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Large E-Modules Exist

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INTRODUCTION

If R is a ring with 1 and M an R-module then M is called $E(R)$-module (or E-module) if $\mathrm{Hom}_{\mathbb{Z}}(R, M) = \mathrm{Hom}_R(R, M)$, where R (and M) is considered a right R-module. Moreover, R is called E-ring if the right module $R = R_R$ is an E-module. It is easy to see that E-rings are commutative. E-rings were introduced by Schultz in [S] and studied further in [BS]. Torsion-free E-rings of finite rank play an important role in some investigations of torsion-free abelian groups of finite rank; cf. [APRVW]. We refer to [Pi] for a discussion of $E(R)$-modules. In the present paper we want to answer a question of C. Visonhaler's: If R is an E-ring, are there arbitrarily large indecomposable $E(R)$-modules? Since this question makes sense only for E-rings R without nontrivial idempotents we may rephrase Vinsonhaler's question: If R is an E-ring, are there arbitrarily large $E(R)$-modules M such that $\mathrm{End}_{\mathbb{Z}}(M) = R$? A partial answer to that question may be found in [DG1]: The Main Theorem in [DG1] states that for any cotorsion-free (cf. [DG1]) ring R with $|R| < \kappa$ there exist arbitrarily large, strongly κ-free R-modules M with $\mathrm{End}_{\mathbb{Z}}(M) = R$ if the set theoretic axiom $V = L$ holds (or some weaker consequence of $V = L$). These modules are $E(R)$-modules since submodules of M of cardinality $|R| < \kappa$ are contained in free R-submodules of M. The aim of the present paper is to prove a similar result without using $V = L$ but instead posing some mild restrictions on the (E-) ring R. The following theorem is our main result. (We refer to [FI/II] for undefined notations in the theory of abelian groups). Let κ denote a regular uncountable cardinal and $\aleph_m$ the least measurable cardinal, [J], if there is any measurable cardinal at all.

THEOREM. *Let R be a ring with 1 such that R^+, the additive group of R, is slender and $|R| < \kappa < \aleph_m$. Let $\lambda > \kappa$ be any cardinal such that $\lambda = \lambda^{2^\kappa}$.*

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Then there exists an R-module G with the following properties:

(a) $|G| = \lambda$.

(b) $\mathrm{End}_{\mathbb{Z}}(G) = R$.

(c) *Every R-submodule M of G of cardinality* $< \kappa$ *is R-torsionless, i.e., a submodule of a cartesian product* $\prod R$.

(d) *G is slender* (cf. [FII], Sect. 94]).

As an immediate consequence of (c) *we have*

(e) *If R is an E-ring, then G is an E(R)-module.*

We would like to mention that every R-module G with property (c) is itself R-torsionless if κ is strongly compact (cf. [J], [AE]). This somewhat explains the presence of the restriction $\kappa < \aleph_m$. In [DMV] we constructed for any cotorsion-free ring S with 1 an E-ring R of large cardinality with $S \subseteq R$. All the rings constructed in [DMV] are slender if only S is slender. If R is an E-ring and A an R-algebra such that A^+ is slender and A is an $E(R)$-module then we may apply our Main Theorem with R replaced by A, and we obtain many examples of $E(R)$-modules G with "pathological decompositions." We refer to [DG1] for examples of such R-algebras A.

1. Preliminaries

In all that follows, let A be a torsion-free reduced group and κ a (regular) uncountable cardinal. Let $A^\kappa = \prod_{\alpha<\kappa} A$ be the cartesian product of κ copies of A. Each element $a \in A^\kappa$ is a map from κ into A and we identify a with $(a(i))_{i<\kappa}$. We will work with the following canonical subgroups of A^κ. First, let

$$A^{<\kappa} = \{a \in A^\kappa \mid |\{\alpha < \kappa \mid a(\alpha) \neq 0\}| < \kappa\}$$

and

$$A^{(\kappa)} = \{a \in A^\kappa \mid \{\alpha < \kappa \mid a(\alpha) \neq 0\} \text{ finite}\}.$$

