Computers and Mathematics with Applications 62 (2011) 4708–4716

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Synchronization between a fractional-order system and an integer order system

Li-xin Yang*, Wan-sheng He, Xiao-jun Liu

School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, Gansu, China

ARTICLE INFO

Article history:
Received 14 March 2011
Received in revised form 4 August 2011
Accepted 24 October 2011

Keywords:
Fractional-order chaotic system
Chaotic system of integer order
Stability theory
Tracking control

ABSTRACT

This paper investigates chaotic synchronization between fractional-order chaotic systems and integer-order chaotic systems. Based on the idea of tracking control and the stability theory of the linear fractional-order system, we design the effective controller to realize the synchronization between fractional-order and integer-order chaotic systems. Theory analysis and numerical simulation results show that the method is effective and feasible.

1. Introduction

The theory of fractional calculus is a 300-year-old topic which can trace back to Leibniz, Riemann, Liouville, Grünwald, and Letnikov [1,2]. However, the fractional calculus did not attract much attention for a long time. Nowadays, the past three decades have witnessed significant progress on fractional calculus, because the applications of fractional calculus were found in more and more scientific fields, covering mechanics, physics, engineering, informatics, and materials. The list of such applications is long, for instance, it includes viscoelasticity [3,4], colored noise, dielectric polarization [5], electrode–electrolyte polarization [6], electromagnetic waves [7] and so on.

On the other hand, since Pecora and Carroll synchronizing two identical chaotic systems with different initial conditions [8], chaotic synchronization has been intensively and extensively investigated because of its potential applications in a variety of areas, such as in secure communications [9,10], chemical and biological systems [11], human heart beat regulation [12] and so on [13–16]. So far, a number of approaches have been proposed to achieve chaos synchronization such as PC method, OGY method, adaptive method, impulsive control, coupling control, etc. Furthermore, recently, fractional-order chaotic systems have become a hot topic. With respect to some recent representative works on this topic, we refer the reader to [14–19] and references therein. In Ref. [14], the chaos in the Chen system with fractional-order has been addressed. Besides, some synchronization-based strategies have been devised to synchronize fractional chaotic systems [15–17]. In Ref. [19], the synchronization of fractional-order chaotic systems has been presented. Moreover, some methods such as PC method and nonlinear control are employed to synchronize two fractional-order chaotic systems [20,21].

It is widely believed that the synchronization between a fractional-order hyper-chaotic system and a hyper-chaotic system of integer order can be applied in encryption efficiently which can enlarge the key space. In Ref. [22], the synchronization of fractional-order chaotic systems has been presented. In Ref. [23], the nonlinear control is employed to synchronize two fractional-order chaotic systems. However, to the best of authors' knowledge, the results on synchronization between the fractional-order chaotic system and the chaotic system of integer order are limited.

* Corresponding author.
E-mail address: jiaodayanglixin@163.com (L.-x. Yang).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.10.061

As compared with other synchronizations, the synchronization between the fractional-order system and the integer-order system can be summarized in the following aspects.

(1) This kind of synchronization can be used in secure communications since the fractional-order chaotic system has more adjustable variables than the integer order chaotic system. So it can additionally enhance the security of communication.
(2) From a practical perspective, it is very important to synchronize two systems of different orders. The cause of the synchronization between the fractional-order system and the integer-order system can generate hybrid chaotic transient signals before the final states and hard to decryption.

Motivated by the above discussions, in this paper, we consider the problem of the synchronization between the fractional-order chaotic system and the chaotic system of integer order. Based on the tracking control, the synchronization between the fractional-order chaotic system and the chaotic system of integer order is achieved.

This paper is organized as follows. In Section 2, some necessary definitions and notations are given. In Section 3, the problem of the synchronization between the fractional order chaotic system and the chaotic system of integer order is investigated. The corresponding simulation results are provided in Section 3 to demonstrate the effectiveness of the proposed method. Finally, the concluding remarks are given.

