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The Characterization of Generalized Wreath Products*

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1. INTRODUCTION

If G is a permutation group on a set S, and H a permutation group on a set T, then the (complete) *wreath product* $G \wr H$ is defined (see [3]) to be the group of those permutations ϕ on $S \times T$ of the form $(s, t)\phi = (sg_t, th)$ where $g_t \in G$ and $h \in H$. The main use of wreath products in group theory has been to supply examples of groups with special properties, and applications of this sort are too numerous and well known to be listed here. It is well known that the wreath product has a certain universal property, which we may describe in the following way. If G is a transitive permutation group on a set S, if C is a G-congruence on S, and if $x \in S$, then G induces, in a natural way, a permutation group A on the set xC, and a permutation group B on the set S/C. Then the universal property is that G may be embedded in the wreath product $A \wr B = W$. Moreover, there is a W-congruence K defined on $R = xC \times S/C$ by $(a, b) \equiv (c, d) (\text{mod } K)$ if and only if $b = d$, and on any one of the K-classes, W induces a permutation group isomorphic to A, while on R/K, W induces a permutation group isomorphic to B.

In [6], Krasner and Kaloujnine devised a way to define the wreath product (which they called "complete product") of a finite sequence $G_1, G_2, \dots, G_n$ of permutation groups on sets $S_1, S_2, \dots, S_n$, respectively, as a certain subgroup of the group of permutations of $S_1 \times S_2 \times \cdots \times S_n$, and which is isomorphic to the iterated wreath product $(\cdots((G_1 \wr G_2) \wr G_3) \wr \cdots) \wr G_n$. They also showed that the wreath product has the universal property, which, in this more general case, we may roughly describe as follows. If W is the wreath product of transitive permutation groups $G_1, G_2, \dots, G_n$, on sets $S_1, S_2, \dots, S_n$, respectively, and $R = S_1 \times S_2 \times \cdots \times S_n$, then the equivalence relation K^i defined on R by

$$(s_1, s_2, \dots, s_n) \equiv (t_1, t_2, \dots, t_n) (\text{mod } K^i)$$
$$\text{if and only if } s_j = t_j \text{ for all } j > i,$$

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is respected by W; in other words, K^i is a W-congruence for each $i = 0, 1, 2, \ldots, n$. For each $x \in R$, the set $(xK^i)/K^{i-1}$ can be identified with S_i in a natural way. Also, W induces a permutation group on $(xK^i)/K^{i-1}$, namely $W^{i,x}/W_{i,x}$ where

$$W^{i,x} = \{g \in W \mid xg \equiv x(\text{mod } K^i)\}, \text{ and}$$

$$W_{i,x} = \{g \in W \mid zg \equiv z(\text{mod } K^{i-1}) \text{ for all } z \in xK^i\},$$

and the "component" $W^{i,x}/W_{i,x}$ may be identified with G_i in a natural way. Thus, W is constructed from the components G_i, and the components can be retrieved from W and a natural sequence of congruences. Now the universal property of W is this: If G is a transitive permutation group on a set S, if $C^0 \subseteq C^1 \subseteq \cdots \subseteq C^n$ are G-congruences on S, with C^n and C^0 the trivial congruences, and if the components of G with respect to the congruences C^i, defined as above for W with respect to K^i, are isomorphic to G_i, respectively, then there is a "nice" embedding of G in the wreath product W. In short, W has components G_i, and W contains every transitive permutation group with the same (ordered set of) components G_i.

There is nothing in the discussion above that depends in any essential way either on the finiteness of the set of congruences, or on the fact that they form a chain. In fact, if we have any collection of congruences of a transitive permutation group H on a set T, and if $Q_\gamma \subseteq Q^\gamma$ are congruences of the collection such that Q^γ "covers" Q_γ (there is no congruence of the collection which lies between them), then we may define components just as above, say the permutation group G_γ on the set S_γ is the component corresponding to the pair of congruences (Q_γ, Q^γ). There is a natural partial order on the index set $\Gamma = \{\gamma\}$ whereby $\gamma_1 < \gamma_2$ if $Q^{\gamma_1} \subseteq Q_{\gamma_2}$. Now we may ask whether there exists a permutation group which is universal for groups with components G_γ on S_γ, $\gamma \in \Gamma$.

The need to consider possibly infinite sets of congruences arises very naturally. The goal of a theory of this sort is to describe an arbitrary permutation group as composed in some cannonical way of permutation groups of a more elementary type. For instance, in ([*8*], p. 18), Wielandt considers a maximal chain of congruences on a finite permutation group, observes that the components, as we have defined them here, are primitive, and suggests that the relationship between properties of the given group and properties of the components be studied. To obtain similar primitive components in the case of an infinite permutation group obviously requires consideration of maximal chains which may be infinite.

From a synthetic point of view, some authors have found it useful to construct groups using wreath products iterated infinitely many times (for instance [*4*] and [*7*]). Such constructions are convenient only if the index set is well ordered. In [5], P. Hall described a construction for the wreath product of permutation groups $\{G_\gamma \mid \gamma \in \Gamma\}$ where Γ is an arbitrary totally ordered set (we shall describe his construction in detail later). Hall's construction is a generalization of the "restricted wreath product," where the restricted wreath product $G \wr H$ of two permutation groups is defined just as the complete wreath product with the additional requirement that for each permutation, all but finitely many g_t's be the identity.

What we shall do in this paper is define the complete wreath product of a collection $\{G_\gamma \mid \gamma \in \Gamma\}$ of permutation groups, where Γ is an arbitrary partially ordered set. Our wreath product has the universal property, and is, in fact, characterized by it (to within permutation isomorphism). In the case Γ is a finite chain, our wreath product is the same as the "complete product" of Krasner and Kaloujnine, and so if Γ is a two-element chain, it is the ordinary wreath product. If Γ is an arbitrary chain, our wreath product contains the restricted wreath product of P. Hall. And if Γ is trivially ordered, our wreath product is the cartesian product.

To the reader familiar with the work of Krasner and Kaloujnine, it will be apparent that the process of embedding a given transitive permutation group G on S, (with a given finite chain of congruences), into the wreath product of its components is essentially a matter of adjoining to G certain other permutations of S. In the infinite case, however, an added difficulty presents itself: the set S may itself fail to be "complete," and thus have to be enlarged. For example, it may occur that there are congruences $\{C_i \mid i = 1, 2, \ldots\}$ and C_i-classes k_i such that $k_1 \supseteq k_2 \supseteq \cdots$, and yet $\bigcap_{i=1}^{\infty} k_i$ is empty. Whenever this occurs, the set S may be increased by adjoining a point contained in each of the classes k_i, still without changing the components of G. For this and similar reasons, we are led into the necessity to say precisely what we mean by embedding one permutation group in another. Section 2 is devoted to notation, including a general discussion of homomorphism theory for permutation groups, and the precise definition of components. In Section 3 (which does not depend on section 2) we construct the wreath product, show that it satisfies a general associative law, and that it is transitive if each of the factors is. In Section 4, we show that if G is a transitive permutation group on S, if $\mathscr{C}$ is a collection of G-congruences on S, if $\Gamma = \{(C_\gamma, C^\gamma)\}$ is a suitable collection of covering pairs of congruences in $\mathscr{C}$, and if G has components G_γ, $\gamma \in \Gamma$, then G can be embedded in the wreath product W of the components G_γ, $\gamma \in \Gamma$. Moreover, every extension of G with the same components can be embedded in W, and W has no proper extensions with the same components. Thus, W is the unique maximal extension of G with the same components.

For the proofs, we borrow freely from the methods of Conrad [1] and Conrad, Harvey, and the author [2] for abelian groups with valuation, and abelian lattice ordered groups. Indeed, our results here may be thought of as a noncommutative analogue of some of the results in those papers.

2. Some Notation

If S is a set, $P(S)$ denotes the symmetric group on S. If G is a subgroup of $P(S)$, we may use the symbol (G, S) to denote G. If $R \subseteq S$, and if $F \subseteq G$, then $RF = \{rf \mid r \in R, f \in F\}$, where rf denotes the image of r under the permutation f. By an *F-congruence*, or *block system* is meant an equivalence relation E on S such that if $f \in F$ and $x, y \in S$, then $x \equiv y (\text{mod } E)$ if and only if $xf \equiv yf (\text{mod } E)$; in other words, each $f \in F$ permutes the E-classes according to the rule $(xE)f = (xf)E$. An *F-fixed block* is a subset $S' \subseteq S$ such that for each $f \in F$, $S'f = S'$.

Let (G, S) and (H, T) be permutation groups. By a *homomorphism* $\pi : (G, S) \rightarrow (H, T)$ is meant a pair of functions (both denoted by the same letter) $\pi : S \rightarrow T$ and $\pi : G \rightarrow H$, where

(1) $\pi : G \rightarrow H$ is a group homomorphism, and

(2) $(x\pi)(g\pi) = (xg)\pi$ for all $x \in S, g \in G$.

If $\pi : S \rightarrow T$ is onto, (1) follows from (2).

Let $\pi : (G, S) \rightarrow (H, T)$ be a homomorphism. The equivalence relation Q on S defined by $x \equiv y (\text{mod } Q)$ if $x\pi = y\pi$, is a G-congruence, for if $g \in G$, and if $x, y \in S$ with $x\pi = y\pi$, then $(xg)\pi = (x\pi)(g\pi) = (y\pi)(g\pi) = (yg)\pi$.

