Oscillation and nonoscillation theorems for second order nonlinear difference equations

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Abstract

Some new oscillation and nonoscillation theorems are obtained for second order nonlinear difference equations

$$\Delta_a(p_n\Delta_a x_n) + q_n\Delta_b x_n = F(n, x_n, \Delta_b x_n),$$

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1. Introduction

Recently, the study of various difference equations has attracted the interest of many authors. In the field of oscillatory properties of solutions of second order difference equations many important papers are also published. We refer to [1,6,7] for known results established for linear difference equations. For recent results proved for nonlinear second order difference equations, we refer to the references [2,3,5,8–10].
In [9], Popenda proved some oscillation and nonoscillation theorems for the following second order nonlinear generalized difference equation

$$\Delta^2 a_n = F(n, x_n, \Delta_b x_n), \quad n \in N := \{1, 2, 3, \ldots\},$$

where $a (\neq 0)$, $b$ are real numbers, $F: N \times R^2 \to R$, $x: N \to R$, $\Delta k x_n := x_{n+1} - k x_n$ for any real number $k$, $\Delta^2 k x_n := \Delta_k (\Delta_k x_n)$, and $x_n = x(n), n \in N$. For brevity, in the sequel we denote $\Delta^1 x_n$ by $\Delta x_n$.

In the paper, we consider the following more general difference equation

$$\Delta a(p_n \Delta a x_n) + q_n \Delta a x_n = F(n, x_n, \Delta b x_n), \quad n \in N,$$

where $\{p_n\}, \{q_n\}$ are real sequences with $p_n \neq 0$ for all $n \in N$, and $a, b, F$ and $x$ are the same as given in Eq. (P). If $p_n = 1$ and $q_n = 0$ hold for all $n \in N$, then (1) reduces to Eq. (P).

Note that $p_n$ ($n \in N$) may take positive as well as negative value, thus the sequence $\{p_n\}$ may even be an oscillatory one in the sense given below.

By a nontrivial solution of Eq. (1) we mean a sequence $x := \{x_n\}$ with $\sup \{|x_n|: n \geq i\} > 0$ for every $i \in N$. As usual, we consider only the nontrivial solutions of (1). A real sequence $\{y_n\} (n \in N)$ is called nonoscillatory if it is eventually positive or negative. Otherwise it is called oscillatory.

For simplicity of statements, throughout the paper we denote by $X$ the class of nontrivial solutions of (1) and define $X_1 = \{x \in X: \Delta a x_k = 0 \text{ holds for some } k \in N\}$ and $X_2 = X \setminus X_1$.

2. Nonoscillation theorems

**Theorem 1.** If $a > 0$ in Eq. (1) and the condition

$$\begin{cases} F(n, u, v) = 0, & \text{if } v + (b - a)u = 0, \\ \frac{1}{p_{n+1}} \left[F(n, u, v) + (ap_n - q_n)[v + (b - a)u]\right] \geq 0, & \text{if } v + (b - a)u \neq 0, \end{cases} \quad (2)$$

is satisfied, where $n \in N$, $u, v \in R$, then all nontrivial solutions of (1) are nonoscillatory.

**Proof.** Let $x$ be any nontrivial solution of (1). Then either $x \in X_1$ or $x \in X_2$ holds.

(A) If $x \in X_1$, then $\Delta a x_k = x_{k+1} - ax_k = 0$ holds for some $k \in N$. It means that the first relation of (2) is valid, i.e., $F(k, x_k, \Delta b x_k) = 0$.

By (1) with $n = k$ we derive

$$\Delta a(p_k \Delta a x_k) = 0$$

and therefore

$$\Delta a x_{k+1} = \frac{ap_k}{p_{k+1}} \Delta a x_k = 0.$$
By repeating above argument, we obtain $\Delta ax_{k+2} = 0$. By induction, we easily derive $\Delta ax_{k+i} = 0$ for $i \in \mathbb{N}$. Hence we have

$$x_{k+i} = a^i x_k, \quad i \in \mathbb{N}. \quad (3)$$

Because $a > 0$ and $x_k \neq 0$ (otherwise $x \notin X$), $\{x_{k+i}\} (i \in \mathbb{N})$ is eventually positive or negative according to $x_k > 0$ or $x_k < 0$.