2. Fractional derivative and its approximation

Fractional calculus is a generalization of integration and differentiation to a noninteger-order integro-differential operator ${}_aD_t^\alpha$ defined by

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & R(\alpha) > 0 \\ 1 & R(\alpha) = 0 \\ \displaystyle\int_\alpha^t (d\tau)^{-\alpha} & R(\alpha) < 0. \end{cases} \tag{1}$$

There are many definitions of fractional derivatives. Perhaps the best-known one is the Riemann–Liouville definition, which is given by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2}$$

for $n-1 \le \alpha < n$, where $\Gamma(\cdot)$ is the gamma function. The geometric and physical interpretation of the fractional derivative is

$$\int_0^\infty e^{-st} {}_0D_t^\alpha f(t) dt = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k {}_0D_t^{\alpha-k-1} f(t)|_{t=0,} \tag{3}$$

for $n-1 \le \alpha < n$, where $s \equiv jw$ denotes the Laplace variable. Upon considering the initial conditions to be zero, this formula reduces to

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha L\{f(t)\}. \tag{4}$$

Thus, the fractional integral operator order "α" can be represented by the transfer function $F(s) = \frac{1}{s^\alpha}$ in the frequency domain.

The standard definitions of fractional differintegral do not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using standard integer-order operators.

Unlike the numerical algorithm for solving an ordinary differential equation, the numerical simulation of a fractional differential equation is not so easy. In this letter, we use the Caputo version and employ a predictor–corrector algorithm for fractional-order differential equations, which is the generalization of the Adams–Bashforth–Moulton one.

3. Synchronization between a fractional-order system and a system of integer-order

3.1. Synchronization between a fractional-order Lü system and a system of integer order

In this section, a fractional-order Lü system and a new system of integer order are used to demonstrate the effectiveness of the proposed method.

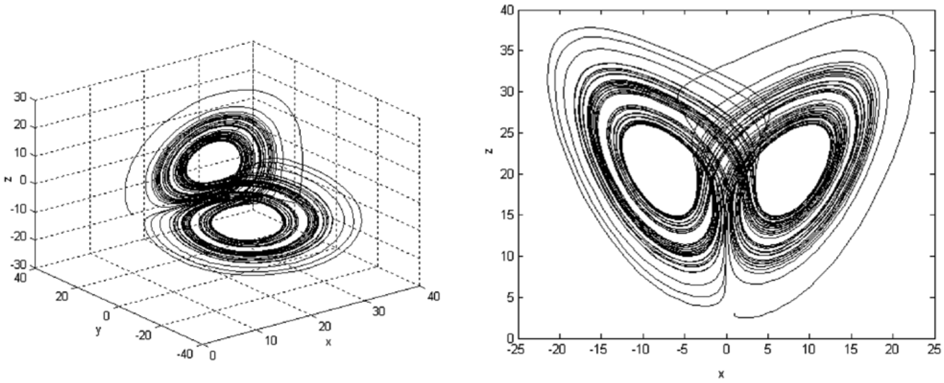

Fig. 1. The attractor of the fractional-order Lü system with $q = (0.985, 0.99, 0.98)$.

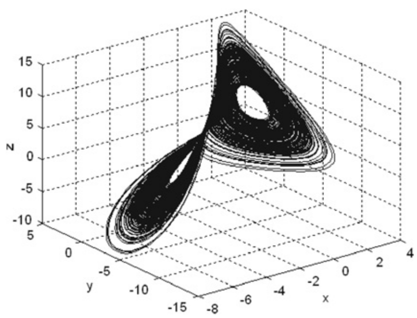

Fig. 2. The attractor of system (6).

The fractional-order Lü dynamical differential equation can be described by

$$\begin{cases} \dfrac{d^{q_1}x}{dt^{q_1}} = -a(y - x) \\ \dfrac{d^{q_2}y}{dt^{q_2}} = -xz + cy \\ \dfrac{d^{q_3}z}{dt^{q_3}} = xy - bz, \end{cases} \tag{5}$$

where $0 < q_1, q_2, q_3 \leq 1, q = (q_1, q_2, q_3)$, with the parameters $(a, b, c) = (36, 3, 20)$, and fractional-order of the system $q = (0.985, 0.99, 0.98)$. Fig. 1 depicts the chaotic attractor of system (5).