If K is the kernel of $\pi : G \rightarrow H$ and Q the equivalence relation defined before, we note that, if $\pi : S \rightarrow T$ is onto,

$$N_Q = N_Q(G) = \{g \in G \mid xg \equiv x (\text{mod } Q) \text{ for all } x \in S\} = K.$$

We shall call Q the *kernel* of $\pi : (G, S) \rightarrow (H, T)$.

A homomorphism $\pi : (G, S) \rightarrow (H, T)$ is an *epimorphism* if each part of π is onto; and π is an *isomorphism* if π is an epimorphism, each part of which of which is one-to-one. It then follows that $\pi : G \rightarrow H$ is a group isomorphism. A homomorphism π is an *embedding* if π is one-to-one on S, and $G\pi$ is faithful on $S\pi$; that is, $(x\pi)(g\pi) = x\pi$ for all $x \in S$ implies $g\pi$ is the identity of H.

If (G, S) is a permutation group and E a G-congruence on S, the *permutation factor group* $(G, S)/E$ is defined to be the permutation group $(G/N_E, S/E)$ where S/E is the set of E-classes, and for $gN_E \in G/N_E$, $xE \in S/E$, $(xE)(gN_E) = (xg)E$. The natural map $\pi : G \rightarrow G/N_E$ and $\pi : S \rightarrow S/E$ is an epimorphism of (G, S) onto $(G, S)/E$. Conversely, if $\pi : (G, S) \rightarrow (H, T)$ is an epimorphism with kernel Q, then $\pi = \alpha\beta$ where $\alpha : (G, S) \rightarrow (G, S)/Q$ is the natural map, and $\beta : (G, S)/Q \rightarrow (H, T)$ is an isomorphism.

Let (H, T) be a permutation group, let F be a subgroup of H, and $R \subseteq T$

an F-fixed block. Then the restriction $F \mid R$ of F to R is a subogroup f $P(R)$; if F is faithful on R, we shall say $(F \mid R, R)$ is a *permutation subgroup* of (H, T). Note that if $\pi : (G, S) \to (H, T)$ is an embedding then $(G\pi, S\pi)$ is a permutation subgroup of (H, T). Also, if G is a subgroup of H, then certainly (G, T) is a permutation subgroup of (H, T).

Let (G, S) be a permutation group and $\mathscr{C}$ a collection of G-congruences on S, and let $C_\gamma, C^\gamma \in \mathscr{C}$. Then C^γ *covers* C_γ (in $\mathscr{C}$) if C^γ properly contains C_γ and no element of $\mathscr{C}$ properly contains C_γ while being properly contained in C^γ. We shall also say that (C_γ, C^γ) is a *covering pair* in $\mathscr{C}$. We shall frequently make the identification $\gamma = (C_\gamma, C^\gamma)$. There is a natural partial order on the set of all covering pairs in $\mathscr{C}$, whereby $\alpha < \beta$ if $C^\alpha \subseteq C_\beta$. Now suppose that (G, S) is a permutation subgroup of (H, T) and let $\mathscr{K}$ be a collection of H-congruences on T. If $K \in \mathscr{K}$, $K \mid S = K \cap (S \times S)$ denotes the restriction of K to S, which is obviously a G-congruence on S. Let $\Delta = \{(K_\gamma, K^\gamma)\}$ be a set of covering pairs in $\mathscr{K}$, and let $\Gamma = \{(K_\gamma \mid S, K^\gamma \mid S)\}$. Then we will say that (H, T, Δ) is an *extension* of (G, S, Γ) if

(i) each $(K_\gamma \mid S, K^\gamma \mid S)$ is a covering pair in $\mathscr{K} \mid S$, the set of restrictions of elements of $\mathscr{K}$ to S, and

(ii) the correspondence $(K_\gamma, K^\gamma) \to (K_\gamma \mid S, K^\gamma \mid S)$ is one-to-one, and an order isomorphism, of Δ on Γ, and

(iii) if $x \in S$, $g \in G$, and $zg \equiv z(\text{mod } K_\gamma)$ for all $z \in xK^\gamma \mid S$, then $zg \equiv z(\text{mod } K_\gamma)$ for all $z \in xK^\gamma$.

Let (G, S) be any permutation group, let $\mathscr{C}$ be a collection of G-congruences on S, let Γ be a set of covering pairs in $\mathscr{C}$, let $x \in S$ and $\gamma \in \Gamma$. Then we define

$$G^{\gamma,x} = \{g \in G \mid xg \equiv x(\text{mod } C^\gamma)\},$$

which is clearly a subgroup of G. Also, xC^γ is a $G^{\gamma,x}$-fixed block. Now we define the *component* of (G, S, Γ) at (γ, x) to be

$$(G_{\gamma,x}, S_{\gamma,x}) = (G^{\gamma,x} \mid xC^\gamma, xC^\gamma)/(C_\gamma \mid xC^\gamma).$$

The component of (G, S, Γ) at (γ, x) may be thought of roughly as the restriction of (a certain subgroup of) G to the congruence class xC^γ, modulo the smaller congruence C_γ. In the sequel we sometimes write $C_\gamma \mid xC^\gamma$ as just C_γ. For example, $S_{\gamma,x} = (xC^\gamma)/C_\gamma$. We shall also omit another restriction bar and write $(G^{\gamma,x} \mid xC^\gamma, xC^\gamma) = (G^{\gamma,x}, xC^\gamma)$. Thus, in the shorter notation,

$$(G_{\gamma,x}, S_{\gamma,x}) = (G^{\gamma,x}, xC^\gamma)/C_\gamma.$$

If (H, T, Δ) is an extension of (G, S, Γ) there should be no confusion if we identify the sets Δ and Γ. We want to show that the components of (G, S, Γ) may be thought of as permutation subgroups of those of (H, T, Γ).

LEMMA 2.1. *If* (H, T, Γ) *is an extension of* (G, S, Γ), *if* $x \in S$ *and* $\gamma \in \Gamma$, *then there is a natural embedding of* $(G_{\gamma,x}, S_{\gamma,x})$ *in* $(H_{\gamma,x}, T_{\gamma,x})$.

Proof. Let (K_γ, K^γ) and $(C_\gamma, C^\gamma) = (K_\gamma \mid S, K^\gamma \mid S)$ be the corresponding covering pairs. There is a subgroup F of H with S an F-fixed block of T and $G = F \mid S$. Let $Q = C_\gamma \mid xC^\gamma$ and N_Q the corresponding normal subgroup of $G^{\gamma,x} \mid xC^\gamma$, and let $P = K_\gamma \mid xK^\gamma$ and N_P the corresponding normal subgroup of $H^{\gamma,x} \mid xK^\gamma$. Then we must establish an embedding of $(G^{\gamma,x}, xC^\gamma)/Q$ in $(H^{\gamma,x}, xK^\gamma)/P$. If $z \in xC^\gamma$, define $(zC_\gamma)\pi = zK_\gamma$. And if $g \in G^{\gamma,x} \mid xC^\gamma$, so that $g = \bar{g} \mid xC^\gamma, \bar{g} \in F \cap H^{\gamma,x}$, define $(gN_Q)\pi = (\bar{g} \mid xK^\gamma)N_P$. If $\bar{g} \mid xC^\gamma \in N_Q$, then $\bar{g} \mid xK^\gamma \in N_P$ by the third condition in the definition of extension, so π is well defined and one-to-one. Also π is a homomorphism, since

$$((zC_\gamma)(gN_Q))\pi = (zgC_\gamma)\pi = (zg)K_\gamma = (z(\bar{g} \mid xK^\gamma))K_\gamma$$
$$= (zK_\gamma)((\bar{g} \mid xK^\gamma)N_P) = (zC_\gamma)\pi(gN_Q)\pi.$$

Finally, if $(gN_Q)\pi$ is the identity on $((xC^\gamma)/Q)\pi$ then for each $z \in xC^\gamma$, $zK_\gamma = (zK_\gamma)(\bar{g}N_P) = (z\bar{g})K_\gamma$, which certainly implies that $zC_\gamma = (z\bar{g})C_\gamma$, and thus that $(gN_Q)\pi$ is the identity on all of $(xK^\gamma)/P$.

LEMMA 2.2. *If* (G, S) *is transitive, and* $\mathscr{C}$ *is a set of G-congruences on* S, *then for each covering pair* γ *in* $\mathscr{C}$, *and for each* $x, y \in S$, $(G_{\gamma,x}, S_{\gamma,x})$ *is isomorphic to* $(G_{\gamma,y}, S_{\gamma,y})$.

Proof. By transitivity, there exists $f \in G$ such that $xf = y$. Let $Q = C_\gamma \mid xC^\gamma$ and $P = C_\gamma \mid yC^\gamma$. Define $\pi : (G_{\gamma,x}, S_{\gamma,x}) \to (G_{\gamma,y}, S_{\gamma,y})$ as follows. For $zC_\gamma \in S_{\gamma,x}$, $(zC_\gamma)\pi = (zf)C_\gamma$; and for $g \in G^{\gamma,x}$,

$$((g \mid xC^\gamma)N_Q)\pi = ((f^{-1}gf) \mid yC^\gamma)N_P.$$

Then it is a straight-forward matter to verify that π is an isomorphism.

