Note

On Monochromatic Paths in $m$-Coloured Tournaments

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We call the tournament $T$ an $m$-coloured tournament if the arcs of $T$ are coloured with $m$ colours. In this paper we have proved that if $T$ is an $m$-coloured tournament which does not contain any tournament of order 3 whose arcs are coloured with three distinct colours then there is a vertex $v$ of $T$ such that for every other vertex $x$ of $T$ there is a monochromatic path from $x$ to $v$.

We call the tournament $T$ an $m$-coloured tournament if the arcs of $T$ are coloured with $m$ colours. Let $T_3$ and $C_3$, respectively, denote the transitive tournament of order 3 and the 3-cycle, both of whose arcs are coloured with three distinct colours. In [1], Sands, Sauer, and Woodrow have proved that every 2-coloured tournament $T$ has a vertex $v$ such that for every other vertex $x$ of $T$ there is a monochromatic path from $x$ to $v$. They also raised the following problem:

**Problem.** Let $T$ be a 3-coloured tournament which does not contain $C_3$. Must $T$ contain a vertex $v$ such that for every other vertex of $T$ there is a monochromatic path from $x$ to $v$?

If in the problem we allow $T$ to contain neither $T_3$ nor $C_3$, the answer will be yes.

The following is our main result.

**Theorem.** Let $T$ be an $m$-coloured tournament which does not contain $T_3$ or $C_3$. Then there is a vertex $v$ of $T$ such that for every other vertex $x$ of $T$ there is a monochromatic path from $x$ to $v$.

**Proof.** We prove this by induction on $n$, the order of $T$. The cases $n = 1$ and $n = 2$ are clear. Suppose that the result holds for all $m$-coloured tour-
nments of order less than \( n \), where \( n > 2 \). So by the induction hypothesis, for each vertex \( v \) of \( T \) there is a vertex, call it \( f(v) \), of \( T \) such that for each vertex \( x \) of \( T - \{ v \} \) there is a monochromatic path from \( x \) to \( f(v) \). If there is \( f(u) = f(v) \), \( u \neq v \), or if for some \( v \) there is a monochromatic path from \( v \) to \( f(v) \), then for each vertex \( x \) of \( T \) there is a monochromatic path from \( x \) to \( f(v) \), and the result holds. So we assume \( f \) is a bijection and further that there is no monochromatic path from \( v \) to \( f(v) \). By the relabelling \( f(v_i) = v_{i+1} \), the vertices of \( T \) are partitioned into cycles

\[
(v_1, v_2, ..., v_n), (v_{n+1}, ..., v_m), ...
\]

where

\[
f(v_1) = v_2, ..., f(v_n) = v_1,
\]

\[
f(v_{n+1}) = v_{n+2}, ..., f(v_m) = v_{m+1}.
\]

If there is more than one cycle, then by the induction hypothesis there is a vertex \( v \) in the set \( \{v_1, v_2, ..., v_{n+1}\} \) such that for every other vertex \( x \) in the set \( \{v_1, v_2, ..., v_{n+1}\} \) there is a monochromatic path from \( x \) to \( v \). This contradicts our assumption, since \( v = f(w) \) for some \( w \in \{v_1, v_2, ..., v_{n+1}\} \).