In 2007, Chu find a new chaotic system, the dynamical differential equation can be described by

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = x_1x_3 - x_2 \\ \dot{x}_3 = b - x_1x_2 - cx_3, \end{cases} \tag{6}$$

where $x_1, x_2, x_3 \in R^3$ represent the state vectors of this system, when $a = 15, c = 1, b = 16$, system (6) display chaotic attractors. Fig. 2 shows the attractor of this system.

Let us consider system (6) of integer order is the drive system, then the response system is the fractional-order Lü system in the form of

$$\begin{cases} \dfrac{d^{q_1}y_1}{dt^{q_1}} = -a(y_2 - y_1) + u_1(x(t)) + U_1(y(t), x(t)) \\ \dfrac{d^{q_2}y_2}{dt^{q_2}} = -y_1y_3 + cy_2 + u_2(x(t)) + U_2(y(t), x(t)) \\ \dfrac{d^{q_3}y_3}{dt^{q_3}} = y_1y_2 - by_3 + u_3(x(t)) + U_3(y(t), x(t)), \end{cases} \tag{7}$$

where $u(x(t)) + U(y(t), x(t))$ is the tracking controller, in the following we design the tracking controller

$$u(x(t)) = \frac{d^q x(t)}{dt^q} - f(x(t)), \tag{8}$$

where $f(x(t)) = \begin{bmatrix} -a(x_2 - x_1) \\ -x_1x_3 + cx_2 \\ x_1x_2 - bx_3 \end{bmatrix}$. (9)

So the controlled fractional-order Lü system (7) can be rewritten as

$$\begin{bmatrix} \frac{d^q y_1}{dt^q} \\ \frac{d^q y_2}{dt^q} \\ \frac{d^q y_3}{dt^q} \end{bmatrix} = \begin{bmatrix} -a(y_2 - y_1) \\ -y_1y_3 + cy_2 \\ y_1y_2 - by_3 \end{bmatrix} \frac{d^q x(t)}{dt^q} - \begin{bmatrix} -a(x_2 - x_1) \\ -x_1x_3 + cx_2 \\ x_1x_2 - bx_3 \end{bmatrix} + U(y(t), x(t)). \tag{10}$$

Denote error

$e_i = y_i - x_i \quad (i = 1, 2, 3).$

Then error system can be obtained from (10) described by

$$\begin{bmatrix} \frac{d^q e_1}{dt^q} \\ \frac{d^q e_2}{dt^q} \\ \frac{d^q e_3}{dt^q} \end{bmatrix} = \begin{bmatrix} a(e_1 - e_2) \\ -x_1e_3 + x_3e_1 - e_1e_3 + ce_2 \\ x_1e_2 + x_2e_1 + e_1e_2 - be_3 \end{bmatrix} + U(y(t), x(t)). \tag{11}$$

The main goal is to find a controller for system (11) such that systems (6) and (7) achieve the chaotic synchronization, it is to say that we should find a suitable controller so that $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - x(t)\| = 0$ is satisfied.

Let $e(t) = \binom{e_1(t)}{e_2(t)}$, where $e_1(t) = e_1$, $e_2(t) = (e_2, e_3)^{\mathrm{T}}$, so we rewrite Eq. (11) in the following form

$$\begin{bmatrix} \frac{d^q e_1}{dt^q} \\ \frac{d^q e_2}{dt^q} \\ \frac{d^q e_3}{dt^q} \end{bmatrix} = \begin{bmatrix} B_1e_1(t) + F_1(x(t), e_1(t), e_2(t)) \\ B_2e_2(t) + F_{21}(x(t), e_1(t), e_2(t)) + F_{22}(x(t), e_2(t)) \end{bmatrix} + U(y(t), x(t)), \tag{12}$$

where $B_1 = a$, $B_2 = \begin{bmatrix} c & 0 \\ 0 & -b \end{bmatrix}$, $F_1(x(t), e_1(t), e_2(t)) = -ae_2$, $F_{21}(x(t), e_1(t), e_2(t)) = \begin{bmatrix} x_3e_1 - e_1e_3 \\ x_2e_1 + e_1e_2 \end{bmatrix}$,

$$F_{22}(x(t), e_2(t)) = \begin{bmatrix} -x_1e_3 \\ x_1e_2 \end{bmatrix}. \tag{13}$$

