ | $bv = ar - bs$ | $u = s + t$ |
| $f_{23}$ | $r = s$ | $bv = (a+c-b)\,u - at$ | $bv = (a-b)\,s$ |
| $f_{24}$ | $s = 0$ | $t = 0$ | $u = 0$ |
| $f_{25}$ | $ar = bs$ | $bv = cu - bt$ | $r = s + t$ |
| $f_{34}$ | $r = u$ | $u = s + t$ | $bv = (a+c-b)\,u$ |
| $f_{35}$ | $t = 0$ | $u = 0$ | $v = 0$ |
| $f_{45}$ | $r = 0$ | $s = 0$ | $v = 0$ |

TABLE II

Quadritangents on the Degree 4 Surface

| | |
|---|---|
| $e_1 e_2 f g_{12}$ | $bv = (a-b) r + (c-b) u$ |
| $e_1 e_3 g f_{13}$ | $c(s-r) = (b-c) t$ |
| $e_1 e_4 g f_{14}$ | $bv = (a+c-b) r - cs + (b-c)(t-u)$ |
| $e_1 e_5 g f_{15}$ | $bcv = (c-b)(cu-bt) + c(ar-bs)$ |
| $e_1 f_{23} f_{14} f_{15}$ | $b(a+c-b) v = (a+c-b)(ar + (c-b) u) - acs + a(b-c) t$ |
| $e_1 f_{24} f_{13} f_{15}$ | $cs = (b-c) t$ |
| $e_1 f_{25} f_{13} f_{14}$ | $cs = (a+c-b) r + (b-c) t$ |
| $e_1 f_{34} f_{12} f_{15}$ | $bv = ar + (c-b) u$ |
| $e_1 f_{35} f_{12} f_{14}$ | $bv = (c-b) u$ |
| $e_1 f_{45} f_{12} f_{13}$ | $r = 0$ |
| $e_2 e_3 g f_{23}$ | $bv = (a-b)(r-s) + (a+c-b) u - at$ |
| $e_2 e_4 g f_{24}$ | $a(t-u) = (b-a) s$ |
| $e_2 e_5 g f_{25}$ | $abv = (a-b)(ar-bs) + a(cu-bt)$ |
| $e_2 f_{13} f_{24} f_{25}$ | $at = (b-a) s$ |
| $e_2 f_{14} f_{23} f_{25}$ | $b(a+c-b) v = (a+c-b)(cu - (b-a) r) - act + c(b-a) s$ |
| $e_2 f_{15} f_{23} f_{24}$ | $at = (b-a) s + (a+c-b) u$ |
| $e_2 f_{34} f_{12} f_{25}$ | $bv = (a-b) r + cu$ |
| $e_2 f_{35} f_{12} f_{24}$ | $u = 0$ |
| $e_2 f_{45} f_{12} f_{23}$ | $bv = (a-b) r$ |
| $e_3 e_4 g f_{34}$ | $bv = (a+c-b)(r-s-t+u)$ |
| $e_3 e_5 g f_{35}$ | $bcv = (a+c-b)(cu-bt)$ |
| $e_3 f_{12} f_{34} f_{35}$ | $bv = (a+c-b) u$ |
| $e_3 f_{14} f_{23} f_{35}$ | $bv = (a+c-b) u - at$ |
| $e_3 f_{15} f_{23} f_{34}$ | $bv = (a+c-b) u + a(r-s-t)$ |
| $e_3 f_{24} f_{13} f_{35}$ | $t = 0$ |
| $e_3 f_{25} f_{13} f_{34}$ | $r = s + t$ |
| $e_3 f_{45} f_{13} f_{23}$ | $r = s$ |
| $e_4 e_5 g f_{45}$ | $abv = (a+c-b)(ar-bs)$ |
| $e_4 f_{12} f_{34} f_{45}$ | $bv = (a+c-b) r$ |
| $e_4 f_{13} f_{24} f_{45}$ | $s = 0$ |
| $e_4 f_{15} f_{24} f_{34}$ | $u = s + t$ |
| $e_4 f_{23} f_{14} f_{45}$ | $bv = (a+c-b) r - cs$ |
| $e_4 f_{25} f_{14} f_{34}$ | $bv = (a+c-b) r + c(u-s-t)$ |
| $e_4 f_{35} f_{14} f_{24}$ | $u = t$ |
| $e_5 f_{12} f_{35} f_{45}$ | $v = 0$ |
| $e_5 f_{13} f_{25} f_{45}$ | $ar = bs$ |
| $e_5 f_{14} f_{25} f_{35}$ | $bv = cu - bt$ |
| $e_5 f_{23} f_{15} f_{45}$ | $bv = ar - bs$ |
| $e_5 f_{25} f_{15} f_{35}$ | $cu = bt$ |
| $e_5 f_{34} f_{15} f_{25}$ | $bv = ar - bs + cu - bt$ |

