J. Symbolic Computation (1995) **20**, 1–41

Modular Properties of Composable Term Rewriting Systems

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(Received 12 June 1995)

In this paper we prove several new modularity results for unconditional and conditional term rewriting systems. Most of the known modularity results for the former systems hold for disjoint or constructor-sharing combinations. Here we focus on a more general kind of combination: so-called composable systems. As far as conditional term rewriting systems are concerned, all known modularity result but one apply only to disjoint systems. Here we investigate conditional systems which may share constructors. Furthermore, we refute a conjecture of Middeldorp (1990, 1993).

1. Introduction

Term rewriting has applications in various fields of computer science such as symbolic computation, functional programming, abstract data type specifications, program verification, program synthesis, and automated theorem proving. In an outstanding paper, Knuth and Bendix (1970) describe a completion procedure which can often be successfully used to transform a given set of equations into a complete term rewriting system (TRS) which defines the same equational theory. Thus TRSs provide an operational model of algebraic specifications of abstract data types. Large specifications, however, must be written in a modular way according to the one page principle of Mark Ardis: "A specification that will not fit on one page of 8.5×11 inch paper cannot be understood".

Modularity is a well-known programming paradigm in computer science. Programmers should design their programs in a modular way, that is, as a combination of small programs. These so-called modules are implemented separately and are then integrated to form the whole program. Since TRSs have important applications in computer science, it is – not only from a theoretical viewpoint but also from a practical point of view – of utmost importance to know under which conditions a combined system inherits desirable properties from its constituent systems. For this reason modular aspects of term rewriting have been receiving increasing attention. A property $\mathcal{P}$ of TRSs (like confluence, termination etc.) is called *modular* if whenever $\mathcal{R}_1$ and $\mathcal{R}_2$ are TRSs both satisfying $\mathcal{P}$, then their combined system $\mathcal{R}_1 \cup \mathcal{R}_2$ also satisfies $\mathcal{P}$. The knowledge that (perhaps under certain conditions) a property $\mathcal{P}$ is modular facilitates program synthesis because it allows an incremental development of programs. On the other hand, it provides a divide and conquer approach to establishing properties of TRSs. If one wants to know whether a large TRS has a certain modular property $\mathcal{P}$, then this system can be decomposed

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into small subsystems and one merely has to check whether each of these subsystems has property $\mathcal{P}$.

As all interesting properties are in general not modular, the starting-point of research were *disjoint unions*, combinations of TRSs having no function symbols in common. Toyama (1987*a*) proved that confluence is modular for disjoint systems. In contrast to that, termination and completeness lack a modular behavior (see Toyama, 1987*b*). Kurihara and Ohuchi (1992) investigated *constructor-sharing systems*: constructors are function symbols that do not occur at the root position of the left-hand side of any rewrite rule, the others are called defined symbols. Among other things, they showed that confluence is not modular for constructor-sharing systems. Middeldorp and Toyama (1993) introduced *composable systems* which have to contain all rewrite rules that define a defined symbol whenever that symbol is shared. The authors, however, restricted their investigations to constructor systems (where no proper subterm of a left-hand side of a rewrite rule is allowed to contain defined symbols). Their main result states that completeness is modular for composable constructor systems. We drop the constructor system requirement, so the composable systems we consider are a proper generalization of constructor-sharing systems. It is worthwhile to investigate combinations of composable systems because they correspond to the union of specifications with common subparts which exist in most specification languages.

The title of this paper reflects that the combination of composable systems is the most general kind of combination which will be investigated here. It will be shown that for those systems semi-completeness is modular, termination is modular for layer-preserving and for non-duplicating systems, completeness is modular for overlay systems, and that the simplifying property is modular as well. We stress the fact that it is possible to compute in the combined system of pairwise composable complete systems. More precisely, the unique normal form of a term can be obtained by any innermost reduction strategy. Then conditional term rewriting systems (CTRSs) are studied. The rewrite rules of those systems may possess conditions, and such a conditional rewrite rule is only applicable if its conditions are fulfilled. We focus on the most prominent kind of CTRSs, the so-called join or standard systems. It should be pointed out that conditional term rewriting is inherently more complicated than unconditional term rewriting. So it is not surprising that most of our results apply only to constructor-sharing systems — as a matter of fact, up until now no positive modularity result is known for the combination of composable CTRSs which may have extra variables in their conditions. Middeldorp (1990, 1993) was the first to investigate modular properties of (disjoint) CTRSs. Among other things, he showed that for disjoint conditional term rewriting systems confluence and semi-completeness are modular whereas local confluence and normalization lack a modular behavior. So the best one can hope for when considering constructor-sharing CTRSs is the modularity of semi-completeness (all other above-mentioned properties cannot be modular for those systems since they already fail to be modular for more restricted systems). We prove that semi-completeness is indeed modular for constructor-sharing CTRSs. Middeldorp has also shown that termination is modular for non-collapsing, and completeness is modular for non-duplicating disjoint CTRSs. Furthermore, he conjectured (see Middeldorp, 1990, 1993) that the disjoint union of two terminating join CTRSs is terminating if one of them contains neither collapsing nor duplicating rules and the other is confluent. We will refute this conjecture by a simple counterexample. Moreover, it will be shown that his results also hold, mutatis mutandis, in the presence of shared constructors. We point out that our proof (though based on the ideas of Middeldorp, 1993) is considerably simpler than that of Middeldorp (1993). Then we investigate finite and decreasing CTRSs. Since these systems exactly capture the finiteness of recursive evaluation of conditions, they have been studied by many authors. Our main result in this context states that it is possible to compute in the combined system of decreasing, confluent, and pairwise constructor-sharing CTRSs. Finally, it is shown that the related simplifying property is modular, even for composable CTRSs.

Since very many new modularity results have been published recently, we cannot render a detailed account of those here. Instead, the reader is referred to Marchiori (1995), Ohlebusch (1993,1995*a, b*) and Gramlich (1994*c*) for recent results on *disjoint* (conditional) TRSs, to Dershowitz (1994), Gramlich (1994*a, b*), and Ohlebusch (1994*a*) which deal with *constructor-sharing* TRSs, and to Kurihara and Ohuchi (1995) as well as Middeldorp (1994*a*) which contain results for *composable* (conditional) TRSs. In this paper we do not investigate so-called *hierarchical combinations* of rewrite systems but refer to Section 7 for a brief discussion of this related work. The paper is organized as follows. First we briefly recall the basic notions of term rewriting. In Section 3 we specify the different kinds of combination. Then the basic notions of composable systems are introduced. Section 5 contains our results about composable TRSs, while Section 6 is concerned with constructor-sharing CTRSs. The paper is concluded with a brief discussion of related work and open problems.

2. Preliminaries

This section contains a concise introduction to term rewriting. The reader is referred to the surveys of Dershowitz and Jouannaud (1990) and Klop (1992) for more detail.

A *signature* is a countable set $\mathcal{F}$ of *function symbols* or *operators*, where every $f \in \mathcal{F}$ is associated with a natural number denoting its arity. Nullary operators are called *constants*. The set $\mathcal{T}(\mathcal{F},\mathcal{V})$ of *terms* built from a signature $\mathcal{F}$ and a countable set of *variables* $\mathcal{V}$ with $\mathcal{F} \cap \mathcal{V} = \emptyset$ is the smallest set such that $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F},\mathcal{V})$ and if $f \in \mathcal{F}$ has arity n and $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$, then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})$. We write f instead of $f(\,)$ whenever f is a constant. The set of variables appearing in a term $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$ is denoted by $\mathcal{V}ar(t)$. For $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$, we define $root(t)$ by $root(t) = t$ if $t \in \mathcal{V}$, and $root(t) = f$ if $t = f(t_1,\ldots,t_n)$. $|t|$ denotes the *size* of t, i.e. $|t| = 1$ if $t \in \mathcal{V}$, and $|t| = 1 + |t_1| + \ldots + |t_n|$ if $t = f(t_1,\ldots,t_n)$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. This set is called the *domain* of σ and will be denoted by $\mathcal{D}om(\sigma)$. Occasionally, we present a substitution σ as $\{x \mapsto \sigma(x) \mid x \in \mathcal{D}om(\sigma)\}$. The substitution with empty domain will be denoted by ϵ. Substitutions extend uniquely to morphisms from $\mathcal{T}(\mathcal{F},\mathcal{V})$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$, that is, $\sigma(f(t_1,\ldots,t_n)) = f(\sigma(t_1),\ldots,\sigma(t_n))$ for every n-ary function symbol f and terms $t_1,\ldots,t_n$. We call $\sigma(t)$ an *instance* of t. We also write $t\sigma$ instead of $\sigma(t)$.

Let $\Box$ be a special constant. A *context* is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\},\mathcal{V})$. $C[,\ldots,]$ denotes a context which contains at least one occurrence of $\Box$ and may be equal to $\Box$, $C\langle,\ldots,\rangle$ stands for a context which contains zero or more occurrence of $\Box$ and may be equal to $\Box$, while $C\{,\ldots,\}$ denotes a context which contains zero or more occurrence of $\Box$ and is different from $\Box$. If $C[,\ldots,]$ is a context with n occurrences of $\Box$ and $t_1,\ldots,t_n$ are terms, then $C[t_1,\ldots,t_n]$ is the result of replacing from left to right the occurrences of $\Box$ with $t_1,\ldots,t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[\,]$. A term t is a *subterm* of a term s if there exists a context $C[\,]$ such that $s = C[t]$. A subterm t of s is *proper*, denoted by $s \rhd t$, if $s \neq t$. By abuse of notation we write $\mathcal{T}(\mathcal{F},\mathcal{V})$ for

$\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$, interpreting $\Box$ as a special constant which is always available but used only for the aforementioned purpose.

Let $\rightarrow$ be a binary relation on terms, i.e. $\rightarrow \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$. The reflexive transitive closure of $\rightarrow$ is denoted by $\rightarrow^*$. If $s \rightarrow^* t$, we say that s *reduces* to t and we call t a *reduct* of s. We write $s \leftarrow t$ if $t \rightarrow s$; likewise for $s \;{}^*\!\!\leftarrow t$. The transitive closure of $\rightarrow$ is denoted by $\rightarrow^+$, and $\leftrightarrow$ denotes the symmetric closure of $\rightarrow$ (i.e. $\leftrightarrow \,=\, \rightarrow \cup \leftarrow$). The reflexive transitive closure of $\leftrightarrow$ is called *conversion* and denoted by $\leftrightarrow^*$. If $s \leftrightarrow^* t$, then s and t are *convertible*. Two terms t_1, t_2 are *joinable*, denoted by $t_1 \downarrow t_2$, if there exists a term t_3 such that $t_1 \rightarrow^* t_3 \;{}^*\!\!\leftarrow t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. The relation $\downarrow$ is called *joinability*. A term s is a *normal form* w.r.t. $\rightarrow$ if there is no term t such that $s \rightarrow t$. A term s has a normal form if $s \rightarrow^* t$ for some normal form t. The set of all normal forms of $\rightarrow$ is denoted by $NF(\rightarrow)$. The relation $\rightarrow$ is *normalizing* if every term has a normal form; it is *terminating*, if there is no infinite reduction sequence $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \ldots$. In the literature, the terminology *weakly normalizing* and *strongly normalizing* is often used instead of normalizing and terminating, respectively. The relation $\rightarrow$ is *confluent* if for all terms s, t_1, t_2 with $t_1 \;{}^*\!\!\leftarrow s \rightarrow^* t_2$ we have $t_1 \downarrow t_2$. It is well-known that $\rightarrow$ is confluent if and only if every pair of convertible terms is joinable. The relation $\rightarrow$ is *locally confluent* if for all terms s, t_1, t_2 with $t_1 \leftarrow s \rightarrow t_2$ we have $t_1 \downarrow t_2$. If $\rightarrow$ is confluent and terminating, it is called *complete* or *convergent*. The famous Newman's Lemma states that termination and local confluence imply confluence. If $\rightarrow$ is confluent and normalizing, then it is called *semi-complete*. Sometimes this property is called *unique normalization* because it is equivalent to the property that every term has a unique normal form.

A *term rewriting system* (TRS) is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subset \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of *rewrite rules* or *reduction rules*. Every rewrite rule (l, r) must satisfy the following two constraints: (i) the left-hand side l is not a variable, and (ii) variables occurring in the right-hand side r also occur in l. Rewrite rules (l, r) will be denoted by $l \rightarrow r$. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). The rewrite rules of a TRS $(\mathcal{F}, \mathcal{R})$ define a *rewrite relation* $\rightarrow_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ as follows: $s \rightarrow_{\mathcal{R}} t$ if there exists a rewrite rule $l \rightarrow r$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s = C[l\sigma]$ and $t = C[r\sigma]$. We say that s rewrites to t by *contracting* redex $l\sigma$. We call $s \rightarrow_{\mathcal{R}} t$ a *rewrite step* or *reduction step*. A TRS $(\mathcal{F}, \mathcal{R})$ has one of the above properties (e.g. termination) if its rewrite relation has the respective property. Let $(\mathcal{F}, \mathcal{R})$ be an arbitrary TRS. A function symbol $f \in \mathcal{F}$ is called a *defined symbol* if there is a rewrite rule $l \rightarrow r \in \mathcal{R}$ such that $f = root(l)$. Function symbols from $\mathcal{F}$ which are not defined symbols are called *constructors*. The set of normal forms of $(\mathcal{F}, \mathcal{R})$ will also be denoted by $NF(\mathcal{F}, \mathcal{R})$. We often simply write $\mathcal{R}$ instead of $(\mathcal{F}, \mathcal{R})$ if there is no ambiguity about the underlying signature $\mathcal{F}$. A rewrite rule $l \rightarrow r$ of a TRS $\mathcal{R}$ is *collapsing* if r is a variable, and *duplicating* if r contains more occurrences of some variable than l. A TRS $\mathcal{R}$ is *non-duplicating* (non-collapsing, respectively) if it does not contain duplicating (collapsing, respectively) rewrite rules.

In a *join conditional term rewriting system* (CTRS for short) $(\mathcal{F}, \mathcal{R})$, the rewrite rules of $\mathcal{R}$ have the form $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ with $l, r, s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. $s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ are the *conditions* of the rewrite rule. If a rewrite rule has no conditions, we write $l \rightarrow r$. We impose the same restrictions on conditional rewrite rules as on unconditional rewrite rules. That is, we allow *extra variables* in the conditions but not on right-hand sides of rewrite rules. The rewrite relation associated with $(\mathcal{F}, \mathcal{R})$ is defined by: $s \rightarrow_{\mathcal{R}} t$ if there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}$, a substitution

$\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, and a context $C[\,]$ such that $s = C[l\sigma], t = C[r\sigma]$, and $s_j\sigma \downarrow_{\mathcal{R}} t_j\sigma$ for all $j \in \{1, \ldots, n\}$. For every CTRS $\mathcal{R}$, we inductively define TRSs $\mathcal{R}_i$, $i \in \mathbb{N}$, by:

$$\begin{aligned} \mathcal{R}_0 &= \{l \to r \mid l \to r \in \mathcal{R}\} \\ \mathcal{R}_{i+1} &= \{l\sigma \to r\sigma \mid l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n \in \mathcal{R} \text{ and} \\ &\qquad s_j\sigma \downarrow_{\mathcal{R}_i} t_j\sigma \text{ for all } j \in \{1, \ldots, n\}\}. \end{aligned}$$

Note that $\mathcal{R}_i \subseteq \mathcal{R}_{i+1}$ for all $i \in \mathbb{N}$. Furthermore, $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}_i} t$ for some $i \in \mathbb{N}$. The *depth* of a rewrite step $s \to_{\mathcal{R}} t$ is defined to be the minimal i with $s \to_{\mathcal{R}_i} t$. Depths of reduction sequences $s \to_{\mathcal{R}}^* t$, conversions $s \leftrightarrow_{\mathcal{R}}^* t$, and valleys $s \downarrow_{\mathcal{R}} t$ are defined analogously. All notions defined previously for TRSs extend to CTRSs.

A *partial ordering* $(A, >)$ is a pair consisting of a set A and a binary irreflexive and transitive relation $>$ on A. A partial ordering is called *well-founded* if there are no infinite sequences $a_1 > a_2 > a_3 > \ldots$ of elements from A. A *multiset* is a collection in which elements are allowed to occur more than once. If A is a set, then the set of all finite multisets over A is denoted by $\mathcal{M}(A)$. The *multiset extension* of a partial ordering $(A, >)$ is the partial ordering $(\mathcal{M}(A), >^{mul})$ defined as follows: $M_1 >^{mul} M_2$ if $M_2 = (M_1 \backslash X) \cup Y$ for some multisets $X, Y \in \mathcal{M}(A)$ that satisfy (i) $\emptyset \neq X \subseteq M_1$ and (ii) for all $y \in Y$ there exists an $x \in X$ such that $x > y$. Dershowitz and Manna (1979) proved that the multiset extension of a well-founded partial ordering is a well-founded partial ordering.

A *simplification ordering* $>$ is a partial ordering on terms which (i) is *closed under contexts* (i.e. $s > t$ implies $C[s] > C[t]$ for every context $C[\,]$), (ii) *closed under substitutions* (i.e. $s > t$ implies $s\sigma > t\sigma$ for every substitution σ), and (iii) has the *subterm property* (i.e. $C[t] > t$ for all contexts $C[\,] \neq \Box$).

3. Various combinations

Very simple examples show that in general all interesting properties are lost under arbitrary combinations of TRSs. Thus, several restricted kinds of combinations have been proposed in the literature. The next definition specifies these combinations.

DEFINITION 3.1. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be TRSs. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote their respective sets of defined symbols and let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote their respective sets of constructors. Their *combined system* is their union $(\mathcal{F}, \mathcal{R}) = (\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$. Its set of defined symbols is obviously $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ and its set of constructors is $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$.

(1) $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are *disjoint* if they do not share function symbols, that is, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (or equivalently $\mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{C}_2 = \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$). In the literature, $(\mathcal{F}, \mathcal{R})$ is sometimes called the *direct sum* of $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$.

(2) $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are *constructor-sharing* if they at most share constructors, i.e., $\mathcal{F}_1 \cap \mathcal{F}_2 \subseteq \mathcal{C}$ (or equivalently $\mathcal{C}_1 \cap \mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{C}_2 = \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$).

(3) $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are *composable* if $\mathcal{C}_1 \cap \mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{C}_2 = \emptyset$ and both systems contain all rewrite rules that define a defined symbol whenever that symbol is shared, more precisely, $\mathcal{S} = \{l \to r \in \mathcal{R} \mid root(l) \in \mathcal{D}_1 \cap \mathcal{D}_2\} \subseteq \mathcal{R}_1 \cap \mathcal{R}_2$. In this situation, the set $\mathcal{S}$ is said to be the set of *shared rules* of $\mathcal{R}_1$ and $\mathcal{R}_2$.

The different kinds of combinations are illustrated in Figure 1.