It is to see that

$$\lim_{e_1(t)\to 0} F_1(x(t), e_1(t), e_2(t)) = \lim_{e_1(t)\to 0} \begin{bmatrix} x_3e_1 - e_1e_3 \\ x_2e_1 + e_1e_2 \end{bmatrix} = 0. \tag{14}$$

For simplicity, we can rewrite Eq. (12) in the following form

$$\begin{bmatrix} \frac{d^q e_1(t)}{dt^q} \\ \frac{d^q e_2(t)}{dt^q} \end{bmatrix} = \begin{bmatrix} B_1e_1(t) + F_1(x(t), e_1(t), e_2(t)) \\ B_2e_2(t) + F_{21}(x(t), e_1(t), e_2(t)) + F_2(x(t), e_2(t)) \end{bmatrix} + U(y(t), x(t)). \tag{15}$$

Let us choose the control function as follows

$$U(y(t), x(t)) = \begin{bmatrix} A_1e_1(t) - F_1(x(t), e_1(t), e_2(t)) \\ A_2e_2 - F_{22}(x(t), e_2(t)) \end{bmatrix}, \tag{16}$$

where A_1, A_2 are matrices to be determined, from Eqs. (14)–(16), it is easy to get that

$$\begin{cases} \frac{d^q e_1(t)}{dt^q} = (a + A_1)e_1(t) \\ \frac{d^q e_2(t)}{dt^q} = \left(\begin{bmatrix} c & 0 \\ 0 & -b \end{bmatrix} + A_2 \right) e_2(t) + F_{21}(x(t), e_1(t), e_2(t)). \end{cases} \tag{17}$$

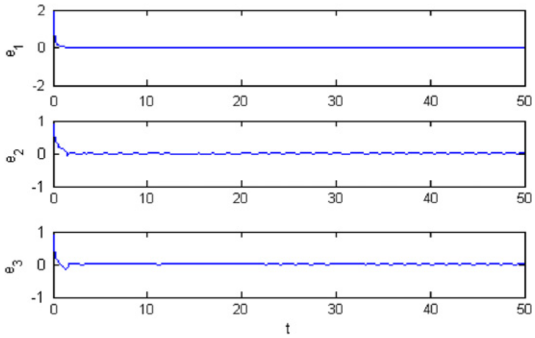

Fig. 3. The synchronization errors between the Lü system and the Chu system.

Before we give our main results, the following lemma should be given first.

Lemma 1 (*See [24]*). *The following autonomous system*

$$D^q X = AX, \quad X(0) = X_0,$$

with $0 < q < 1, X \in R^n A \in R^{n\times n}$*, is asymptotically stable if and only if* $|\arg(\lambda)| > \frac{q\pi}{2}$ *is satisfied for all eigenvalues* (λ) *of matrix A. Also, this system is stable if and only if* $|\arg(\lambda)| \geq \frac{q\pi}{2}$ *is satisfied for all eigenvalues of matrix A and those critical eigenvalues which satisfy* $|\arg(\lambda)| > \frac{q\pi}{2}$ *have geometric multiplicity one.*

Theorem 1. *If we choose appropriate matrices* $A_1, A_2 \in R^{2\times 2}$*, so that* $A_1 + a < 0$ *and all eigenvalues of matrix* $\left(\begin{bmatrix} c & 0 \\ 0 & -b \end{bmatrix} + A_2\right)$ *satisfy Lemma 1, then* $\lim_{t\to\infty} e_i = \lim_{t\to\infty} y_i - x_i = 0$ $(i = 1, 2, 3)$ *is satisfied, which means that drive system (6) and response system (7) have achieved the synchronization via controller.*

In the following steps, we would like to give the numerical simulations to verify the effectiveness of the above-designed controller. The initial conditions are $x(0) = (0, 1, 1)^{\mathrm{T}}, y(0) = (2, 3, 5)^{\mathrm{T}}$. $A_1 = -16, A_2 = \begin{bmatrix} -19 & 11 \\ -10 & -7 \end{bmatrix}$, the error curve is shown in Fig. 3.

