Noninner Automorphisms of Order $p$ of Finite $p$-Groups

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The main result of this paper shows that if $G$ is a finite nonabelian $p$-group and if $C_G(Z(\Phi(G))) \neq \Phi(G)$, then $G$ has a noninner automorphism of order $p$ which fixes $\Phi(G)$. This reduces the verification of the longstanding conjecture that every finite nonabelian $p$-group $G$ has a noninner automorphism of order $p$ to the degenerate case in which $C_G(Z(\Phi(G))) = \Phi(G)$. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Let $G$ denote a finite nonabelian $p$-group. A longstanding conjecture asserts that

(C) $G$ possesses at least one noninner automorphism of order $p$.

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The statement of (C) is a sharpened version of a celebrated theorem of W. Gaschütz [1] asserting that finite nonabelian \( p \)-groups have noninner automorphisms of \( p \)-power order. As far as we know, the conjecture (C) is still open even for \( p \)-groups of class 2.

Regular \( p \)-groups satisfy (C): this follows from a cohomological result of Peter Schmid [6]. Hans Liebeck [4] has shown that odd order \( p \)-groups of class 2 must have a noninner automorphism of order \( p \) fixing the Frattini subgroup. For 2-groups the situation is different: Liebeck produced an example of a 2-group \( G \) of class 2 and order \( 2^7 \) with the property that all automorphisms of order 2 fixing its Frattini subgroup are inner.

The main result of this paper is the following

**Theorem.** Let \( G \) be a finite nonabelian \( p \)-group such that \( \Phi(G) \neq C_G(Z(\Phi(G))) \). Then \( G \) has a noninner automorphism of order \( p \) which fixes \( \Phi(G) \).

Since \( \Phi(G) \leq C_G(Z(\Phi(G))) \), the theorem reduces the task of verifying (C) to considering only \( p \)-groups \( G \) satisfying \( \Phi(G) = C_G(Z(\Phi(G))) \), which can be viewed as a sort of degenerate case. In particular, a strengthened version of the conjecture (C) is true for all \( p \)-groups \( G \) such that \( C_G(\Phi(G)) \neq Z(\Phi(G)) \), as, for example, \( p \)-groups \( G \) having an abelian maximal subgroup or \( p \)-groups \( G \) with \( Z(G) \) not contained in \( \Phi(G) \).

P. Schmid, in his elegant paper [5], called \( p \)-groups \( G \) satisfying \( C_G(Z(\Phi(G))) = \Phi(G) \) strongly Frattinian and he proved, by using cohomological methods, that strongly Frattinian \( p \)-groups have noninner automorphisms of \( p \)-power order fixing both \( Z(\Phi(G)) \) and \( G/\Phi(G) \). Unfortunately, the methods used in [5] do not shed any light on the truth of (C) for these groups.

It is perhaps worth mentioning here that Liebeck’s example, namely \( G = \langle a, b | a^4 = 1, b^8 = [a, b], [a, b, a] = 1 \rangle \), is a strongly Frattinian group and as such it is one of the groups to which our theorem does not apply. This group has a noninner automorphism of order 2 which fixes its center. All groups considered in this paper are finite. The unexplained notation is standard and follows that of Gorenstein [2].

### 2. PROOF OF THE THEOREM

We proceed by contradiction and assume that \( G \) is a counterexample to the theorem. The set of all maximal subgroups of \( G \) will be denoted by \( \mathcal{M}(G) \). The first remark is probably well known; it is a particular case of the theorem.

**Remark 1.** \( Z(G) \leq \Phi(G) \).
Proof. Suppose the contrary, select \( M \in \mathcal{H}(G) \) such that \( Z(G) \not\subseteq M \) and pick an element \( g \in Z(G) \setminus M \), so that \( G = M(g) \). If one fixes a nontrivial element \( u \in \Omega_1(Z(M)) \leq \Omega_1(Z(G)) \), then the map defined by \( m \mapsto m \) for all \( m \in M \) and \( g \rightarrow ug \) extends to an automorphism \( \alpha \) of \( G \) of order \( p \) which fixes \( M \). This automorphism is noninner, because \( g \) is central and \( \alpha(g) = ug \neq g \). This contradicts the fact that \( G \) is a counterexample to the theorem.