TABLE III

Intersections of Lines on the Degree 4 Surface

| | |
|---|---|
| $e_1 \cap g$ | $(0: a(b-c): ac: b(a+c-b): (c-b)(a+c-b))$ |
| $e_1 \cap f_{12}$ | $(0: b-c: c: 0: 0)$ |
| $e_1 \cap f_{13}$ | $(0: 0: 0: b: c-b)$ |
| $e_1 \cap f_{14}$ | $(0: b(b-c): bc: bc: c(c-b))$ |
| $e_1 \cap f_{15}$ | $(0: b-c: c: b: c-b)$ |
| $e_2 \cap g$ | $(b(a+c-b): ac: c(b-a): 0: (a-b)(a+c-b))$ |
| $e_2 \cap f_{12}$ | $(0: a: b-a: 0: 0)$ |
| $e_2 \cap f_{23}$ | $(ab: ab: b(b-a): 0: a(a-b))$ |
| $e_2 \cap f_{24}$ | $(b: 0: 0: 0: a-b)$ |
| $e_2 \cap f_{25}$ | $(b: a: b-a: 0: a-b)$ |
| $e_3 \cap g$ | $(ab: ab: 0: b(a-b): (a-b)(a+c-b))$ |
| $e_3 \cap f_{13}$ | $(0: 0: 0: b: a+c-b)$ |
| $e_3 \cap f_{23}$ | $(b(a+c-b): b(a+c-b): 0: b(a-b): (a-b)(a+c-b))$ |
| $e_3 \cap f_{34}$ | $(b: b: 0: b: a+c-b)$ |
| $e_3 \cap f_{35}$ | $(1: 1: 0: 0: 0)$ |
| $e_4 \cap g$ | $(b(c-b): 0: bc: bc: (c-b)(a+c-b))$ |
| $e_4 \cap f_{14}$ | $(b(c-b): 0: b(a+c-b): b(a+c-b): (c-b)(a+c-b))$ |
| $e_4 \cap f_{24}$ | $(b: 0: 0: 0: a+c-b)$ |
| $e_4 \cap f_{34}$ | $(b: 0: b: b: a+c-b)$ |
| $e_4 \cap f_{45}$ | $(0: 0: 1: 1: 0)$ |
| $e_5 \cap g$ | $(b(b-c): a(b-c): c(a-b): b(a-b): 0)$ |
| $e_5 \cap f_{15}$ | $(b(b-c): a(b-c): ac: ab: 0)$ |
| $e_5 \cap f_{25}$ | $(bc: ac: c(b-a): b(b-a): 0)$ |
| $e_5 \cap f_{35}$ | $(b: a: 0: 0: 0)$ |
| $e_5 \cap f_{45}$ | $(0: 0: c: b: 0)$ |
| $f_{12} \cap f_{34}$ | $(0: -1: 1: 0: 0)$ |
| $f_{12} \cap f_{35}$ | $(0: 1: 0: 0: 0)$ |
| $f_{12} \cap f_{45}$ | $(0: 0: 1: 0: 0)$ |
| $f_{13} \cap f_{24}$ | $(0: 0: 0: 0: 1)$ |
| $f_{13} \cap f_{25}$ | $(0: 0: 0: b: c)$ |
| $f_{13} \cap f_{45}$ | $(0: 0: 0: 1: 0)$ |
| $f_{14} \cap f_{23}$ | $(b(c-b): b(c-b): b(a-b): b(a-b): (a-b)(c-b))$ |
| $f_{14} \cap f_{25}$ | $(b^2: ab: b(b-a): b(b-a): (c-b)(b-a))$ |
| $f_{14} \cap f_{35}$ | $(c: a+c-b: 0: 0: 0)$ |
| $f_{15} \cap f_{23}$ | $(b(b-c): b(b-c): bc: b^2: (b-a)(c-b))$ |
| $f_{15} \cap f_{24}$ | $(b: 0: 0: 0: a)$ |
| $f_{15} \cap f_{34}$ | $(b: b-c: c: b: a+c-b)$ |
| $f_{23} \cap f_{45}$ | $(0: 0: a+c-b: a: 0)$ |
| $f_{24} \cap f_{35}$ | $(1: 0: 0: 0: 0)$ |
| $f_{25} \cap f_{34}$ | $(b: a: b-a: b: a+c-b)$ |