3.2. Synchronization between the fractional-order Lorenz hyper-chaotic system and the Chen hyper-chaotic system of integer order

The fractional-order hyper-chaotic Lorenz dynamical differential equation can be described by

$$\begin{cases} \dfrac{d^q x}{dt^a} = a(y - x) + w \\ \dfrac{d^q y}{dt^a} = cx - y - xz \\ \dfrac{d^q z}{dt^a} = xy - bz \\ \dfrac{d^q w}{dt^a} = -yz - rw, \end{cases} \tag{18}$$

where x, y, z, w represent the state vectors of the fractional-order Lorenz hyper-chaotic system. When $q = 0.98$ is the fractional-order of the system, with the parameters $a = 10, b = 8/3, c = 28, r = 1$, the fractional-order Lorenz hyper-chaotic system has a hyper-chaotic attractor as shown in Fig. 4.

The integer-order hyper-chaotic Chen dynamical differential equation can be described by

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + x_4 \\ \dot{x}_2 = dx_1 - x_1x_3 + cx_2 \\ \dot{x}_3 = x_1x_2 - bx_3 \\ \dot{x}_4 = x_2x_3 + rx_4, \end{cases} \tag{19}$$

where $X = (x_1, x_2, x_3, x_4)^{\mathrm{T}} \in R^{4\times 1}$ represent the state vectors of the Chen hyper-chaotic system, $a, b, c, d, r \in R$ are the parameters, when $a = 35, b = 3, c = 12, d = 7, r = 0.5$, system (19) display hyper-chaotic attractors in Fig. 5.

In order to observe the synchronization between the integer-order hyper-chaotic Chen system and the fractional-order Lorenz system, we define the response system is the fractional-order Lorenz system in the form of

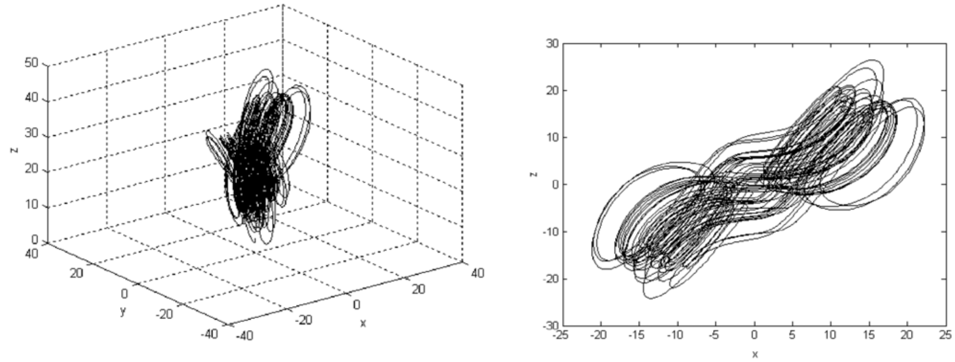

Fig. 4. The attractor of the Lorenz fractional-order system with $q = 0.98$.

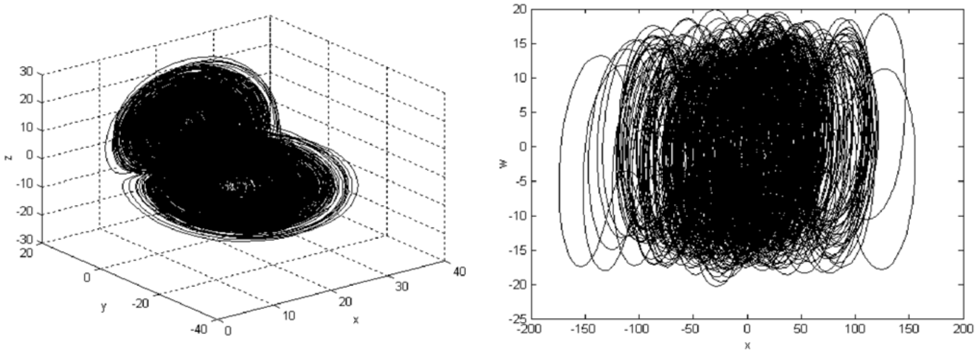

Fig. 5. The attractors of the Chen hyper-chaotic system.