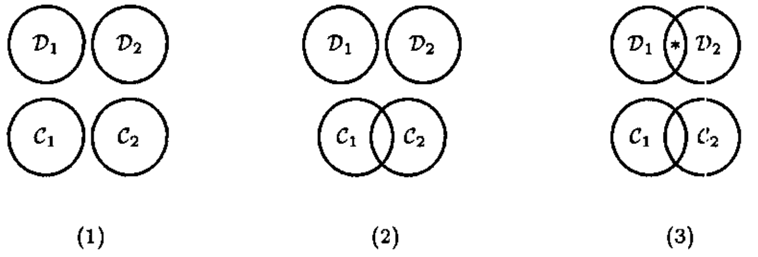

Figure 1 Different combinations.

DEFINITION 3.2. A property $\mathcal{P}$ is *modular for composable TRSs* if, for all composable TRSs $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$, their union $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$ has the property $\mathcal{P}$ if and only if both $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ have the property $\mathcal{P}$.

We will also use the phrases *modular for constructor-sharing TRSs* and *modular for disjoint TRSs*. The meanings of these phrases are obvious. We are of course not only interested in the combination of two TRSs. It should also be possible to deal with situations where more than two systems are combined. The next proposition shows that the combination of n TRSs, $n \geq 2$, can be reduced to the case $n = 2$ by successively combining two systems. The simple proof is omitted.

PROPOSITION 3.3. Let $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ be n, $n \geq 2$, pairwise composable TRSs. Then the term rewriting systems $(\bigcup_{j=1}^{n-1} \mathcal{F}_j, \bigcup_{j=1}^{n-1} \mathcal{R}_j)$ and $(\mathcal{F}_n, \mathcal{R}_n)$ are composable.

4. Composable systems: basic notions

In this section, the basic notions concerning the combination of two composable term rewriting systems $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are introduced. These notions will easily be identified with those already introduced for disjoint systems (see e.g. Middeldorp, 1990) and constructor-sharing systems (see e.g. Kurihara and Ohuchi, 1992). So from now on we tacitly assume that $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are two composable TRSs and that $(\mathcal{F}, \mathcal{R})$ denotes their combined system. In the sequel $\to \; = \; \to_{\mathcal{R}} \; = \; \to_{\mathcal{R}_1 \cup \mathcal{R}_2}$. First of all, we introduce the chromatic terminology which is now common.

DEFINITION 4.1. The set $\mathcal{F}_1 \cap \mathcal{F}_2$ of *shared function symbols*, i.e. function symbols that occur in *both* signatures, is denoted by $\mathcal{B}$. $\mathcal{A}_1 = \mathcal{F}_1 \setminus \mathcal{B}$ is called the set of *alien function symbols* for $\mathcal{F}_2$ and $\mathcal{B}$ because $\mathcal{A}_1 \cap \mathcal{F}_2 = \emptyset$ and $\mathcal{A}_1 \cap \mathcal{B} = \emptyset$. $\mathcal{A}_2$ is defined analogously. Note that $\mathcal{F} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \mathcal{B}$. In order to enhance readability, function symbols from $\mathcal{A}_1$ are called *black*, those from $\mathcal{A}_2$ *white*, and shared function symbols as well as variables are called *transparent*. A term s is called *top black (top white, top transparent)* if $root(s)$ is black (white, transparent). In a term every transparent symbol c acts like a chameleon, that is, it changes its color to match the surrounding: If there is no black or white symbol above c (there is no surrounding so to speak), then it remains transparent. Otherwise, its color is the same as the color of its parent (the definition applies recursively if the parent is a shared symbol). If a term does not contain white (black) function symbols, we speak of a *black (white) term*. A term containing both black and white function symbols is called a *mixed term*. A term is said to be *transparent* if it only contains shared function symbols and variables.

Please notice a subtlety in the preceding definition: A transparent term may be regarded as black or white; this is very convenient for later purposes.

EXAMPLE 4.2. Let $\mathcal{F}_1 = \{add, mult, S, 0\}$ and

$$\mathcal{R}_1 = \begin{cases} add(0,x) & \to & x \\ add(S(x),y) & \to & S(add(x,y)) \\ mult(0,x) & \to & 0 \\ mult(S(x),y) & \to & add(mult(x,y),y). \end{cases}$$

Moreover, let $\mathcal{F}_2 = \{add, fib, S, 0\}$ and

$$\mathcal{R}_2 = \begin{cases} add(0,x) & \to & x \\ add(S(x),y) & \to & S(add(x,y)) \\ fib(0) & \to & 0 \\ fib(S(0)) & \to & S(0) \\ fib(S(S(x))) & \to & add(fib(S(x)),fib(x)). \end{cases}$$

It is apparent that $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ are composable systems. *mult* is the only black symbol, *fib* is the only white symbol, and the symbols $add, S, 0$ are transparent. Consider the mixed term $s = add(0, add(fib(S(mult(0,0))), mult(0, mult(fib(0), fib(0)))))$. Figure 2 shows how s can be decomposed into an outer transparent and further inner black and white parts. We will next specify this decomposition.

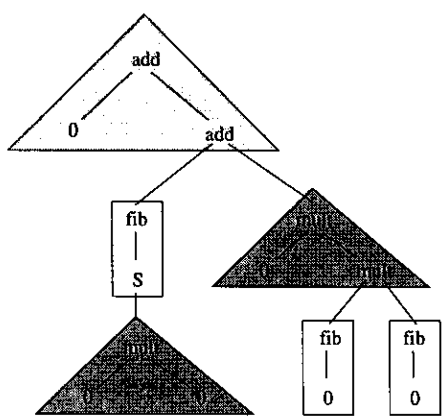

Figure 2 A colored term.

LEMMA 4.3. Every term $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ has unique representations

$$s = \begin{cases} C^t\langle s_1,\ldots,s_l\rangle, & \text{where } C^t\langle,\ldots,\rangle \in \mathcal{T}(\mathcal{B},\mathcal{V}) \ , \ root(s_j) \in \mathcal{A}_1 \uplus \mathcal{A}_2 \\ C^b\langle t_1,\ldots,t_m\rangle, & \text{where } C^b\langle,\ldots,\rangle \in \mathcal{T}(\mathcal{A}_1 \uplus \mathcal{B},\mathcal{V}) \ , \ root(t_j) \in \mathcal{A}_2 \\ C^w\langle u_1,\ldots,u_n\rangle, & \text{where } C^w\langle,\ldots,\rangle \in \mathcal{T}(\mathcal{A}_2 \uplus \mathcal{B},\mathcal{V}) \ , \ root(u_j) \in \mathcal{A}_1. \end{cases}$$

PROOF. Routine. □

DEFINITION 4.4. In the situation of Lemma 4.3, we will use the following shorthands for the unique representations of the term s:

$$s = \begin{cases} C^t\langle\langle s_1, \ldots, s_l\rangle\rangle \\ C^b\langle\langle t_1, \ldots, t_m\rangle\rangle \\ C^w\langle\langle u_1, \ldots, u_n\rangle\rangle. \end{cases}$$

If we have some more information about a context $C\langle\langle, \ldots, \rangle\rangle$, then we also use different notations. If we know that $C\langle\langle, \ldots, \rangle\rangle \neq \Box$, then we write $C\{\!\{, \ldots, \}\!\}$. If it is known that furthermore $C\{\!\{, \ldots, \}\!\}$ contains at least one occurrence of □, then we write $C[\![, \ldots,]\!]$. Moreover, we define

$$S_1(s) = [s_1, \ldots, s_n], \quad S_P^w(s) = [t_1, \ldots, t_m], \quad S_P^b(s) = [u_1, \ldots, u_n].$$

$t_1, \ldots, t_m$ ($u_1, \ldots, u_n$, respectively) are called *white (black, respectively) principal subterms* of s. The *topmost black homogeneous part* $top^b(s)$ of s is the term $C^b\langle, \ldots, \rangle$ in the unique representation $s = C^b\langle\langle t_1, \ldots, t_m\rangle\rangle$. The *topmost white homogeneous part* $top^w(s)$ and the *topmost transparent homogeneous part* $top^t(s)$ of s are defined analogously.

EXAMPLE 4.5. In the situation of Example 4.2, s has representations

$$C^t[\![fib(S(mult(0,0))), mult(0, mult(fib(0), fib(0)))]\!], \; C^t[, \ldots,] = add(0, add(\Box, \Box))$$

$$C^b[\![fib(S(mult(0,0))), fib(0), fib(0)]\!], \; C^b[, \ldots,] = add(0, add(\Box, mult(0, mult(\Box, \Box))))$$

$$C^w[\![mult(0,0), mult(0, mult(fib(0), fib(0)))]\!], \; C^w[, \ldots,] = add(0, add(fib(S(\Box)), \Box)).$$

To exemplify the above, we have for instance $S_P^w(s) = [fib(S(mult(0,0))), fib(0), fib(0)]$ and $top^b(s) = add(0, add(\Box, mult(0, mult(\Box, \Box))))$.

The term $t = mult(0,0)$ has representations $t = C^t\langle\langle t\rangle\rangle$, $t = C^w\langle\langle t\rangle\rangle$, and $t = C^b\langle\langle, \ldots, \rangle\rangle$, where $C^t\langle, \ldots, \rangle = C^w\langle, \ldots, \rangle = \Box$ and $C^b\langle, \ldots, \rangle$ contains no occurrence of □, i.e. is equal to $mult(0,0)$. We have for instance $S_P^w(t) = [\,]$, $S_P^b(t) = [t]$, $top^w(t) = \Box$, and $top^b(t) = mult(0,0)$.

The term $u = add(fib(0), fib(0))$ has representations $u = C^t\langle\langle fib(0), fib(0)\rangle\rangle$, $u = C^b\langle\langle fib(0), fib(0)\rangle\rangle$, and $u = C^w\langle\langle, \ldots, \rangle\rangle$, where $C^t\langle, \ldots, \rangle = C^b\langle, \ldots, \rangle = add(\Box, \Box)$ and $C^w\langle, \ldots, \rangle$ contains no occurrence of □, i.e. is equal to u. We have for instance $S_P^b(u) = [\,]$, $S_P^w(u) = [fib(0), fib(0)]$, and $top^w(u) = u$.

DEFINITION 4.6. The *rank* of a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows.
If t is a top black or top white term, then

$$rank(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}(\mathcal{A}_1 \uplus \mathcal{B}, \mathcal{V}) \cup \mathcal{T}(\mathcal{A}_2 \uplus \mathcal{B}, \mathcal{V}) \\ 1 + max\{rank(t_j) \mid 1 \leq j \leq n\} & \text{if } t = C^b[\![t_1, \ldots, t_n]\!] \text{ or } t = C^w[\![t_1, \ldots, t_n]\!]. \end{cases}$$

If t is a top transparent term, then

$$rank(t) = \begin{cases} 0 & \text{if } t \in \mathcal{T}(\mathcal{B}, \mathcal{V}) \\ max\{rank(t_j) \mid 1 \leq j \leq m\} & \text{if } t = C^t[\![t_1, \ldots, t_m]\!]. \end{cases}$$

The term s of Example 4.5 has rank 2. Several definitions and considerations are symmetrical in the colors black and white. Therefore, we often state the respective definitions and considerations only for one color (the same applies mutatis mutandis for the other color).

DEFINITION 4.7. Let $s \to t$ by an application of a rewrite rule $l \to r \in \mathcal{R}$.
If $s = C^b[\![s_1, \ldots, s_n]\!]$ is a top black term, then we write

$s \to^i t$ if $t = C^b[s_1, \ldots, s_{j-1}, t_j, s_{j+1}, \ldots, s_n]$ and $s_j \to t_j$ for some $j \in \{1, \ldots, n\}$.
$s \to^o_{\mathcal{A}_1} t$ otherwise.

If $s = C^t[\![s_1, \ldots, s_n]\!]$ is a top transparent term, then we write

$s \to^i t$ if $t = C^t[\![s_1, \ldots, s_{j-1}, t_j, s_{j+1}, \ldots, s_n]\!]$ and $s_j \to^i t_j$ for some $j \in \{1, \ldots, n\}$.
$s \to^o_{\mathcal{A}_1} t$ if $t = C^t[s_1, \ldots, s_{j-1}, t_j, s_{j+1}, \ldots, s_n]$ and $s_j \to^o_{\mathcal{A}_1} t_j$ for a top black s_j.
$s \to^o_{\mathcal{A}_2} t$ if $t = C^t[s_1, \ldots, s_{j-1}, t_j, s_{j+1}, \ldots, s_n]$ and $s_j \to^o_{\mathcal{A}_2} t_j$ for a top white s_j.
$s \to^t t$ otherwise.

The relations $\to^i$, $\to^o_{\mathcal{A}_1}$, $\to^o_{\mathcal{A}_2}$, $\to^o = \to^o_{\mathcal{A}_1} \cup \to^o_{\mathcal{A}_2}$, and $\to^t$ are called *inner, black outer, white outer, outer*, and *transparent* reduction, respectively. We will also use the abbreviations $\to^{t,o}_{\mathcal{A}_1} = \to^t \cup \to^o_{\mathcal{A}_1}$, $\to^{t,o}_{\mathcal{A}_2} = \to^t \cup \to^o_{\mathcal{A}_2}$, and $\to^{t,o} = \to^t \cup \to^o$. Note that every reduction step $s \to t$ is classified by the above definition: it is either an inner or a black outer or a white outer or a transparent reduction step. Moreover, if s is top black (top white), then the reduction step cannot be a transparent or white (black) outer reduction step.

For disjoint and constructor-sharing systems, respectively, we will use the common $\to^o_{\mathcal{R}_1}$ instead of $\to^o_{\mathcal{A}_1}$ because for those system the reduction in a "black part" of a term implies that the applied rule stems exclusively from $\mathcal{R}_1$. As to composable TRSs, this is not true in general. If s is for instance a top black term, then the topmost black homogeneous part may very well be reduced by a rule from $\mathcal{R}_2$ (but never at the root!). However, since $\mathcal{R}_1$ and $\mathcal{R}_2$ are composable, we know that this rule is also contained in $\mathcal{R}_1$. So if the topmost black homogeneous part is reduced, we may w.l.o.g. assume that the applied rule stems from $\mathcal{R}_1$ (notwithstanding the fact that it may also stem from $\mathcal{R}_2$).

EXAMPLE 4.8. Once again, consider the TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ as well as the term s of Example 4.2.

$$
\begin{array}{lll}
s & = & add(0, add(fib(S(mult(0,0))), mult(0, mult(fib(0), fib(0))))) \\
 & \to^t & add(fib(S(mult(0,0))), mult(0, mult(fib(0), fib(0)))) \\
 & \to^i & add(fib(S(0)), mult(0, mult(fib(0), fib(0)))) \\
 & \to^o_{\mathcal{A}_2} & add(S(0), mult(0, mult(fib(0), fib(0)))) \\
 & \to^i & add(S(0), mult(0, mult(0, fib(0)))) \\
 & \to^o_{\mathcal{A}_1} & add(S(0), 0).
\end{array}
$$

DEFINITION 4.9. Let s be a top black term. A rewrite step $s \to t$ is *destructive at level 1* if the root symbols of s and t have different colors (that is to say, $root(t) \in \mathcal{A}_2 \uplus \mathcal{B} \uplus \mathcal{V}$). It is *destructive at level* $m+1$, $m \geq 1$, if $s \to^i t$, where $s = C^b[\![s_1, \ldots, s_j, \ldots, s_n]\!]$, $t = C^b[s_1, \ldots, t_j, \ldots, s_n]$, and $s_j \to t_j$ is destructive at level m. For a top transparent term s, a rewrite step $s \to t$ is *destructive at level 0* if the root symbols of s and t have different colors (that is to say, $root(t) \in \mathcal{A}_1 \uplus \mathcal{A}_2$). It is *destructive at level m* for some $m \geq 1$ if it has a representation of the form $s = C^t[\![s_1, \dots, s_j, \dots, s_n]\!] \to C^t[s_1, \dots, t_j, \dots, s_n] = t$ with $s_j \to t_j$ destructive at level m.

In the reduction sequence of Example 4.8, the third and the last rewrite steps are destructive at level 1, whereas the second and fourth are destructive at level 2.

LEMMA 4.10. Let $s \to t$ by an application of some rule $l \to r \in \mathcal{R}$.

(1) If $s \to t$ is destructive at level 0, then $l \to r$ is a shared collapsing rule.
(2) If $s \to t$ is destructive at level $m > 0$, then $root(l) \in \mathcal{A}_1 \uplus \mathcal{A}_2$ and $root(r) \in \mathcal{B} \uplus \mathcal{V}$.

PROOF. Straightforward. □

In order to code certain special subterms by variables and to cope with transparent or outer rewrite steps using non-left-linear rules, the following notation is convenient.

DEFINITION 4.11. Let $s_1, \dots, s_n$ and $t_1, \dots, t_n$ be sequences of terms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We write $s_1, \dots, s_n \propto t_1, \dots, t_n$ if $s_i = s_j$ implies $t_i = t_j$ for all $1 \leq i < j \leq n$. If we have both $s_1, \dots, s_n \propto t_1, \dots, t_n$ and $t_1, \dots, t_n \propto s_1, \dots, s_n$, then we write $s_1, \dots, s_n \infty t_1, \dots, t_n$.

We omit the simple proofs of the following lemmata.

LEMMA 4.12. The pair $(\mathcal{B}, \mathcal{S})$ is a term rewriting system.

LEMMA 4.13. Let s, t be terms such that $s \to t$ by an application of some rule $l \to r \in \mathcal{R}$. Then $s \in \mathcal{T}(\mathcal{B}, \mathcal{V})$ implies $l \to r \in \mathcal{S}$ and $t \in \mathcal{T}(\mathcal{B}, \mathcal{V})$. Moreover, the restrictions of $\to_{\mathcal{S}}$ and $\to_{\mathcal{R}}$ to $\mathcal{T}(\mathcal{B}, \mathcal{V})$ coincide.

The following facts will be heavily used in the sequel (sometimes without being explicitly mentioned).

LEMMA 4.14. If $s \to^t t$, then $s = C^t\{\!\{s_1, \dots, s_n\}\!\}, t = \hat{C}^t\langle\!\langle s_{i_1}, \dots, s_{i_m}\rangle\!\rangle$ for some transparent contexts $C^t\{, \dots, \}$, $\hat{C}^t\langle, \dots, \rangle$, $i_1, \dots, i_m \in \{1, \dots, n\}$, and terms $s_1, \dots, s_n$ with $root(s_j) \in \mathcal{A}_1 \uplus \mathcal{A}_2$. If $s \to^t t$ is not destructive at level 0, then $t = \hat{C}^t\{\!\{s_{i_1}, \dots, s_{i_m}\}\!\}$. If $C^t\{\!\{s_1, \dots, s_n\}\!\} \to^t \hat{C}^t\langle\!\langle s_{i_1}, \dots, s_{i_m}\rangle\!\rangle$, by application of some rule $l \to r \in \mathcal{R}$, then we also have $C^t\{t_1, \dots, t_n\} \to \hat{C}^t\langle t_{i_1}, \dots, t_{i_m}\rangle$ by an application of the same rule $l \to r$ for all terms $t_1, \dots, t_n$ with $s_1, \dots, s_n \propto t_1, \dots, t_n$. Moreover, $l \to r \in \mathcal{S}$. Please note that analogous statements hold for $s \to^{t,o}_{\mathcal{A}_1} t$ and $s \to^{t,o}_{\mathcal{A}_2} t$.