Thus, in the transitive case, which is the case that mainly concerns us, we shall usually drop the subscript x, and write

$$(G_\gamma, S_\gamma) = (G_{\gamma,x}, S_{\gamma,x})$$

for any x, and call it the *component of* (G, S, Γ) *at* γ.

If (H, T) is a permutation group, if $\mathscr{K}$ is a set of H-congruences on T, and if Δ is a set of covering pairs in $\mathscr{K}$, then Δ is said to be *plenary* if

(i) if $x, y \in T$ with $x \neq y$, then there exists $\gamma \in \Delta$ such that $x \equiv y (\text{mod } C^\gamma)$ and $x \not\equiv y (\text{mod } C_\gamma)$, and

(ii) if $x, y \in T$ and $\alpha \in \Delta$ such that $x \not\equiv y (\text{mod } C^\alpha)$, then there exists $\beta \in \Delta$, $\beta > \alpha$, such that $x \equiv y (\text{mod } C^\beta)$ and $x \not\equiv y (\text{mod } C_\beta)$.

Such a γ described in (i) is said to be a *value* of the pair (x, y). The terminology is that of [2], and the idea stems from general valuation theory [1]. Plenary sets arise in many ways. We list three examples, in which Δ is plenary.

1. If every chain in $\mathscr{K}$ is finite, if $\mathscr{K}$ contains the trivial congruences, and if Δ is the set of all covering pairs in $\mathscr{K}$.

2. If $\mathscr{K}$ is the set of all H-congruences on T and Δ is the set of all covering pairs in $\mathscr{K}$.

3. If $\mathscr{K}$ is the set of all H-congruences on T and Δ is the set of those covering pairs γ in $\mathscr{K}$ such that there exist $x, y \in T$ with γ a value of (x, y).

We will say that (H, T, Δ) is an *immediate extension* of (G, S, Γ) if

(i) Δ is plenary

(ii) (H, T, Δ) is an extension of (G, S, Γ)

(iii) the natural embedding described in Lemma 2.1 is onto for each component of (G, S, Γ) (there may be components of (H, T, Δ) in which no component of (G, S, Γ) is naturally embedded), and

(iv) the H-orbit of every $t \in T$ contains an element of S. (It follows that if (G, S) is transitive, so is (H, T).)

Roughly in words, (H, T, Δ) is an immediate extension of (G, S, Γ) if it is an extension with the same components.

3. Construction of the Wreath Product

Let Γ be a partially ordered set, and for each $\gamma \in \Gamma$, let (G_γ, S_γ) be a permutation group with S_γ more than one point. We wish to construct a permutation group (G, S) with a suitable set of G-congruences on S, and a set Γ of covering pairs of congruences such that the components of (G, S, Γ) are exactly the groups (G_γ, S_γ).

First, consider the elements of the cartesian product $\prod_{\gamma \in \Gamma} S_\gamma$ as functions defined on Γ, so that if $x \in \prod S_\gamma$, $x(\gamma)$ denotes the γth coordinate of x. Fix, for the remainder of this discussion, an arbitrary element $0 \in \prod S_\gamma$. If $x \in \prod S_\gamma$, the *support* of x is $\{\gamma \in \Gamma \mid x(\gamma) \neq 0(\gamma)\}$. Now let S consist of those $x \in \prod S_\gamma$ such that the support of x satisfies the maximum condition; that is, every nonempty subset of the support of x has a maximal element, or equivalently, every strictly increasing chain in the support of x is finite. It follows that for every $x_1, x_2 \in S$, the set $\{\gamma \in \Gamma \mid x_1(\gamma) \neq x_2(\gamma)\}$ satisfies the maximum condition.

For each $\gamma \in \Gamma$, let K_γ be the equivalence relation on S defined by

$$x_1 \equiv x_2 (\text{mod } K_\gamma) \quad \text{if} \quad x_1(\alpha) = x_2(\alpha) \qquad \text{for all} \quad \alpha \geqslant \gamma.$$

We shall also abbreviate $x_1 \equiv x_2 (\text{mod } K_\gamma)$ by $x_1 \equiv_\gamma x_2$. Let

$$G' = \{ g \in P(S) \mid K_\gamma \text{ is a } g\text{-congruence for each } \gamma \in \Gamma \}.$$

Then G' is readily seen to be a subgroup of the symmetric group $P(S)$. For each $\gamma \in \Gamma$, let

$$K^\gamma = \bigcap_{\beta > \gamma} K_\beta, \qquad (\text{and } K^\gamma = S \times S \text{ if there is no } \beta > \gamma),$$

so that

$$x_1 \equiv x_2 (\text{mod } K^\gamma) \quad \text{if} \quad x_1(\beta) = x_2(\beta) \qquad \text{for all} \quad \beta > \gamma.$$

We shall abbreviate $x_1 \equiv x_2 (\text{mod } K^\gamma)$ by $x_1 \equiv^\gamma x_2$. Then each K^γ is a G'-congruence.

LEMMA 3.1. *For each subset Σ of Γ, define an equivalence relation $E(\Sigma)$ on S by $x_1 \equiv x_2 (\text{mod } E(\Sigma))$ if $x_1(\beta) = x_2(\beta)$ for each $\beta \in \Sigma$. Then the correspondence $\Sigma \to E(\Sigma)$ is one-to-one. Also, $\Sigma \subseteq \Phi$ if and only if $E(\Sigma) \supseteq E(\Phi)$.*

Proof. For each $\gamma \in \Gamma$, choose $b_\gamma(\gamma) \in S_\gamma$, $b_\gamma(\gamma) \neq 0(\gamma)$, which can be done since each S_γ has more than one point; otherwise, define $b_\gamma(\alpha) = 0(\alpha)$ for all $\alpha \neq \gamma$. Then the support of b_γ is $\{\gamma\}$, so $b_\gamma \in S$. Now, if Σ and Φ are subsets of Γ and $\Sigma \not\subseteq \Phi$, there exists $\alpha \in \Sigma \backslash \Phi$. Then $b_\alpha \equiv 0(\text{mod } E(\Phi))$ but $b_\alpha \not\equiv 0(\text{mod } E(\Sigma))$ so $E(\Sigma) \not\supseteq E(\Phi)$. On the other hand, it is obvious that if $\Sigma \subseteq \Phi$ then $E(\Sigma) \supseteq E(\Phi)$. Thus, the second conclusion of the lemma is true. Also, if $\Sigma \neq \Phi$ then we may suppose $\Sigma \not\subseteq \Phi$, so $E(\Sigma) \not\supseteq E(\Phi)$; in particular, $E(\Sigma) \neq E(\Phi)$, and the correspondence is one-to-one.

COROLLARY 3.2. *In the collection of equivalences*

$$\mathscr{K} = \{K_\gamma \mid \gamma \in \Gamma\} \cup \{K^\gamma \mid \gamma \in \Gamma\},$$

the pairs (K_γ, K^γ) are covering pairs.

Proof. Obviously the set $\Phi_\gamma = \{\beta \in \Gamma \mid \beta \geqslant \gamma\}$ covers the set $\Phi^\gamma = \{\beta \in \Gamma \mid \beta > \gamma\}$ in the set of subsets of Γ. And $K^\gamma = E(\Phi^\gamma)$, while $K_\gamma = E(\Phi_\gamma)$. The result follows from Lemma 3.1 and the observation that for any $K \in \mathscr{K}$, there exists a subset $\Psi \subseteq \Gamma$ such that $E(\Psi) = K$.

Let $g \in G'$, $\gamma \in \Gamma$, and $x \in S$. We define a function $g_{\gamma,x}$ on S_γ, which will turn out to be a permutation of S_γ, and which we will think of as the "component of g at (γ, x)," in the following way. If $a \in S_\gamma$, define $x' \in S$ by $x'(\beta) = x(\beta)$, if $\beta > \gamma$; $x'(\gamma) = a$; and $x'(\beta) = 0(\beta)$ if $\beta \ngtr \gamma$. Now, we define

$$a g_{\gamma,x} = (x'g)(\gamma).$$

The first thing to observe is that if $x \equiv^\gamma y$, then $g_{\gamma,x} = g_{\gamma,y}$, because if x' is defined as above, and y' is defined similarly, then $x' = y'$ since if $\beta > \gamma$, $x'(\beta) = x(\beta) = y(\beta) = y'(\beta)$. Hence $ag_{\gamma,x} = (x'g)(\gamma) = (y'g)(\gamma) = ag_{\gamma,y}$. Next, we see that $(x(\gamma))\, g_{\gamma,x} = (xg)(\gamma)$, for if $x'(\beta) = x(\beta)$, $\beta \geqslant \gamma$, and $x'(\beta) = 0(\beta)$ otherwise, then $x \equiv_\gamma x'$ and so $xg \equiv_\gamma x'g$, and

$$(x(\gamma))\, g_{\gamma,x} = (x'g)(\gamma) = (xg)(\gamma).$$

Lemma 3.3. $g_{\gamma,x} \in P(S_\gamma)$.