Moreover,

$$A^{[\kappa]} = \{a \in A^\kappa \mid \{\alpha < \kappa \mid a(\alpha) \neq 0\} \text{ finite or countable}\}$$

and

$$A^{\langle\kappa\rangle} = \{a \in A^\kappa \mid \{a(\alpha) \mid \alpha < \lambda\} \text{ finite}\}.$$

Kaup and Keane [KK] generalized a celebrated result due to Nöbeling (cf. [FII, Sect. 97]) by showing that $A^{\langle\kappa\rangle}$ is isomorphic to a direct sum of copies of A. Moreover, $A^{\langle\kappa\rangle} \cong \bigoplus_{x\in I} h_x A$, where h_x, $X \subseteq \kappa$, is the characteristic function of X and $A^{<\kappa} \cap A^{\langle\kappa\rangle}$ is a direct summand of $A^{\langle\kappa\rangle}$ with a complement generated by a characteristic A-basis; cf. [FII, Theorem 97.5].

Since A is reduced and $\kappa = cf(\kappa) > \omega$, all groups A^κ, $A^{<\kappa}$, $A^\kappa/A^{<\kappa}$, $(A^{\langle\kappa\rangle} + A^{<\kappa})/A^{<\kappa}$ are Hausdorff in the $\mathbb{Z}$-adic topology and all inclusions are pure. For G any reduced group, let $G^\wedge = \hat{G}$ be the $\mathbb{Z}$-adic completion of G. Then $(A^{\langle\kappa\rangle})^\wedge \subseteq (A^\kappa)^\wedge$ and we define $A^\#_\kappa = (A^{\langle\kappa\rangle})^\wedge \cap A^\kappa$. The pure subgroup $A^\#_\kappa$ consists of all the elements $a = (a(\alpha))_{\alpha<\kappa}$ of A such that

(1) $\{a(\alpha) \mid \alpha < \kappa\}$ is at most countable,

(2) $\forall n < \omega$, $\{a(\alpha) \mid \alpha < \kappa,\ a(\alpha) \notin n!\, A\}$ is finite.

We are now ready to state a stronger version of Los' theorem (cf. [FII, Theorem 94.4]) that is crucial for us.

THEOREM 1.1. *Let G be a slender group and $\eta \in \operatorname{Hom}_{\mathbb{Z}}(A^\#_\kappa, G)$. If κ is nonmeasurable and $\eta(A^{(\kappa)}) = 0$, then $\eta = 0$.*

Proof. Our proof will be essentially the same as the proof of Theorem 4.4 in [FII]. The group $A^\#_\kappa / A^{\langle\kappa\rangle}$ is divisible and therefore $\eta = 0$ if and only if $\eta(A^{\langle\kappa\rangle}) = 0$. By way of contradiction, assume that $\eta(a) \neq 0$ for some $a \in A^{\langle\kappa\rangle}$. Since $\{a(\alpha) \mid \alpha < \kappa\}$ is finite we may assume that $a(\alpha) \in \{0, b\}$ for some fixed $0 \neq b \in A$. For any subset J of κ define $a_J \in A^{\langle\kappa\rangle}$ by $a_J(\alpha) = a(\alpha)$ for $\alpha \in J$ and $a_J(\alpha) = 0$ for $\alpha \notin J$. Note that $a_\kappa = a$ and $a_J \in A^{\langle\kappa\rangle}$ for all $J \subseteq \kappa$. We define a G-valued measure ν on κ by $\nu(J) = \eta(a_J)$. Then $\nu(\kappa) = \eta(a) \neq 0$ and $\nu(\{\alpha\}) = 0$ for all $\alpha < \kappa$. Note that ν is additive since η is a homomorphism. In order to show that ν is countably additive, let J_n, $n < \omega$, be pairwise disjoint subsets of κ and set $a^{(n)} = a_{J_n}$. Then $P = \prod_{n<\omega} (a^{(n)} n!)\mathbb{Z}$ is contained in $A^\#_\kappa$ and $P \cong \mathbb{Z}^\omega$. Since G is slender we obtain $n_0 < \omega$ and $\eta(a^{(n)} n!) = n!\, \eta(a^{(n)}) = 0$ for all $n \geqslant n_0$. Let

$$y = \sum_{k=n_0}^{\infty} a^{(k)} \in A^{\langle\kappa\rangle}.$$

Then

$$\nu\left(\bigcup_{n<\omega} J_n\right) = \eta\left(\sum_{k=0}^{m_0-1} a^{(k)} + y\right) = \nu\left(\bigcup_{k=0}^{n_0-1} J_k\right) + \eta(y).$$