(B) If $x \in X_2$, the second relation of (2) always holds. Provided that $x$ is oscillatory, then

(i) $x_m > 0$, $x_{m+1} \leq 0$, or
(ii) $x_m \geq 0$, $x_{m+1} < 0$

holds for some $m \in \mathbb{N}$. In the case (i) we have

$$\Delta ax_m < 0. \quad (4)$$

Rewrite (1) in the form

$$\Delta ax_{n+1} = \frac{1}{p_{n+1}} \left[ F(n, x_n, \Delta bx_n) + (ap_n - q_n) \Delta ax_n \right], \quad n \in \mathbb{N}. \quad (1')$$

Letting $n = m$ in (1'), then multiplying the both sides by $\Delta ax_m$, we obtain

$$\Delta ax_m \Delta ax_{m+1} \geq 0, \quad (5)$$

since $\Delta ax_m = \Delta bx_m + (b-a)x_m < 0$ from (4) and the second relation of (2). In view of $\Delta ax_m \neq 0$ from $x \in X_2$, by (4) and (5) we have

$$\Delta ax_{m+1} < 0. \quad (6)$$

Repeating the same argument from (4) to (6), we derive $\Delta ax_{m+i} < 0$ for all $i \in \mathbb{N}$. Since $a > 0$ and $x_{m+1} \leq 0$, we easily obtain

$$x_{m+i+2} < a_{i+1} x_{m+1} \leq 0, \quad i \in \mathbb{N}. \quad (7)$$

This contradicts to the assumed oscillatory property of $x$. The case (ii) can be proved similarly. □

**Theorem 2.** If $a > 0$ in Eq. (1), and the condition

$$\frac{v + bu}{p_{n+1}} \left[ F(n, u, v) + (ap_n + q_n) \left[ v + (b-a)u \right] \right] \geq 0 \quad (8)$$

holds for $n \in \mathbb{N}$ and all $u, v \in \mathbb{R}$ with $v + bu \neq 0$, then all nontrivial solutions of (1) are nonoscillatory.

**Proof.** Taking $u = x_n$, $v = \Delta bx_n$, we have $v + bu = x_{n+1} = \Delta ax_n$. Let $x \in X$ be any nontrivial solution of (1). Then for some $m \in \mathbb{N}$ we have $x_{m+1} \neq 0$ and hence $\Delta bx_m + bx_m \neq 0$. 
Letting \( n = m \) in (1′) and multiplying the both sides by \( x_{m+1} \), we obtain

\[ x_{m+1} \Delta a x_{m+1} \geq 0, \tag{9} \]

here we used the inequality (8) with \( n = m \).

(i) If \( x_{m+1} > 0 \), it follows from (9) that \( \Delta a x_{m+1} \geq 0 \) and hence \( x_{m+2} \geq a x_{m+1} > 0 \). By induction, we have

\[ x_{m+i+1} \geq a^i x_{m+1} > 0, \quad i \in \mathbb{N}. \]

It means that, in this case the solution \( x \) is eventually positive.

(ii) If \( x_{m+1} < 0 \), from (9) we have \( \Delta a x_{m+1} \leq 0 \) and hence \( x_{m+2} \leq a x_{m+1} < 0 \). By induction, we have

\[ x_{m+i+1} \leq a^i x_{m+1} < 0, \quad i \in \mathbb{N}. \]

Hence, in this case the solution \( x \) is eventually negative. \( \square \)

### 3. Oscillation theorems

**Theorem 3.** If \( a < 0 \) in Eq. (1) and for \( n \in \mathbb{N} \) and \( u, v \in \mathbb{R} \) the condition

\[
\begin{cases}
F(n, u, v) = 0, & \text{if } v + (b - a)u = 0, \\
\frac{v + (b-a)u}{p_{n+1}} \left[ F(n, u, v) + (ap_n - q_n) \left[ v + (b-a)u \right] \right] \leq 0, & \text{if } v + (b-a)u \neq 0,
\end{cases}
\tag{10}
\]

is satisfied, then all nontrivial solutions of (1) are oscillatory.

**Proof.** Let \( x \) be any nontrivial solutions of (1). Then, \( x \in X_1 \) or \( x \in X_2 \) holds.

(A) If \( x \in X_1 \), by using the same argument as in the part (A) of the proof of Theorem 1, we can derive that \( x_{k+i} = a^i x_k \) holds for some \( k \in \mathbb{N} \) with \( x_k \neq 0 \). Since \( a < 0 \), we have \( x_{k+i} x_{k+i+1} = a^{2i+1} x_k^2 < 0 \) for \( i \in \mathbb{N} \). Hence, \( x \) is oscillatory.