Proof. $g_{\gamma,x}$ is one-to-one, for if $a \neq b$, $a, b \in S_\gamma$, and if $x_1(\beta) = x_2(\beta) = x(\beta)$ for $\beta > \gamma$, if $x_1(\beta) = x_2(\beta) = 0(\beta)$ for $\beta \ngeqslant \gamma$, and if $x_1(\gamma) = a$ and $x_2(\gamma) = b$, then $x_1 \equiv^\gamma x_2$ but $x_1 \not\equiv_\gamma x_2$, so $x_1 g \equiv^\gamma x_2 g$ but $x_1 g \not\equiv_\gamma x_2 g$. Hence $ag_{\gamma,x} = (x_1 g)(\gamma) \neq (x_2 g)(\gamma) = bg_{\gamma,x}$.

$g_{\gamma,x}$ maps S_γ onto itself, for if $a \in S_\gamma$, define $x_3 \in S$ by $x_3(\beta) = (xg)(\beta)$ for $\beta > \gamma$, $x_3(\beta) = 0(\beta)$ for $\beta \ngeqslant \gamma$, and $x_3(\gamma) = a$. Then $x_3 \equiv^\gamma xg$ so $x_3 g^{-1} \equiv^\gamma x$. Moreover, $(x_3 g^{-1})(\gamma) \in S_\gamma$. Hence

$$((x_3 g^{-1})(\gamma))\, g_{\gamma,x} = ((x_3 g^{-1})(\gamma))\, g_{\gamma,x_3 g^{-1}} = ((x_3 g^{-1})\, g)(\gamma) = x_3(\gamma) = a.$$

Definition. The *wreath product* of the permutation groups (G_γ, S_γ), $\gamma \in \Gamma$, over the partially ordered index set Γ is the set of those permutations $g \in G'$ such that $g_{\gamma,x} \in G_\gamma$ for every $\gamma \in \Gamma$ and $x \in S$.

We note the following important rules for computation.

1. $(x(\gamma))\, g_{\gamma,x} = (xg)(\gamma)$.
2. If $x \equiv^\gamma y$ then $g_{\gamma,x} = g_{\gamma,y}$.
3. $(gh)_{\gamma,x} = g_{\gamma,x} h_{\gamma,xg}$.
4. $(g^{-1})_{\gamma,x} = (g_{\gamma,xg^{-1}})^{-1}$.

Of these, 1. and 2. have already been proved, 3. follows from a straightforward computation, and 4. is an immediate consequence of 3. From 1. it is clear that each $g \in G'$ is completely determined by the set $\{g_{\gamma,x} \mid \gamma \in \Gamma, x \in S\}$. And from 3. and 4., we have

Theorem 3.4. *The wreath product of the permutation groups* (G_γ, S_γ) *is a (permutation) subgroup of the symmetric group* $P(S)$.

Let us look at some special cases. When every chain in Γ is inversely well-ordered (and only then), $S = \prod S_\gamma$. In particular, if $\Gamma = \{\alpha, \beta\}$, $\alpha < \beta$, then the wreath product of permutation groups (G_γ, S_γ), $\gamma \in \Gamma$, is the classical wreath product $G_\alpha \wr G_\beta$.

The *cartesian product* of a set of permutation groups (G_γ, S_γ) is the permutation group $\prod (G_\gamma, S_\gamma) = (\prod G_\gamma, \prod S_\gamma)$ such that

$$(..., s_\gamma, ...)(..., g_\gamma, ...) = (..., s_\gamma g_\gamma, ...).$$

In case Γ is trivially ordered, the wreath product of permutation groups (G_γ, S_γ), $\gamma \in \Gamma$, is just the cartesian product, because $x \equiv^\gamma y$ for all $x, y \in S$, $\gamma \in \Gamma$ (there are no $\beta > \gamma$), and thus $g_{\gamma,x} = g_{\gamma,y} = g_\gamma$, say; and $(gh)_\gamma = g_\gamma h_\gamma$. Thus the mapping $s \to s$ of S $(= \prod S_\gamma)$ onto itself, and $g \to (..., g_\gamma, ...)$ of G to $\prod G_\gamma$ is an isomorphism of the wreath product onto the cartesian product.

If Γ is an arbitrary totally ordered set, then the wreath product of permutation groups (G_γ, S_γ), $\gamma \in \Gamma$, contains as a permutation subgroup, the wreath product $(\bar{G}, \bar{S})$ defined by P. Hall [5], which may be described as follows. Let $\bar{S}$ consist of those elements of S with finite support, and $\bar{G}$ those $g \in G$ such that $g_{\gamma,x}$ is the identity for all but finitely many pairs (γ, x) (where, for each γ, we choose only one x from each K^γ-class).

The equivalences K_γ and K^γ, being G'-congruences, are also G-congruences. Also, by Corollary 3.2, the pairs (K_γ, K^γ) are covering pairs. Thus, we may ask: What are the components of (G, S, Γ)?

LEMMA 3.5. *The component of* (G, S, Γ) *at* (γ, x) *is isomorphic to* (G_γ, S_γ).

Proof. If $z \in xK^\gamma$, let $(zK_\gamma)\pi = z(\gamma) \in S_\gamma$. Then $\pi : (xK^\gamma)/K_\gamma \to S_\gamma$ is one-to-one and well-defined. Moreover, π is onto, since for each $x \in S$ and $a \in S_\gamma$ there exists $z \in S$ such that $z \equiv^\gamma x$ and $z(\gamma) = a$ (see the definition of $g_{\gamma,x}$). If $g \in G$ such that $x \equiv^\gamma xg$, let $(gN_{K_\gamma})\pi = g_{\gamma,x}$. Then $\pi : G^{\gamma,x}/N_{K_\gamma} \to G_\gamma$ is well-defined, one-to-one, and onto. Moreover,

$$\begin{aligned}((zK_\gamma)(gN_{K_\gamma}))\pi &= ((zg)K_\gamma)\pi = (zg)(\gamma) = (z(\gamma))\, g_{\gamma,z} \\ &= (z(\gamma))\, g_{\gamma,x} = (zK_\gamma)\,\pi(gN_{K_\gamma})\pi.\end{aligned}$$

Hence, π is an isomorphism from the component $(G_{\gamma,x}, S_{\gamma,x})$ of (G, S), onto (G_γ, S_γ).

There is a certain amount of arbitrariness in the construction of the wreath product arising from the choice of the element $0 \in \prod S_\gamma$. In fact, the wreath products resulting from different choices of 0 may fail to be isomorphic, as the following example shows.

Example 3.6. Let Γ be the naturally ordered set of positive integers, and for each $\gamma \in \Gamma$, let $S_\gamma = \{0', 1', 2'\}$, and G_γ the 2-element subgroup of $P(S_\gamma)$ which fixes $0'$. If we construct the wreath product (G, S) using the element $0 \in \prod S_\gamma$ such that $0(\gamma) = 0'$ for each γ, then 0 is a fixed point for (G, S) since $(0g)(\gamma) = (0(\gamma))\, g_{\gamma,0} = 0(\gamma)$ for all $g \in G$. However, if we construct the wreath product (G^*, S^*) of the same permutation groups using the element $0^* \in \prod S_\gamma$ such that $0^*(\gamma) = 1'$ for each γ, then (G^*, S^*) has no fixed points, for let $x \in S^*$, so that the support of x (with respect to 0^*) satisfies the maximum condition. Then for some γ, $x(\gamma) \neq 0'$. Now define g componentwise by letting $g_{\alpha,y}$ be the identity when $\alpha \neq \gamma$, $g_{\gamma,y}$ the identity when $y \not\equiv^\gamma x$, and $g_{\gamma,x}$ the 2-cycle $(1', 2')$. Then g is an element of (G^*, S^*), and since $(xg)(\gamma) = x(\gamma)\, g_{\gamma,x} \neq x(\gamma)$, then $xg \neq x$.

However, in the case of main interest, when each (G_γ, S_γ) is transitive, the wreath product is independent of the choice of 0. In fact, something slightly more general is true.

LEMMA 3.7. *Let Γ be a partially ordered set and for each $\gamma \in \Gamma$, (G_γ, S_γ) a permutation group such that the normalizer of G_γ in $P(S_\gamma)$ is transitive on S_γ. Let (G, S) be the wreath product constructed using the element $0 \in \prod S_\gamma$ and (G^*, S^*) the wreath product constructed using the element $0^* \in \prod S_\gamma$. Then (G, S) is isomorphic to (G^*, S^*).*

Proof. For each γ there is a permutation p_γ in the normalizer of G_γ such that $(0(\gamma))\, p_\gamma = 0^*(\gamma)$. Now define a mapping $p : S \to S^*$ by $(xp)(\gamma) = (x(\gamma))\, p_\gamma$. From the fact that the support of x with respect to 0 is the same as the support of xp with respect to 0^*, it is seen that p is a one-to-one map of S onto S^*. And now let p on G be the naturally induced map $g \to p^{-1}gp$. It is easily seen that $p^{-1}gp$ respects K_γ if g does. Thus, the lemma will be proved if we can show that the components of $p^{-1}gp$ are in the right place. We claim that $(p^{-1}gp)_{\gamma,xp} = p_\gamma^{-1} g_{\gamma,x} p_\gamma \in G_\gamma$ since $g_{\gamma,x} \in G_\gamma$ and p_γ is in the normalizer of G_γ. To see the equality, if $a \in S_\gamma$, and if $z \in S^*$ such that $z \equiv^\gamma xp$ and $z(\gamma) = a$, so that $zp^{-1} \equiv^\gamma x$, then

$$a\,(p^{-1}gp)_{\gamma,xp} = (zp^{-1}gp)(\gamma) = (zp^{-1}g)(\gamma)\, p_\gamma = (zp^{-1})(\gamma)\, g_{\gamma,x}p_\gamma$$
$$= z(\gamma)\, p_\gamma^{-1} g_{\gamma,x}p_\gamma = ap_\gamma^{-1} g_{\gamma,x}p_\gamma\,.$$

If Γ is a partially ordered set, if (G_γ, S_γ) is a permutation group for each $\gamma \in \Gamma$, and if $0 \in \prod S_\gamma$ then ${}^*\prod^0_{\gamma\in\Gamma} (G_\gamma, S_\gamma)$ will denote the wreath product of the permutation groups (G_γ, S_γ) constructed using the element $0 \in \prod S_\gamma$. When it can be done without confusion, we shall omit the superscript 0.