We will show that $\eta(y) = 0$. Set $y_n = \sum_{k=n_0}^{n} a^{(k)}$ and let $\pi = \sum_{n=0}^{\infty} n!\, z_n \in \hat{\mathbb{Z}}$. Then $w = \sum_{n=0}^{\infty} (y - y_{n+n_0})\, n!\, z_n \in A^\#_\kappa$ and since η is continuous in the $\mathbb{Z}$-adic topology we have $\eta(w) = \sum_{n=0}^{\infty} \eta(y - y_{n+n_0})\, n!\, z_n = \sum_{n=0}^{\infty} \eta(y)\, n!\, z_n = \eta(y)\, \pi$. This shows that $\eta(y)\hat{\mathbb{Z}} \subseteq G$ and we conclude $\eta(y) = 0$ since the slender group G is also cotorsion-free. This shows $\nu(\bigcup_{n<\omega} J_n) =$

$\sum_{n=0}^{n_0-1} \nu(J_n) = \sum_{n<\omega} \nu(J_n)$ and ν is countably additive. Now consider the countably additive ideal $\mathbb{I} = \{J \subseteq \kappa \mid \nu(J') = 0 \text{ for all } J' \subseteq J\}$ and let $\mathbb{B}$ be the boolean algebra of all subsets of κ. As in the proof of Theorem 94.4 in [FII] it follows that $\mathbb{B}/\mathbb{I}$ is finite. Thus ν gives rise to a countably additive, $\{0, 1\}$-measure on κ and we arrived at a contradiction since κ is non-measurable.

The next definition may be found in [BS] and we refer to [Pi] for easy reference.

DEFINITION 1.2. Let R be a ring. If for each $\varphi \in \mathrm{End}_{\mathbb{Z}}(R)$ there is an $r \in R$ such that $\varphi(x) = xr$ for all $x \in R$, then R is called an E-ring. Note that E-rings are commutative. An R-module M is called an E-module (or $E(R)$-module) if $\mathrm{Hom}_{\mathbb{Z}}(R, M) = \mathrm{Hom}_{R}(R, M)$. Note that R is an E-ring iff the R-module R_R is an $E(R)$-module. Finally, an R-module M is called R-torsionless if M is isomorphic to an R-submodule of R^{κ} for some cardinal κ.

Remark 1.3. If R is an E-ring then every R-torsionless module is an $E(R)$-module, and the class of $E(R)$-modules is closed with respect to submodules, cartesian products, and direct sums.

Remark 1.4. Let $\aleph_{\mathrm{m}}$ be the least measurable cardinal (if it exists), κ a regular cardinal $< \aleph_{\mathrm{m}}$, and R an E-ring of cardinality $< \kappa$. Then $R^{\kappa}/R^{<\kappa}$ is an $E(R)$-module.

Proof. By the Wald–Los lemma (cf. [DG2]), every submodule of $R^{\kappa}/R^{<\kappa}$ of cardinality $< \kappa$ is R-torsionless. Since $|R| < \kappa$ this implies that $R^{\kappa}/R^{<\kappa}$ (and all its submodules) are $E(R)$-modules.

In order to prove the theorem mentioned in the Introduction, we will have to construct an R-module G with $\mathrm{End}(G) = R$. This will be done using a "Black Box" construction similar to the one in [DMV]. Therefore, we will be a little sketchy at times and point out only the major differences.