$$
\begin{cases}
\dfrac{d^q y_1}{dt^a} = a(y_2 - y_1) + y_4 + u_1(x(t)) + U_1(y(t), x(t)) \\
\dfrac{d^q y_2}{dt^a} = cy_1 - y_2 - y_1 y_3 + u_2(x(t)) + U_2(y(t), x(t)) \\
\dfrac{d^q y_3}{dt^a} = y_1 y_2 - by_3 + u_3(x(t)) + U_3(y(t), x(t)) \\
\dfrac{d^q y_4}{dt^a} = -y_2 y_3 - ry_4 + u_4(x(t)) + U_4(y(t), x(t)),
\end{cases}
\tag{20}
$$

where $u(x(t)) + U(y(t), x(t))$ is the tracking controller, the tracking controller can be proposed as follows

$$
u(x(t)) = \frac{d^q x(t)}{dt^q} - f(x(t)),
\tag{21}
$$

$$
\text{where } f(x(t)) = \begin{bmatrix} a(x_2 - x_1) + x_4 \\ cx_1 - xy_2 - x_1 x_3 \\ x_1 x_2 - bx_3 \\ -x_2 x_3 - rx_4 \end{bmatrix}.
\tag{22}
$$

So the controlled fractional-order Lorenz system (20) can be rewritten as

$$
\begin{bmatrix} \dfrac{d^q y_1}{dt^q} \\ \dfrac{d^q y_2}{dt^q} \\ \dfrac{d^q y_3}{dt^q} \\ \dfrac{d^q y_4}{dt^q} \end{bmatrix} = \begin{bmatrix} a(y_2 - y_1) + y_4 \\ cy_1 - y_2 - y_1 y_3 \\ y_1 y_2 - by_3 \\ -y_2 y_3 - ry_4 \end{bmatrix} \frac{d^q x(t)}{dt^q} - \begin{bmatrix} a(x_2 - x_1) + x_4 \\ cx_1 - x_2 - x_1 x_3 \\ x_1 x_2 - bx_3 \\ -x_2 x_3 - rx_4 \end{bmatrix} + U(y(t), x(t)).
\tag{23}
$$

Let the synchronization error

$$e_i = y_i - x_i \quad (i = 1, 2, 3, 4).$$

The error system can be obtained from (23) described by

$$\begin{bmatrix} \frac{d^q e_1}{dt^q} \\ \frac{d^q e_2}{dt^q} \\ \frac{d^q e_3}{dt^q} \\ \frac{d^q e_4}{dt^q} \end{bmatrix} = \begin{bmatrix} a(e_2 - e_1) \\ ce_1 - e_2 - x_3e_1 - x_1e_3 - e_1e_3 \\ x_2e_1 + x_1e_2 + e_1e_2 - be_3 \\ -re_4 - x_3e_2 - x_2e_3 - e_2e_3 \end{bmatrix} + U(y(t), x(t)). \tag{24}$$

Our aim is to achieve the chaotic synchronization between the Lorenz system and the Chen system via the controller, so we should find a suitable controller so that $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - x(t)\| = 0$ is satisfied.

Let $e(t) = \begin{pmatrix} e_1(t) \\ e_2(t), \end{pmatrix}$ where $e_1(t) = e_1$, $e_2(t) = (e_2, e_3, e_4)^T$, so Eq. (24) can be rewritten as

$$\begin{bmatrix} \frac{d^q e_1}{dt^q} \\ \frac{d^q e_2}{dt^q} \\ \frac{d^q e_3}{dt^q} \\ \frac{d^q e_4}{dt^q} \end{bmatrix} = \begin{bmatrix} B_1e_1(t) + F_1(x(t), e_1(t), e_2(t)) \\ B_2e_2(t) + F_{21}(x(t), e_1(t), e_2(t)) + F_{22}(x(t), e_2(t)) \end{bmatrix} + U(y(t), x(t)), \tag{25}$$