The classical wreath product is well known to be associative in the sense that $A \wr (B \wr C)$ is isomorphic to $(A \wr B) \wr C$. The wreath product of P. Hall satisfies a generalized associative law [5]. The wreath product we have defined satisfies an associative law which generalizes both the classical one and that of Hall. Let $\{\Delta_\gamma \mid \gamma \in \Gamma\}$ be a collection of partially ordered sets, and suppose Γ is also partially ordered. The *lexicographic union* of the sets $\{\Delta_\gamma\}$ is the disjoint union of those sets, partially ordered by letting $a < b$ if either $a \in \Delta_\alpha$, $b \in \Delta_\beta$ and $\alpha < \beta$, or $a, b \in \Delta_\alpha$ and $a < b$ in the order of Δ_α.

THEOREM 3.8. (*The associative law*) *Let Γ be a partially ordered set, and for each $\gamma \in \Gamma$ let $(G_\gamma, S_\gamma) = {}^*\prod_{\delta\in\Delta_\gamma} (G_{\gamma,\delta}, S_{\gamma,\delta})$ be a wreath product. Let Δ be the lexicographic union of the sets $\{\Delta_\gamma \mid \gamma \in \Gamma\}$, and let $0 \in \prod_{\gamma\in\Gamma} S_\gamma$. Then ${}^*\prod^0_{\gamma\in\Gamma} (G_\gamma, S_\gamma)$ is isomorphic to ${}^*\prod^{0^*}_{\delta\in\Delta} (G_{\gamma,\delta}, S_{\gamma,\delta})$ for a suitable $0^* \in \prod_{\delta\in\Delta} S_{\gamma,\delta}$.*

Proof. Let $(G, S) = {}^*\prod_{\gamma\in\Gamma}(G_\gamma, S_\gamma)$. There is a natural one-to-one mapping of the set $\prod_{\gamma\in\Gamma} S_\gamma$ into $\prod_{\delta\in\Delta} S_{\gamma,\delta}$, namely for $s \in \prod S_\gamma$, let $s^* \in \prod S_{\gamma,\delta}$ be defined by $s^*(\delta) = (s(\gamma))(\delta)$, where $\delta \in \Delta_\gamma$. This merely expresses the associative law for the cartesian product of sets. Let $s \in \prod S_\gamma$ differ from $0 \in \prod S_\gamma$ on a set $D \subseteq \Gamma$ satisfying the maximum condition. Let D^* be the subset of Δ where 0^* and s^* differ. Then $D = \{\gamma \in \Gamma \mid$ there exists $\delta \in D^*$ with $\delta \in \Delta_\gamma\}$. Now let E^* be any nonvoid subset of D^* and let

$$E = \{\gamma \in \Gamma \mid \text{there exists } \delta \in E^* \text{ with } \delta \in \Delta_\gamma\}.$$

Then E is a nonvoid subset of D, and so E contains a maximal element γ_m. Let $F = E^* \cap \Delta_{\gamma_m}$. Then

$$\{\delta \in \Delta_{\gamma_m} \mid (0(\gamma_m))(\delta) \neq (s(\gamma_m))(\delta)\}$$

satisfies the maximum condition, and contains F as a nonvoid subset. Therefore, F has a maximal element, δ_m. Clearly, δ_m is a maximal element of E^*. Thus D^* satisfies the maximum condition. Thus, if the support of s with respect to 0 satisfies the maximum condition, the support of s^* with respect to 0^* does too. That the converse is true, can be seen in a similar way. Then $*$ maps S onto S^* where S is those elements of $\prod_{\gamma\in\Gamma} S_\gamma$ whose support with respect to 0 satisfies the maximum condition, and S^* is those elements of $\prod_{\delta\in\Delta} S_{\gamma,\delta}$ whose support with respect to 0^* satisfies the maximum condition.

Let $(G^*, S^*) = {}^*\prod_{\delta\in\Delta}^{0^*}(G_{\gamma,\delta}, S_{\gamma,\delta})$. For $g \in P(S)$, define

$$g^* = *^{-1}g* \in P(S^*).$$

Then for $s \in S$, $s^*g^* = (sg)^*$, and $*$ is certainly one-to-one on G. We must show that G^*, which we have defined independently above, is actually the image of G under $*$.

Let $g \in G$ and $\delta \in \Delta$. We first show that K_δ is respected by g^*. Let $s, t \in S$ with $s^* \equiv_\delta t^*$; say $\delta \in \Delta_{\gamma_\delta}$. Then for each $\beta \in \Delta$ with $\beta \geqslant \delta$, say $\beta \in \Delta_{\gamma_\beta}$, we have $s^*(\beta) = t^*(\beta)$, or $(s(\gamma_\beta))(\beta) = (t(\gamma_\beta))(\beta)$. In particular, if $\gamma_\beta > \gamma_\delta$ then $(s(\gamma_\beta))(\alpha) = (t(\gamma_\beta))(\alpha)$ for each $\alpha \in \Delta_{\gamma_\beta}$, since $\gamma_\beta = \gamma_\alpha$. Thus, $s(\gamma_\beta) = t(\gamma_\beta)$, if $\gamma_\beta > \gamma_\delta$. This means that $s \equiv^{\gamma_\delta} t$, and so $sg \equiv^{\gamma_\delta} tg$, or $(sg)(\gamma_\beta) = (tg)(\gamma_\beta)$, if $\gamma_\beta > \gamma_\delta$. In particular,

$$((sg)(\gamma_\beta))(\beta) = ((tg)(\gamma_\beta))(\beta), \quad \text{if} \quad \gamma_\beta > \gamma_\delta. \tag{1}$$

Also, $(s(\gamma_\delta))(\beta) = (t(\gamma_\delta))(\beta)$ for all $\beta \in \Delta_{\gamma_\delta}$, $\beta \geqslant \delta$. This means $s(\gamma_\delta) \equiv_\delta t(\gamma_\delta)$, and as $g_{\gamma_\delta,s} \in G_\gamma$, $s(\gamma_\delta)\, g_{\gamma_\delta,s} \equiv_\delta t(\gamma_\delta)\, g_{\gamma_\delta,s}$. Hence

$$\begin{aligned} ((sg)(\gamma_\delta))(\beta) &= (s(\gamma_\delta)\, g_{\gamma_\delta,s})(\beta) = (t(\gamma_\delta)\, g_{\gamma_\delta,s})(\beta) \\ &= ((tg)(\gamma_\delta))(\beta) \quad \text{if} \quad \beta \in \Delta_{\gamma_\delta}, \quad \beta \geqslant \delta. \end{aligned} \tag{2}$$

Combining (1) and (2), we have for all $\beta \geqslant \delta$, $\beta \in \Delta$,

$$(s^*g^*)(\beta) = (sg)^*(\beta) = ((sg)(\gamma_\beta))(\beta) = ((tg)(\gamma_\beta))(\beta) \\ = (tg)^*(\beta) = (t^*g^*)(\beta),$$

so $s^*g^* \equiv_\delta t^*g^*$.

Now we need to show that $(g^*)_{\delta,s^*} \in G_{\gamma,\delta}$. This will follow if $(g^*)_{\delta,s^*} = (g_{\gamma,s})_{\delta,s(\gamma)}$ where $\delta \in \Delta_\gamma$. To establish the last equality,

$$((s^*)(\delta))(g_{\gamma,s})_{\delta,s(\gamma)} = ((s(\gamma))(\delta))(g_{\gamma,s})_{\delta,s(\gamma)} = ((s(\gamma))\, g_{\gamma,s})(\delta) = ((sg)(\gamma))(\delta) \\ = ((sg)^*)(\delta) = ((s^*)(g^*))(\delta) = ((s^*)(\delta))(g^*)_{\delta,s^*}.$$

Since for any $z \in S_{\gamma,\delta}$, there is an $s' \in S$ with $s'^* \equiv^\delta s^*$ (and hence $s'(\gamma) \equiv^\delta s(\gamma)$, and $s \equiv^\gamma s'$) and $(s'^*)(\delta) = (s'(\gamma))(\delta) = z$, it follows that

$$z(g_{\gamma,s})_{\delta,s(\gamma)} = ((s'^*)(\delta))(g_{\gamma,s'})_{\delta,s'(\gamma)} = ((s'^*)(\delta))(g^*)_{\delta,s'^*} = z(g^*)_{\delta,s^*},$$

so that $(g_{\gamma,s})_{\delta,s(\gamma)} = (g^*)_{\delta,s^*} \in G_{\gamma,\delta}$, as claimed. This shows that * maps G into G^*.