TABLE IV

Cross Ratios of Degree 4 Surfaces

| Pairs of Intersecting Lines | Cross Ratio |
|---|---|
| $e_1 g, f_{23} f_{45}, f_{24} f_{35}, f_{25} f_{34}$<br>$e_2 f_{12}, e_3 f_{13}, e_4 f_{14}, e_5 f_{15}$ | $\frac{c}{b}$ |
| $e_2 g, f_{13} f_{45}, f_{14} f_{35}, f_{15} f_{34}$<br>$e_1 f_{12}, e_3 f_{23}, e_4 f_{24}, e_5 f_{25}$ | $\frac{a}{b}$ |
| $e_3 g, f_{12} f_{45}, f_{14} f_{25}, f_{15} f_{24}$<br>$e_1 f_{13}, e_2 f_{23}, e_4 f_{34}, e_5 f_{35}$ | $\frac{a}{a+c-b}$ |
| $e_4 g, f_{12} f_{35}, f_{13} f_{25}, f_{15} f_{23}$<br>$e_1 f_{14}, e_2 f_{24}, e_3 f_{34}, e_5 f_{45}$ | $\frac{c}{a+c-b}$ |
| $e_5 g, f_{12} f_{34}, f_{13} f_{24}, f_{14} f_{23}$<br>$e_1 f_{15}, e_2 f_{25}, e_3 f_{35}, e_4 f_{45}$ | $\frac{ac}{b(a+c-b)}$ |

automorphism $\Gamma: F \to F$ of the entire family, with $\Gamma \in W(D_5)$. The first observation is that, on $M$, we have $\gamma(m) = \bar{m}$. (The renaming of lines of the surface comes from conjugating the equations, which means conjugating the parameters.) This allows us to determine which $E$-points of $M$ permit $E$-automorphisms for descent to $k$. To then find the $k$-equations of the descended surface, we make the linear action $\Gamma$ agree with the galois action $\gamma$.

The particular action we are interested in is number IV in Manin's list [9]. The orbits of the lines consist of eight pairs of intersecting lines. We may assume that the galois group of the splitting extension is $\mathbb{Z}/2\mathbb{Z}$, with nontrivial element acting through

$$\gamma = (e_1 f_{15})(e_2 f_{25})(e_3 f_{35})(e_4 f_{45})(e_5 g)(f_{12} f_{34})(f_{13} f_{24})(f_{14} f_{23}).$$

Manin has shown that $H^1(G, \operatorname{Pic}(X_E))$, for a surface with this action, is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since a detailed description of this group is essential to an explicit calculation of the group of zero cycles on these surfaces, we give a proof of this result.

**Proposition 4.1.** *Let $X$ be a $k$-rational del Pezzo surface of degree 4 with quadratic splitting field $E$ and galois group $\langle \gamma \rangle = G$. Then*

$$H^1(G, \operatorname{Pic}(X_E)) \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

TABLE V

Action of the Weyl Group

| Root | Action on moduli space |
|---|---|
| Linear action on $\mathbb{P}^4$ | |

$r_1 = (e_1 e_2)$ $\qquad (a\colon\ b\colon\ c) \to (b-c\colon\ b\colon\ b-a)$

$$\begin{pmatrix} c(c-b) & 0 & 0 & 0 & b(c-b) \\ (a+c-b)(c-b) & (b-c)(a-b) & 0 & 0 & b(c-b) \\ 0 & 0 & (b-c)(a-b) & (a-b)(a+c-b) & b(a-b) \\ 0 & 0 & 0 & a(a-b) & b(a-b) \\ 0 & 0 & 0 & 0 & (a-b)(c-b) \end{pmatrix}$$

$r_2 = (e_2 e_3)$ $\qquad (a\colon\ b\colon\ c) \to (a+c-b\colon\ c\colon\ b)$

$$\begin{pmatrix} -b & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & b \\ a-b & 0 & 0 & b & -b \\ -b & b & b & 0 & 0 \\ -(a+c-b) & c & 0 & 0 & 0 \end{pmatrix}$$