2. THE BLACK BOX

Let $\kappa < \aleph_{\mathrm{m}}$ be a regular cardinal and λ a cardinal $\geqslant \kappa$ with $\lambda^{(2^{\kappa})} = \lambda$. Moreover, let R be a cotorsion-free ring such that $|R| < \kappa$. Let $F = \bigoplus_{\alpha < \lambda} \bigoplus_{k < 2^{\kappa}} ((R^{\langle\kappa\rangle} + R^{<\kappa})/R^{<\kappa})(\alpha, k)$, where $((R^{\langle\kappa\rangle} + R^{<\kappa})/R^{<\kappa})(\alpha, k)$ is a copy of $R^{\langle\kappa\rangle} + R^{<\kappa}/R^{<\kappa}$ labelled by (α, k), $\alpha < \lambda$, $k < 2^{\kappa}$. We set $B = (R^{\langle\kappa\rangle} + R^{<\kappa})/R^{<\kappa}$ and $\bar{B} = (R^{\#}_{\kappa} + R^{<\kappa})/R^{<\kappa} \subseteq R^{\kappa}/R^{<\kappa}$. Note that B is a free R-module and because of 1.1, $\mathrm{Hom}_{\mathbb{Z}}(\bar{B}, G) = 0$ for any slender group G. Moreover, $\bar{B}/B$ is divisible (as abelian group). Eventually, we will construct an R-module M such that

$$\bigoplus_{\alpha<\lambda} \bigoplus_{k<2^\kappa} B(\alpha, k) = F \subseteq M \subseteq \left(\bigoplus_{\alpha<\lambda} \bigoplus_{k<2^\kappa} \bar{B}(\alpha, k) \right) + (B^{(\lambda+2^\kappa)})^- = \tilde{F},$$

where $(B^{(\lambda\times 2^k)})^-$ is the $\mathbb{Z}$-adic closure of $B^{(\lambda\times 2^\kappa)}$ in $B^{\lambda\times 2^\kappa} = \prod_{\alpha<\lambda} \prod_{k<2^\kappa} B(\alpha, k)$. Note that $\tilde{F}/F$ is divisible and $\tilde{F} \subseteq \bar{B}^{\lambda\times 2^\kappa} \subseteq \prod_{\lambda\times 2^\kappa} (R^\kappa/R^{<\kappa})$. Thus every submodule L of $\tilde{F}$ of cardinality $<\kappa$ is R-torsionless. Moreover, for each $\varphi: F \to \tilde{F}$ there exists a unique extension $\tilde{\varphi}: \tilde{F} \to (\tilde{F})^\wedge$ and we will identify φ and $\tilde{\varphi}$. For $x \in \tilde{F}$ let $[x] = \{(\alpha, k) \in \lambda \times 2^\kappa \mid x(\alpha, k) \neq 0\}$ and $[\![x]\!] = \{\alpha < \lambda \mid \exists k < 2^\kappa((\alpha, k) \in [x])\}$. If P is any R-submodule of $\tilde{F}$, we define $\|P\| = \sup\{[\![x]\!] \mid x \in P\}$.

Definition 2.1. (1) A canonical submodule P of $\tilde{F}$ is any R-submodule P of $\tilde{F}$ such that for some fixed $I \subseteq \kappa$, $|I| \leqslant 2^\kappa$, we have $x \in P$ iff $[\![x]\!] \in I$.

(2) A trap is a triple (f, P, φ), where f is a sequence of ordinals $f(0) < f(1) < \cdots < \lambda$; P is a canonical submodule such that $\bar{B}(\alpha, k) \subseteq P$ for any $(\alpha, k) \in \lambda \times 2^\kappa$ whenever $\alpha = f(n)$ for some $n < \omega$ and $\|P\| = \sup\{f(n) \mid n < \omega\}$ and $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(P \cap F, P)$.

Without proof we state Shelah's principle, the so-called "Black Box." A proof may be found in [DMV, Theorem 2.6].

Theorem 2.2. *For some ordinal λ^* there exists a sequence of traps $\tau_\alpha = (f_\alpha, P_\alpha, \varphi_\alpha)$, $\alpha < \lambda^*$, such that*

(1) $\|P_\alpha\| \leqslant \|P_\beta\|$ *if* $\alpha \leqslant \beta$.

(2) *If* $\alpha \neq \beta$ *then* $\mathrm{Im}\, f_\alpha \cap \mathrm{Im}\, f_\beta$ *is finite.*

(3) *If* $\beta + (2^\kappa)^\omega \leqslant \alpha$ *then for all* $e: \omega \to 2^\kappa$ *there exists* $n_0 < \omega$ *such that* $\prod_{n<n_0} \bar{B}(f_\alpha(n), e(n)) \cap P_\beta = 0$.