Next we show that * maps G onto G^*. Let $f \in G^*$, and $g = {}^*f^{*-1} \in P(S)$. Then if $\gamma \in \Gamma$, we want to show that g respects K_γ. Suppose $s \equiv_\gamma t$, $s, t \in S$. Then $s(\beta) = t(\beta)$ for all $\beta \in \Gamma$, $\beta \geqslant \gamma$. Hence for all $\delta \in \Delta_\beta$, $\beta \geqslant \gamma$,

$$s^*(\delta) = (s(\beta))(\delta) = (t(\beta))(\delta) = t^*(\delta),$$

$$s^* \equiv_\delta t^*, \qquad s^*f \equiv_\delta t^*f, \qquad \text{and} \qquad (s^*f)(\delta) = (t^*f)(\delta).$$

Hence, for all $\beta \in \Gamma$, $\beta \geqslant \gamma$, and all $\delta \in \Delta_\beta$,

$$((sg)(\beta))(\delta) = ((s^*f^{*-1})(\beta))(\delta) = (s^*f)(\delta) = (t^*f)(\delta) \\ = ((t^*f^{*-1})(\beta))(\delta) = ((tg)(\beta))(\delta),$$

so

$$(sg)(\beta) = (tg)(\beta),$$

and therefore

$$sg \equiv_\gamma tg.$$

Finally, since $(g_{\gamma,s})_{\delta,s(\gamma)} = f_{\delta,s^*} \in G_{\gamma,\delta}$ for all γ, s, and $\delta \in \Delta_\gamma$, we have that $g_{\gamma,s} \in G_\gamma$, and so $g \in G$, completing the proof.

THEOREM 3.9. *If each (G_γ, S_γ) is transitive, then so is ${}^*\prod (G_\gamma, S_\gamma) = (G, S)$ transitive.*

Proof. Let $s, t \in S$. For each γ, there exists $g_\gamma \in G_\gamma$ such that $s(\gamma)\, g_\gamma = t(\gamma)$, and we may assume that if $s(\gamma) = t(\gamma)$ then g_γ is the identity. Now define a function g on S by $(xg)(\gamma) = x(\gamma) g_\gamma$. Clearly $sg = t$. Also, the set

$$R = \{\gamma \mid s(\gamma) \neq t(\gamma)\} = \{\gamma \mid g_\gamma \text{ is not the identity}\}$$

satisfies the maximum condition. Then

$$\{\gamma \mid (xg)(\gamma) \neq x(\gamma)\} \subseteq R,$$

and so

$$\text{support } (xg) \subseteq R \cup \text{support } (x),$$

which implies that the support of xg satisfies the maximum condition. Therefore, $xg \in S$, so $g : S \to S$. By symmetry, g^{-1}, which is defined by $(xg^{-1})(\gamma) = x(\gamma) g_\gamma^{-1}$, also maps S into S, and thus $g \in P(S)$. It is immediate that g respects each K_γ, and that $g_{\gamma,x} = g_\gamma \in G_\gamma$. Thus $g \in G$, and (G, S) is transitive.

Of course, higher degrees of transitivity do not carry over in the same way, since if Γ has more than one point, (G, S) has the proper congruence K_γ.

We make one final observation.

LEMMA 3.10. *The set* $\Gamma = \{(K_\gamma, K^\gamma)\}$ *is plenary.*

Proof. If $s, t \in S$, $s \neq t$, then there is a maximal element α in the set $\{\gamma \in \Gamma \mid s(\gamma) \neq t(\gamma)\}$. Then $s \equiv^\alpha t$ and $s \not\equiv_\alpha t$. Also, if $s \not\equiv^\beta t$, then there is a maximal element δ in the set $\{\gamma \in \Gamma \mid \gamma > \beta, s(\gamma) \neq t(\gamma)\}$. Then $s \equiv^\delta t$ but $s \not\equiv_\delta t$.

4. An Embedding Theorem

Let (G, S) be a transitive permutation group, $\mathscr{C}$ a set of G-congruences on S, and $\Gamma = \{(C_\gamma, C^\gamma)\}$ a set of covering pairs from $\mathscr{C}$. Our aim is to show that if there are enough elements of Γ, then there is a nice embedding of (G, S) in the wreath product $^*\prod_{\gamma\in\Gamma} (G_\gamma, S_\gamma)$ of the components of (G, S, Γ).

Let (H, T) be a permutation group and Σ a set of pairs of H-congruences on T. Let $\phi : (G, S) \to (H, T)$ be an embedding, and let $\Gamma\phi$ denote the corresponding set of covering pairs of $G\phi$-congruences on $S\phi$. We shall say that ϕ is an *immediate embedding* of (G, S, Γ) in (H, T, Σ) if (H, T, Σ) is an immediate extension of $(G\phi, S\phi, \Gamma\phi)$.

We wish to warn the reader at this point that we are going to change our convention and denote the wreath product by (W, R) instead of (G, S) as in the last section; and we shall avoid any notational distinction between $\Gamma = \{(C_\gamma, C^\gamma)\}$ as a set of covering pairs of G-congruences on S, and $\Gamma = \{(K_\gamma, K^\gamma)\}$ as the natural set of covering pairs of W-congruences on R.

THEOREM 4.1. *Let (G, S) be a transitive permutation group, $\mathscr{C}$ a set of G-congruences on S, and Γ a plenary set of covering pairs in $\mathscr{C}$. Then there is an immediate embedding of (G, S, Γ) in (W, R, Γ), where $(W, R) = {}^*\prod_{\gamma\in\Gamma}(G_\gamma, S_\gamma)$ is the wreath product of the components of (G, S, Γ).*

Proof. In the proof of the corresponding theorem in [2] the crucial role is played by a lemma of Banaschewski. Here, the corresponding lemma is much simpler.

LEMMA 4.2. *Let G be any group. Then there exists a set $\{T(A) \mid A$ is a subgroup of $G\}$ such that $T(A)$ consists of exactly one element from each right coset of A in G ($T(A)$ is a "transversal of A"), $T(A) \cap A$ is the identity, and if A and B are subgroups of G with $A \subseteq B$, then $T(A) \supseteq T(B)$.*

Proof. Well-order the elements of G with the identity element smallest. Then let $T(A)$ consist of the smallest element in each right coset of A in G.

Returning to the proof of the theorem, we first choose an arbitrary $0 \in S$. Remembering that $G^{\gamma,0} = \{g \in G \mid 0g \equiv 0(\text{mod } C^\gamma)\}$, we choose a set of transversals as in Lemma 4.2, and in accordance with Lemma 2.2 and the remark after it, we take $(G_\gamma, S_\gamma) = (G_{\gamma,0}, S_{\gamma,0})$. We will use the transversals to "coordinatize" each of the sets $(sC^\gamma)/C_\gamma$ by the set $(0C^\gamma)/C_\gamma$. Now we define a function $\phi : S \to \prod S_\gamma$ in the following way. Let $s \in S$. By transitivity, there exists $h \in G$ such that $0h = s$. There is exactly one element $g \in T(G^{\gamma,0}) \cap G^{\gamma,0}h$. Let

$$(s\phi)(\gamma) = sg^{-1}C_\gamma \in S_\gamma .$$

This definition is independent of the choice of h, for if also $0h' = s$, then $G^{\gamma,0}h = G^{\gamma,0}h'$.

Because of the transitivity of G, it is easily seen that that the components (G_γ, S_γ) are also transitive. Thus, by Lemma 3.7, we have a certain amount of freedom in the construction of the wreath product. We are going to use as a reference point, the point $0\phi \in \prod S_\gamma$. Thus, R will consist of those elements of $\prod S_\gamma$ whose support with respect to 0ϕ satisfies the maximum condition. We note that $(0\phi)(\gamma) = 0C_\gamma$.

To see that ϕ maps S into R, suppose s and h are as before, and P is the support of $s\phi$. Suppose that $\gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \cdots$ is an ascending sequence in P. Let $A = \bigcup G^{\gamma_i,0}$. Then since $G^{\gamma_1,0} \subseteq G^{\gamma_2,0} \subseteq G^{\gamma_3,0} \subseteq \cdots$, A is a subgroup of G. Hence, there is exactly one $a \in T(A) \cap Ah$. Now, $ha^{-1} \in A$, so for some n, $ha^{-1} \in G^{\gamma_n,0}$. If, for some i, $\gamma_i > \gamma_n$, then $C^{\gamma_n} \subseteq C_{\gamma_i} \subseteq C^{\gamma_i}$, and $G^{\gamma_n,0} \subseteq G^{\gamma_i,0} \subseteq A$. Hence $T(G^{\gamma_n,0}) \supseteq T(G^{\gamma_i,0}) \supseteq T(A)$. There is just one element $c \in T(G^{\gamma_i,0}) \cap G^{\gamma_i,0}h$. Now, $ha^{-1} \in G^{\gamma_n,0} \subseteq G^{\gamma_i,0}$, so $a \in G^{\gamma_i,0}h$. Also, $a \in T(A) \subseteq T(G^{\gamma_i,0})$. Therefore, $a = c$. Thus, $hc^{-1} = ha^{-1} \in G^{\gamma_n,0}$. Hence $sc^{-1} \equiv 0(\text{mod } C^{\gamma_n})$, which implies $sc^{-1} \equiv 0(\text{mod } C_{\gamma_i})$. We conclude that