$r_3 =$ quadratic $\qquad (a\colon\ b\colon\ c) \to (ac\colon\ (a-b)c\colon\ (a-b)b)$

$$\begin{pmatrix} -b^2c & b^2c & abc & -b^2(a+c-b) & (a-b)\,bc \\ 0 & b^2c & abc & -abc & 0 \\ 0 & 0 & 0 & (a-b)\,bc & 0 \\ 0 & 0 & (a-b)\,bc & 0 & 0 \\ 0 & 0 & (a-b)(a+c-b)\,c & -b(a-b)(a+c-b) & (a-b)^2c \end{pmatrix}$$

$r_4 = (e_3 e_4)$ $\qquad (a\colon\ b\colon\ c) \to (b-a\colon\ b\colon\ b-c)$

$$\begin{pmatrix} 0 & 0 & 0 & c(c-b) & b(c-b) \\ 0 & 0 & (b-a)(c-b) & (c-b)(a+c-b) & b(c-b) \\ (a-b)(a+c-b) & (b-a)(c-b) & 0 & 0 & b(a-b) \\ a(a-b) & 0 & 0 & 0 & b(a-b) \\ 0 & 0 & 0 & 0 & (a-b)(c-b) \end{pmatrix}$$

$r_5 = (e_4 e_5)$ $\qquad (a\colon b\colon c) \to (ac\colon (a-b)\,c\colon (a-b)\,b)$

$$\begin{pmatrix} b^2c & 0 & 0 & 0 & 0 \\ abc & -(a-b)\,bc & 0 & 0 & 0 \\ (b-a)\,bc & (a-b)\,bc & (a-b)\,bc & 0 & 0 \\ b^2c & 0 & 0 & (a-b)\,b^2 & -bc(a-b) \\ bc(a+c-b) & 0 & 0 & 0 & -c^2(a-b) \end{pmatrix}$$

*Proof.* We begin by choosing a $G$-invariant set of divisors which generates the Picard group. Take

$$e_4, f_{45}, e_3, f_{35}, e_2, f_{25}, e_5, g.$$

Let $L$ be the free abelian group on this set. Define the set $R$ of relations by the exact sequence

$$0 \to R \to L \to \mathrm{Pic}(X_E) \to 0.$$

Then $R$ is generated by the $G$-invariant elements

$$\phi = e_3 + f_{35} - e_4 - f_{45}$$
$$\psi = e_2 + f_{25} - e_4 - f_{45}.$$

Since $L$ is a free $\mathbb{Z}[G]$ module, it is clear that $H^i(G, L) = 0$ for all $i > 0$. So, the cohomology of the above exact sequence gives an isomorphism

$$H^1(G, \mathrm{Pic}(X_E)) \approx H^2(G, R) = R^G/N(R).$$

But, as we saw above, $R = R^G$. The norms are

$$N(\phi) = 2\phi$$
$$N(\psi) = 2\psi.$$

So, $H^1(G, \mathrm{Pic}\, X_E) \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, with generators $\phi$, $\psi$.

*Remark.* We may think of $\phi$, $\psi$ as rational functions with the specified divisors. Then the injection $R \to E(X)^*/E^*$ allows us to interpret elements of $H^2(G, R)$ as Azumaya algebras on $X$. Suppose now that $k$ is a local field. As we have seen, zero cycles $x - y$ of degree zero are tested by pairing against elements of $H^2(G, R)$. In this case, pairing reduces to looking at

$$\frac{\phi(x)}{\phi(y)} \text{ and } \frac{\psi(x)}{\psi(y)} \text{ in } k^*/NE^* = H^2(G, E^*) = \mathbb{Z}/2\mathbb{Z}.$$

We now return to the problem of constructing type IV surfaces. We first determine which $E$-points of $\mathbb{P}^2$ can give rise to such surfaces.

LEMMA 4.2. *Let $m \in \mathbb{P}^2(E)$ be such that $F_m$ is $E$-isomorphic to the extension of a $k$-surface $X$ of type IV. Then $m$ is actually a $k$-rational point.*

*Proof.* From Table IV of Section 3, we see that the element $\Gamma \in W(D_5)$ representing the type IV action must fix all the cross ratios. If we let $\gamma \in \mathrm{Gal}(E/k)$ be the nontrivial element, then $\gamma(m) = \Gamma(m) = m$. So, $m$ is actually a $k$-point.