(4) *If* $A \subseteq \tilde{F}$, $|A| \leqslant 2^\kappa$ *and if* $\varphi \in \mathrm{Hom}(F, \tilde{F})$ *then there exists* $\alpha < \lambda^*$ *such that* $A \subseteq P_\alpha$, $\|A\| \leqslant \|P_\alpha\|$ *and* $\varphi \restriction (F \cap P_\alpha) = \varphi_\alpha$.

3. The Construction

Let $1(\alpha, k)$ be the element in $B(\alpha, k)$ induced by $a \in R^{\langle\kappa\rangle}$ with $a(i) = 1$ for all $i < k$, i.e., $1(\alpha, k) = (a + R^{<\kappa})(\alpha, k)$.

We will utilize the sequence of traps $(f_\alpha, P_\alpha, \varphi_\alpha)$, $\alpha < \lambda^*$. For $\alpha < \lambda^*$ and a sequence $e_\alpha(0) < e_\alpha(1) < e_\alpha(2) < \cdots < \kappa$ let

$$\bar{e}_\alpha = \sum_{n<\omega} 1(f_\alpha(n), e_\alpha(n))\, n! \in P_\alpha.$$

Further, choose $b_\alpha \in P_\alpha$ such that $\|b_\alpha\| < \|P_\alpha\| = \sup\{f_\alpha(n): n<\omega\}$ and set

$$a_\alpha = \bar{e}_\alpha + b_\alpha \in P_\alpha. \qquad (*)$$

Using any elements of this form we obtain a transfinite chain of pure subgroups G_α of $\tilde{F}$, which are also R-modules, satisfying

(I_0) $G_0 = F$;

(I_α) $G_\alpha = \bigcup_{\eta<\alpha} G_{\eta+1}$;

(II_α) $G_{\alpha+1} = \langle G_\alpha, a_\alpha R\rangle_*$, the pure submodule of $\tilde{F}$ generated by G_α and a_α.

We will set $G = \bigcup_{\alpha<\lambda^*} G_\alpha$. We will specify later how to choose the functions $e_\alpha: \omega \to 2^\kappa$ and the elements b_α in $(*)$ to ensure $\mathrm{End}_{\mathbb{Z}}(G) = R$.

LEMMA 3.1. (a) *For each* $g \in G$ *there exist* $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \lambda^*$, $r_i \in R$ *and* $m \in \mathbb{N}$, $u \in F$ *such that* $mg = \sum_{i=1}^n a_{\alpha_i} r_i + u$. *Moreover, the pair* (m, g) *determines* α_i, r_i, $1 \leqslant i \leqslant n_1$, *and* u *uniquely.*

(b) *If* R *is cotorsion-free, then* G *is cotorsion-free.*

(c) *If* R *is slender, then* G *is slender.*

Proof. (a) The existence of m, $r_i \in R$, $1 \leqslant i \leqslant n$, and $u \in F$ is obvious and the uniqueness follows from 2.2(2). A proof of (b) may be found in [CG] or [DMV] and is omitted here. We will prove (c): Let $\mathbb{Z}^\omega = \prod_{n<\omega} d_n \mathbb{Z}$, where $d_n(i) = \delta_{ni}$ and let $\varphi: \mathbb{Z}^\omega \to G$ be a homomorphism of groups. We may assume that $0 \neq \varphi(d_n) = \sum_{i=1}^{k_n} a_{\alpha_{i,n}} r_{i,n} + u^{(n)}$, where the $\alpha_{i,n}$, $1 \leqslant i \leqslant k_n$, are distinct and $r_{i,n} \neq 0$ for all i, n.

Case 1. $\{\alpha_{i,n} \mid n<\omega,\ 1 \leqslant i \leqslant k_n\}$ *is infinite.* W.l.o.g. we may assume $\{\alpha_{1,n} \mid n<\omega\}$ is infinite and $\alpha_{1,n} \neq \alpha_{1,m}$ for $n \neq m$. By induction on n we find a cofinite subset $T_n \subseteq [a_{\alpha_{1,n}}]$ and $k_{n-1} < k_n \in \mathbb{N}$ such that $T_m \subseteq [(\sum_{n<\omega} \varphi(d_n) k_n!)]$ for all $m<\omega$. Thus $\bigcup_{m<\omega} T_m \subseteq [\sum_{n<\omega} \varphi(d_n) k_n!]$, which contradicts (a) and 2.2(2). Thus we may assume