LEMMA 4.15. Let $s = C^b[\![s_1, \dots, s_n]\!]$. If $s \to^o_{\mathcal{A}_2} t$ or $s \to^i t$, then there is a $j \in \{1, \dots, n\}$ such that $s_j \to t_j$ for some term t_j. That is, $t = C^b[s_1, \dots, t_j, \dots, s_n]$. If the reduction step is non-destructive, then $t = C^b[\![s_1, \dots, t_j, \dots, s_n]\!]$.

Now it is possible to prove that the rank of a term is never increased by a reduction step $s \to t$. This can be done by induction on $rank(s)$ and further distinguishing the cases $s \to^i t$, $s \to^o_{\mathcal{A}_j} t$, and $s \to^t t$.

PROPOSITION 4.16. If $s \to^* t$, then $rank(s) \geq rank(t)$.

DEFINITION 4.17. Let σ and τ be substitutions. We write $\sigma \propto \tau$ if $x\sigma = y\sigma$ implies $x\tau = y\tau$ for all $x, y \in \mathcal{V}$. The notation $\sigma \to^* \tau$ is used if $x\sigma \to^* x\tau$ for all $x \in \mathcal{V}$. Note that $\sigma \to^* \tau$ implies $t\sigma \to^* t\tau$ for all $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Moreover, σ is said to be in *normal form* or $\to$ *normalized* if $x\sigma \in NF(\to)$ for every $x \in \mathcal{V}$. A substitution σ is called *black* if $x\sigma$ is black for all $x \in \mathcal{D}om(\sigma)$ and it is said to be *top black* if $x\sigma$ is top black for all $x \in \mathcal{D}om(\sigma)$.

PROPOSITION 4.18. Every substitution σ can be decomposed into $\sigma_2 \circ \sigma_1$ such that σ_1 is black and σ_2 is top white and $\sigma_2 \propto \epsilon$ (recall that ϵ denotes the empty substitution).

PROOF. Essentially the same as for disjoint systems, see Middeldorp (1990, 1993). □

5. Modular properties of composable systems

5.1. SEMI-COMPLETENESS

Our first result is the modularity of semi-completeness for composable TRSs. From our point of view, this result is very important because semi-completeness is one of the most desirable properties of TRSs. Let us make this more precise. A TRS is a kind of applicative program that computes by reducing terms to other terms. The point of a computation is, of course, its result which consists of an irreducible term. If the TRS under consideration is confluent, then we know that a computed result is uniquely determined. That is, the normal form obtained is independent of the strategy used to compute it. If the TRS is also normalizing, then we know in addition that every term has a normal form. Thus, if the TRS is semi-complete, then every term has a unique normal form and all we further need is a (good) normalizing reduction strategy to compute that unique normal form (efficiently). On the other hand, there is hardly a method to prove semi-completeness of a TRS. In practise, one always tries to apply the following technique to prove *completeness*: At first, termination of the TRS is proved (mostly by some simplification ordering), and then convergence of all critical pairs is checked. Moreover, many complete TRSs are obtained via Knuth-Bendix completion. Since completeness is not modular (even for disjoint systems), the combination of pairwise composable complete TRSs does not yield a complete system. However, it yields a semi-complete and innermost terminating TRS (see below). So in this very important case the unique normal form w.r.t. the combined system can be obtained by any innermost reduction strategy.

LEMMA 5.1. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be composable TRSs.

(1) If one of the systems is confluent, then $(\mathcal{B}, \mathcal{S})$ is confluent.
(2) If one of the systems is normalizing, then $(\mathcal{B}, \mathcal{S})$ is normalizing.
(3) If one of the systems is semi-complete, then $(\mathcal{B}, \mathcal{S})$ is semi-complete.

PROOF. Routine. □

The basic proof idea of Theorem 5.2 is illustrated in Figure 3. For each term in a

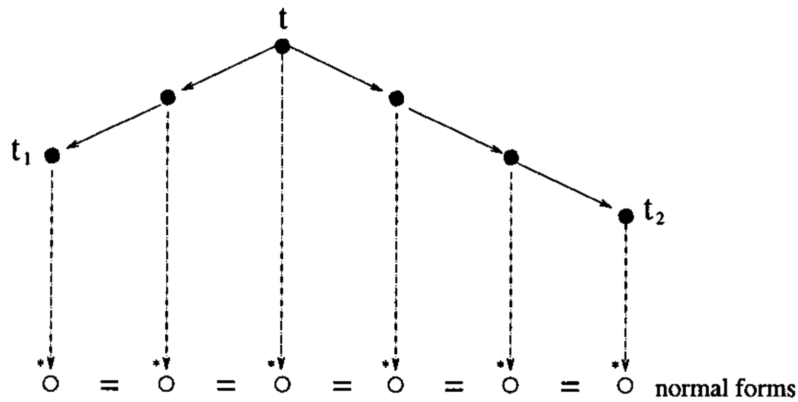

Figure 3 The proof idea of Theorem 5.2.

conversion t_1 ${}^*\!\!\leftarrow t \rightarrow^* t_2$ we construct a normal form† and then show that all these normal forms are identical. Hence every term t has a unique normal form. The simplified proof of the modularity of confluence for disjoint TRSs given by Klop *et al.* (1994) is based on a similar idea. There, every term in a conversion t_1 ${}^*\!\!\leftarrow t \rightarrow^* t_2$ is first reduced to a so-called witness and then it is shown that these witnesses have a common reduct. As a matter of fact, their approach has been extended to composable systems. In Ohlebusch (1994*b*), it is shown that confluence is modular for composable systems provided that a certain collapsing reduction relation $\rightarrow_c$ is normalizing. In the case of semi-complete constructor-sharing TRSs, we know that $\rightarrow_c$ is normalizing, see Ohlebusch (1994*a*, *b*). However, it is unknown whether $\rightarrow_c$ is also normalizing for semi-complete composable TRSs. Thus we use a different approach.

THEOREM 5.2. Semi-completeness is a modular property of composable TRSs.

PROOF. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be two composable TRSs. It has to be shown that their combined system $(\mathcal{F}, \mathcal{R})$ is semi-complete if and only if $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are semi-complete. The only-if direction is straightforward, so suppose that $(\mathcal{F}, \mathcal{R})$ is the combined system of two semi-complete composable TRSs $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$. We show by induction on $rank(t) = k$ that every term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ has a unique norm form w.r.t. $\mathcal{R}$. In the base case, $k = 0$ implies $t \in \mathcal{T}(\mathcal{B}, \mathcal{V})$. Here the claim follows from the semi-completeness of $(\mathcal{B}, \mathcal{S})$ and Lemma 4.13. So let $k \geq 1$ and consider a conversion t_1 ${}^*\!\!\leftarrow t \rightarrow^* t_2$.

Case (i): t is top black.
Let u be any term in the conversion t_1 ${}^*\!\!\leftarrow t \rightarrow^* t_2$. With u we associate terms $\tilde{u}$ and $\hat{u}$ which are defined as follows. If $rank(u) < k$, then u has a unique normal form $u\!\downarrow$ according to the induction hypothesis and we set $\tilde{u} = \hat{u} = u\!\downarrow$. If $rank(u) = k$, then u cannot be top white, hence it can be written as $u = C^b\{\!\{s_1, \ldots, s_n\}\!\}$. Since $rank(s_j) < k$, it follows from the induction hypothesis that, for every $j \in \{1, \ldots, n\}$, the white principal subterm s_j has a unique normal form $s_j\!\downarrow$. The result of replacing each white principal

† One of the referees has observed that a similar construction appeared in Middeldorp (1994*a*).

subterm with its unique normal form is denoted by $\tilde{u}$, i.e. $\tilde{u} = C^b\{s_1\downarrow, \ldots, s_n\downarrow\}$. Note that $u \to^* \tilde{u}$. Moreover, $\tilde{u}$ has a unique representation $\tilde{u} = \tilde{C}^b\{\!\{u_1, \ldots, u_m\}\!\}$ in which the u_i, $i \in \{1, \ldots, m\}$, are top white normal forms. Choose variables $x_1, \ldots, x_m$ not occurring in $\tilde{u}$ satisfying $u_1, \ldots, u_m \propto x_1, \ldots, x_m$. Since $\tilde{C}^b\{x_1, \ldots, x_m\} \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ and the TRS $(\mathcal{F}_1, \mathcal{R}_1)$ is semi-complete, it follows that $\tilde{C}^b\{x_1, \ldots, x_m\}$ rewrites to its unique $(\mathcal{F}_1, \mathcal{R}_1)$ normal form $\hat{C}^b\langle x_{i_1}, \ldots, x_{i_l}\rangle$. We set $\hat{u} = \hat{C}^b\langle u_{i_1}, \ldots, u_{i_l}\rangle$. It is easy to verify that $\hat{u} \in NF(\mathcal{F}, \mathcal{R})$. Observe that $\tilde{u} \to^*_{\mathcal{R}_1} \hat{u}$ and hence $u \to^* \hat{u}$.

Let $u_1 \to u_2$ be a step in the conversion $t_1 \ {}^*\!\!\leftarrow t \to^* t_2$. We show that $\hat{u_1} = \hat{u_2}$. If $rank(u_1) < k$, then $rank(u_2) < k$ as well. Hence $\hat{u}_1 = u_1\downarrow = u_2\downarrow = \hat{u}_2$. If $rank(u_1) = k$, then u_1 is a top black or top transparent term, i.e. $u_1 = C_1^b\{\!\{s_1, \ldots, s_n\}\!\}$. Here we have the following subcases.

(a) If $u_1 \xrightarrow{t,o}_{\mathcal{A}_1} u_2$, then u_2 can be written as $u_2 = C_2^b\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$. It follows that $\tilde{u}_1 = C_1^b\{s_1\downarrow, \ldots, s_n\downarrow\}$ and $\tilde{u}_2 = C_2^b\langle s_{i_1}\downarrow, \ldots, s_{i_l}\downarrow\rangle$. We obtain $\tilde{u}_1 \to_{\mathcal{R}_1} \tilde{u}_2$ (cf. Lemma 4.14). Every $s_j\downarrow$ has a representation $s_j\downarrow = \bar{C}_j^b\langle\!\langle u_1^j, \ldots, u_{m_j}^j\rangle\!\rangle$. Hence

$$\tilde{u}_1 = C_1^b\{\bar{C}_1^b\langle\!\langle u_1^1, \ldots, u_{m_1}^1\rangle\!\rangle, \ldots, \bar{C}_n^b\langle\!\langle u_1^n, \ldots, u_{m_n}^n\rangle\!\rangle\} \to_{\mathcal{R}_1}$$

$$C_2^b\langle \bar{C}_{i_1}^b\langle\!\langle u_1^{i_1}, \ldots, u_{m_{i_1}}^{i_1}\rangle\!\rangle, \ldots, \bar{C}_{i_l}^b\langle\!\langle u_1^{i_l}, \ldots, u_{m_{i_l}}^{i_l}\rangle\!\rangle\rangle = \tilde{u}_2.$$

Choose fresh variables $x_1^1, \ldots, x_{m_n}^n$ satisfying $u_1^1, \ldots, u_{m_n}^n \propto x_1^1, \ldots, x_{m_n}^n$ and note that this implies $u_1^{i_1}, \ldots, u_{m_{i_l}}^{i_l} \propto x_1^{i_1}, \ldots, x_{m_{i_l}}^{i_l}$. Another application of Lemma 4.14 yields

$$C_1^b\{\bar{C}_1^b\langle x_1^1, \ldots, x_{m_1}^1\rangle, \ldots, \bar{C}_n^b\langle x_1^n, \ldots, x_{m_n}^n\rangle\} \to_{\mathcal{R}_1}$$

$$C_2^b\langle \bar{C}_{i_1}^b\langle x_1^{i_1}, \ldots, x_{m_{i_1}}^{i_1}\rangle, \ldots, \bar{C}_{i_l}^b\langle x_1^{i_l}, \ldots, x_{m_{i_l}}^{i_l}\rangle\rangle.$$

Since both terms are trivially joinable, they reduce to the same unique $(\mathcal{F}_1, \mathcal{R}_1)$ normal form $\hat{C}^b\langle y_1, \ldots, y_p\rangle$, where $y_1, \ldots, y_p \in \{x_1^1, \ldots, x_{m_n}^n\}$. Hence $\hat{u}_1 = \sigma(\hat{C}^b\langle y_1, \ldots, y_p\rangle) = \hat{u}_2$ where $\sigma = \{x_i^j \mapsto u_i^j \mid j \in \{1, \ldots, n\}, i \in \{1, \ldots, m_j\}\}$.

(b) If $u_1 \to^o_{\mathcal{A}_2} u_2$ or $u_1 \to^i u_2$, then we have $u_1 = C_1^b\{\!\{s_1, \ldots, s_j, \ldots, s_n\}\!\} \to C_1^b\{s_1, \ldots, s_j', \ldots, s_n\} = u_2$, where $s_j \to s_j'$. If the rewrite step is not destructive, then we conclude from $s_j\downarrow = s_j'\downarrow$ that $\tilde{u}_1 = \tilde{u}_2$ and thus $\hat{u}_1 = \hat{u}_2$. If the rewrite step is destructive, then s_j' has a representation $s_j' = C_j'^b\{\!\{v_1, \ldots, v_p\}\!\}$. Clearly, $s_j' \to^* C_j'^b\{v_1\downarrow, \ldots, v_p\downarrow\}$. Furthermore, $C_j'^b\{v_1\downarrow, \ldots, v_p\downarrow\}$ can be written as $\tilde{C}_j^b\{\!\{u_1', \ldots, u_q'\}\!\}$, where $\tilde{C}_j^b\{\ldots,\}$ is a black context and u_i' are top white terms in $NF(\mathcal{F}, \mathcal{R})$. Now we know from the induction hypothesis that s_j and s_j' must have the same normal form $s_j\downarrow$. Hence $\tilde{C}_j^b\{\!\{u_1', \ldots, u_q'\}\!\} \xrightarrow{t,o}{}^*_{\mathcal{A}_1} s_j\downarrow$. Choose fresh variables $y_1, \ldots y_q$ such that $u_1', \ldots, u_q' \propto y_1, \ldots, y_q$. Repeated application of Lemma 4.14 yields $\tilde{C}_j^b\{y_1, \ldots, y_q\} \to^*_{\mathcal{R}_1} \bar{C}_j^b\langle y_{i_1}, \ldots, y_{i_l}\rangle$ for some black context $\bar{C}_j^b\langle, \ldots,\rangle$ as well as $s_j\downarrow = \bar{C}_j^b\langle\!\langle u_{i_1}', \ldots, u_{i_l}'\rangle\!\rangle$. Every $s_i\downarrow$, $i \neq j$, has a representation $s_i\downarrow = \bar{C}_i^b\langle\!\langle u_1^i, \ldots, u_{m_i}^i\rangle\!\rangle$. We obtain the following reduction sequence.

$$\tilde{u}_2 = C_1^b\{\bar{C}_1^b\langle\!\langle u_1^1, \ldots, u_{m_1}^1\rangle\!\rangle, \ldots, \tilde{C}_j^b\{\!\{u_1', \ldots, u_q'\}\!\}, \ldots, \bar{C}_n^b\langle\!\langle u_1^n, \ldots, u_{m_n}^n\rangle\!\rangle\} \to^*_{\mathcal{R}_1}$$

$$C_1^b\{\bar{C}_1^b\langle\!\langle u_1^1, \ldots, u_{m_1}^1\rangle\!\rangle, \ldots, \bar{C}_j^b\langle\!\langle u_{i_1}', \ldots, u_{i_l}'\rangle\!\rangle, \ldots, \bar{C}_n^b\langle\!\langle u_1^n, \ldots, u_{m_n}^n\rangle\!\rangle = \tilde{u}_1.$$

Choose fresh variables $x_1^1, \ldots, x_{m_{j-1}}^{j-1}, y_1, \ldots, y_q, x_1^{j+1}, \ldots, x_{m_n}^n$ satisfying

$u_1^1, \ldots, u_{m_{j-1}}^{j-1}, u_1', \ldots, u_q', u_1^{j+1}, \ldots, u_{m_n}^n \propto x_1^1, \ldots, x_{m_{j-1}}^{j-1}, y_1, \ldots, y_q, x_1^{j+1}, \ldots, x_{m_n}^n$.

Note that this implies

$u_1^1, \ldots, u_{m_{j-1}}^{j-1}, u_{i_1}', \ldots, u_{i_l}', u_1^{j+1}, \ldots, u_{m_n}^n \propto x_1^1, \ldots, x_{m_{j-1}}^{j-1}, y_{i_1}, \ldots, y_{i_l}, x_1^{j+1}, \ldots, x_{m_n}^n$.

We derive from $\tilde{C}_j^b\{y_1, \ldots, y_q\} \to_{\mathcal{R}_1}^* \bar{C}_j^b\langle y_{i_1}, \ldots, y_{i_l}\rangle$ that

$$C_1^b\{\bar{C}_1^b\langle x_1^1, \ldots, x_{m_1}^1\rangle, \ldots, \tilde{C}_j^b\{y_1, \ldots, y_q\}, \ldots, \bar{C}_n^b\langle x_1^n, \ldots, x_{m_n}^n\rangle\} \to_{\mathcal{R}_1}^*$$

$$C_1^b\{\bar{C}_1^b\langle x_1^1, \ldots, x_{m_1}^1\rangle, \ldots, \bar{C}_j^b\langle y_{i_1}, \ldots, y_{i_l}\rangle, \ldots, \bar{C}_n^b\langle x_1^n, \ldots, x_{m_n}^n\rangle\}.$$

Since both terms are joinable, they reduce to the same unique $(\mathcal{F}_1, \mathcal{R}_1)$ normal from $\hat{C}^b\langle z_1, \ldots, z_r\rangle$ where $z_1, \ldots, z_r \in \{x_1^1, \ldots, x_{m_{j-1}}^{j-1}, y_1, \ldots, y_q, x_1^{j+1}, \ldots, x_{m_n}^n\}$. It follows as above that $\hat{u}_1 = \hat{u}_2$.

All in all, $\hat{t} = \hat{t_1} = \hat{t_2}$ is the unique normal form of t w.r.t. $(\mathcal{F}, \mathcal{R})$.

Case (ii): t is top white. The assertion follows from similar arguments as in case (i).