where $B_1 = -a$, $B_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -r \end{bmatrix}$, $F_1(x(t), e_1(t), e_2(t)) = ae_2$, $F_{21}(x(t), e_1(t), e_2(t)) = \begin{bmatrix} -x_3e_1 - e_1e_3 \\ e_1e_2 + x_2e_1 \\ 0, \end{bmatrix}$ $F_{22}(x(t), e_2(t)) = \begin{bmatrix} -x_1e_3 \\ x_1e_2 \\ -x_3e_2 - x_2e_3 - e_2e_3 \end{bmatrix}$, it is to see that

$$\lim_{e_1(t)\to 0} F_1(x(t), e_1(t), e_2(t)) = \lim_{e_1(t)\to 0} \begin{bmatrix} -x_3e_1 - e_1e_3 \\ e_1e_2 + x_2e_1 \\ 0 \end{bmatrix} = 0.$$

For the sake of simplicity, Eq. (25) can be rewritten in the following form

$$\begin{bmatrix} \frac{d^q e_1(t)}{dt^q} \\ \frac{d^q e_2(t)}{dt^q} \end{bmatrix} = \begin{bmatrix} B_1e_1(t) + F_1(x(t), e_1(t), e_2(t)) \\ B_2e_2(t) + F_{21}(x(t), e_1(t), e_2(t)) + F_2(x(t), e_2(t)) \end{bmatrix} + U(y(t), x(t)), \tag{26}$$

we design the controller as following

$$U(y(t), x(t)) = \begin{bmatrix} A_1e_1(t) - F_1(x(t), e_1(t), e_2(t)) \\ A_2e_2 - F_{22}(x(t), e_2(t)) \end{bmatrix}, \tag{27}$$

where A_1, A_2 are matrices to be determined, it is easy to get that

$$\begin{cases} \frac{d^q e_1(t)}{dt^q} = (-a + A_1)e_1(t) \\ \frac{d^q e_2(t)}{dt^q} = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -r \end{bmatrix} + A_2 \right) e_2(t) + F_{21}(x(t), e_1(t), e_2(t)). \end{cases} \tag{28}$$

Theorem 2. *If we select appropriate matrix A_1, so that $(-a + A_1) < 0$, appropriate matrix $A_2 \in R^{2\times 2}$, so that all eigenvalues of matrix $\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -r \end{bmatrix} + A_2 \right)$ satisfy Lemma 1, then $\lim_{t\to\infty} e_i = \lim_{t\to\infty} y_i - x_i = 0$ $(i = 1, 2, 3, 4)$ is satisfied, which means that drive system (19) and response system (20) have achieved the synchronization via controller (27).*

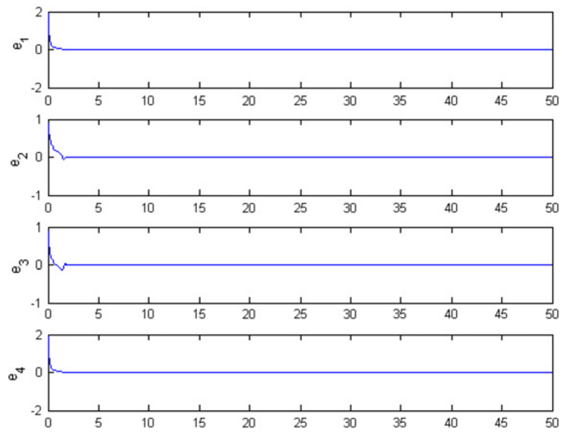

Fig. 6. The synchronization errors between the fractional-order Lorenz and Chen systems.

Analogously, we also would like to give the numerical simulations to verify the effectiveness of the above-designed controller. We select initial values as $x(0) = (0, 1, 2, 3)^T$, $y(0) = (0, 1, 3, 5)^T$, respectively. Without loss of generality, we select the matrices $A_1 = 8, A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, the errors between the Lorenz and Chen systems are shown in Fig. 6.

4. Conclusions

In this paper, we study the synchronization between the fractional-order chaotic system and the chaotic system of integer order in detail. By using stability criteria of the fractional-order system, based on the tracking control, we design the suitable controller, which is simple and applicable. Finally, the simulation results demonstrate the effectiveness of the proposed schemes.