Case 2. $\{\alpha_{i,n} \mid n<\omega,\ 1 \leqslant i \in k_n\}$ *is finite.* Restricting ourselves to an infinite subset of ω, we may assume $\alpha_{i,n} = \alpha_{i,1}$ for all $1 \leqslant i \leqslant k_1$, and $k_1 = k_n$ for all $n<\omega$. Then there exists $j_0 < \omega$ such that $f_{\alpha_{1,1}}(j) \notin [\![a_{\alpha_{i,1}}]\!]$ for all $j \geqslant j_0$ and $i > 1$. Thus

$$\left(\varphi\left(\sum_n d_n z_n\right)\right)(f_{\alpha_{11}}(j), e_{\alpha_{11}}(j)) = (1(f_{\alpha_{11}}(j), e_{\alpha_{11}}(j)))\left(j! \sum_{n<\omega} r_n z_n\right)$$

for every $j \geqslant j_0$ and every $\mathbb{Z}$-adic zero-sequence $\{z_n \mid n<\omega\}$.

This implies that $\sum_{n<\omega} r_n z_n \in R$ for every $\mathbb{Z}$-adic zero-sequence $\{z_n \mid n<\omega\}$. This implies (since R is slender and hence cotorsion-free) that $r_n = 0$ for almost all n. This cotradicts our assumption $0 \neq \varphi(d_n)$ for all n.

Case 3. *$\varphi(d_n) \in F$ for almost all n.* W.l.o.g. we may assume that $\varphi(d_n) \in F$ for all $n<\omega$.

Case 3.1. *$\bigcup_{n<\omega} [\varphi(d_n)]$ is infinite.* By induction and substituting ω by an infinite subset if necessary we find a sequence $(\alpha_n, k_n) \in \lambda \times 2^{\kappa}$ such that $(\alpha_n, k_n) \in [\varphi(d_n)]$ and $(\alpha_n, k_n) \notin [\varphi(d_i)]$ for $1 \leqslant i \leqslant n-1$. Moreover we may assume that $\alpha_n \leqslant \alpha_{n+1}$ for all $n<\omega$. Suppose $\sum_{n<\omega} \varphi(d_n)\, t_n! \in G$, where $t_n \in \mathbb{N}$ is a sequence such that $(\varphi(d_n)\, t_n!)(\alpha_n, k_n) \notin t_{n+1}!\, \bar{B}(\alpha_n, k_n)$ and $n \cdot t_n < t_{n+1}!$ for all n. This implies that there exists $\alpha < \lambda^*$ such that for a subsequence (α_{n_i}, k_{n_i}), $i<\omega$, we have $(\alpha_{n_i}, k_{n_i}) = (f_\alpha(i), e_\alpha(i))$, $i \geqslant i_0$, and $(\varphi(d_{n_i})\, t_{n_i}!)(\alpha_{n_i}, k_{n_i}) \equiv i!\, r \cdot 1(f_\alpha(i), e_\alpha(i)) \bmod t_{n_i+1}!\, \bar{B}(\alpha_{n_i}, k_{n_i})$. We now pick another sequence $\{\tilde{t}_n \mid n<\omega\}$ such that $t_n < \tilde{t}_n$ and $\tilde{t}_{n_i} > 2i$. If again $\sum \varphi(d_n)\, \tilde{t}_n! \in G$ we obtain for some $s \in R$,

$$(\varphi(d_{n_i})\, \tilde{t}_{n_i}!)(\alpha_{n_i}, k_{n_i}) \equiv i!\, s \cdot 1(f_\alpha(i), e_\alpha(i)) \bmod \tilde{t}_{n_i+1}!\, \bar{B}(\alpha_{n_i}, k_{n_i})$$

for all $i \geqslant \tilde{i}_0$. The latter implies $s \in \tilde{t}_{n_i}!\, (i!)^{-1} R$ and since $\{t_{n_i} - i \mid i<\omega\}$ is unbounded we arrive at the contradiction $s = 0$. Thus we have to consider

Case 3.2. *$\bigcup_{n<\omega} [\varphi(d_n)]$ finite.* In this case φ gives rise to a map $\varphi\colon \mathbb{Z}^{(\omega)} \to \bigoplus_{\text{finite}} B$ since the only elements in G with finite support are the elements of F. By continuity of φ we have : $\prod d_n n!\, \mathbb{Z} \to F$. Since F is a free R-module and R is slender we obtain $\varphi(d_n) = 0$ for all but finitely many n. This shows that G is slender.