The second remark provides the groundwork to apply a result of P. Schmid.

Remark 2. \( Z(G) < Z(M) \) for every \( M \in \mathcal{H}(G) \).

Proof. By Remark 1, \( Z(G) \leq \Phi(G) \). Let \( M \in \mathcal{H}(G) \), so that \( C_G(M) = Z(M) \). Let \( x \in G \setminus M \). If \( u \in \Omega_1(Z(G)) \), then one can construct an automorphism \( \alpha \) of \( G \) by insisting that \( \alpha(m) = m \) for \( m \in M \) and \( \alpha(x) = ux \). It is easy to verify that \( \alpha \) has order \( p \). Since \( \alpha \) fixes \( M \), by the original assumption on \( G \) one must have \( \alpha \in \Inn(G) \). Thus \( \alpha \) must be an inner automorphism of \( G \) induced by conjugation by an element in \( Z(M) \setminus Z(G) \). This completes the proof.

At this point we pause for a moment to analyze a useful class of nonabelian \( p \)-groups. Recall that a minimal nonabelian \( p \)-group (or Rédei \( p \)-group) is a nonabelian \( p \)-group all whose maximal subgroups are abelian. Of course, Rédei \( p \)-groups have class 2; these groups are described in [3, Aufgabe 22, p. 309]. The next remark shows that Rédei \( p \)-groups satisfy a sharpened version of (C):

Remark 3. If \( X \) is a Rédei \( p \)-group, then \( X \) has a noninner automorphism of order \( p \) fixing \( \Phi(X) = Z(X) \).

Proof. Since \( X \) has class 2, Liebeck’s result mentioned above disposes of the case where \( p \) is odd; we only have to prove the statement if \( p = 2 \).

If \( X \) is a Rédei 2-group, then \( X \) is isomorphic to one of the following groups:

\[
\begin{align*}
(i) & \quad X_{m,n} = \langle a, b | a^{2^m} = b^{2^n} = 1, [a, b] = a^{2^{m-1}} \rangle, \quad m \geq 2, \ n \geq 1, \\
(ii) & \quad X_{m,n,1} = \langle a, b | a^{2^m} = b^{2^n} = [a, b] = [a, b, a] = [a, b, b] = 1 \rangle, \quad m \geq n \geq 1, \ \text{or} \\
(iii) & \quad Q_8 = \langle a, b | a^4 = 1, a^2 = b^2 = [a, b] \rangle.
\end{align*}
\]

The groups given in the above list are mutually nonisomorphic, with one exception: \( X_{2,1} \cong X_{1,1,1} \) is the dihedral group of order 8. To avoid subsequent problems, it is convenient to divide these groups into six smaller classes. In each case we will define an automorphism of order 2 which fixes the center. If \( X \) is a Rédei 2-group, then \( \{a^2, b^2, [a, b]\} \) is a generating set.
for \(Z(X) = \Phi(X)\)—albeit not always a minimal one and \(X' = \langle [a, b]\rangle\) has order 2. Although by a remark of H. Liebeck [4] it suffices to check only the groups with cyclic center, we will cover all cases, since the amount of extra work is minimal.

We shall perform the verifications in detail only in the first case; in the other cases we shall just indicate the images of the automorphism on the generators.

Cover all cases, since the amount of extra work is minimal.

Case 1: \(X \cong X_{m,n}, m \geq 2, n \geq 2\).

Define \(\alpha \in \text{Aut}(X)\) by \(\alpha(a) = a^{2^{m-1}+1}, \alpha(b) = b^{2^{n-1}+1}\). Clearly \(X = \langle \alpha(a), \alpha(b) \rangle\) and the images verify the relations defining \(X\). Moreover, \(\alpha\) fixes the generators \(a^2, b^2\) of \(Z(X)\) and \(\alpha^2\) fixes both \(a\) and \(b\). Thus \(\alpha\) has order 2 and \(\alpha\) is not inner because \(b^{-1}a(b) = b^{2^{n-1}}\) is not a commutator.

Case 2: \(X \cong X_{m,1}, m \geq 3\).

Here \(\alpha(a) = a^{2^{m-2}+1}b, \alpha(b) = a^{2^{m-1}}b\).