Now fix a $k$-point $m=(a\colon b\colon c)\in M$. The next step is to determine the $E$-automorphism $\Gamma$ which has the right action on the lines. We use the fact that linear maps of $\mathbb{P}^4$ are determined by their action on the points $f_{24}\cap f_{35}$, $f_{12}\cap f_{35}$, $f_{12}\cap f_{45}$, $f_{13}\cap f_{45}$, $f_{13}\cap f_{24}$, $e_3\cap f_{35}$, $f_{12}\cap f_{34}$, $e_4\cap f_{45}$, $f_{13}\cap f_{25}$. A glance at Table III reveals that these points are chosen to make calculation of the matrices extremely easy. Since we know explicitly the action on the points, straightforward calculation shows that $\Gamma$ is represented by the action of the following matrix on $\mathbb{P}^4$

$$\Gamma=\begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ -\dfrac{a+c-b}{b} & \dfrac{a+c-b}{b} & \dfrac{a+c-b}{b} & \dfrac{a+c-b}{b} & 1 \end{pmatrix}$$

PROPOSITION 4.3. *Let $E=k(\sqrt{d})$. Every $E$-split $k$-surface of type IV is $k$-isomorphic to one defined by*

$$a(dR^2-T^2)=b(V+S)(S+U)$$
$$c(dR^2-S^2)=b(V+T)(T-U).$$

*Proof.* By descent theory, the $k$-forms of an $E$-split type IV surface $E$-isomorphic to $F_m$ are classified by $H^1(G,\operatorname{Aut}_E(F_m))=\mathbb{Z}/2\mathbb{Z}$, where $m=(a\colon b\colon c)$ is a $k$-point. The nontrivial form associated to $F_m$ is found by imposing the condition $\Gamma(p)=\bar{p}$, where $\Gamma$ is the above matrix and $p\to\bar{p}$ is the nontrivial element of $G=\mathbb{Z}/2\mathbb{Z}$. This leads to the system of equations

$$\bar{r}=s+t-u$$
$$\bar{s}=s$$
$$\bar{t}=t$$
$$\bar{u}=s+t-r$$
$$\bar{v}=\frac{a+c-b}{b}(s+t-r-u)+v.$$

In addition to the rationality of $s$ and $t$, these equations imply

$$\bar{r}-r=\bar{u}-u=\frac{b}{a+c-b}(\bar{v}-v)$$

and

$$\bar{r} + r + \bar{u} + u = 2(s + t).$$

This set of conditions is met by the change of variables

$$\begin{aligned} r &= T + \sqrt{d}R \\ s &= S + U \\ t &= T - U \\ u &= S + \sqrt{d}R \\ bv &= cS + aT + bV + (a + c - b)\sqrt{d}R \end{aligned}$$

which leads to the above set of equations.

The subgroup $\langle \gamma \rangle$ is invariant under conjugation by the group $S_4$ of permutations of $\{e_1, e_2, e_3, e_4\}$. So, renaming the lines by any permutation in this $S_4$ yields a $k$-isomorphism of type IV surfaces listed in Proposition 4.3. The moduli space of $E$-split type IV $k$-surfaces should therefore be thought of as the quotient of $N$ by $S_4$. Type IV surfaces can thus degenerate in three ways, by $(a\colon b\colon c)$ moving

($e$) to one of the lines $e_i$,

($f$) to one of the lines $f_{ij}$,

($p$) to an intersection point $e_i \cap f_{ij}$.

For calculation purposes, we may choose to which of the lines $e_i$ or $f_{ij}$ the point $(a\colon b\colon c)$ degenerates.

Recall that equivalence of points is tested by evaluation at the rational functions (with divisors):

$$F = \frac{t}{s} = \frac{T - U}{S + U} \qquad (e_3 + f_{35} - e_4 - f_{45})$$

$$G = \frac{at + (a - b)s}{s} = \frac{a(S + T) - b(S + U)}{S + U} \qquad (e_2 + f_{25} - e_4 - f_{45}).$$