$(s\phi)(\gamma_i) = sc^{-1}C_{\gamma_i} = 0C_{\gamma_i} = 0\phi(\gamma_i)$ so that γ_i is not in the support of $s\phi$, a contradiction. Hence, $\gamma_i \leqslant \gamma_n$ for all i. Thus, P satisfies the maximum condition, which proves

$$\phi : S \to R. \tag{1}$$

If $s \neq t$, $s, t \in S$, there is a $\gamma \in \Gamma$ such that $s \equiv t(\text{mod } C^\gamma)$ but $s \not\equiv t(\text{mod } C_\gamma)$, because Γ is plenary. If $0h_1 = s$ and $0h_2 = t$, with $h_1, h_2 \in G$, then

$$0h_1h_2^{-1} \equiv sh_2^{-1} \equiv th_2^{-1} \equiv 0(\text{mod } C^\gamma),$$

so $h_1h_2^{-1} \in G^{\gamma,0}$. If g is the unique element of

$$T(G^{\gamma,0}) \cap G^{\gamma,0}h_1 = T(G^{\gamma,0}) \cap G^{\gamma,0}h_2,$$

$(s\phi)(\gamma) = sg^{-1}C_\gamma \neq tg^{-1}C_\gamma = (t\phi)(\gamma)$, since $sC_\gamma \neq tC_\gamma$. Hence, $s\phi \neq t\phi$, and

$$\phi \text{ is one-to-one on } S. \tag{2}$$

Next, let $\{(K_\gamma, K^\gamma)\}$ be the natural covering pairs for the wreath product. We aim to show that

$$s \equiv t(\text{mod } C_\gamma) \quad \text{if and only if} \quad s\phi \equiv t\phi(\text{mod } K_\gamma). \tag{3}$$

First, suppose $s \equiv t(\text{mod } C_\gamma)$. Then for every $\beta \geqslant \gamma$, $s \equiv t(\text{mod } C_\beta$ and $C^\beta)$. If $0h_1 = s$ and $0h_2 = t$, then $G^{\beta,0}h_1 = G^{\beta,0}h_2$ and if $g \in T(G^{\beta,0}) \cap G^{\beta,0}h_1$,

$$(s\phi)(\beta) = sg^{-1}C_\beta = tg^{-1}C_\beta = (t\phi)(\beta).$$

Therefore, $s\phi \equiv t\phi(\text{mod } K_\gamma)$. Conversely, if $s \not\equiv t(\text{mod } C_\gamma)$, then because Γ is plenary, there is a $\beta \geqslant \gamma$ such that $s \equiv t(\text{mod } C^\beta)$ but $s \not\equiv t(\text{mod } C_\beta)$, and as before,

$$(s\phi)(\beta) = sg^{-1}C_\beta \neq tg^{-1}C_\beta = (t\phi)(\beta),$$

so $s\phi \not\equiv t\phi(\text{mod } K_\gamma)$.

Let $r \in R$ such that $r \equiv^\gamma 0\phi$ for some $\gamma \in \Gamma$. Then $r(\gamma) \in S_\gamma = (0C^\gamma)/C_\gamma$, so for some $s \in 0C^\gamma$, $r(\gamma) = sC_\gamma$. If $0h = s$, $h \in G$, then $h \in G^{\gamma,0}$, and if $g \in T(G^{\gamma,0}) \cap G^{\gamma,0}$, g is the identity, so $(s\phi)(\gamma) = sC_\gamma = r(\gamma)$. Hence the natural correspondence $sC_\gamma \to (s\phi)K_\gamma$ between $(0C^\gamma)/C_\gamma$ and $((0\phi)K^\gamma)/K_\gamma$ is one-to-one and onto. By transitivity, the same is true of the natural correspondence between any $(tC^\gamma)/C_\gamma$ and $((t\phi)K^\gamma)/K_\gamma$.

Now let $g \in G$, $\gamma \in \Gamma$, and $x \in R$. Then g induces a permutation of S_γ, which we shall formally denote by $(g\phi)_{\gamma,x}$, as follows. First, if $xK^\gamma \cap S\phi$ is empty, let $(g\phi)_{\gamma,x}$ be the identity of $P(S_\gamma)$. However, if there exists $t \in S$ with $t\phi \in xK^\gamma$, then for each $a \in S_\gamma$ there exists $x' \in xK^\gamma$ such that $x'(\gamma) = a$, and by the preceding paragraph there exists $s \in tC^\gamma$ such that $(s\phi) K_\gamma = x'K_\gamma$. In other words, $s\phi \equiv^\gamma x$ and $(s\phi)(\gamma) = x'(\gamma) = a$. Now define

$$a(g\phi)_{\gamma,x} = ((sg)\phi)(\gamma).$$

This is well defined, for if also $x'' \in xK^\gamma$ and $x''(\gamma) = a$ then $x' \equiv_\gamma x''$; and if $(s_1\phi) K_\gamma = x'K_\gamma = (s_2\phi) K_\gamma$, then $s_1\phi \equiv_\gamma s_2\phi$ and so by (3), $s_1 \equiv s_2(\text{mod } C_\gamma)$, which implies $s_1 g \equiv s_2 g(\text{mod } C_\gamma)$ and by (3) again, $(s_1 g)\phi \equiv_\gamma (s_2 g)\phi$. There is another way to view this definition. There is a unique $f_1 \in T(G^{\gamma,0})$ such that $0f_1 \equiv t(\text{mod } C^\gamma)$, and there is a unique $f_2 \in T(G^{\gamma,0})$ such that $0f_2 \equiv tg(\text{mod } C^\gamma)$. Then for $a \in S_\gamma$, $a(g\phi)_{\gamma,x} = af_1 gf_2^{-1}$. Since $f_1 gf_2^{-1} \in G^{\gamma,0}$, it is apparent that

$$(g\phi)_{\gamma,x} \in G_\gamma = G^{\gamma,0}/N_{C_\gamma}. \tag{4}$$

Now we define $g\phi$ as a function on R by

$$(r(g\phi))(\gamma) = (r(\gamma))(g\phi)_{\gamma,r}.$$

The first thing to observe is that the notation is consistant—the permutation $(g\phi)_{\gamma,x}$ which has just been defined is the same as the (γ, x)-component of $g\phi$, since they agree on $x(\gamma)$. Next, we see that

$$((s\phi)(g\phi))(\gamma) = ((s\phi)(\gamma))(g\phi)_{\gamma,s\phi} = ((sg)\phi)(\gamma)$$

by definition, so that

$$(s\phi)(g\phi) = (sg)\phi. \tag{5}$$

To see that $g\phi \in W$, we must first show that for all $r \in R$, $r(g\phi) \in R$, that is, that the support of $r(g\phi)$ satisfies the maximum condition. Let Δ be a non-empty subset of the support of $r(g\phi)$. If for every $\delta \in \Delta$, there is no $s \in S$ with $s\phi \equiv^\delta r$, then for each $\delta \in \Delta$,

$$0(\delta) \neq (r(g\phi))(\delta) = (r(\delta))(g\phi)_{\delta,r} = r(\delta),$$

so Δ is a subset of the support of r, and consequently has a maximal element. In the other case, there exists $\alpha \in \Delta$ and $s \in S$ such that $s\phi \equiv_\alpha r$, and for all $\beta \geqslant \alpha$, $\beta \in \Delta$,

$$0(\beta) \neq (r(g\phi))(\beta) = (r(\beta))(g\phi)_{\beta,r} = ((s\phi)(\beta))(g\phi)_{\beta,s\phi} = ((sg)\phi)(\beta).$$

Thus $\{\beta \in \Delta \mid \beta \geqslant \alpha\}$ is a non-empty subset of the support of $(sg)\phi$ which must, therefore, posess a maximal element β_1, and β_1 is clearly a maximal element of Δ. This shows that $g\phi : R \to R$.

Next, we show that

$$\text{if} \quad (s\phi)(g\phi) = s\phi \quad \text{for all} \quad s \in S, \quad \text{then} \quad x(g\phi) = x \quad \text{for all} \quad x \in R. \tag{6}$$

If there is no $s \in S$ with $s\phi \equiv_\gamma x$, then $(x(g\phi))(\gamma) = (x(\gamma)(g\phi)_{\gamma,x} = x(\gamma)$. On the other hand, if there is such an s, then

$$(x(\gamma))(g\phi)_{\gamma,x} = ((sg)\phi)(\gamma) = ((s\phi)(g\phi))(\gamma) = (s\phi)(\gamma) = x(\gamma).$$

Thus, in either case $(x(g\phi))(\gamma) = x(\gamma)$, so $x(g\phi) = x$.