Case (iii): t is top transparent, i.e. $t = C^t[\![s_1, \ldots, s_n]\!]$. We proceed in a similar way as above. Let u be any term in the conversion $t_1 \,{}^*\!\leftarrow t \to^* t_2$. With u we associate terms $\tilde{u}$ and $\hat{u}$ which are defined as follows. If $rank(u) < k$, then u has a unique normal form $u\!\downarrow$ according to the induction hypothesis and we set $\tilde{u} = \hat{u} = u\!\downarrow$. If $rank(u) = k$ and u is top black or top white, then u has a unique normal form $u\!\downarrow$ according to cases (i) and (ii). Again, we set $\tilde{u} = \hat{u} = u\!\downarrow$. If $rank(u) = k$ and u is top transparent, then it can be written as $u = C^t[\![s_1, \ldots, s_n]\!]$. It follows from the foregoing that every s_j, $j \in \{1, \ldots, n\}$, has a unique normal form $s_j\!\downarrow$. The result of replacing each s_j with its unique normal form is denoted by $\tilde{u}$, i.e. $\tilde{u} = C^t[s_1\!\downarrow, \ldots, s_n\!\downarrow]$. Note that $u \to^* \tilde{u}$. Moreover, $\tilde{u}$ has a representation $\tilde{u} = \tilde{C}^t\{\!\{u_1, \ldots, u_m\}\!\}$ in which the u_i, $i \in \{1, \ldots, m\}$, are top black or top white normal forms w.r.t. $(\mathcal{F}, \mathcal{R})$. Choose variables $x_1, \ldots, x_m$ not occurring in $\tilde{u}$ satisfying $u_1, \ldots, u_m \propto x_1, \ldots, x_m$. Since $\tilde{C}^t\{x_1, \ldots, x_m\} \in \mathcal{T}(\mathcal{B}, \mathcal{V})$ and the TRS $(\mathcal{B}, \mathcal{S})$ is semi-complete, it follows that $\tilde{C}^t\{x_1, \ldots, x_m\}$ rewrites to its unique $(\mathcal{B}, \mathcal{S})$ normal form $\hat{C}^t\langle x_{i_1}, \ldots, x_{i_l}\rangle \in NF(\mathcal{B}, \mathcal{S}) = NF(\mathcal{F}, \mathcal{R}) \cap \mathcal{T}(\mathcal{B}, \mathcal{V})$. We set $\hat{u} = \hat{C}^t\langle u_{i_1}, \ldots, u_{i_l}\rangle$. It is easy to verify that $\hat{u} \in NF(\mathcal{F}, \mathcal{R})$. Observe that $\tilde{u} \to^* \hat{u}$ and hence $u \to^* \hat{u}$.

Let $u_1 \to u_2$ be a step in the conversion $t_1 \,{}^*\!\leftarrow t \to^* t_2$. Again, we show that $\hat{u_1} = \hat{u_2}$. If $rank(u_1) < k$, then $rank(u_2) < k$ as well. Hence $\hat{u}_1 = u_1\!\downarrow = u_2\!\downarrow = \hat{u}_2$. If $rank(u_1) = k$ and u_1 is top black or top white, then $\hat{u_1} = u_1\!\downarrow$ is the unique normal form of u_1 w.r.t. $(\mathcal{F}, \mathcal{R})$. Since $u_1 \to u_2 \to^* \hat{u_2} \in NF(\mathcal{F}, \mathcal{R})$, it follows $\hat{u_1} = \hat{u_2}$. If $rank(u_1) = k$ and u_1 is top transparent, then $u_1 = C_1^t[\![s_1, \ldots, s_n]\!]$. Consider the following subcases.

(a) If $u_1 \to^t u_2$, then the assertion follows as in case (i) (a).
(b) If $u_1 \to^o u_2$, then the assertion follows as in case (i) (b).
(c) If $u_1 \to^i u_2$, then $u_1 = C^t[\![s_1, \ldots, s_j, \ldots, s_n]\!]$ and $u_2 = C^t[\![s_1, \ldots, t_j, \ldots, s_n]\!]$, where $s_j \to^i t_j$. Hence $\hat{u}_1 = u_1\!\downarrow = u_2\!\downarrow = \hat{u}_2$.

Again, $\hat{t} = \hat{t_1} = \hat{t_2}$ is the unique normal form of t w.r.t. $\mathcal{R}$. This concludes the proof. □

It has already been mentioned that given a semi-complete TRS $(\mathcal{F}, \mathcal{R})$, we also have to solve the problem how to find the unique normal form of a term t. If $(\mathcal{F}, \mathcal{R})$ is finitely branching, then the normal form of t can always be obtained by traversing the reduction graph of t breadth first. This is well-known from logic programming. For efficiency reasons, however, the searching strategy of almost all Prolog implementations is depth first. Fortunately, in the aforementioned very important special case, we can use an innermost reduction strategy. A reduction step $s \to_{\mathcal{R}} t$ is *innermost* if no proper subterm of the contracted redex is itself a redex. An *innermost reduction sequence* consists only of innermost reduction steps. The TRS $(\mathcal{F}, \mathcal{R})$ is *innermost normalizing* if, for every term s, there is an innermost reduction sequence $s \to_{\mathcal{R}}^* t$ so that $t \in NF(\to_{\mathcal{R}})$. It is *innermost terminating* if there is no infinite innermost reduction sequence. The notions are related as follows: termination $\Rightarrow$ innermost termination $\Rightarrow$ innermost normalization $\Rightarrow$ normalization.

PROPOSITION 5.3. The following properties are modular for composable TRSs:

(1) Local confluence.
(2) Normalization.
(3) Innermost normalization.
(4) Innermost termination.

PROOF. (1) In essence, this follows from the Critical Pair Lemma since the set of all critical pairs of $\mathcal{R}$ coincides with the union of the sets of all critical pairs of $\mathcal{R}_1$ and $\mathcal{R}_2$ (cf. Middeldorp, 1990, and Ohlebusch, 1994*b*).

(2) Every term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ can be rewritten to normal form reducing layer by layer in a bottom up fashion. That is, first the bottom layer of t is reduced to normal form, then the same is done with the layer above the bottom layer and so on. Eventually the top layer is reduced to normal form; the term obtained is a normal form of t (cf. Ohlebusch, 1994*b*, and Middeldorp, 1990). Note that even if the top layer is transparent, it can be normalized by $(\mathcal{B}, \mathcal{S})$ according to Lemma 5.1.

(3) Analogous to (2).

(4) It is not too difficult to prove this by structural induction (see Ohlebusch, 1994*b*; cf. also Gramlich, 1994*b*, Krishna Rao, 1993).

□

COROLLARY 5.4. The combined system $\mathcal{R}$ of two complete composable TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ is semi-complete and innermost terminating.

5.2. TERMINATION

As far as termination is concerned, the first modularity results were obtained by investigating the distribution of collapsing and duplicating rules among the TRSs. Rusinowitch (1987) showed that termination is modular for non-collapsing and non-duplicating disjoint TRSs, respectively. Furthermore, Middeldorp (1989) proved that termination is preserved under disjoint union if one of the systems contains neither collapsing nor duplicating rules. A simple proof for all *three* results can be found in Ohlebusch (1993). These results extend, *mutatis mutandis*, to constructor-sharing TRSs, see Ohlebusch (1995*a*)[†].

[†] A similar proof sketch was given independently in Dershowitz (1994).

The basic underlying idea of Ohlebusch (1993, 1995*a*) can be used to establish the next result, namely the generalization of the above-mentioned results to composable TRSs. First, we need a few prerequisites.

DEFINITION 5.5. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be composable TRSs. Let $j \in \{1,2\}$. The system $(\mathcal{F}_j, \mathcal{R}_j)$ is called *layer-preserving*, if for all $l \to r \in \mathcal{R}_j$ we have $root(r) \in \mathcal{A}_j$ whenever $root(l) \in \mathcal{A}_j$.

Disjoint TRSs are layer-preserving if and only if they are non-collapsing. Constructor-sharing TRSs are layer-preserving if and only if they contain neither collapsing nor constructor-lifting rules (constructor-lifting rules are rewrite rules in which the right-hand side has a shared constructor at its root position).

LEMMA 5.6. Let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $s \to^t t$ is a non-duplicating reduction step. Then $S_1(t) \subseteq S_1(s)$. In particular, $S_P^w(t) \subseteq S_P^w(s)$ and $S_P^b(t) \subseteq S_P^b(s)$.

PROOF. According to Lemma 4.14, $s = C^t[\![s_1, \ldots, s_n]\!] \to^t t = C^t\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$ by some rule $l \to r \in \mathcal{S}$. Let $l = C[x_1, \ldots, x_k]$ with all variables displayed. Then r has a representation $r = C'[x_{i_1}, \ldots, x_{i_j}]$. The multiset inclusion $[x_{i_1}, \ldots, x_{i_j}] \subseteq [x_1, \ldots, x_k]$ holds because the rule is non-duplicating. It is easy to verify that this implies $S_1(t) \subseteq S_1(s)$. Consequently, we also have $S_1^w(t) \subseteq S_1^w(s)$ and $S_1^b(t) \subseteq S_1^b(s)$. Now we infer $S_2^w(t) \subseteq S_2^w(s)$ from $S_1^b(t) \subseteq S_1^b(s)$. All in all, it follows $S_P^w(t) = S_1^w(t) \cup S_2^w(t) \subseteq S_1^w(s) \cup S_2^w(s) = S_P^w(s)$. The remaining inclusion $S_P^b(t) \subseteq S_P^b(s)$ is proved analogously. □

LEMMA 5.7. Let $s \to_{\mathcal{A}_1}^{t,o} t$ be a non-duplicating reduction step. Then $S_P^w(t) \subseteq S_P^w(s)$.

PROOF. Similar to the proof of Lemma 5.6. □

PROPOSITION 5.8. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two terminating composable TRSs such that their combined system $\mathcal{R}$ does not terminate. Then either statement (i) holds or, if (i) does not hold, then statement (ii) must hold.

(i) There is an infinite $\mathcal{R}$ derivation D starting from a non-top-transparent, say top black, term such that:

(1) There is no top white term in D.
(2) There are infinitely many $\to_{\mathcal{A}_1}^{t,o}$ reduction steps in D.
(3) There are infinitely many $\to_{\mathcal{R}_2}$ reduction steps in D which are destructive at level 1 or level 2.
(4) There are infinitely many duplicating $\to_{\mathcal{A}_1}^{t,o}$ reduction steps in D.

(ii) There is an infinite $\mathcal{R}$ derivation D such that:

(1) D consists solely of top transparent terms.
(2) There are infinitely many $\to^t$ reduction steps in D.
(3) There are infinitely many $\to^o$ reduction steps in D destructive at level 1.
(4) There are infinitely many duplicating $\to^t$ reduction steps in D.

PROOF. (i) Suppose there is an infinite $\mathcal{R}$ derivation D starting from a top black term. Let the rank of a derivation $D : s_1 \to s_2 \to s_3 \to \ldots$ be defined by $rank(D) = rank(s_1)$. W.l.o.g., we may assume that D is of minimal rank. In other words, if $rank(D) = k$, then $\to_{\mathcal{R}}$ is terminating on $\mathcal{T}^{<k} = \{t \in \mathcal{T}(\mathcal{F},\mathcal{V}) \mid rank(t) < k\}$. Clearly, this implies that every white principal subterm occurring in D has a rank less than k.

(1) If there was a top white term in D, then there would be an infinite rewrite derivation starting from a term with a rank less than k – a contradiction.
(2) Suppose that there are only finitely many $\to_{\mathcal{A}_1}^{t,o}$ reduction steps in D. W.l.o.g. we may assume that D contains no $\to_{\mathcal{A}_1}^{t,o}$ reduction steps at all. Thus, if $s_1 = C^b[\![t_1,\ldots,t_n]\!]$, then there must be an infinite $\mathcal{R}$ derivation starting from some $t_l \in S_{\mathcal{P}}^w(s_1)$. But this contradicts the minimality assumption on $rank(D)$ since $rank(t_l) < rank(s_1)$.
(3) As above we suppose that there is no $\to_{\mathcal{R}_2}$ reduction step in D which is destructive at level 1 or level 2. In this case we have for any reduction step $s_j \to_{\mathcal{A}_1}^{t,o} s_{j+1}$ in D that $top^b(s_j) \to_{\mathcal{R}_1} top^b(s_{j+1})$ using the same rule from $\mathcal{R}_1$ and for every reduction step $s_j \to_{\mathcal{A}_2}^{o} s_{j+1}$ or $s_j \to^{i} s_{j+1}$, we have $top^b(s_j) = top^b(s_{j+1})$. Hence we conclude by (2) that $\mathcal{R}_1$ is non-terminating – a contradiction.
(4) Let $> = (\to_{\mathcal{R}} \cup \rhd)^+$. Since $\to_{\mathcal{R}}$ is closed under contexts, it is not too difficult to prove that $(\mathcal{T}^{<k}, >)$ is a well-founded ordering. Let $(\mathcal{M}(\mathcal{T}^{<k}), >^{mul})$ denote its well-founded multiset extension. Note that every multiset $S_{\mathcal{P}}^w(s_j)$ is an element of $\mathcal{M}(\mathcal{T}^{<k})$. As above, we may suppose that there is no duplicating $\to_{\mathcal{A}_1}^{t,o}$ reduction step in D. We distinguish between two cases:

    If $s_j \to_{\mathcal{A}_1}^{t,o} s_{j+1}$, then by Lemma 5.7 $S_{\mathcal{P}}^w(s_{j+1}) \subseteq S_{\mathcal{P}}^w(s_j)$ because the reduction step is non-duplicating. Clearly, this implies $S_{\mathcal{P}}^w(s_j) \geq^{mul} S_{\mathcal{P}}^w(s_{j+1})$.

    If $s_j \to_{\mathcal{A}_2}^{o} s_{j+1}$ or $s_j \to^{i} s_{j+1}$, then there exists a white principal subterm $u \in S_{\mathcal{P}}^w(s_j)$ such that $u \to v$ for some v, i.e. $s_j = C^b[\![,\ldots,u,\ldots,]\!] \to C^b[,\ldots,v,\ldots,] = s_{j+1}$. Thus we have $S_{\mathcal{P}}^w(s_{j+1}) = (S_{\mathcal{P}}^w(s_j)\setminus[u]) \cup S_{\mathcal{P}}^w(v)$. It follows from $u \to v$ in conjunction with $v = w$ or $v \rhd w$ for any principal subterm $w \in S_{\mathcal{P}}^w(v)$ that $u > w$ for any $w \in S_{\mathcal{P}}^w(v)$. Therefore $S_{\mathcal{P}}^w(s_j) >^{mul} S_{\mathcal{P}}^w(s_{j+1})$.

    We conclude from the well-foundedness of $(\mathcal{M}(\mathcal{T}^{<k}), >^{mul})$ that only finitely many $\to_{\mathcal{A}_2}^{o}$ and $\to^{i}$ steps can occur in the derivation D. This contradicts (3).

(ii) Suppose that there are no infinite rewrite derivations starting from top black or top white terms. Let D be an infinite $\mathcal{R}$ derivation. Clearly, this implies that every black or white principal subterm occurring in D is terminating. W.l.o.g., we may assume that D is of minimal rank k.

(1) If there was a top black or top white term in D, then there would be an infinite $\mathcal{R}$ derivation starting from a top black or top white term – a contradiction.
(2) Suppose that there are only finitely many $\to^{t}$ reduction steps in D. W.l.o.g. we may assume that D contains no $\to^{t}$ reduction steps at all. Thus, if $s_1 = C^t[\![t_1,\ldots,t_n]\!]$, then there must be an infinite rewrite derivation starting from some top black or top white term $t_l \in S_1(s_1)$ which is a contradiction to (1).
(3) Suppose that there is no $\to^{o}$ reduction step in D which is destructive at level 1. In

this case we have for any reduction step $s_j \to^t s_{j+1}$ in D that $top^t(s_j) \to^t top^t(s_{j+1})$ using the same rule from $\mathcal{S}$ and for every $s_j \to^o s_{j+1}$ or $s_j \to^i s_{j+1}$ step we have $top^t(s_j) = top^t(s_{j+1})$. Hence we conclude that $\mathcal{S}$ is non-terminating which contradicts the termination of $\mathcal{R}_1$.

(4) Let $T = \{t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid rank(t) = k$ and $root(t) \in \mathcal{A}_1 \uplus \mathcal{A}_2$ or $rank(t) < k\}$. It follows from our assumptions that $\to_\mathcal{R}$ is terminating on T. Let $> = (\to_\mathcal{R} \cup \rhd)^+$. Again, $(T, >)$ is a well-founded ordering. Let $(\mathcal{M}(T), >^{mul})$ denote its well-founded multiset extension. Note that $S_1(s_j) \in \mathcal{M}(T)$. Again, we may suppose that there is no duplicating $\to^t$ reduction step in D at all. We distinguish between two cases:

If $s_j \to^t s_{j+1}$, then by Lemma 5.6 $S_1(s_{j+1}) \subseteq S_1(s_j)$ because the reduction step is non-duplicating. Clearly, this implies $S_1(s_j) \geq^{mul} S_1(s_{j+1})$.

If $s_j \to^o s_{j+1}$ or $s_j \to^i s_{j+1}$, then there is a black or white principal subterm $u \in S_1(s_j)$ such that $u \to v$ for some v, i.e. $s_j = C^t[\![, \ldots, u, \ldots,]\!] \to C^t[, \ldots, v, \ldots,] = s_{j+1}$. Thus we have $S_1(s_{j+1}) = (S_1(s_j) \setminus [u]) \cup S_1(v)$. It follows from $u \to v$ in conjunction with $v = w$ or $v \rhd w$ for any term $w \in S_1(v)$ that $u > w$ for any $w \in S_1(v)$. Therefore $S_1(s_j) >^{mul} S_1(s_{j+1})$.

We conclude from the well-foundedness of $(\mathcal{M}(T), >^{mul})$ that only finitely many $\to^o$ and $\to^i$ steps can occur in the derivation D. This contradicts (3).

□

The following example illustrates that case (ii) of Proposition 5.8 may occur.

EXAMPLE 5.9. Let $\mathcal{R}_1 = \{F(x, C_1, C_2, y, D_1, D_2) \to F(x, x, x, y, y, y), A \to C_1, A \to C_2\}$ and $\mathcal{R}_2 = \{F(x, C_1, C_2, y, D_1, D_2) \to F(x, x, x, y, y, y), b \to D_1, b \to D_2\}$. We have the cyclic derivation

$$t = F(A, C_1, C_2, b, D_1, D_2) \to^t F(A, A, A, b, b, b) \xrightarrow{o}^*_{\mathcal{A}_1} F(A, C_1, C_2, b, b, b) \xrightarrow{o}^*_{\mathcal{A}_2} t.$$

THEOREM 5.10. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two terminating composable TRSs. Their combined system $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is terminating provided that one of the following conditions is satisfied:

(1) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are layer-preserving.
(2) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-duplicating.
(3) One of the systems is both layer-preserving and non-duplicating.

PROOF. (1) If both systems are layer-preserving, then there can be no rewrite step which is destructive at level 1 or level 2; so neither case (i) (3) nor case (ii) (3) is possible.