(B) If \( x \in X_2 \), by the second relation of (10) we have

\[ \frac{\Delta a x_n}{p_{n+1}} \left[ F(n, x_n, \Delta b x_n) + (ap_n - q_n) \Delta a x_n \right] \leq 0, \quad n \in \mathbb{N}, \tag{11} \]

since \( \Delta b x_n + (b - a)x_n = \Delta a x_n \neq 0 \).

Suppose that \( x \) is nonoscillatory. Then it is either eventually positive or eventually negative.

(i) If \( x \) is eventually positive, we may assume that \( x_n > 0 \) for \( n \geq m \) where \( m \) is an even integer. By letting \( n = m \) in Eq. (1′) and then multiplying the both sides by \( \Delta a x_m (\neq 0) \), we derive

\[ \Delta a x_m \Delta a x_{m+1} \leq 0, \tag{12} \]
here we used the relation (11) with \( n = m \). From (12) we have
\[
\Delta \left( \frac{x_m}{a^m} \right) \Delta \left( \frac{x_{m+1}}{a^{m+1}} \right) \geq 0, \tag{13}
\]
since \( a < 0 \) and \( \Delta_a x_m = a^{m+1} \Delta (x_m/a^m) \). We observe that \( \Delta (x_m/a^m) < 0 \) is valid. Otherwise, if \( \Delta (x_m/a^m) \geq 0 \), then \( x_{m+1}/a^{m+1} \geq x_m/a^m > 0 \) holds. Therefore, we obtain \( x_{m+1} < 0 \), which is a contradiction to the assumption that \( x_n > 0 \) holds for \( n \geq m \). Hence, we derived from (13) \( \Delta (x_{m+1}/a^{m+1}) \leq 0 \), i.e.,
\[
\frac{x_{m+2}}{a^{m+2}} \leq \frac{x_{m+1}}{a^{m+1}}. \tag{14}
\]
Since \( a < 0 \), \( x_{m+1} > 0 \) and \( m \in N \) is an even integer, (14) ensures that
\[
\frac{x_{m+2}}{a^{m+2}} < 0.
\]
It follows from the last inequality the desired contradiction \( x_{m+2} < 0 \), since \( m \) is an even integer.

The case (ii) when \( x \) is eventually negative can be proved similarly. \( \Box \)

**Theorem 4.** If \( a < 0 \) in Eq. (1) and the condition
\[
v + bu \leq \frac{F(n, u, v) + (ap_n + q_n)[v + (b - a)u]}{p_{n+1}} \leq 0 \tag{15}
\]
holds for \( n \in N \) and \( u, v \in R \) with \( v + bu \neq 0 \), then all nontrivial solutions of (1) are oscillatory.

**Proof.** Because the relations \( v + bu = x_{n+1}, v + (b - a)u = \Delta_x x_n \) hold when \( u = x_n \) and \( v = \Delta_x x_n, n \in N \). We observe from (15) that
\[
\frac{x_{n+1}}{p_{n+1}} \left[ F(n, x_n, \Delta_x x_n) + (ap_n - q_n) \Delta_x x_n \right] \leq 0 \tag{16}
\]
holds for all \( n \in N \) with the property \( x_{n+1} \neq 0 \).

Let \( x \in X \) be any nontrivial solution of (1). Then there is at least one \( m \in N \) such that \( x_{m+1} \neq 0 \). Letting \( n = m \) in Eq. (1’) and then multiplying the both sides by \( x_{m+1} \), we derive
\[
x_{m+1} \Delta_a x_{m+1} \leq 0; \tag{17}
\]
here we used the inequality (16) with \( n = m \).

(i) If \( x_{m+1} > 0 \), by (17) we have \( \Delta_a x_{m+1} \leq 0 \), and hence \( x_{m+2} \leq ax_{m+1} < 0 \).

(ii) If \( x_{m+1} < 0 \), by (17) we have \( \Delta_a x_{m+1} \geq 0 \), and hence \( x_{m+2} \geq ax_{m+1} > 0 \).

It is not difficult to see that the relation
\[
x_{m+i} x_{m+i+1} < 0
\]
is true for all \( i \in N \). By definition, the solution \( \{x_n\} \) of Eq. (1) is oscillatory. \( \Box \)
4. Consequences for (P)

We show that the theorems proved in Sections 2 and 3 generalize and improve all main results given in [9].