It is immediate from the definition that if $x \equiv_\gamma y$, then $x(g\phi) \equiv_\gamma y(g\phi)$. On the other hand, if $x \not\equiv_\gamma y$, then there exists $\alpha \geqslant \gamma$ with $x \equiv^\alpha y$ and $x \not\equiv_\alpha y$. Then

$$(x(g\phi))(\alpha) = (x(\alpha))(g\phi)_{\alpha,x} \neq (y(\alpha))(g\phi)_{\alpha,x} = (y(\alpha))(g\phi)_{\alpha,y} = (y(g\phi))(\alpha).$$

Hence $x(g\phi) \not\equiv_\alpha y(g\phi)$. Next we show that for $g, h \in G$, $(g\phi)(h\phi) = (gh)\phi$. Let $x \in R$ and $\gamma \in \Gamma$. First, suppose there exists $s \in S$ with $s \equiv_\gamma x$. Then for $\beta \geqslant \gamma$, $(x(g\phi))(\beta) = (x(\beta))(g\phi)_{\beta,x} = ((sg)\phi)(\beta)$, so $x(g\phi) \equiv_\gamma (sg)\phi$. Then a straight-forward computation shows that $(x((g\phi)(h\phi)))(\gamma) = ((sgh)\phi)(\gamma) = (x((gh)\phi))(\gamma)$. On the other hand, if there is no $s \in S$ with $s\phi \equiv_\gamma x$, then neither is there any $t \in S$ with $t\phi \equiv_\gamma x(g\phi)$, since $(tg^{-1})\phi \not\equiv_\gamma x$ implies $t\phi = ((tg^{-1})g)\phi = ((tg^{-1})\phi)(g\phi) \not\equiv_\gamma x(g\phi)$. Hence

$$\begin{aligned}(x((g\phi)(h\phi)))(\gamma) &= ((x(g\phi))(\gamma))(h\phi)_{\gamma,x(g\phi)} = ((x(\gamma))(g\phi)_{\gamma,x}) = x(\gamma)\\ &= x(\gamma)((gh)\phi)_{\gamma,x} = (x((gh)\phi))(\gamma).\end{aligned}$$

Thus in either case $(gh)\phi = (g\phi)(h\phi)$. Hence $(g\phi)(g^{-1}\phi) = (g^{-1}\phi)(g\phi) =$ the identity function. Thus, we finally see that $g\phi : R \to R$ is one-to-one and onto, and so is a permutation of R. Hence $g\phi \in W$. Moreover, (5) and (6) show that $\phi : (G, S) \to (W, R)$ is a homomorphism, and by (2), ϕ is one-to-one on S. Hence ϕ is an embedding of (G, S) in (W, R).

Now we must show that (W, R, Γ) is an extension of $(G\phi, S\phi, \Gamma)$. By (3), $K_\gamma \mid S\phi = C_\gamma\phi$ ($s\phi \equiv t\phi (\text{mod } C_\gamma\phi)$ if $s \equiv t (\text{mod } C_\gamma)$). Hence, conditions (i) and (ii) in the definition of extension are satisfied. As for condition (iii), suppose that $(s\phi)(g\phi) \equiv_\gamma s\phi$ for all $s\phi \in (x\phi) K^\gamma$, and let $z \in (x\phi) K^\gamma$. Then $z \equiv_\gamma s\phi$ for some s, and

$$z(g\phi) \equiv_\gamma (s\phi)(g\phi) \equiv_\gamma s\phi \equiv_\gamma z.$$

Hence, in this case too, $z(g\phi) \equiv_\gamma z$.

Now we check that the extension is immediate. By Lemma 3.10, Γ is plenary for (W, R). Also, since (G, S) is transitive, each component (G_γ, S_γ) is transitive, and hence by Theorem 3.9 (W, R) is transitive, which makes condition (iv) in the definition of immediate extension trivial. There remains condition (iii). We have already seen that the natural set correspondence $sC_\gamma \to (s\phi) K_\gamma$ from $(0C^\gamma)/C_\gamma$ to $((0\phi) K^\gamma)/K_\gamma$ is onto. In order to show that the group part of the natural correspondence is also onto, we must show that for each $w \in W^{\gamma,0\phi}$ there exists $g \in G^{\gamma,0}$ such that for each $s \in 0C^{\gamma}$, $(s\phi)(g\phi) \equiv_{\gamma} (s\phi)w$. Now $w_{\gamma,0\phi} \in G_{\gamma} = G^{\gamma,0}/N_{C_\gamma}$, so there exists $g \in G^{\gamma,0}$ such that $w_{\gamma,0\phi} = gN_{C_\gamma}$. Then if $s \in 0C^{\gamma}$,

$$(s\phi)w \equiv^{\gamma} s\phi \equiv^{\gamma} (sg)\phi = (s\phi)(g\phi),$$

and

$$((s\phi)\, w)(\gamma) = ((s\phi)(\gamma))\, w_{\gamma,0\phi} = (sC_\gamma)\, gN_{C_\gamma} = (sg)\, C_\gamma$$
$$= ((sg)\, \phi)(\gamma) = ((s\phi)(g\phi))(\gamma),$$

so $(s\phi)w \equiv_{\gamma} (s\phi)(g\phi)$.

Thus, (W, R, Γ) is an immediate extension $(G\phi, S\phi, \Gamma)$ and the proof is complete.

If (G, S) is transitive, if $\mathscr{C}$ is the collection of *all* G-congruences on S, and Γ the set of *all* covering pairs in $\mathscr{C}$, then clearly Γ is plenary. And in this case, the components (G_γ, S_γ) are primitive, that is, (G_γ, S_γ) has no proper congruences. For if K is a G_γ-congruence on $S_\gamma = S_{\gamma,x} = (xC^{\gamma})/C_\gamma$, then there is a natural $G^{\gamma,x}$-congruence K' on xC^{γ} defined by $z \equiv y (\mathrm{mod}\ K')$ if $zC_\gamma \equiv yC_\gamma (\mathrm{mod}\ K)$. And K' has a natural extension to a G-congruence K'' on S defined by $a \equiv b (\mathrm{mod}\ K'')$ if there exists $g \in G$ such that $ag \equiv bg (\mathrm{mod}\ K')$. Then K'' lies between C_γ and C^{γ}, and since (C_γ, C^{γ}) is a covering pair in the set of all G-congruences, either $K'' = C_\gamma$ or $K'' = C^{\gamma}$. This implies that K is trivial on S_γ. Thus we have the following.

THEOREM 4.3A. *If (G, S) is a transitive permutation group, (G, S) can be embedded in a wreath product of primitive permutation groups (G_γ, S_γ).*

We can refine this theorem a bit more. If $\mathscr{C}$ is a maximal chain of G-congruences on S, and Γ the set of all covering pairs in $\mathscr{C}$, then Γ is plenary and as before, each component (G_γ, S_γ) is primitive.

THEOREM 4.3B. *If (G, S) is a transitive permutation group, (G, S) can be embedded in a wreath product $*\prod_{\gamma\in\Gamma} (G_\gamma, S_\gamma)$ such that each (G_γ, S_γ) is primitive and Γ is totally ordered.*

Now we need a slight refinement of Theorem 4.1.

THEOREM 4.4. *Let (G, S) be a transitive permutation group, $\mathscr{C}$ a set of G-congruences on S, Γ a plenary set of covering pairs in $\mathscr{C}$, and $\phi : (G, S, \Gamma) \to (W, R, \Gamma)$ an immediate embedding, where (W, R) is the wreath product of the components of (G, S, Γ). Let $\psi : (G, S, \Gamma) \to (H, T, \Gamma)$*

be an immediate embedding. Then there exists an immediate embedding $\phi^* : (H, T, \Gamma) \to (W, R, \Gamma)$ *such that* $\psi\phi^* = \phi$ *on* S, *and for each* $g \in G$, $g\psi\phi^* \mid S\phi = g\phi \mid S\phi$.

We only outline the proof. If $t \in T$, $\gamma \in \Gamma$, and if there is an $s \in S$ such that $t \equiv_\gamma s\psi$, then let $(t\phi^*)(\gamma) = (s\phi)(\gamma)$. If there is no such s, we define $(t\phi^*)(\gamma)$ as in the proof of Theorem 4.1. It is easy to check that $\phi^* : T \to R$ and $\psi\phi^* = \phi$ on S. The rest of the proof follows the lines of the proof of Theorem 4.1.

We shall say a permutation group (G, S, Γ) is *immediately closed* if every immediate embedding $\psi : (G, S, \Gamma) \to (H, T, \Gamma)$ is an isomorphism.

COROLLARY. *If, in addition to the hypotheses of Theorem* 4.4, (H, T, Γ) *is immediately closed, then* (H, T, Γ) *is isomorphic to* (W, R, Γ).

Proof. The isomorphism is ϕ^*.

THEOREM 4.5. *If* (W, R, Γ) *is a transitive wreath product then* (W, R, Γ) *is immediately closed.*

Proof. The identity map $i : (W, R, \Gamma) \to (W, R, \Gamma)$ is an immediate embedding. If $\psi : (W, R, \Gamma) \to (H, T, \Gamma)$ is an immediate embedding, then by Theorem 4.4, there exists an immediate embedding

$$\phi^* : (H, T, \Gamma) \to (W, R, \Gamma)$$

such that $\psi\phi^* = i$ on R (and so $R\psi = T$ since $\phi^* : T \to R$ is one-to-one), and for each $w \in W$, $w\psi\phi^* \mid Ri = w \mid Ri$, or $w\psi\phi^* = w$ (and so $W\psi = H$ since $\phi^* : H \to W$ is one-to-one). Thus ψ is an isomorphism.

Combining the previous Corollary and Theorem, we have the following characterization of wreath products.

THEOREM 4.6. *If* (G, S, Γ) *and* (W, R, Γ) *are as in Theorem* 4.4, *the wreath product* (W, R, Γ) *is the unique (to within isomorphism) immediately closed immediate extension of* (G, S, Γ). (G, S, Γ) *is immediately closed if and only if* (G, S, Γ) *is isomorphic to* (W, R, Γ).

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