(2) If both systems are non-duplicating, then neither case (i) (4) nor case (ii) (4) is possible.

(3) Let $\mathcal{R}_1$ be layer-preserving and non-duplicating. The existence of an infinite derivation starting from a top black term is ruled out by (i) (4). Also, no infinite derivation starting from a top white term is possible because of (the adjusted version of) case (i) (3). Thus, if $\mathcal{R}$ were not terminating, then there would be an infinite derivation

starting from a top transparent term. However, this possibility is excluded by (ii) (4).
□

An equivalent formulation of Theorem 5.10 reads as follows: If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two terminating composable TRSs such that their combined system $\mathcal{R}_1 \cup \mathcal{R}_2$ is non-terminating, then $\mathcal{R}_1$ is duplicating and $\mathcal{R}_2$ is not layer-preserving or vice versa.

5.3. COMPLETENESS

The next theorem is due to Gramlich (1994*b*). His original proof was, however, rather complicated. In the meantime, several authors have independently given simpler proofs for this theorem (which resemble one another), see Dershowitz and Hoot (1994), Middeldorp (1994*b*) and Ohlebusch (1994*b*). Thereby, a TRS $\mathcal{R}$ is called an *overlay system* if every critical pair between rules of $\mathcal{R}$ is obtained by overlapping left-hand sides of rules at root positions.

THEOREM 5.11. An overlay system is complete if and only if it is locally confluent and innermost terminating.

COROLLARY 5.12. Completeness is modular for composable overlay systems.

PROOF. Immediate consequence of Theorem 5.11 and Proposition 5.3. □

The main result of Middeldorp and Toyama (1993) stating that completeness is a modular property of composable constructor systems follows from Corollary 5.12 because a constructor system is an overlay system and the combined system of two composable constructor systems is again a constructor system. Also, modularity of termination (or equivalently completeness) for non-overlapping TRSs is a consequence of Corollary 5.12 because those systems are locally confluent overlay systems and the combined system of two non-overlapping composable systems is non-overlapping. The same is true for orthogonal systems which are (left-linear and) non-overlapping.

5.4. THE SIMPLIFYING PROPERTY

A TRS is called *simplifying* if its rewrite relation is contained in some simplification ordering. This property is important because every finite simplifying TRS is terminating (cf. Dershowitz, 1982) and virtually all termination proofs are based on this fact. Kurihara and Ohuchi (1992) have proved that the simplifying property is modular for constructor-sharing TRSs (please note that they used the phrase "simply terminating" instead of "simplifying"). We will next generalize their result to composable systems by combining the techniques of Kurihara and Ohuchi (1992) and Gramlich (1994*a*)†. Again we need some preparatory lemmata.

† This generalization has been claimed independently by Krishna Rao (1994), it is stated there without proof.

DEFINITION 5.13. Let $\mathcal{F}$ be a signature. The TRS $\mathcal{F}^{arg}$ consists of all rewite rules

$$f(x_1, \ldots, x_n) \to x_j,$$

where $f \in \mathcal{F}$ is a function symbol of arity $n \geq 1$ and $j \in \{1, \ldots, n\}$.

LEMMA 5.14. A TRS $(\mathcal{F}, \mathcal{R})$ is simplifying if and only if $\to^+_{\mathcal{R} \cup \mathcal{F}^{arg}}$ is irreflexive.

PROOF. See Kurihara and Ohuchi (1992). □

LEMMA 5.15. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be composable TRSs. If one of them is simplifying, then $(\mathcal{B}, \mathcal{S})$ is simplifying.

PROOF. Let $>$ be a simplification ordering on $\mathcal{T}(\mathcal{F}_j, \mathcal{V})$, $j \in \{1,2\}$, such that $\to_{\mathcal{R}_j} \subseteq >$. The restriction $>|_{\mathcal{T}(\mathcal{B},\mathcal{V})}$ of $>$ to $\mathcal{T}(\mathcal{B}, \mathcal{V})$ is a simplification ordering and the inclusion $\to_{\mathcal{S}}|_{\mathcal{T}(\mathcal{B},\mathcal{V})} \subseteq >|_{\mathcal{T}(\mathcal{B},\mathcal{V})}$ holds. This means that $(\mathcal{B}, \mathcal{S})$ is simplifying. □

THEOREM 5.16. The simplifying property is a modular property of composable TRSs.

PROOF. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be composable TRSs and let $(\mathcal{F}, \mathcal{R})$ be their combined system. It has to be proved that $(\mathcal{F}, \mathcal{R})$ is simplifying if and only if $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are simplifying.

"only-if": Let $(\mathcal{T}(\mathcal{F}, \mathcal{V}), >)$ be a simplification ordering with $\to_{\mathcal{R}} \subseteq >$. It is not too difficult to prove that $(\mathcal{T}(\mathcal{F}_j, \mathcal{V}), >|_{\mathcal{T}(\mathcal{F}_j,\mathcal{V})})$ is a simplification ordering and that furthermore $\to_{\mathcal{R}_j}|_{\mathcal{T}(\mathcal{F}_j,\mathcal{V})} \subseteq >|_{\mathcal{T}(\mathcal{F}_j,\mathcal{V})}$. In other words, $(\mathcal{F}_j, \mathcal{R}_j)$ is simplifying.

"if": First of all, note that $\mathcal{R}_1 \cup \mathcal{F}_1^{arg}$ and $\mathcal{R}_2 \cup \mathcal{F}_2^{arg}$ are composable systems. According to Lemma 5.14, it must be shown that $\to^+_{\mathcal{R} \cup \mathcal{F}^{arg}}$ is irreflexive. Assuming that $\to^+_{\mathcal{R} \cup \mathcal{F}^{arg}}$ is not irreflexive, we will derive a contradiction. So suppose that there is a cyclic derivation

$$D: \quad t = t_1 \to_{\mathcal{R} \cup \mathcal{F}^{arg}} \ldots \to_{\mathcal{R} \cup \mathcal{F}^{arg}} t_n = t,$$

$n > 1$, of terms $t_j \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $j \in \{1, \ldots, n\}$. W.l.o.g. we may assume that z is the only variable occurring in D. We may further assume that $rank(t) = k$ is minimal, i.e., there is no cyclic derivation $s = s_1 \to_{\mathcal{R} \cup \mathcal{F}^{arg}} \ldots \to_{\mathcal{R} \cup \mathcal{F}^{arg}} s_m = s$, $m > 1$, with $\text{rank}(s) < k$. Consequently, $\to^+_{\mathcal{R} \cup \mathcal{F}^{arg}}$ is irreflexive on $\mathcal{T}^{<k}$. Note that $k > 1$ by Lemma 5.15.

Case (i): t is top black. Obviously, every term in D must have rank k. Therefore, each term in D is either top black or top transparent. Let

$$\mathcal{T}_D = \{s \in \mathcal{T}(\mathcal{F}, \{z\}) \mid s \text{ is a subterm of a term occurring in } D\}.$$

Note that $\mathcal{T}_D$ is finite. Let $Cons$ be a new binary function symbol not occurring in $\mathcal{F}$ and let $\mathcal{C}_\mathcal{E} = \{Cons(x,y) \to x,\ Cons(x,y) \to y\}$. The proof idea is to define a transformation function $\Phi_b^D : \mathcal{T}_D \to \mathcal{T}(\mathcal{F}_1 \uplus \{Cons\}, \{z\})$ such that

$$\Phi_b^D(D): \quad \Phi_b^D(t) = \Phi_b^D(t_1) \to^*_{(\mathcal{R}_1 \cup \mathcal{F}_1^{arg}) \uplus \mathcal{C}_\mathcal{E}} \cdots \to^*_{(\mathcal{R}_1 \cup \mathcal{F}_1^{arg}) \uplus \mathcal{C}_\mathcal{E}} \Phi_b^D(t_n) = \Phi_b^D(t)$$

is a non-empty cyclic derivation of terms from $\mathcal{T}(\mathcal{F}_1 \uplus \{Cons\}, \{z\})$. This contradicts the irreflexivity of $\to^+_{\mathcal{R}_1 \cup \mathcal{F}_1^{arg}}$ because one can prove that $\to^+_{(\mathcal{R}_1 \cup \mathcal{F}_1^{arg}) \uplus \mathcal{C}_\mathcal{E}}$ is irreflexive on $\mathcal{T}(\mathcal{F}_1 \uplus \{Cons\}, \{z\})$ if and only if $\rightarrow^+_{\mathcal{R}_1 \cup \mathcal{F}_1^{arg}}$ is irreflexive on $\mathcal{T}(\mathcal{F}_1, \{z\})$. In order to define Φ_b^D we need the following definitions. The *inner subterm occurrences of* D are those terms which are subterms of a white principal subterm occurring in D. The others are called *outer subterm occurrences of* D. Let O_b^D denote the set of all outer subterm occurrences of D. Furthermore, let $S_P^w(D)$ denote the set of all white principal subterms appearing in D. Observe that both sets are finite and that every element of $S_P^w(D)$ has a rank less than k. Moreover, for $s \in S_P^w(D)$, we define $\Delta_b^D(s) = \{u \in O_b^D \mid s \rightarrow^+_{\mathcal{R} \cup \mathcal{F}^{arg}} u\}$. It is important to notice that $\Delta_b^D(s)$ is finite for any $s \in S_P^w(D)$. Let $\succ$ be a total ordering on $\mathcal{T}(\mathcal{F} \uplus \{Cons\}, \{z\})$. Let

$$\Phi_b^D(s) = \begin{cases} C^b\{\Phi_b^D(s_1), \ldots, \Phi_b^D(s_m)\} & \text{if } s = C^b\{\!\{s_1, \ldots, s_m\}\!\} \\ \\ Sort(\{\Phi_b^D(u) \mid u \in \Delta_b^D(s)\}) & \text{if } root(s) \in \mathcal{A}_2, \end{cases}$$

where $Sort(\{t_1, \ldots, t_n\}) = \langle t_{\pi(1)}, \ldots, t_{\pi(n)} \rangle$ such that $t_{\pi(j)} \succ t_{\pi(j+1)}$ for $1 \le j < n$. Here $\langle t_{\pi(1)}, \ldots, t_{\pi(n)} \rangle$ stands for the term $Cons(t_{\pi(1)}, Cons(t_{\pi(2)}, \ldots, Cons(t_{\pi(n)}, z) \ldots))$. Note that $Sort(\{\Phi_b^D(u) \mid u \in \Delta_b^D(s)\}) = z$ if $root(s) \in \mathcal{A}_2$ and $\Delta_b^D(s) = \emptyset$. It is easy to verify that the transformation function Φ_b^D is well-defined.

We show next that $t_j \rightarrow_{\mathcal{R} \cup \mathcal{F}^{arg}} t_{j+1}$ implies $\Phi_b^D(t_j) \rightarrow^*_{(\mathcal{R}_1 \cup \mathcal{F}_1^{arg}) \uplus \mathcal{C}_\mathcal{E}} \Phi_b^D(t_{j+1})$, using $\rightarrow$ as a shorthand for $\rightarrow_{\mathcal{R} \cup \mathcal{F}^{arg}} = \rightarrow_{(\mathcal{R}_1 \cup \mathcal{F}_1^{arg})} \cup \rightarrow_{(\mathcal{R}_2 \cup \mathcal{F}_2^{arg})}$. There are the following cases.

(a) If $t_j \rightarrow^{t,o}_{\mathcal{A}_1} t_{j+1}$ by some rewrite rule $l \rightarrow r$, then we have $l \rightarrow r \in \mathcal{R}_1 \cup \mathcal{F}_1^{arg}$, $t_j = C^b[\![s_1, \ldots, s_m]\!]$, and $t_{j+1} = \hat{C}^b\langle\!\langle s_{i_1}, \ldots, s_{i_l}\rangle\!\rangle$. Applying Φ_b^D, we obtain $\Phi_b^D(t_j) = C^b[\Phi_b^D(s_1), \ldots, \Phi_b^D(s_m)]$ and $\Phi_b^D(t_{j+1}) = \hat{C}^b\langle\Phi_b^D(s_{i_1}), \ldots, \Phi_b^D(s_{i_l})\rangle$. It follows from $s_1, \ldots, s_m \propto \Phi_b^D(s_1), \ldots, \Phi_b^D(s_m)$ that the rule $l \rightarrow r$ that reduced t_j to t_{j+1} also reduces $\Phi_b^D(t_j)$ to $\Phi_b^D(t_{j+1})$ (cf. Lemma 4.14).

(b) If $t_j \rightarrow^o_{\mathcal{A}_2} t_{j+1}$ or $t_j \rightarrow^i t_{j+1}$, then we have $t_j = C^b[\![s_1, \ldots, s_l, \ldots, s_m]\!]$ as well as $t_{j+1} = C^b[s_1, \ldots, s_l', \ldots, s_m]$ for some $l \in \{1, \ldots, m\}$ and some term s_l', where $s_l \rightarrow s_l'$. Clearly, $\Phi_b^D(t_j) = C^b[\Phi_b^D(s_1), \ldots, \Phi_b^D(s_m)]$. We consider the following subcases.

(b1) If $root(s_l') \in \mathcal{A}_2$, then t_{j+1} has a representation $t_{j+1} = C^b[\![s_1, \ldots, s_l', \ldots, s_m]\!]$ and $\Phi_b^D(t_{j+1}) = C^b[\Phi_b^D(s_1), \ldots, \Phi_b^D(s_l'), \ldots \Phi_b^D(s_m)]$. Therefore, it is sufficent to show $\Phi_b^D(s_l) \rightarrow^*_{(\mathcal{R}_1 \cup \mathcal{F}_1^{arg}) \uplus \mathcal{C}_\mathcal{E}} \Phi_b^D(s_l')$. Now it follows from $\Delta_b^D(s_l') \subseteq \Delta_b^D(s_l)$ that

$$\begin{aligned} \Phi_b^D(s_l) &= Sort(\{\Phi_b^D(u) \mid u \in \Delta_b^D(s_l)\}) \\ &= \langle u_1, \ldots, u_n \rangle \\ &\rightarrow^*_{\mathcal{C}_\mathcal{E}} \langle u_{i_1}, \ldots, u_{i_p} \rangle \\ &= Sort(\{\Phi_b^D(u) \mid u \in \Delta_b^D(s_l')\}) \\ &= \Phi_b^D(s_l'), \end{aligned}$$

where $u_{i_1}, \ldots, u_{i_p}$ is a subsequence of $u_1, \ldots, u_n$.

(b2) If $root(s_l') \notin \mathcal{A}_2$, then $s_l' = \bar{C}^b\{\!\{u_1, \ldots, u_p\}\!\}$ and we have $\Phi_b^D(t_{j+1}) = \hat{C}^b\{\Phi_b^D(s_1), \ldots, \Phi_b^D(s_{l-1}), \Phi_b^D(u_1), \ldots, \Phi_b^D(u_p), \Phi_b^D(s_{l+1}), \ldots, \Phi_b^D(s_m)\}$, where $\hat{C}^b\{, \ldots, \} = C^b[, \ldots, \bar{C}^b\{, \ldots\}, \ldots,]$. Now it is a consequence of $s_l' \in \Delta_b^D(s_l)$

that $\Phi_b^D(s_l')$ occurs in the term $\Phi_b^D(s_l)$ and hence

$$\begin{aligned}\Phi_b^D(s_l) &= Sort(\{\Phi_b^D(u) \mid u \in \Delta_b^D(s_l)\}) \\ &= \langle \dots, \Phi_b^D(s_l'), \dots \rangle \\ &\to_{\mathcal{C}_\mathcal{E}}^+ \Phi_b^D(s_l') \\ &= C^b\{\Phi_b^D(u_1), \dots, \Phi_b^D(u_p)\}.\end{aligned}$$

All in all, $\Phi_b^D(t_j) \to_{\mathcal{C}_\mathcal{E}}^* \Phi_b^D(t_{j+1})$ which concludes case (b).

Since there must be at least one $\to^t$ or $\to_{\mathcal{A}_1}^o$ step in D, we obtain a non-empty cyclic derivation $\Phi_b^D(D)$ of terms from $\mathcal{T}(\mathcal{F}_1 \uplus \{Cons\}, \{z\})$. This is a contradiction to the fact that $\mathcal{R}_1$ is simplifying, so case (i) is proved.

Case (ii): t is top white. Here the above proof applies with appropriate notational changes.

Case (iii): t is top transparent. If one of the terms in

$$D: \quad t = t_1 \to_{\mathcal{R} \cup \mathcal{F}^{arg}} \dots \to_{\mathcal{R} \cup \mathcal{F}^{arg}} t_j \to_{\mathcal{R} \cup \mathcal{F}^{arg}} \dots \to_{\mathcal{R} \cup \mathcal{F}^{arg}} t_n = t,$$

is top black, say t_j, then there exists a non-empty cyclic reduction derivation $t_j \to_{\mathcal{R} \cup \mathcal{F}^{arg}} \dots \to_{\mathcal{R} \cup \mathcal{F}^{arg}} t_j$ starting from the top black term t_j and the assertion follows from case (i). So we may assume that every term in D is top transparent. We use the same proof idea as in (i). Since the proofs are very much alike, we only sketch the construction. Now the *inner subterm occurrences of D* are those terms which are subterms of a black or white principal subterm occurring in D. The others are called *outer subterm occurrences of D*. Let O_t^D denote the set of all outer subterm occurrences of D. Furthermore, let $S_P(D)$ denote the set of all black or white principal subterms appearing in D. Moreover, for $s \in S_P(D)$, define $\Delta_t^D(s) = \{u \in O_t^D \mid s \to_{\mathcal{R} \cup \mathcal{F}^{arg}}^+ u\}$. Let

$$\Phi_t^D(s) = \begin{cases} C^t\{\Phi_t^D(s_1), \dots, \Phi_t^D(s_m)\} & \text{if } s = C^t\{\!\{s_1, \dots, s_m\}\!\} \\ \\ Sort(\{\Phi_t^D(u) \mid u \in \Delta_t^D(s)\}) & \text{if } root(s) \in \mathcal{A}_1 \uplus \mathcal{A}_2 \end{cases}$$

where $Sort$ is defined as in case (i). Again, note that $Sort(\{\Phi_t^D(u) \mid u \in \Delta_t^D(s)\}) = z$ if $root(s) \in \mathcal{A}_1 \uplus \mathcal{A}_2$ and $\Delta_t^D(s) = \emptyset$. It follows from similar arguments as above that $t_j \to_{\mathcal{R} \cup \mathcal{F}^{arg}} t_{j+1}$ implies $\Phi_t^D(t_j) \to_{(\mathcal{S} \cup \mathcal{B}^{arg}) \uplus \mathcal{C}_\mathcal{E}}^* \Phi_t^D(t_{j+1})$. We just have to consider the following cases (using $\to$ as a shorthand for $\to_{\mathcal{R} \cup \mathcal{F}^{arg}}$).

(a) If $t_j \to^t t_{j+1}$, then show $\Phi_t^D(t_j) \to_{\mathcal{S} \cup \mathcal{B}^{arg}} \Phi_t^D(t_{j+1})$ using arguments similar to those given in case (i) (a).