By putting \( p_n = 1 \) and \( q_n = 0 \) \((n \in \mathbb{N})\) in Theorems 1–4, we derive the next two corollaries.

**Corollary 1.** If \( a > 0 \) in Eq. (P), and the condition

\[
\begin{align*}
&\text{(i) for } n \in \mathbb{N} \text{ and } u, v \in \mathbb{R}, \\
&\quad \left\{ \\
&\quad \quad F(n, u, v) = 0, \quad \text{if } v + (b - a)u = 0, \\
&\quad \quad [v + (b - a)u][F(n, u, v) + a[v + (b - a)u]] \geq 0, \\
&\quad \quad \text{if } v + (b - a)u \neq 0,
\end{align*}
\]

(18)

or

\[
\begin{align*}
&\text{(ii) for } n \in \mathbb{N} \text{ and } u, v \in \mathbb{R} \text{ with } v + bu \neq 0, \\
&\quad (v + bu)\left\{ F(n, u, v) + a[v + (b - a)u] \right\} \geq 0
\end{align*}
\]

(19)

is satisfied, then all nontrivial solutions of Eq. (P) are nonoscillatory.

**Corollary 2.** If \( a < 0 \) in Eq. (P), and the condition

\[
\begin{align*}
&\text{(i) for } n \in \mathbb{N} \text{ and } u, v \in \mathbb{R}, \\
&\quad \left\{ \\
&\quad \quad F(n, u, v) = 0, \quad \text{if } v + (b - a)u = 0, \\
&\quad \quad [v + (b - a)u][F(n, u, v) + a[v + (b - a)u]] \leq 0, \\
&\quad \quad \text{if } v + (b - a)u \neq 0,
\end{align*}
\]

(20)

or

\[
\begin{align*}
&\text{(ii) for } n \in \mathbb{N} \text{ and } u, v \in \mathbb{R} \text{ with } v + bu \neq 0, \\
&\quad (v + bu)\left\{ F(n, u, v) + a[v + (b - a)u] \right\} \leq 0
\end{align*}
\]

(21)

is satisfied, then all nontrivial solutions of Eq. (P) are nonoscillatory.

**Remark 1.**

(a) In Corollary 1, (i) and (ii) by replacing the weaker inequality symbol \( \geq \) with the stronger one \( > \), Theorems 1 and 2 of [9] follow, respectively;

(b) In Corollary 2, (i) and (ii), by replacing \( \leq \) with \( < \), Theorems 4 and 5 of [9] follow, respectively. Thus, the above mentioned two corollaries are slightly improved versions of the main results given in Papenda [9].
5. Further corollaries

By different choice of the numbers $a (\neq 0)$, $b$ and the real sequences \{\(p_n\) and \(q_n\)}, a number of interesting consequences can also be given for some special cases of Eq. (1). For instance, letting $p_n = 1$ and $q_n = a$ for all $n \in \mathbb{N}$, then (1) reduces to

\[ \Delta a x_{n+1} = F(n, x_n, \Delta b x_n), \quad n \in \mathbb{N}. \tag{22} \]

The following two results can be derived from Theorems 1, 2 and Theorems 3, 4, directly.

**Corollary 3.** If $a > 0$ in Eq. (22), and the condition

(i) for $n \in \mathbb{N}$ and $u, v \in \mathbb{R}$,

\[
\begin{cases}
F(n, u, v) = 0, & \text{if } v + (b - a)u = 0, \\
F(n, u, v)[v + (b - a)u] \geq 0, & \text{if } v + (b - a)u \neq 0,
\end{cases}
\tag{23}
\]

or

(ii) for $n \in \mathbb{N}$ and $u, v \in \mathbb{R}$ with $v + (b - a)u \neq 0$,

\[(v + bu)F(n, u, v) \geq 0 \tag{24}\]

is satisfied, then all nontrivial solutions of (22) are nonoscillatory.