We introduce a symbol $\infty \notin \tilde{F}$ and refine the construction of G_α by constructing, together with $a_\alpha = b_\alpha + \bar{e}_\alpha$, elements $t_\alpha \in \tilde{F} \cup \{\infty\}$ such that, in addition to (I_0), (I_α), (II_α) we have

(III_α) $t_\beta \notin G_\alpha$ for $\beta < \alpha$.

(IV_α) (a) If for all $b \in P_\alpha \cap \oplus_{(a,k) \in \lambda \times 2^{\kappa}} \bar{B}(\alpha, k)$ with $\|b\| \leqslant \|P_\alpha\|$, $t_\beta \notin \langle G_\alpha, (b + \bar{e}_\alpha) R\rangle_*$ when $\beta < \alpha$ but $(b + \bar{e}_\alpha)\, \varphi_\alpha \in \langle G_\alpha, (b + \bar{e}_\alpha) R\rangle_*$ then we set $t_\alpha = \infty$ and $a_\alpha = \bar{e}_\alpha$, where $\bar{e}_\alpha = \sum_{n<\omega} 1(f_\alpha(n), e(n)) n!$

(b) If (a) does not hold we set $t_\alpha = \varphi_\alpha(a_\alpha)$ and set $a_\alpha = b + \bar{e}_\alpha$, where b is a witness for the failure of (a).

LEMMA 3.2. *There exists a sequence of triples $(G_\alpha, a_\alpha, t_\alpha)$, $\alpha < \lambda^*$, such that (I_0), (I_α)–(IV_α) hold.*

Proof. [DMV, Lemma 3.4].

LEMMA 3.3. *If one chooses $(G_\alpha, a_\alpha, t_\alpha)$, $\alpha < \lambda^*$, as in 3.2, then $G = \bigcup_{\alpha < \lambda^*} G_\alpha$ is an R-module, $F \subseteq G \subseteq \tilde{F}$, and* $\mathrm{End}_{\mathbb{Z}}(G) = R$.

Proof. Let $\varphi\colon G \to G$ be a $\mathbb{Z}$-homomorphism and $F_\alpha = F \cap P_\alpha$. Assume that for each $\alpha < \lambda^*$ with $\varphi \restriction F_\alpha = \varphi_\alpha$ we have some $r_\alpha \in R$ such that $\varphi \restriction F_\alpha = \varphi_\alpha = r_\alpha$. If $\varphi \notin R$ then there is some $\beta < \lambda^*$ with $r_\alpha \neq r_\beta$ and by (2.2) there exists $\gamma < \lambda^*$ such that $\varphi \restriction F_\gamma = \varphi_\gamma = r_\gamma$ and $F_\alpha, F_\beta \subseteq F_\gamma$. Let $B(\alpha^0, 0) \subseteq F_\alpha$ and $B(\beta^0, 0) \subseteq F_\beta$. Then

$$\varphi(1(\alpha^0, 0)) = (1(\alpha^0, 0))\, r_\alpha = (1(\alpha^0, 0))\, r_\gamma$$

and

$$\varphi(1(\beta^0, 0)) = (1(\beta^0, 0))\, r_\beta = (1(\beta^0, 0))\, r_\gamma.$$

These equations imply $r_\alpha = r_\gamma = r_\beta$. Thus $\varphi \in R$. Therefore we may assume that for some $\alpha < \lambda^*$ we have $\varphi \restriction F_\alpha = \varphi_\alpha$ and $\varphi \restriction F_\alpha \notin R \restriction F_\alpha$. We will show that this ordinal $\alpha < \lambda^*$ satisfies (IV_α)(b).