To see what values may be taken on by these functions, we first make the substitution

$$T - U = \phi(S + U).$$

The pair of equations becomes

$$a(dR^2 - T^2) = \frac{b}{1 + \phi}(V + S)(S + T)$$

$$c(dR^2 - S^2) = \frac{b\phi}{1 + \phi}(V + T)(S + T).$$

Eliminate $R$ from the equations and cancel a common factor of $S+T$. Then,

$$b(\phi a-c)\,V=(c+c\phi-b\phi)\,aT+(b-a-a\phi)\,cS.$$

Substituting in the first equation, we find

$$(1+\phi)(a\phi-c)(dR^2-T^2)=(b\phi-c-c\phi)(S^2-T^2)$$

or

$$\left[\frac{(1+\phi)(a\phi-c)}{b\phi-c-c\phi}\right]dR^2+\left[1-\frac{(1+\phi)(a\phi-c)}{b\phi-c-c\phi}\right]T^2=S^2.$$

This conic has a $k$-rational point, and hence the surface on which it lies has a $k$-point $x$ with $F(x)=\phi$, and only if

$$\Phi\equiv 1-\frac{(1+\phi)(a\phi-c)}{b\phi-c-c\phi}=\frac{\phi(b-a-a\phi)}{b\phi-c-c\phi}\in NE^*.$$

We now do the same analysis for the rational function $G$. Set

$$\psi(S+U)=a(T+S)-b(S+U).$$

Then

$$dR^2-T^2=\frac{b}{b+\psi}(V+S)(S+T)$$

$$dR^2-S^2=\frac{b(b+\psi-a)}{c(b+\psi)}(V+T)(S+T).$$

Again eliminating $R^2$ and dividing by $S+T$ gives

$$b(b+\psi-a-c)\,V=(cb+c\psi-b^2-b\psi+ab)\,T-\psi cS.$$

After substituting in the first equation, we find

$$\left[\frac{(b+\psi)(b+\psi-a-c)}{b(b-a-c)+\psi(b-c)}\right]dR^2+\left[1-\frac{(b+\psi)(b+\psi-a-c)}{b(b-a-c)+\psi(b-c)}\right]T^2=S^2.$$

Hence, there is a $k$-point with $G(x)=\psi$ if and only if

$$\Psi=\frac{\psi(a-b-\psi)}{b(b-a-c)+\psi(b-c)}\in NE^*.$$

Finally, we run through this calculation for the quotient

$$\frac{G}{F}=\frac{a(S+T)-b(S+U)}{T-U}=\frac{a(T-U)+(a-b)(S+U)}{T-U}.$$

We replace $U$ using the relation

$$(\chi-a)(T-U)=(b-a)(S+U).$$

After eliminating $R^2$ and dividing by $S+T$ we are left with

$$b[a(a+c-b)-c\chi]\ V=a[c\chi-b(a+c-b)]\ T+c\chi(b-a)\ S.$$

Substituting for $V$, the first equation becomes

$$(1-X)\ dR^2+XT^2=S^2$$

with

$$X=\frac{\chi}{(\chi-a)(c\chi+b(b-a-c))}.$$

So, there is a point $x$ on $X$ with $(G/F)(x)=\chi$ if and only if $X\in NE^*$.

Now specialize to the case of $k$ a nonarchimedean (characteristic zero) local field with residue characteristic $>2$. Assume that $E/k$ is an unramified quadratic extension. A type IV surface has bad reduction if and only if the parametrizing point specializes onto one of the lines of $N$. We will refer to surfaces of type IV$e$, IV$f$, IV$p$ according to the type of bad reduction.

THEOREM 4.4. *Let $E=k(\sqrt{d})$ be an unramified extension of local fields. Let $X$ be an $E$-split $k$-surface of type* IV$f$. *Then $A_0(X)=\mathbb{Z}/2\mathbb{Z}$.*

*Proof.* Take a pair of lines $Y$ on $X$ which are interchanged by the Galois action. Firstly, their intersection point is a rational point on $X$. Next, there is a one-dimensional family of hyperplanes in $\mathbb{P}^4$ containing $Y$. This pencil gives a conic bundle structure on $X$ with few bad fibres. We can thus apply the result of Colliot-Thélène and Coray [4] that $A_0(X)$ is generated by differences of rational points. So it is enough to use the criteria determined above regarding the possible values of $F$, $G$. Observe that $\lambda\in k^*$ is a norm only if $v(\lambda)$ is even, where $v$ is the usual valuation on $k$.