(b) If $t_j \to t_{j+1}$ is not a transparent rewrite step, then $t_j = C^t[\![s_1, \dots, s_l, \dots, s_m]\!]$ and $t_{j+1} = C^t[s_1, \dots, s_l', \dots, s_m]$ for some $l \in \{1, \dots, m\}$ and some term s_l', where $s_l \to s_l'$. Now check whether $root(s_l') \in \mathcal{A}_1 \uplus \mathcal{A}_2$ holds. If so, then case (i) (b1) applies, if not, then case (i) (b2) applies, making only notational changes. Thus $\Phi_t^D(t_j) \to_{\mathcal{C}_\mathcal{E}}^* \Phi_t^D(t_{j+1})$.

Since there must be at least one $\to^t$ step in D, we obtain a non-empty cyclic derivation $\Phi_t^D(D)$ of terms from $\mathcal{T}(\mathcal{B} \uplus \{Cons\}, \{z\})$. This is a contradiction to the fact that $(\mathcal{B}, \mathcal{S})$ is simplifying according to Lemma 5.15. □

6. Conditional term rewriting systems†

When dealing with the combination $\mathcal{R}$ of composable TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$, we have tacitly used the fundamental property that $s \to_{\mathcal{R}} t$ implies $s \to_{\mathcal{R}_1} t$ or $s \to_{\mathcal{R}_2} t$. It has been stressed by Middeldorp (1990, 1993) that this basic property does not hold true for CTRSs. Consequently, it is much more subtle to prove the modularity of a certain property for CTRSs.

If $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are composable or constructor-sharing CTRSs, then $(\mathcal{F}, \mathcal{R}_1)$ and $(\mathcal{F}, \mathcal{R}_2)$ are also CTRSs, where $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. In order to avoid misunderstandings, we write $\Rightarrow_{\mathcal{R}_i}$ for the rewrite relation associated with $(\mathcal{F}_i, \mathcal{R}_i)$ and $\to_{\mathcal{R}_i}$ for the rewrite relation associated with $(\mathcal{F}, \mathcal{R}_i)$, where $i \in \{1, 2\}$. If $s, t \in \mathcal{T}(\mathcal{F}_i, \mathcal{V})$ and $s \Rightarrow_{\mathcal{R}_i} t$, then we clearly have $s \to_{\mathcal{R}_i} t$. A priori, it is not clear at all whether the converse is also true. For, if $s \to_{\mathcal{R}_i} t$, then there exists a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_i$, a substitution $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, and a context $C[\,]$ such that $s = C[l\sigma], t = C[r\sigma]$, and $s_j\sigma \downarrow_{\mathcal{R}_i} t_j\sigma$ for all $j \in \{1, \ldots, n\}$. And $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ may substitute mixed terms for extra-variables occurring in the conditions.

6.1. SEMI-COMPLETENESS

Our next goal is to show that semi-completeness is modular for constructor-sharing CTRSs. So in this subsection we tacitly assume that $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are given constructor-sharing CTRSs. We use the structure and the ideas of the proof showing that confluence is modular for disjoint CTRSs, see Middeldorp (1990, 1993). The basic proof idea is to construct two rewrite relations $\to_1$ and $\to_2$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that their union is semi-complete, and reduction in the combined system $(\mathcal{F}, \mathcal{R})$ corresponds to joinability with respect to $\to_{1,2} = \to_1 \cup \to_2$. From these two properties and the equality $NF(\mathcal{F}, \mathcal{R}) = NF(\to_{1,2})$, the modularity of semi-completeness for CTRSs with shared constructors follows.

DEFINITION 6.1. The rewrite relation $\to_1$ is defined by: $s \to_1 t$ if there exists a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a substitution $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, and a context $C[\,]$ such that $s = C[l\sigma], t = C[r\sigma]$, and $s_j\sigma \downarrow_1^o t_j\sigma$ for $j \in \{1, \ldots, n\}$. Here the superscript o in $s_j\sigma \downarrow_1^o t_j\sigma$ means that $s_j\sigma$ and $t_j\sigma$ are joinable using only *outer* $\to_1$ reduction steps. The relation $\to_2$ is defined analogously. The union of $\to_1$ and $\to_2$ is denoted by $\to_{1,2}$.

EXAMPLE 6.2. Let $\mathcal{R}_1 = \{F(x, c) \to G(x) \Leftarrow x \downarrow c\}$ and $\mathcal{R}_2 = \{a \to c\}$. We have $F(a, c) \to_{\mathcal{R}} G(a)$ but neither $F(a, c) \to_1 G(a)$ nor $F(a, c) \to_2 G(a)$. However, the terms are joinable with respect to $\to_{1,2}$: $F(a, c) \to_2 F(c, c) \to_1 G(c) \leftarrow_2 G(a)$.

The simple proofs of the next two lemmata are omitted.

LEMMA 6.3. If $s \to_{1,2} t$, then $s \to_{\mathcal{R}} t$.

LEMMA 6.4. Let s be a black term and let σ be a top white substitution such that $s\sigma \to_1^o t$. Then there is a black term u such that $t = u\sigma$.

† Parts of the material presented in this section originate from Ohlebusch (1994c).

LEMMA 6.5. Let s, t be black terms and let σ be a top white substitution with $s\sigma \to_1^o t\sigma$. If τ is a substitution with $\sigma \propto \tau$, then $s\tau \to_1^o t\tau$.

PROOF. The lemma is proved by induction on the depth of $s\sigma \to_1^o t\sigma$. The case of zero depth is straightforward. Let the depth of $s\sigma \to_1^o t\sigma$ equal $d+1$, $d \geq 0$. There is a context $C[\,]$, a substitution $\rho : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$ such that $s\sigma = C[l\rho]$, $t\sigma = C[r\rho]$ and $s_j\rho \downarrow_1^o t_j\rho$ is of depth $\leq d$ for every $j \in \{1, \ldots, n\}$. According to Proposition 4.18, ρ has a decomposition $\rho = \rho_2 \circ \rho_1$ such that ρ_1 is black, ρ_2 is top white, and $\rho_2 \propto \epsilon$. We define a substitution ρ' by $\rho'(x) := \tau(y)$ for every $x \in \mathcal{D}om(\rho_2)$ and $y \in \mathcal{D}om(\sigma)$ satisfying $\rho_2(x) = \sigma(y)$. ρ' is well-defined because $\sigma \propto \tau$. It follows from $\rho_2 \propto \epsilon$ and $\epsilon \propto \rho'$ that $\rho_2 \propto \rho'$. By Lemma 6.4, for any $j \in \{1, \ldots, n\}$, we may write

$$\rho_2(\rho_1(s_j)) = \rho_2(u_1) \to_1^o \cdots \to_1^o \rho_2(u_k) = \rho_2(v_l) \;{}_1^o\!\leftarrow \cdots {}_1^o\!\leftarrow \rho_2(v_1) = \rho_2(\rho_1(t_j))$$

for some black terms $u_1, \ldots, u_k, v_1, \ldots, v_l$. Now repeated application of the induction hypothesis yields

$$\rho'(\rho_1(s_j)) = \rho'(u_1) \to_1^o \cdots \to_1^o \rho'(u_k) = \rho'(v_l) \;{}_1^o\!\leftarrow \cdots {}_1^o\!\leftarrow \rho'(v_1) = \rho'(\rho_1(t_j))$$

Thus $\rho'(\rho_1(l)) \to_1^o \rho'(\rho_1(r))$. Let $\hat{C}[\,]$ be the context obtained from $C[\,]$ by replacing every white principal subterm which must be of the form $\sigma(x)$ for some variable $x \in \mathcal{D}om(\sigma)$ by the corresponding $\tau(x)$. (This is a slight abuse of notation because $\hat{C}[\,]$ contains in general more that one occurrence of $\Box$.) It is fairly simple to verify that $s\tau = \hat{C}[\rho'(\rho_1(l))]$ and $t\tau = \hat{C}[\rho'(\rho_1(r))]$. Hence $s\tau \to_1^o t\tau$. $\Box$

LEMMA 6.6. The restriction of $\to_1$ to $\mathcal{T}(\mathcal{F}_1, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ and $\Rightarrow_{\mathcal{R}_1}$ coincide.

PROOF. "$\supseteq$" Trivial.
"$\subseteq$" Let $s, t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ with $s \to_1 t$. In order to show that $s \Rightarrow_{\mathcal{R}_1} t$, we proceed by induction on the depth of $s \to_1^o t$. The case of zero depth is straightforward. So suppose that the depth of $s \to_1^o t$ equals $d+1$, $d \geq 0$. Then there exists a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a substitution $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, and a context $C[\,]$ such that $s = C[l\sigma], t = C[r\sigma]$, and $s_j\sigma \downarrow_1^o t_j\sigma$ with depth $\leq d$ for $j \in \{1, \ldots, n\}$. According to Proposition 4.18, σ can be decomposed into $\sigma_2 \circ \sigma_1$ such that σ_1 is black, σ_2 is top white, and $\sigma_2 \propto \epsilon$. Induction on the number of rewrite steps in $s_j\sigma \downarrow_1^o t_j\sigma$ in combination with Lemma 6.5 yields $\sigma_1(s_j) \downarrow_1^o \sigma_1(t_j)$ for $j \in \{1, \ldots, n\}$. Since every term in the conversion $\sigma_1(s_j) \downarrow_1^o \sigma_1(t_j)$ is black, we obtain $\sigma_1(s_j) \Downarrow_{\mathcal{R}_1} \sigma_1(t_j)$ by repeated application of the induction hypothesis. Consequently, we have $\sigma_1(l) \Rightarrow_{\mathcal{R}_1} \sigma_1(r)$. Now $s \Rightarrow_{\mathcal{R}_1} t$ follows from $s = C[l\sigma] = C[l\sigma_1]$ and $t = C[r\sigma] = C[r\sigma_1]$ because s and t are black. $\Box$

PROPOSITION 6.7. If $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are semi-complete, then the relation $\to_{1,2}$ is semi-complete as well.

PROOF. We define two unconditional TRSs $(\mathcal{F}_1, \mathcal{S}_1)$ and $(\mathcal{F}_2, \mathcal{S}_2)$ by

$$\mathcal{S}_i = \{u \to v \mid u, v \in \mathcal{T}(\mathcal{F}_i, \mathcal{V}), root(u) \notin \mathcal{C}_1 \cap \mathcal{C}_2 \text{ and } u \to_i v\}.$$

First of all note that $(\mathcal{F}_1, \mathcal{S}_1)$ and $(\mathcal{F}_2, \mathcal{S}_2)$ are constructor-sharing TRSs. By Lemma 6.6, the restriction of $\to_i$ to $\mathcal{T}(\mathcal{F}_i, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_i, \mathcal{V})$ and $\Rightarrow_{\mathcal{R}_i}$ coincide. It is easy to show that

$\rightarrow_{\mathcal{S}_i}$ and the restriction of $\rightarrow_i$ to $\mathcal{T}(\mathcal{F}_i,\mathcal{V}) \times \mathcal{T}(\mathcal{F}_i,\mathcal{V})$ are also the same. Hence $\rightarrow_{\mathcal{S}_i}$ and $\Rightarrow_{\mathcal{R}_i}$ coincide on $\mathcal{T}(\mathcal{F}_i,\mathcal{V}) \times \mathcal{T}(\mathcal{F}_i,\mathcal{V})$. In particular, the TRS $(\mathcal{F}_i,\mathcal{S}_i)$ is semi-complete because $(\mathcal{F}_i,\mathcal{R}_i)$ is semi-complete. It follows from Theorem 5.2 that $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{S}_1 \cup \mathcal{S}_2)$ is also semi-complete.

We next show that the relations $\rightarrow_{\mathcal{S}_i}$ and $\rightarrow_i$ are also the same on $\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$.
"$\subseteq$" Straightforward.
"$\supseteq$" Without loss of generality, let $i = 1$. If $s \rightarrow_1 t$, then there exist a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a substitution $\sigma : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F},\mathcal{V})$, and a context $C[\,]$ such that $s = C[l\sigma], t = C[r\sigma]$, and $s_j\sigma \downarrow_1^o t_j\sigma$ for $j \in \{1,\ldots,n\}$. Note that particularly $l\sigma \rightarrow_1 r\sigma$. According to Proposition 4.18, σ has a decomposition $\sigma = \sigma_2 \circ \sigma_1$ such that σ_1 is black, σ_2 is top white, and $\sigma_2 \propto \epsilon$. Now we apply Lemma 6.5: $\sigma_1(l)$ and $\sigma_1(r)$ are black terms and σ_2 is a top white substitution with $\sigma_2(\sigma_1(l)) \rightarrow_1 \sigma_2(\sigma_1(r))$ and ϵ is a substitution with $\sigma_2 \propto \epsilon$. Consequently, we obtain $\sigma_1(l) = \epsilon(\sigma_1(l)) \rightarrow_1 \epsilon(\sigma_1(r)) = \sigma_1(r)$. Since $\sigma_1(l)$ and $\sigma_1(r)$ are black terms and $root(\sigma_1(l)) = root(l) \notin \mathcal{C}_1 \cap \mathcal{C}_2$, it follows that $\sigma_1(l) \rightarrow \sigma_1(r)$ is a rewrite rule of $\mathcal{S}_1$. Thus $s = C[\sigma_2(\sigma_1(l))] \rightarrow_{\mathcal{S}_1} C[\sigma_2(\sigma_1(r))] = t$.

With the above results, it further follows from

$$\rightarrow_{\mathcal{S}_1 \cup \mathcal{S}_2} \;=\; \rightarrow_{\mathcal{S}_1} \cup \rightarrow_{\mathcal{S}_2} \;=\; \rightarrow_1 \cup \rightarrow_2 \;=\; \rightarrow_{1,2}$$

that $\rightarrow_{1,2}$ is semi-complete on $\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$. □

Note that the above approach fails for composable CTRSs because (the accordingly defined) sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are in general not composable.

DEFINITION 6.8. If $\rightarrow_{1,2}$ is semi-complete, then every term t has a unique normal form w.r.t. $\rightarrow_{1,2}$. In the sequel, this normal form will be denoted by $t^{\rightarrow}$. Furthermore, for any substitution σ, $\sigma^{\rightarrow}$ denotes the substitution $\{x \mapsto \sigma(x)^{\rightarrow} \mid x \in \mathcal{D}om(\sigma)\}$.

LEMMA 6.9. Let $\rightarrow_{1,2}$ be semi-complete. If s and t are black terms and σ is a top white $\rightarrow_{1,2}$ normalized substitution such that $s\sigma \downarrow_{1,2} t\sigma$, then $s\sigma \downarrow_1^o t\sigma$.

PROOF. We show that $s\sigma \rightarrow_{1,2} u$ implies $s\sigma \rightarrow_1^o u$. Since $u = v\sigma$ for some black term v by Lemma 6.4, the lemma then follows by a straightforward induction on the length of the valley. In order to prove the claim, we use induction on the depth of $s\sigma \rightarrow_{1,2} u$. The case of zero depth is trivial. So suppose that the depth of $s\sigma \rightarrow_{1,2} u$ equals $d+1$, $d \geq 0$. Since σ is a top white $\rightarrow_{1,2}$ normalized substitution, there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a substitution $\rho : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F},\mathcal{V})$, and a context $C[\,]$ such that $s\sigma = C[l\rho]$, $u = C[r\rho]$, and $s_j\rho \downarrow_{1,2} t_j\rho$ with depth $\leq d$ for $j \in \{1,\ldots,n\}$. By Proposition 4.18, ρ can be decomposed into $\rho_2 \circ \rho_1$ such that ρ_1 is black, ρ_2 is top white, and $\rho_2 \propto \epsilon$. Note that for every $x \in \mathcal{D}om(\rho_2) \cap \mathcal{V}ar(l\rho_1)$, we have $\rho_2(x) \in NF(\rightarrow_{1,2})$. Nevertheless, we do not have $\rho_2(x) \in NF(\rightarrow_{1,2})$ in general because of possible extra variables. Since $\rightarrow_{1,2}$ is semi-complete, $\rho_2 \rightarrow_{1,2}^* \rho_2^{\rightarrow}$. Thus $\rho_2^{\rightarrow}(\rho_1(s_j)) \;{}_{1,2}^{*}\!\leftarrow s_j\rho \downarrow_{1,2} t_j\rho \rightarrow_{1,2}^* \rho_2^{\rightarrow}(\rho_1(t_j))$. The confluence of $\rightarrow_{1,2}$ guarantees $\rho_2^{\rightarrow}(\rho_1(s_j)) \downarrow_{1,2} \rho_2^{\rightarrow}(\rho_1(t_j))$ for every $j \in \{1,\ldots,n\}$. By Proposition 4.18, $\rho_2^{\rightarrow}$ can be decomposed into $\rho_4 \circ \rho_3$ such that ρ_3 is black, ρ_4 is top white, and $\rho_4 \propto \epsilon$. Evidently, $\rho_3(\rho_1(s_j)), \rho_3(\rho_1(t_j))$ are black terms and ρ_4 is a top white $\rightarrow_{1,2}$ normalized substitution. Hence the induction hypothesis yields $\rho_4(\rho_3(\rho_1(s_j))) \downarrow_1^o \rho_4(\rho_3(\rho_1(t_j)))$. In other words, $\rho_2^{\rightarrow}(\rho_1(s_j)) \downarrow_1^o \rho_2^{\rightarrow}(\rho_1(t_j))$, and we obtain as a consequence that $\rho_2^{\rightarrow}(\rho(l)) \rightarrow_1^o \rho_2^{\rightarrow}(\rho_1(r))$

and $C[\rho_2^{\rightarrow}(\rho_1(l))] \rightarrow_1^{o} C[\rho_2^{\rightarrow}(\rho_1(r))]$. Clearly, $s\sigma = C[\rho_2^{\rightarrow}(\rho_1(l))]$ and $u = C[\rho_2^{\rightarrow}(\rho_1(r))]$ because $\rho_2(x) \in NF(\rightarrow_{1,2})$ for every $x \in \mathcal{D}om(\rho_2) \cap \mathcal{V}ar(l\rho_1)$. This proves the claim. □

LEMMA 6.10. Let $\rightarrow_{1,2}$ be semi-complete and let $s_1, \ldots, s_n, t_1, \ldots, t_n$ be black terms. If σ is a substitution with $s_j\sigma \downarrow_{1,2} t_j\sigma$ for every $j \in \{1, \ldots, n\}$, then $\sigma^{\rightarrow}(s_j) \downarrow_1^{o} \sigma^{\rightarrow}(t_j)$ for every $j \in \{1, \ldots, n\}$.