Suppose $\varphi_\alpha(\bar{e}_\alpha) \in \langle G_\alpha, \bar{e}_\alpha R\rangle_*$. This implies $n\varphi_\alpha(\bar{e}_\alpha) \equiv \bar{e}_\alpha r \bmod G_\alpha$. Since R is pure in $\mathrm{Hom}(F, \tilde{F})$ and $\varphi_\alpha \notin R$ we conclude $n\varphi_\alpha \notin R$ and therefore $n\varphi_\alpha - r\colon F_\alpha \to P_\alpha$ is not the zero map. Thus there is some $(\alpha, k) \in \lambda \times 2^\kappa$ such that $B(\alpha, k) \subseteq F_\alpha$ and $(n\varphi_\alpha - r) \restriction B(\alpha, k) \neq 0$. Since G_α is slender, there is some $b \in \bar{B}(\alpha, k)$ such that $(n\varphi_\alpha - r)(b) \notin G_\alpha$.

Now set $a_\alpha = b + \bar{e}_\alpha$. If $m\varphi_\alpha(a_\alpha) \in \langle G_\alpha, a_\alpha R\rangle_*$ for some $m \in \mathbb{N}$, then

$$m\varphi_\alpha(a_\alpha) \equiv (b + \bar{e}_\alpha)\, s \qquad \bmod G_\alpha$$

and we obtain $nm\varphi_\alpha(b) \equiv bns + \bar{e}_\alpha(ns - mr) \bmod G_\alpha$. Since $\|b\|$, $\|\varphi_\alpha(b)\| < \|P_\alpha\|$ we conclude that $ns = mr$ and $n\varphi_\alpha(b) \equiv br \bmod G_\alpha$, which contradicts our choice of b. Therefore $(\mathrm{IV}_\alpha)(a)$ does not hold and by Lemma 3.2 we have that (IV_α)(b) holds and $t_\alpha = \varphi_\alpha(a_\alpha) = \varphi(a_\alpha) \notin G$. This contradicts $\varphi \in \mathrm{End}(G)$ and $\mathrm{End}(G) = R$ follows.

Remark. As in [DMV, Theorem 5.1] we may also construct rigid systems $\{G^{(\alpha)} \mid \alpha < 2^\lambda\}$ of such modules, i.e., $\mathrm{Hom}_{\mathbb{Z}}(G^{(\alpha)}, G^{(\beta)}) = \delta_{\alpha\beta} R$ for all $\alpha \neq \beta < 2^\lambda$.

References

[AE] Y. Abe and I. Eda, Compact cardinals and abelian groups, *Tsukuba J. Math.* **11** (1987), 353–360.

[APRVW] D. Arnold, R. S. Pierce, J. D. Reid, C. Vinsonhaler, and W. Wickless, Torsion-free abelian groups of finite rank projective as modules over their endomorphism rings, *J. Algebra* **71** (1981), 1–10.

[BS] R. A. Bowshell and P. H. Schultz, Unital rings whose additive endomorphisms commute, *Math. Ann.* **228** (1977), 197–214.

[CG] A. L. S. Corner and R. Göbel, Prescribing endomorphism algebras, a unified treatment, *Proc. London Math. Soc.* (3) **50**, 447–479.

[DG1] M. DUGAS AND R. GÖBEL, Every cotorsion-free ring is an endomorphism ring, *Proc. London Math. Soc.* (3) **45** (1982), 319–336.

[DG2] M. DUGAS AND R. GÖBEL, On radicals and products, *Pacific J. Math.* **118** (1985), 79–104.

[DMV] M. DUGAS, A. MADER, AND C. VINSONHALER, Large E-rings exist, *J. Algebra* **108** (1987), 88–101.

[FI/II] L. FUCHS, "Infinite Abelian Groups," Vols. I, II, Academic Press, New York, 1970, 1973.

[J] T. JECH, "Set Theory," Academic Press, New York, 1978.

[KK] L. KAUP AND M. S. KEANE, Induktive Limiten endlich erzeugter freier Moduln, *Manuscripta Math.* **1** (1969), 9–21.

[Pi] R. S. PIERCE, E-modules, *Contemp. Math.* **87** (1989), 221–240.

[S] P. SCHULTZ, The endomorphis ring of the additive group of a ring, *J. Austral. Math. Soc.* **15** (1973), 60–69.