Since $X$ has type IV$f$ reduction, we may assume that $v(b)>0$ and $v(a)=v(c)=0$. That is, we let $(a\colon b\colon c)$ specialize onto $f_{12}$. We claim first that the rational function $F$ does not distinguish between points on $X$, so that $A_0(X)$ is at most $\mathbb{Z}/2\mathbb{Z}$. Now $F$ takes on the value $\phi$ iff $v(\Phi)$ is even. But $v(\phi)\neq 0$ forces $v(\Phi)=v(\phi)$. This is true since

$\Phi = \phi(b-a-a\phi)/(b\phi - c - c\phi)$ and both expressions in parentheses are units. This means that every rational point $x \in X$ satisfies

$$v(F(x)) = v(\phi) = v(\Phi) = 0 \bmod 2.$$

We now know that equivalence of points on $X$ is tested by the rational function $G$. Suppose $\psi \in k^*$ is such that $0 \leqslant v(\psi) \leqslant v(b)$. If $v(\psi) = v(b)$, take $\psi$ so that

$$v(b(b-a-c) + \psi(b-c)) = v(\psi).$$

Then

$$v(\Psi) = v\left(\frac{\psi(a-b-\psi)}{b(b-a-c)+\psi(b-c)}\right) = v\left(\frac{\psi \cdot \text{unit}}{b \cdot \text{unit} + \psi \cdot \text{unit}}\right) = 0.$$

So, there are points on $X$ taking on all values $0 \leqslant v(\psi) \leqslant v(b)$. In particular, $v(G(x)) = v(\psi)$ takes on both odd and even values, so that $A_0(X) = \mathbb{Z}/2\mathbb{Z}$.

The situation is more complicated for surfaces of type IVe and IV$p$ as $A_0(X)$ depends on the parity of the valuations of the parameters. By permuting $e_i$'s, we may assume that IVe and IV$p$ surfaces arise because the parameters satisfy

$$0 = v(c) < v(b) = v(b-a) \leqslant v(a).$$

Using this model, we have

THEOREM 4.5. *Let $X$ be a type IV surface split by $E/k$, an unramified quadratic extension of local fields. Then*

$$\begin{aligned} A_0(X) &= \mathbb{Z}/2\mathbb{Z} && v(b) \text{ even} < v(a) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} && v(b) \text{ odd} < v(a) \\ &= 0 && v(b) \text{ even} = v(a) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} && v(b) \text{ odd} = v(a). \end{aligned}$$

*Proof.* Observe that

$$\begin{aligned} &\text{if } \phi = -1 && \text{then } v(\Phi) = 0 \\ &\text{if } \psi = 1 && \text{then } v(\Psi) = 0 \\ &\text{if } \chi = 1 && \text{then } v(X) = 0. \end{aligned}$$

So, to find different classes of points on the surface, we must find values for $\phi$, $\psi$, $\chi$ with odd valuations which make $v(\Phi)$, $v(\Psi)$, $v(X)$ even.

We claim first that if $v(\psi) \neq v(b)$, then $v(\Psi) = v(\psi)$. This follows because

$$\begin{aligned} v(a-b-\psi) &= v(b+\psi) \\ &= v((b-c)(b+\psi)-ab) \\ &= v(b(b-a-c)+\psi(b-c)). \end{aligned}$$

So, given any rational point $x \in X$ with $v(G(x)) = v(\psi) \neq v(b)$, we find that $v(\psi)$ is even. If $v(b)$ is actually even, then $v(\psi)$ takes on only even values and $A_0(X)$ is at most $\mathbb{Z}/2\mathbb{Z}$. If $v(b)$ is odd, however, take $\psi$ so that $v(b+\psi) \gg 0$. Then

$$v(\Psi) = v(a\psi/ab) = v(\psi) - v(b) = 0.$$

In this case, $v(\psi)$ takes on odd values and $A_0(X) \supset \mathbb{Z}/2\mathbb{Z}$.