In further works, on one hand, we would like to investigate delayed fractional-order systems, on the other hand, we want to study the identification of parameters of the fractional-order system. In addition, we intend to study the synchronization of complex networks whose nodes are composed by fractional-order chaotic systems.

Acknowledgments

The authors would like to thank the anonymous reviewers for their helpful comments and suggestions.

This research is supported by Gansu Province Government of China (0808-04) and Scientific Research Foundations of Tianshui Normal University of China (TSA0938).

References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[2] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, New Jersey, 2001.
[3] R.L. Bagley, R.A. Calico, Fractional order state equations for the control of viscoelastically damped structures, J. Guid. Control Dyn. 14 (1991) 304–311.
[4] R.C. Koeller, Application of fractional calculus to the theory of viscoelasticity, J. Appl. Mech. 51 (1984) 199.
[5] H.H. Sun, A.A. Abdelwahed, B. Onaral, Linear approximation for transfer function with a pole of fractional order, IEEE Trans. Automat. Control 29 (1984) 441–444.
[6] M. Ichise, Y. Nagayanagi, T. Kojima, An analog simulation of noninteger order transfer functions for analysis of electrode process, J. Electroanal Chem. 33 (1971) 253–265.
[7] O. Heaviside, Electromagnetic Theory, Chelsea, New York, 1971.
[8] L.M. Pecora, T.L. Carroll, Synchronization in chaotic systems, Phys. Rev. Lett. 64 (1990) 821–825.
[9] B. Nana, P. Woafo, S. Domngang, Chaotic synchronization with experimental application to secure communications, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 2266–2276.
[10] U.E. Vincent, R. Guo, A simple adaptive control for full and reduced-order synchronization of uncertain time-varying chaotic systems, Commun. Nonlinear Sci. Numer. Simulat. 14 (11) (2009) 3925–3932.
[11] C.J. Luo Albert, A theory for synchronization of dynamical systems, Commun. Nonlinear Sci. Numer. Simulat. 14 (5) (2009) 1901–1951.
[12] J.M.V. Grzybowski, M. Rafikov, J.M. Balthazar, Synchronization of the unified chaotic system and application in secure communication, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 2793–2806.
[13] Y. Tang, J. Fang, Q. Miao, Synchronization of stochastic delayed neural networks with Markovian switching and its application, Int. J. Neural. Syst. 19 (2009) 43–56.
[14] C. Li, G. Peng, Chaos in Chen's system with a fractional order, Chaos Solitons Fractals 22 (2004) 443–450.
[15] Z.M. Ge, A.R. Zhang, Chaos in a modified van der Pol system and in its fractional order systems, Chaos Solitons Fractals 32 (2007) 1791–1822.
[16] Z.M. Ge, C.Y. Ou, Chaos synchronization of fractional order modified duffing systems with parameters excited by a chaotic signal, Chaos Solitons Fractals 35 (2008) 705–717.
[17] L.J. Sheu, H.K. Chen, J.H. Chen, et al., Chaos in the Newton–Leipnik system with fractional order, Chaos Solitons Fractals 36 (2008) 98–103.
[18] J.W. Wang, Y.B. Zhang, Designing synchronization schemes for chaotic fractional-order unified systems, Chaos Solitons Fractals 30 (2006) 1265–1272.

[19] C.G. Li, X.X. Liao, J.B. Yu, Synchronization of fractional order chaotic systems, Phys. Rev. E 68 (2003) 067203.
[20] J.G. Lu, Chaotic dynamics of the fractional-order Lü system and its synchronization, Phys. Lett. A 354 (2006) 305–311.
[21] C.P. Li, J.P. Yan, The synchronization of three fractional differential systems, Chaos Solitons Fractals 32 (2007) 751–757.
[22] C.G. Li, X.X. Liao, J.B. Yu, Synchronizaiton of fractional order chaotic systems, Phys. Rev. E 68 (2003) 067–078.
[23] J.G. Lu, Chaotic dynamical of the fractional order Lu system and its synchronization, Phys. Lett. A 354 (2006) 305–311.
[24] D. Matignon, Stability results of fractional differential equationa with applications to control processing. in: IMACS, IEEE-SMC, Lille, France, pp. 963–968.