**Corollary 4.** If $a < 0$, then all nontrivial solutions of Eq. (22) are oscillatory provided one of the following two conditions is satisfied:

(i) for $n \in \mathbb{N}$ and $u, v \in \mathbb{R}$,

\[
\begin{cases}
F(n, u, v) = 0, & \text{if } v + (b - a)u = 0, \\
F(n, u, v)[v + (b - a)u] \leq 0, & \text{if } v + (b - a)u \neq 0,
\end{cases}
\tag{25}
\]

or

(ii) for $n \in \mathbb{N}$ and $u, v \in \mathbb{R}$ with $v + (b - a)u \neq 0$,

\[F(n, u, v)[v + (b - a)u] \leq 0. \tag{26}\]

Another interesting special case of Eq. (1) usually occurred in the literature is of the form

\[ \Delta(p_n \Delta x_n) + q_n \Delta x_n = F(n, x_n, \Delta x_n), \quad n \in \mathbb{N}. \tag{27} \]

The following corollary for Eq. (27) follows from above Theorems 1–4 directly.

**Corollary 5.** If $p_n > 0$ ($n \in \mathbb{N}$), then all nontrivial solutions of Eq. (27) are nonoscillatory provided one of the following two conditions is satisfied:
(i) for \( n \in \mathbb{N} \) and \( u, v \in \mathbb{R} \),
\[ F(n, u, 0) \equiv 0 \text{ and } v[F(n, u, v) + (p_n - q_n)v] \geq 0 \text{ for } v \neq 0, \]
(ii) for \( n \in \mathbb{N} \) and \( u, v \in \mathbb{R} \),
\[ (v + u)[F(n, u, v) + (p_n - q_n)v] \geq 0. \] \hfill (28)

**Remark 2.** Note that the Theorem 3 of Popenda [9] can be treated as a special case of Corollary 5(ii) when \( p_n = q_n = 1 \) \((n \in \mathbb{N})\). In fact, in this case Eq. (27) can be rewritten in the form
\[ \Delta^2 x_n = f(n, x_n, \Delta x_n), \quad n \in \mathbb{N}, \]
where \( f(n, x_n, \Delta x_n) := F(n, x_n, \Delta x_n) - \Delta x_n \). By (28), it follows the sufficient condition for nonoscillation proved in [9]:
\[ (u + v)f(n, u, v) \geq 0, \quad \text{for } (n, u, v) \in \mathbb{N} \times \mathbb{R}^2. \]

The last three corollaries are useful in many situations and they have not appeared in the literature.

### 6. Examples

**Example 1.** \( \Delta^2 x_n + \Delta x_n = f(n)g(x_n, \Delta x_n), n \in \mathbb{N}. \)

If \( f : \mathbb{N} \to \mathbb{R}_+ = [0, \infty) \), \( g : \mathbb{R}^2 \to \mathbb{R} \) with \( g(u, 0) \equiv 0 \) and \( vg(u, v) \geq 0 \), then all nontrivial solutions of this difference equation are nonoscillatory. In fact, under the assumed conditions this equation is in the condition (i) of Corollary 1, where \( a = b = 1 \) and \( F(n, u, v) = f(n)g(u, v) \).

**Example 2.** \( x_{n+2} - (1 + x_n^2) x_{n+1} + x_n^3 = 0, n \in \mathbb{N}. \)

By rewriting this equation in the form
\[ \Delta^2 x_n = F(n, x_n, \Delta x_n), \quad n \in \mathbb{N}, \]
where \( F(n, u, v) := u^2v - v \), we see that the condition (i) of Corollary 1 with \( a = b = 1 \) is satisfied. Hence all nontrivial solutions of this equation are nonoscillatory.

**Example 3.** \( x_{n+2} + (1 + p)x_{n+1} + px_n = 0, n \in \mathbb{N}, p > 0. \)

It is a simple matter to check that all solutions of this second order linear difference equation are oscillatory. Actually, the given equation can be rewritten in the form
\[ \Delta ax_{n+1} = F(n, x_n, \Delta x_n), \quad n \in \mathbb{N}, \] \hfill (29)
where \( a = -p < 0 \), \( F(n, u, v) = -v - (1 + p)u \). Clearly, the condition (i) of Corollary 4 with \( a = -p \) and \( b = 1 \) is satisfied by Eq. (29). Hence the desired conclusion is valid.
**Example 4.** \( \Delta[(2^n + n) \Delta x_n] + 2^n \Delta x_n = nx_n, \quad n \in \mathbb{N} \).

Letting \( F(n, u, v) = nu, \quad p_n = 2^n + n, \quad q_n = 2^n \), by Corollary 5(ii), we infer that all nontrivial solutions of this equation are nonoscillatory.

Note that the difference equations considered in above examples cannot be treated by means of the known results given in Refs. [1–10].

**References**