Now turn to the values of $v(\phi)$. If $v(\phi) > 0$, then $v(\Phi) = v(\phi) + v(b-a-a\phi) = v(\phi) + v(b)$. If $v(b)$ is odd, then all positive odd values of $v(\phi)$ correspond to rational points on $X$. This contributes another factor of $\mathbb{Z}/2\mathbb{Z}$ to the group of zero cycles. To complete the odd cases of the theorem, we must check that the rational functions $F$ and $G$ (and their values $\phi, \psi$) do not encode the same information. That is, we must see that the ratio $G/F$ is not always a norm. It suffices, therefore, to check that $v(\chi)$ takes on odd values. If $v(a)$ is odd, choose $\chi$ so that $v(\chi - a) = v(\chi) + 1$. Then

$$\begin{aligned} v(X) &= v(\chi) - v(\chi - a) - v(c\chi + b(b-a-c)) \\ &= v(\chi) - v(\chi) - 1 - v(b) \\ &= -1 - v(b) \end{aligned}$$

which is even. On the other hand, if $v(a)$ is even, take $v(\chi) > v(a)$ and odd, so that

$$v(X) = v(\chi) - v(a) - v(b)$$

is again even.

We may now assume $v(b)$ is even. We have seen that $v(\psi)$ is always even, so that $A_0(X) \subset \mathbb{Z}/2\mathbb{Z}$ and equivalence of points is tested by $v(\phi)$. We have also seen that $v(\phi) > 0$ forces $v(\Phi) = v(\phi) + v(b)$ and therefore positive $v(\phi)$ are always even. So, suppose $v(\phi) < 0$. Then

$$v(\Phi) = v(\phi) + v\left(\frac{b-a}{\phi} - a\right).$$

If $v(a)=v(b)=v(b-a)$, then $v(\Phi)< =v(\phi)+v(a)=v(\phi)+v(b)$. Since $v(b)$ is even, this forces $v(\phi)$ to be even and $A_0(X)=0$. If, however, $v(a)>v(b)$, choose $\phi$ so that $v(\phi)=-1$ and $v((b-a)/\phi-1)=v(b-a)+1$. Then $v(\Phi)=v(b-a)$ is even, and $A_0(X)=\mathbb{Z}/2\mathbb{Z}$.

## Acknowledgments

We would like to thank S. Bloch for bringing this problem to our attention and for many helpful discussions during our work. We would also like to thank M. Sullivan for typing the manuscript.

## References

1. S. Bloch, On the Chow groups of certain rational surfaces, *Ann. Sci. École Norm. Sup.* **14** (1981), 41–59.
2. S. Bloch, "Lectures on Algebraic Cycles," Duke Math. Series, Duke University, Durham, N.C., 1980.
3. J.-L. Colliot-Thélène, Hilbert's theorem 90 for $K_2$, with application to the Chow groups of rational surfaces, *Invent. Math.* **71** (1983), 1–20.
4. J.-L. Colliot-Thélène and D. Coray, L'équivalence rationnelle sur les points fermes des surfaces rationnelles fibrées en coniques, *Compositio Math.* **39** (1979), 301–332.
5. J.-L. Colliot-Thélène and J.-J. Sansuc, On the Chow groups of certain rational surfaces: a sequel to a paper of S. Bloch, *Duke Math. J.* **48** (1981), 421–447.
6. J.-L. Colliot-Thélène and J.-J. Sansuc, La *R*-equivalence sur les tores, *Ann. Sci. École Norm. Sup.* **10** (1977), 175–230.
7. F. Enriques, Sulle irrazionalita da cui puo farsi dipendere la risoluzione d'un equazione $f(xyz)=0$ con funzioni razionali di due parametri, *Math. Ann.* **49** (1897), 1–23.
8. V. A. Iskovskih, Minimal models of rational surfaces over arbitrary fields, *Izv. Akad. Nauk. SSSR* **43** (1979), 19–43, 237. (Engl. trans. *Math. USSR-Izv.* **14** (1980), 17–39.)
9. J. I. Manin, Rational surfaces over perfect fields I, *Publ. Math. IHES* **30** (1966), 55–113. (Engl. trans. *Amer. Math. Soc. Transl.* **84** (1969), 137–186.)
10. J. I. Manin, "Cubic forms," North-Holland, Amsterdam, 1974.
11. M. Nagata, On rational surfaces I, *Mem. Coll. Sci. Kyoto, Ser. A.* **32** (1960), 351–370.
12. I. Naruki, Cross ratio variety as a moduli space of cubic surfaces, *Proc. London Math. Soc.* **45** (1982), 1–30.
13. H. P. F. Swinnerton-Dyer, Rational points on del Pezzo surfaces of degree 5, *in* "Algebraic Geometry, Oslo 1970," pp. 287–290.
14. G. Timms, The nodal cubic surfaces and the surfaces from which they are derived by projection, *Proc. Roy. Soc. London* **119** (1928), 213–248.