Case 3: \(X \cong X_{m,n,1}, m \geq n \geq 2\).

Here \(\alpha(a) = a^{2^{m-1}+1}, \alpha(b) = b^{2^{n-1}+1}\).

Case 4: \(X \cong X_{m,1,1}, m \geq 2\).

Here \(\alpha(a) = a^{2^{m-1}+1}, \alpha(b) = b\).

Case 5: \(X \cong D_8 = \langle a, b | a^4 = b^2 = 1, [a, b] = a^2 \rangle\).

Here \(\alpha(a) = a^3, \alpha(b) = ab\).

Case 6: \(X \cong Q_8\).

Here \(\alpha(a) = a^3, \alpha(b) = ab\).

This completes the verifications in the case \(p = 2\).

We need one more auxiliary remark, whose proof is immediate:

Remark 4. Let \(X\) be a central product of subgroups \(A, B\); i.e., \(X = AB\) and \([A, B] = 1\). Suppose that \(\alpha \in \text{Aut}(A)\) and \(\beta \in \text{Aut}(B)\) agree on \(A \cap B\). Then \(\alpha\) and \(\beta\) admit a common extension \(\gamma \in \text{Aut}(X)\). In particular, if \(A\) has a noninner automorphism of order \(p\) which fixes \(Z(A)\), then \(X\) has a noninner automorphism of order \(p\) which fixes \(Z(A)\) and \(B\).

We return now to our counterexample \(G\). Recall that our aim is to show that \(C_G(Z(\Phi(G))) = \Phi(G)\): once this is proved, one obtains a contradiction with the hypothesis.

By Remark 1, \(Z(G) \leq \Phi(G)\) and by Remark 2, \(Z(G) < Z(M)\) for every \(M \in \mathcal{M}(G)\). P. Schmid [5] studied finite \(p\)-groups \(G\) satisfying the condition that \(Z(G) \neq Z(M)\) for every \(M \in \mathcal{M}(G)\) and called these groups\ Frattinian\ \(p\)-groups. The counterexample \(G\) is thus Frattinian. By elementary arguments, Schmid proved that if \(G\) is Frattinian, then either

(a) \(G = E_1E_2 \cdots E_s,\) where \([E_i, E_j] = 1\) for all \(i \neq j, |E_i| = p^2|Z(G)|,\) and \(Z(G) = Z(E_i)\) for all \(1 \leq i \leq s,\) or
(b) \( G = EF \), where \( E = C_G(F) \), \( C_F(Z(\Phi(F))) = \Phi(F) \), \( \Phi(E) \leq Z(E) \), and both \( E \) and \( F \) are Frattinian \( p \)-groups.

Moreover, in case (b), either \( E = Z(G) \) (and therefore \( G = F \)), or \( E \) is a central product as in case (a).

Suppose first that our counterexample \( G \) is a central product as in (a). Since the factors \( E_i \) are Rédei \( p \)-groups, we have that \( Z(E_i) = \Phi(E_i) = Z(G) \) for \( 1 \leq i \leq s \), which implies \( Z(G) = \Phi(G) \). Remarks 3 and 4 imply the existence of a noninner automorphism of order \( p \) which fixes \( \Phi(G) \), a contradiction.

Thus \( G \) must have the structure indicated in case (b). The proof will be complete if one could show that \( G = F \). Indeed, if \( G = F \), then \( C_G(Z(\Phi(G))) = \Phi(G) \), contradicting the hypothesis.

Suppose the contrary; i.e., \( G \) is as in (b) and \( G \neq F \). Then \( G = EF \), where \( E = C_G(F) = E_1E_2\cdots E_s \) and the factors \( E_i \) satisfy the conditions in (a). In this case \( E \cap F \leq Z(E) \); moreover, as was shown above, \( E \) has a noninner automorphism of order \( p \) which fixes \( Z(E) \), a contradiction.

Observe next that \( \Phi(G) = Z(G)\Phi(F) = Z(E)\Phi(F) \). Apply now Remark 4 to the central product \( G = EF \) to obtain a noninner automorphism of order \( p \) which fixes \( Z(E) \) and \( F \). This automorphism certainly fixes \( \Phi(G) \), contradicting our assumption on \( G \). The proof of the theorem is now complete.

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