PROOF. We have $\sigma^{\rightarrow}(s_j) \ {}_{1,2}^{*}\!\!\leftarrow s_j\sigma \downarrow_{1,2} t_j\sigma \rightarrow_{1,2}^{*} \sigma^{\rightarrow}(t_j)$. The confluence of $\rightarrow_{1,2}$ implies $\sigma^{\rightarrow}(s_j) \downarrow_{1,2} \sigma^{\rightarrow}(t_j)$. Proposition 4.18 yields a decomposition of $\sigma^{\rightarrow}$ into $\sigma_2 \circ \sigma_1$ such that σ_1 is black and σ_2 is top white. Evidently, $\sigma_1(s_j)$, $\sigma_1(t_j)$ are black terms and σ_2 is a top white $\rightarrow_{1,2}$ normalized substitution. According to Lemma 6.9, we eventually derive $\sigma^{\rightarrow}(s_j) = \sigma_2(\sigma_1(s_j)) \downarrow_1^{o} \sigma_2(\sigma_1(t_j)) = \sigma^{\rightarrow}(t_j)$. □

PROPOSITION 6.11. If $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are semi-complete and $s \rightarrow_{\mathcal{R}} t$, then $s \downarrow_{1,2} t$.

PROOF. We proceed by induction on the depth of $s \rightarrow t$. The case of zero depth is trivial. So suppose that the depth of $s \rightarrow t$ equals $d+1$, $d \geq 0$. Then there is a context $C[\,]$, a substitution $\sigma : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$, and a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}$ such that $s = C[l\sigma]$, $t = C[r\sigma]$, and $s_j\sigma \downarrow t_j\sigma$ is of depth less than or equal to d for every $j \in \{1, \ldots, n\}$. Figure 4 depicts how the induction hypothesis and confluence of $\rightarrow_{1,2}$ yield $s_j\sigma \downarrow_{1,2} t_j\sigma$ for every $j \in \{1, \ldots, n\}$ (where (1) signals an application of the induction hypothesis and (2) stands for an application of Proposition 6.7). W.l.o.g. we may assume that the applied rewrite rule stems from $\mathcal{R}_1$. By Lemma 6.10, we have $\sigma^{\rightarrow}(s_j) \downarrow_1^{o} \sigma^{\rightarrow}(t_j)$ for $j \in \{1, \ldots, n\}$ and thus $\sigma^{\rightarrow}(l) \rightarrow_1 \sigma^{\rightarrow}(r)$. Finally, we obtain $s \downarrow_{1,2} t$ from $s = C[l\sigma] \rightarrow_{1,2}^{*} C[\sigma^{\rightarrow}(l)] \rightarrow_1 C[\sigma^{\rightarrow}(r)] \ {}_{1,2}^{*}\!\!\leftarrow C[r\sigma] = t$. □

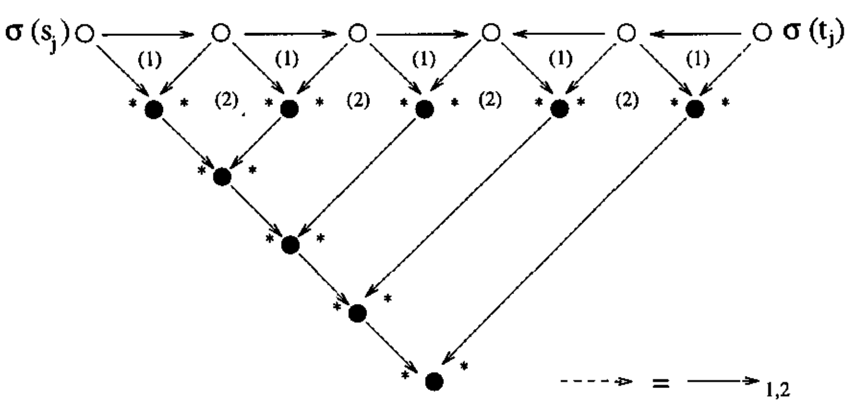

Figure 4 The proof idea of Proposition 6.11.

PROPOSITION 6.12. If $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are semi-complete, then the relations $\leftrightarrow_{\mathcal{R}}^{*}$ and $\downarrow_{1,2}$ coincide.

PROOF. This is a consequence of Lemma 6.3 and Propositions 6.11 and 6.7. □

LEMMA 6.13. If $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are semi-complete, then

(1) $NF(\mathcal{F}, \mathcal{R}) = NF(\mathcal{F}, \mathcal{R}_1) \cap NF(\mathcal{F}, \mathcal{R}_2)$.
(2) $NF(\mathcal{F}, \mathcal{R}) = NF(\rightarrow_{1,2})$.

PROOF. We will only prove the first statement because the proof of the second statement is essentially the same.
"$\subseteq$" Trivial.
"$\supseteq$" If $NF(\mathcal{F}, \mathcal{R}_1) \cap NF(\mathcal{F}, \mathcal{R}_2) \not\subseteq NF(\mathcal{F}, \mathcal{R})$, then there is a term s with $s \notin NF(\mathcal{F}, \mathcal{R})$ and $s \in NF(\mathcal{F}, \mathcal{R}_1) \cap NF(\mathcal{F}, \mathcal{R}_2)$. W.l.o.g. we may assume that s is of minimal size (i.e., $|s|$ is minimal). Hence s is a redex and every proper subterm of s is irreducible by $\rightarrow_{\mathcal{R}}$. Therefore, there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}$ and a substitution $\sigma : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $s = l\sigma, t = r\sigma$, and $s_j\sigma \downarrow_{\mathcal{R}} t_j\sigma$ for all $j \in \{1, \ldots, n\}$. Note that for every variable $x \in \mathcal{D}om(\sigma) \cap \mathcal{V}ar(l)$, we have $\sigma(x) \in NF(\mathcal{F}, \mathcal{R})$ because $\sigma(x)$ is a proper subterm of s. W.l.o.g. we may further assume that the applied rewrite rule originates from $\mathcal{R}_1$. By Proposition 6.12, $s_j\sigma \downarrow_{1,2} t_j\sigma$ which, in conjunction with Lemma 6.10, yields $\sigma^{\rightarrow}(s_j) \downarrow_1^o \sigma^{\rightarrow}(t_j)$. It follows $s = \sigma(l) = \sigma^{\rightarrow}(l) \rightarrow_1^o \sigma^{\rightarrow}(r)$ because $\sigma(x) = \sigma^{\rightarrow}(x)$ for every $x \in \mathcal{V}ar(l)$. This means that $s \notin NF(\mathcal{F}, \mathcal{R}_1)$, a contradiction. □

THEOREM 6.14. Semi-completeness is modular for constructor-sharing CTRSs.

PROOF. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be CTRSs with shared constructors. We have to show that their combined system $(\mathcal{F}, \mathcal{R})$ is semi-complete if and only if both $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are semi-complete. In order to show the if case, we consider a conversion $t_1 \;{}_{\mathcal{R}}^{*}\!\!\leftarrow s \rightarrow_{\mathcal{R}}^{*} t_2$. According to Proposition 6.12 we have $t_1 \downarrow_{1,2} t_2$. Since $\rightarrow_{1,2}$ is semi-complete, $t_1 \rightarrow_{1,2}^{*} t_3$ and $t_2 \rightarrow_{1,2}^{*} t_3$, where t_3 is the unique normal form of s, t_1, and t_2. Now Lemma 6.3 implies $t_1 \rightarrow_{\mathcal{R}}^{*} t_3 \;{}_{\mathcal{R}}^{*}\!\!\leftarrow t_2$. Thus $(\mathcal{F}, \mathcal{R})$ is confluent. It remains to show normalization of $\rightarrow_{\mathcal{R}}$. Let $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Since $\rightarrow_{1,2}$ is normalizing, $s \rightarrow_{1,2}^{*} t$ for some $t \in NF(\rightarrow_{1,2})$. By Lemma 6.3, $s \rightarrow_{\mathcal{R}}^{*} t$. It follows from Lemma 6.13 (2) that $t \in NF(\mathcal{F}, \mathcal{R})$. Hence $(\mathcal{F}, \mathcal{R})$ is also normalizing. This all proves that $(\mathcal{F}, \mathcal{R})$ is semi-complete. The only-if case follows straightforwardly from Lemma 6.15. □

LEMMA 6.15. Let $(\mathcal{F}, \mathcal{R})$ be the combined system of two constructor-sharing CTRSs $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ such that $(\mathcal{F}, \mathcal{R})$ is semi-complete. If s is a black term and $s \rightarrow_{\mathcal{R}} t$, then $s \Rightarrow_{\mathcal{R}_1} t$.

PROOF. We show the following stronger claim, where the rewrite relation associated with $(\mathcal{F}_1 \cup \{\Box\}, \mathcal{R}_1)$ is also denoted by $\Rightarrow_{\mathcal{R}_1}$.

Claim: If s is a black term and σ is a top white $\rightarrow_{\mathcal{R}}$ normalized substitution such that $s\sigma \rightarrow_{\mathcal{R}} t\sigma$, then $s\sigma^{\Box} \Rightarrow_{\mathcal{R}_1} t\sigma^{\Box}$, where $\sigma^{\Box} = \{x \mapsto \Box \mid x \in \mathcal{D}om(\sigma)\}$.

Since $\mathcal{R}$ is semi-complete, every term t has a unique normal form $t\downarrow$ w.r.t. $\mathcal{R}$. Furthermore, for any substitution σ, let $\sigma\downarrow$ denote the substitution $\{x \mapsto \sigma(x)\downarrow \mid x \in \mathcal{D}om(\sigma)\}$. The claim is proved by induction on the depth of $s\sigma \rightarrow t\sigma$. The case of zero depth is straightforward. Let the depth of $s\sigma \rightarrow t\sigma$ equal $d+1$, $d \geq 0$. There is a context $C[\,]$, a substitution $\rho : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$, and a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$ such that

$s\sigma = C[l\rho], t\sigma = C[r\rho]$ and $s_j\rho \downarrow t_j\rho$ is of depth $\leq d$ for every $j \in \{1, \ldots, n\}$. According to Proposition 4.18, ρ can be decomposed into $\rho_2 \circ \rho_1$ such that ρ_1 is black and ρ_2 is top white. Note that for every variable $x \in \mathcal{D}om(\rho_2) \cap \mathcal{V}ar(l\rho_1)$, we have $\rho_2(x) \in NF(\rightarrow)$. Nevertheless, we do not have $\rho_2(x) \in NF(\rightarrow)$ in general because of possible extra variables. Since $\rightarrow$ is semi-complete, $\rho_2 \rightarrow^* \rho_2\!\downarrow$. Thus $\rho_2\!\downarrow(\rho_1(s_j)) \;{}^*\!\leftarrow s_j\rho \downarrow t_j\rho \rightarrow^* \rho_2\!\downarrow(\rho_1(t_j))$. The confluence of $\rightarrow$ guarantees $\rho_2\!\downarrow(\rho_1(s_j)) \downarrow \rho_2\!\downarrow(\rho_1(t_j))$ for every $j \in \{1, \ldots, n\}$. By Proposition 4.18, $\rho_2\!\downarrow$ can be decomposed into $\rho_4 \circ \rho_3$ such that ρ_3 is black and ρ_4 is top white. Evidently, $\rho_3(\rho_1(s_j))$ and $\rho_3(\rho_1(t_j))$ are black terms and ρ_4 is a top white $\rightarrow$ normalized substitution. Repeated application of the induction hypothesis yields $\rho_4^\square(\rho_3(\rho_1(s_j))) \Downarrow_{\mathcal{R}_1} \rho_4^\square(\rho_3(\rho_1(t_j)))$. We obtain as a consequence that $\rho_4^\square(\rho_3(\rho_1(l))) \Rightarrow_{\mathcal{R}_1} \rho_4^\square(\rho_3(\rho_1(r)))$. Clearly, $s\sigma = C[\rho_2\!\downarrow(\rho_1(l))]$ and $t\sigma = C[\rho_2\!\downarrow(\rho_1(r))]$ because $\rho_2(x) \in NF(\rightarrow)$ for every $x \in \mathcal{D}om(\rho_2) \cap \mathcal{V}ar(l\rho_1)$. Let $\hat{C}[\;]$ be the context obtained from $C[\;]$ by replacing every white principal subterm which must be of the form $\sigma(x)$ for some variable $x \in \mathcal{D}om(\sigma)$ with $\square$. It is fairly simple to verify that $s\sigma^\square = \hat{C}[\rho_4^\square(\rho_3(\rho_1(l)))]$ and $t\sigma^\square = \hat{C}[\rho_4^\square(\rho_3(\rho_1(r)))]$. Thus $s\sigma^\square \Rightarrow_{\mathcal{R}_1} t\sigma^\square$ and we are done. $\square$

6.2. TERMINATION AND COMPLETENESS

Middeldorp (1990, 1993) conjectured that the disjoint union of two terminating join CTRSs is terminating if one of them contains neither collapsing nor duplicating rules and the other is confluent. The next example disproves this conjecture. The function symbols have been chosen in resemblance to other known counterexamples.

EXAMPLE 6.16. Let

$$\mathcal{R}_1 = \left\{ \begin{array}{ccccc} 0 & & & & 1 \\ \downarrow & \searrow & & \swarrow & \downarrow \\ A & & 2 & & B \end{array} \right. \qquad F(x) \rightarrow F(x) \Leftarrow x \downarrow A,\ x \downarrow B$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} g\,(x, y, y) \rightarrow x \\ g\,(y, y, x) \rightarrow x. \end{array} \right.$$

Clearly, $\mathcal{R}_1$ is non-collapsing, non-duplicating, and terminating (there is no $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ which rewrites to both A and B). Note that $\mathcal{R}_1$ is not confluent. Moreover, the CTRS $\mathcal{R}_2$ is evidently terminating and confluent. However, their disjoint union $\mathcal{R} = \mathcal{R}_1 \uplus \mathcal{R}_2$ is not terminating. Since

$$B \;{}_{\mathcal{R}}\!\leftarrow 1 \;{}_{\mathcal{R}}\!\leftarrow g(0,0,1) \rightarrow_{\mathcal{R}} g(0,2,1) \rightarrow_{\mathcal{R}} g(0,2,2) \rightarrow_{\mathcal{R}} 0 \rightarrow_{\mathcal{R}} A,$$

there is the cyclic reduction "sequence" $F(g(0,0,1)) \rightarrow_{\mathcal{R}} F(g(0,0,1))$.

Note that the above example also shows (the known fact) that termination is not modular for non-duplicating disjoint CTRSs. Middeldorp (1990, 1993) has given sufficient conditions for the modularity of termination. It will next be shown that his results also hold, *mutatis mutandis*, in the presence of shared constructors. We emphasize that our proof is considerably simpler than that of Middeldorp (1990, 1993).

As in the previous subsection, let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be constructor-sharing join

CTRSs. It is not difficult to verify that the CTRS $(\mathcal{F}_i \cup \{\Box\}, \mathcal{R}_i)$ is terminating if and only if $(\mathcal{F}_i, \mathcal{R}_i)$ is terminating. Again, we also denote the rewrite relation associated with $(\mathcal{F}_i \cup \{\Box\}, \mathcal{R}_i)$ by $\Rightarrow_{\mathcal{R}_i}$ (by abuse of notation).

PROPOSITION 6.17. Let $(\mathcal{F}_2, \mathcal{R}_2)$ be layer-preserving.

(1) If $s \to^o t$ by some rule from $\mathcal{R}_1$, then $top^b(s) \Rightarrow_{\mathcal{R}_1} top^b(t)$.
(2) If $s \to^o t$ by some rule from $\mathcal{R}_2$ or $s \to^i t$, then $top^b(s) = top^b(t)$.

PROOF. We proceed by induction on the depth of $s \to t$. The case of zero depth is straightforward. So suppose that the depth of $s \to t$ equals $d+1$, $d \geq 0$. The induction hypothesis covers the statement that $u \to v$ implies $top^b(u) \Rightarrow^*_{\mathcal{R}_1} top^b(v)$ whenever $u \to v$ is of depth less than or equal to d.

(1) If $s \to^o t$ by some rule from $\mathcal{R}_1$, then $s = C^b\{\!\{u_1, \ldots, u_p\}\!\}$ and $t = \hat{C}^b\langle\!\langle u_{i_1}, \ldots, u_{i_q}\rangle\!\rangle$, where $i_1, \ldots, i_q \in \{1, \ldots, p\}$. Moreover, there is a context $C[\;]$, a substitution σ and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n \in \mathcal{R}_1$ such that $s = C[l\sigma]$, $t = C[r\sigma]$ and $s_j\sigma \downarrow t_j\sigma$ is of depth less than or equal to d for every $j \in \{1, \ldots, n\}$. We first show that $top^b(s_j\sigma) \Downarrow_{\mathcal{R}_1} top^b(t_j\sigma)$ for every $j \in \{1, \ldots, n\}$. Fix j. Let w be the common reduct of $s_j\sigma$ and $t_j\sigma$. Clearly, it suffices to show that $top^b(s_j\sigma) \Rightarrow^*_{\mathcal{R}_1} top^b(w)$ and $top^b(t_j\sigma) \Rightarrow^*_{\mathcal{R}_1} top^b(w)$. W.l.o.g. we consider only the former claim. The claim is proved by induction on the length of $s_j\sigma \to^* w$. The case of zero length is trivial, so let $s_j\sigma \to v \to^l w$ with $l \geq 0$. The induction hypothesis (on l) yields $top^b(v) \Rightarrow^*_{\mathcal{R}_1} top^b(w)$. Furthermore, the induction hypothesis (on d) yields $top^b(s_j\sigma) \Rightarrow^*_{\mathcal{R}_1} top^b(v)$. This proves the claim. Thus $top^b(w)$ is a common reduct of $top^b(s_j\sigma)$ and $top^b(t_j\sigma)$ w.r.t. $\Rightarrow_{\mathcal{R}_1}$. According to Proposition 4.18, $\sigma = \sigma_2 \circ \sigma_1$, where σ_1 is a black substitution and σ_2 is top white. Recall that $\sigma_2^\Box$ denotes the substitution $\{x \mapsto \Box \mid x \in \mathcal{D}om(\sigma_2)\}$. It is clear that $top^b(s_j\sigma) = \sigma_2^\Box(\sigma_1(s_j))$ and $top^b(t_j\sigma) = \sigma_2^\Box(\sigma_1(t_j))$. Hence $\sigma_2^\Box(\sigma_1(s_j)) \Downarrow_{\mathcal{R}_1} \sigma_2^\Box(\sigma_1(t_j))$ and thus $\sigma_2^\Box(\sigma_1(l)) \Rightarrow_{\mathcal{R}_1} \sigma_2^\Box(\sigma_1(r))$. Let $\hat{C}[\;]$ be the context obtained from $C[\;]$ by replacing all white principal subterms with $\Box$. Now (1) follows from $top^b(s) = \hat{C}[\sigma_2^\Box(\sigma_1(l))]$ and $top^b(t) = \hat{C}[\sigma_2^\Box(\sigma_1(r))]$.
(2) Let $s \to^o t$ by some rule from $\mathcal{R}_2$ or $s \to^i t$. Since $\mathcal{R}_2$ is layer-preserving, we may write $s = C^b\langle\!\langle u_1, \ldots, u_j, \ldots, u_p\rangle\!\rangle$ and $t = C^b\langle\!\langle u_1, \ldots, u'_j, \ldots, u_p\rangle\!\rangle$, where $u_j \to u'_j$. Hence $top^b(s) = top^b(t)$.