Now we assume there is just one cycle \( (v_1, v_2, ..., v_n) \). Since there is no monochromatic path from \( v_i \) to \( v_{i+1} \), this implies that there are arcs \((v_2, v_1), (v_3, v_2), ..., (v_n, v_{n-1}), (v_1, v_n)\). Let their colours be \( a_1, a_2, ..., a_n \), respectively. If \( a_1 = a_2 = \cdots = a_{n-1} \) then \( v_1 \) can reach \( v_i \) via a monochromatic path \( v_n \cdots v_2 v_1 \) with colour \( a_1 \). This contradicts our assumption. Thus \( a_1, a_2, ..., a_{n-1} \) cannot be all equal. There must exist \( a_{s-1} \) and \( a_s \) with \( a_{s-1} \neq a_s \). Without loss of generality, we may assume \( a_{s-1} = 1 \), \( a_s = 2 \). There is a monochromatic path from \( v_{s-1} \) to \( v_{s+1} \) with colour \( b \). It is easy to see \( b \neq 1 \) and \( b \neq 2 \), for otherwise \( v_t \) can reach \( v_{s+1} \) via a monochromatic path with colour 1 or \( v_{s-1} \) can reach \( v_{s} \) via a monochromatic path with colour 2. So we may assume \( b = 3 \). Let the path \( u_1, u_2, ..., u_t \) be a shortest monochromatic path from \( v_{s-1} \) to \( v_{s+1} \) with colour 3. Here \( u_1 = v_{s-1}, u_t = v_{s+1} \). Obviously, this path is a simple path. It is shown in Fig. 1.
Consider the colour of the arc between \( u_i \) and \( u_i \), for \( 1 < i < t \). It cannot be 3, otherwise this would contradict our assumption. It is easy to see that there are edges \( \{v, u_i\} \) and \( \{v, u_{i+1}\} \) with distinct colours, because the edges \( \{v, u_i\} \) and \( \{v, u_t\} \) are coloured by distinct colours. Thus \( v_i u_i u_{i+1} \) is a triangle with three distinct colours. This contradicts the given condition. Hence the result holds.

We can easily obtain two corollaries as follows.

**Corollary 1.** Let \( T \) be a 2-coloured tournament. Then there is a vertex \( v \) of \( T \) such that for every other vertex of \( T \) there is a monochromatic path from \( x \) to \( v \).

This is the same as that in [1] and [2].

**Corollary 2.** Suppose \( T, H_1, H_2, \ldots, H_n \), where \( T = \{v_1, v_2, \ldots, v_n\} \), are \( m \)-coloured tournaments containing no 3-coloured triangle. Let \( T' \) be the tournament formed by replacing each vertex \( v_i \) of \( T \) with \( H_i \) and letting all edges between \( H_i \) and \( H_j \) be the same colour as the edge between \( v_i \) and \( v_j \), but with arbitrary directions. Then \( T' \) contains a vertex \( v \) such that for every other vertex \( x \) of \( T' \) there is a monochromatic path from \( x \) to \( v \).

**Proof.** It is clear that for any three vertices \( v_i, v_j, \) and \( v_k \) the triangle \( v_i v_j v_k \) cannot be a 3-coloured triangle. So the result holds by the theorem.

In the case of \( m = 3 \), this corollary implies Theorem 3 in [1].

If we insist only that \( T \) not contain \( C_3 \) in the theorem, the result will fail.

For example the tournament \( G_5 \) in Fig. 2 is a 5-coloured, of order 5, and contains no 3-coloured 3-cycle. But \( G_5 \) does not contain any vertex \( v \) such that for every other vertex \( x \) of \( G_5 \) there is a monochromatic path from \( x \) to \( v \). In fact, \( v_{i+1} \) cannot reach \( v_i \) via a monochromatic path, where the subscripts \( i + 1 \) are computed mod 5.

We construct larger counterexamples with \( m = 5 \) by adding vertices to \( G_5 \).
one at a time, connecting each new vertex to all previous vertices by an arc coloured 1.

Similarly if we insist only that $T$ not contain $T_3$ in the theorem, the result will also fail. For example, let $D_n$ be a 4-coloured tournament with vertices $v_1, v_2, ..., v_n$ such that the arcs $(v_1, v_2), (v_2, v_3),$ and $(v_3, v_1)$ are coloured with colour 1, 2, and 3, respectively, and all the other edges are coloured with colour 4 and directed as $(v_i, v_j)$, if $i > j$. Obviously, $D_n$ is a 4-coloured tournament containing no $T_3$, but $D_n$ does not contain any vertex $v$ such that for every other vertex $x$ of $D_n$ there is a monochromatic path from $x$ to $v$.

So if $m \geq 5$, the condition in the theorem, "which does not contain $T_3$ or $C_3$," cannot be improved. In a general sense the main result is the best result. But for the cases $m = 3, 4$, we have not found any counterexample. Certainly, the problem mentioned at the beginning is still an open question.

APPENDIX: NOMENCLATURE

$T_3$ the transitive tournament of order 3 whose arcs are coloured with three distinct colours.

$C_3$ the 3-cycle whose arcs are coloured with three distinct colours.

$G_5$ shown in Fig. 2.

REFERENCES