□

In the preceding proposition, the assumption that $(\mathcal{F}_2, \mathcal{R}_2)$ has to be layer-preserving cannot be dropped, as is witnessed by the next example (cf. Middeldorp, 1990, 1993).

EXAMPLE 6.18. Let $\mathcal{R}_1 = \{F(x) \to G(x) \Leftarrow x \downarrow A\}$ and $\mathcal{R}_2 = \{h(x) \to x\}$. Then $F(h(A)) \to^o G(h(A))$ by the only rule of $\mathcal{R}_1$ but $top^b(F(h(a)))) = F(\Box)$ is a normal form w.r.t. $\Rightarrow_{\mathcal{R}_1}$.

Our next goal is to show an analogous statement to Proposition 6.17 (1) without the layer-preservingness requirement on $(\mathcal{F}_2, \mathcal{R}_2)$ but under the additional assumption that $\to_{1,2}$ is semi-complete.

DEFINITION 6.19. Let the rewrite relation $\rightarrow_{1,2}$ be semi-complete. For every term $t = C^b\langle\langle t_1, \ldots, t_m\rangle\rangle$, we define $top^b_{\rightarrow}(t)$ by:

$$top^b_{\rightarrow}(t) = top^b(C^b\langle t_1^{\rightarrow}, \ldots, t_m^{\rightarrow}\rangle)$$

In other words, first the white principal subterms in t are replaced with their unique $\rightarrow_{1,2}$ normal form, and then the topmost black homogeneous part of the term obtained is taken.

LEMMA 6.20. Let $\rightarrow_{1,2}$ be semi-complete. If s, t are black terms and σ is a top white substitution such that $s\sigma \rightarrow^o t\sigma$ by some rule from $\mathcal{R}_1$, then $\sigma^{\rightarrow}(s) \rightarrow_1^o \sigma^{\rightarrow}(t)$.

PROOF. There is a context $C[\;]$, a substitution $\rho : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n \in \mathcal{R}_1$ such that $s\sigma = C[l\rho]$, $t\sigma = C[r\rho]$ and $s_j\rho \downarrow t_j\rho$ for $j \in \{1, \ldots, n\}$. Fix j. From Proposition 6.12 we know that $s_j\rho \downarrow_{1,2} t_j\rho$. According to Proposition 4.18, ρ can be decomposed into $\rho_2 \circ \rho_1$ such that ρ_1 is black and ρ_2 is top white. Since $\rightarrow_{1,2}$ is semi-complete, it follows as in the proof of Lemma 6.9 that $\rho_2^{\rightarrow}(\rho_1(s_j)) \downarrow_{1,2} \rho_2^{\rightarrow}(\rho_1(t_j))$. Applying Lemma 6.10 to the black terms $\rho_1(s_1), \ldots, \rho_1(s_n), \rho_1(t_1), \ldots, \rho_1(t_n)$ and the substitution $\rho_2^{\rightarrow}$ yields $\rho_2^{\rightarrow}(\rho_1(s_j)) \downarrow_1^o \rho_2^{\rightarrow}(\rho_1(t_j))$. Therefore, $\rho_2^{\rightarrow}(\rho_1(l)) \rightarrow_1^o \rho_2^{\rightarrow}(\rho_1(r))$. Let $\hat{C}[\;]$ be the context obtained from $C[\;]$ by replacing all white principal subterms with their respective $\rightarrow_{1,2}$ normal form. It is clear that $\sigma^{\rightarrow}(s) = \hat{C}[\rho_2^{\rightarrow}(\rho_1(l))]$ and $\sigma^{\rightarrow}(t) = \hat{C}[\rho_2^{\rightarrow}(\rho_1(r))]$. Thus $\sigma^{\rightarrow}(s) \rightarrow_1^o \sigma^{\rightarrow}(t)$. □

PROPOSITION 6.21. Let $\rightarrow_{1,2}$ be semi-complete. If $s \rightarrow^o t$ by some rule from $\mathcal{R}_1$, then $top^b_{\rightarrow}(s) \Rightarrow_{\mathcal{R}_1} top^b_{\rightarrow}(t)$.

PROOF. We may write $s = C^b\{\!\{s_1, \ldots, s_n\}\!\}$ and $t = \hat{C}^b\langle\langle s_{i_1}, \ldots, s_{i_m}\rangle\rangle$ for some black contexts $C^b\{, \ldots, \}$, $\hat{C}^b\langle, \ldots, \rangle$, and $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Let $x_1, \ldots, x_n$ be distinct fresh variables and define $\sigma = \{x_j \mapsto s_j \mid 1 \leq j \leq n\}$, $s' = C^b\{x_1, \ldots, x_n\}$, and $t' = \hat{C}^b\langle x_{i_1}, \ldots, x_{i_m}\rangle$. Since σ is top white, we obtain $\sigma^{\rightarrow}(s') \rightarrow_1^o \sigma^{\rightarrow}(t')$ by Lemma 6.20. According to Proposition 4.18, $\sigma^{\rightarrow}$ has a decomposition $\sigma^{\rightarrow} = \sigma_2 \circ \sigma_1$, where σ_1 is black and σ_2 is top white. It follows from Lemma 6.5 that $\sigma_2^{\square}(\sigma_1(s')) \rightarrow_1^o \sigma_2^{\square}(\sigma_1(t'))$ because $\sigma_2 \propto \sigma_2^{\square}$. To verify that $\sigma_2^{\square}(\sigma_1(s')) \Rightarrow_{\mathcal{R}_1} \sigma_2^{\square}(\sigma_1(t'))$ is relatively simple. Now $top^b_{\rightarrow}(s) \Rightarrow_{\mathcal{R}_1} top^b_{\rightarrow}(t)$ is a consequence of

$$top^b_{\rightarrow}(s) = top^b(C^b\{s_1^{\rightarrow}, \ldots, s_n^{\rightarrow}\}) = top^b(\sigma^{\rightarrow}(s')) = top^b(\sigma_2(\sigma_1(s'))) = \sigma_2^{\square}(\sigma_1(s'))$$

and $top^b_{\rightarrow}(t) = \sigma_2^{\square}(\sigma_1(t'))$. □

With the above preparatory considerations, we are now able to prove one of the major results of this subsection. In Theorem 6.22, statement (3) is the interesting new part. For disjoint unions, statements (1) and (2) were already proved in Gramlich (1993). In the context of Theorem 6.22 — but only for finite disjoint unions — Gramlich (1993) showed furthermore that the system $\mathcal{R}_d$ cannot be $\mathcal{C}_{\mathcal{E}}$-terminating, i.e. the system $\mathcal{R}_d \uplus \{Cons(x, y) \rightarrow x, Cons(x, y) \rightarrow y\}$ must be non-terminating. The finiteness requirement results from the special transformation proof technique used in Gramlich (1993).

THEOREM 6.22. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be terminating constructor-sharing CTRSs such that their combined system $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is not terminating. Then the following statements hold (where $d, \bar{d} \in \{1,2\}$ with $d \neq \bar{d}$):

(1) There exists an infinite $\mathcal{R}$ rewrite derivation $D : s_1 \to s_2 \to s_3 \to \dots$ of minimal rank such that D contains infinitely many $s_j \to^o s_{j+1}$ reduction steps where s_j reduces to s_{j+1} by some rule from $\mathcal{R}_d$.
(2) $\mathcal{R}_{\bar{d}}$ is not layer-preserving.
(3) If both systems are confluent, then D contains infinitely many duplicating $s_j \to^o s_{j+1}$ reduction steps such that s_j reduces to s_{j+1} by some rule from $\mathcal{R}_d$.

PROOF. Let D be an infinite $\mathcal{R}$ rewrite derivation of minimal rank, say $rank(D) = k$. Then $rank(s_j) = rank(D)$ for all indices j. Moreover, $\to_\mathcal{R}$ is terminating on $\mathcal{T}^{<k}$. If s_1 is top transparent, say $s_1 = C^t[\![t_1, \dots, t_n]\!]$, then there must be an infinite rewrite derivation starting from some t_l, $l \in \{1, \dots, n\}$ with $rank(t_l) = k$. Therefore, we may assume without loss of generality that s_1 is top black or top white, say top black. It follows for every $j \in \mathbb{N}$ and for each white principal subterm $u \in S_P^w(s_j)$ that $rank(u) < k$.

(1) Suppose that there are only finitely many $\to^o$ reduction steps using a rule from $\mathcal{R}_1$ in D. Then we find an index $j \in \mathbb{N}$ such that the rewrite derivation

$$D' : s_j \to s_{j+1} \to s_{j+2} \to \dots$$

contains no such reduction steps at all. Thus, if $s_j = C^b[\![t_1, \dots, t_n]\!]$, then there must be an infinite rewrite derivation starting from some $t_l \in S_P^w(s_j)$. But this contradicts the minimality assumption on $rank(D)$ since $rank(t_l) < rank(s_j)$.

(2) Suppose that $\mathcal{R}_2$ is layer-preserving, i.e. it contains neither collapsing nor constructor-lifting rules. By Proposition 6.17, we have:

If $s_j \to^o s_{j+1}$ by some rule from $\mathcal{R}_1$, then $top^b(s_j) \Rightarrow_{\mathcal{R}_1} top^b(s_{j+1})$.
If $s_j \to^o s_{j+1}$ by some rule from $\mathcal{R}_2$ or $s_j \to^i s_{j+1}$, then $top^b(s_j) = top^b(s_{j+1})$.

From (1) we know that infinitely many reduction steps of the former kind occur in D. This yields a contradiction to the termination of $\Rightarrow_{\mathcal{R}_1}$.

(3) Let $> \; = \; (\to_\mathcal{R} \cup \rhd)^+$. Then $(\mathcal{T}^{<k}, >)$ is a well-founded ordering. Let $(\mathcal{M}(\mathcal{T}^{<k}), >^{mul})$ denote its multiset extension. Note that $S_P^w(s_j) \in \mathcal{M}(\mathcal{T}^{<k})$. As in the proof of (1), we may suppose that there is no outer reduction step using a duplicating rule from $\mathcal{R}_1$ in D. We distinguish between three cases:

If $s_j \to^o s_{j+1}$ by some rule from $\mathcal{R}_1$, then, by Lemma 5.7, $S_P^w(s_{j+1}) \subseteq S_P^w(s_j)$ because the reduction step is non-duplicating. Clearly, this implies $S_P^w(s_j) \geq^{mul} S_P^w(s_{j+1})$.

If $s_j \to^i s_{j+1}$ by some rule from $\mathcal{R}_1$, then there exists a white principal subterm u of s_j such that $u = C^w[\![u_1, \dots, u_l, \dots, u_n]\!] \to C^w[u_1, \dots, v_l, \dots, u_n] = v$. Evidently, $v \in S_P^w(s_{j+1})$. It follows from $S_P^w(s_{j+1}) = (S_P^w(s_j) \setminus [u]) \cup [v]$ that $S_P^w(s_j) >^{mul} S_P^w(s_{j+1})$.

If $s_j \to s_{j+1}$ by some rule from $\mathcal{R}_2$, then there is a white principal subterm $u \in S_P^w(s_j)$ such that $u \to v$ for some v, i.e. $s_j = C^b[\![, \dots, u, \dots,]\!] \to C^b[, \dots, v, \dots,] = s_{j+1}$. Thus we have $S_P^w(s_{j+1}) = (S_P^w(s_j) \setminus [u]) \cup S_P^w(v)$. It follows from $u \to v$ in conjunction with $v = w$ or $v \rhd w$ for any principal subterm $w \in S_P^w(v)$ that $u > w$ for any $w \in S_P^w(v)$. Therefore $S_P^w(s_j) >^{mul} S_P^w(s_{j+1})$.

We conclude from the well-foundedness of $(\mathcal{M}(\mathcal{T}^{<k}), >^{mul})$ that only a finite number of inner reduction steps as well as reduction steps using a rule from $\mathcal{R}_2$ occur in D. W.l.o.g. we may suppose that there are no reduction steps of that kind in D. Consequently, for all $j \in \mathbb{N}$, we have $s_j \rightarrow^o s_{j+1}$ by some rule from $\mathcal{R}_1$. Now $\rightarrow_{1,2}$ is semi-complete because $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are complete. Proposition 6.21 yields $top^b_{\rightarrow}(s_j) \Rightarrow_{\mathcal{R}_1} top^b_{\rightarrow}(s_{j+1})$ for every $j \in \mathbb{N}$. This is a contradiction to the termination of $\Rightarrow_{\mathcal{R}_1}$.

□

Corollary 6.23. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating CTRSs with shared constructors, then their combined system $\mathcal{R}$ is terminating provided that one of the following conditions is satisfied:

(1) Neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contain either collapsing or constructor-lifting rules.
(2) Both systems are confluent and non-duplicating.
(3) Both systems are confluent and one of the systems contains neither collapsing, constructor-lifting, nor duplicating rules.

Proof. This is an immediate consequence of Theorem 6.22. □

Corollary 6.24.

(1) Termination is modular for layer-preserving constructor-sharing CTRSs.
(2) Completeness is modular for layer-preserving constructor-sharing CTRSs.
(3) Completeness is modular for non-duplicating constructor-sharing CTRSs.

Proof. (1) is an immediate consequence of Corollary 6.23. (2) and (3) follow from Theorem 6.14 in conjunction with Corollary 6.23. □

Clearly, it also follows from the aforementioned that $\mathcal{C}_{\mathcal{E}}$-termination is a modular property of finite disjoint CTRSs; see Gramlich (1993).

6.3. Combining decreasing systems

Simple counterexamples show that innermost termination is not modular for disjoint CTRSs. So in contrast to the unconditional case (see Corollary 5.4), it is not clear *how the unique normal form of a term w.r.t. the combined system of complete pair*wise constructor-sharing CTRSs can be obtained. We will show next how this unique normal form can be computed for finite, decreasing CTRSs. Note that decreasingness is not modular, even for disjoint CTRSs. The counterexample of Toyama (1987*b*) to the modularity of termination for disjoint TRSs applies because every terminating TRS can be regarded as a decreasing CTRS.

Definition 6.25. A CTRS $\mathcal{R}$ is *decreasing* if there exists a well-founded partial ordering $>$ possessing the subterm property such that $>$ contains $\rightarrow_{\mathcal{R}}$ and for every rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n \in \mathcal{R}$ and every substitution σ we have $l\sigma > s_i\sigma$ as well as $l\sigma > t_i\sigma$, where $1 \leq i \leq n$.

Note that decreasing systems do not allow extra variables in the conditions. Decreasing (finite) CTRSs have been investigated by many researchers because all basic properties (like reducibility for instance) are decidable and a critical pair lemma holds for those systems (cf. Dershowitz *et al.* 1988). In order to show how (unique) normal forms w.r.t. the combined system of n finite, decreasing, confluent, and pairwise constructor-sharing CTRSs can be obtained, we recall the modular reduction relation introduced by Kurihara and Ohuchi (1991).

DEFINITION 6.26. Let $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ be pairwise constructor-sharing CTRSs. Let $\mathcal{F} = \bigcup_{j=1}^{n} \mathcal{F}_j$. For $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ define $s \rightsquigarrow_{\mathcal{R}_j} t$ if and only if $s \rightarrow^{+}_{\mathcal{R}_j} t$ and $t \in NF(\mathcal{F}, \mathcal{R}_j)$, where $j \in \{1, \ldots, n\}$. Moreover, define $s \rightsquigarrow t$ if and only if $s \rightsquigarrow_{\mathcal{R}_j} t$ for some $j \in \{1, \ldots, n\}$. $\rightsquigarrow$ is called *modular reduction relation.*

Roughly speaking, reduction steps (including the evaluation of the conditions) have to be performed using the same constituent CTRS $\mathcal{R}_j$ for as long as possible.

THEOREM 6.27. If $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ are pairwise constructor-sharing CTRSs, then the modular reduction relation $\rightsquigarrow$ is terminating on $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

PROOF. The same as for unconditional TRSs; see Kurihara and Ohuchi (1991). □

The proofs of the following results heavily depend on the fact that we are dealing with constructor-sharing systems instead of disjoint unions. This is probably the reason why no such results had been achieved in the investigation of disjoint unions of CTRSs.

LEMMA 6.28. Let $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$, $n \geq 2$, be semi-complete pairwise constructor-sharing CTRSs. Then $NF(\mathcal{F}, \mathcal{R}) = \bigcap_{j=1}^{n} NF(\mathcal{F}, \mathcal{R}_j)$.

PROOF. "$\subseteq$" Trivial.
"$\supseteq$" We use induction on the number n of CTRSs. The case $n = 2$ is covered by Lemma 6.13 (1). So suppose $n > 2$. First of all, by repeated application of Theorem 6.14, we infer that the CTRS $(\bigcup_{j=1}^{n-1} \mathcal{F}_j, \bigcup_{j=1}^{n-1} \mathcal{R}_j)$ is semi-complete. It is immediately obvious that the systems $(\bigcup_{j=1}^{n-1} \mathcal{F}_j, \bigcup_{j=1}^{n-1} \mathcal{R}_j)$ and $(\mathcal{F}_n, \mathcal{R}_n)$ are constructor-sharing; thus, using Lemma 6.13 (1), we derive $NF(\mathcal{F}, \mathcal{R}) = NF(\mathcal{F}, \bigcup_{j=1}^{n-1} \mathcal{R}_j) \cap NF(\mathcal{F}, \mathcal{R}_n)$. The equality

$$NF(\mathcal{F}, \bigcup_{j=1}^{n-1} \mathcal{R}_j) = \bigcap_{j=1}^{n-1} NF(\mathcal{F}, \mathcal{R}_j)$$

remains to be shown. Set $\mathcal{F}' = \mathcal{F} \setminus (\bigcup_{j=1}^{n-1} \mathcal{F}_j)$. It is not difficult to verify that the CTRSs $(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{R}_1), \ldots, (\mathcal{F}_{n-1} \uplus \mathcal{F}', \mathcal{R}_{n-1})$ are semi-complete and pairwise constructor-sharing because $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_{n-1}, \mathcal{R}_{n-1})$ are semi-complete and pairwise constructor-sharing. An application of the induction hypothesis yields

$$NF(\mathcal{F}, \bigcup_{j=1}^{n-1} \mathcal{R}_j) = NF(\bigcup_{j=1}^{n-1} (\mathcal{F}_j \uplus \mathcal{F}'), \bigcup_{j=1}^{n-1} \mathcal{R}_j) = \bigcap_{j=1}^{n-1} NF(\bigcup_{i=1}^{n-1} (\mathcal{F}_i \uplus \mathcal{F}'), \mathcal{R}_j) = \bigcap_{j=1}^{n-1} NF(\mathcal{F}, \mathcal{R}_j)$$

□

PROPOSITION 6.29. If $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ are semi-complete pairwise constructor-sharing CTRSs, then $NF(\mathcal{F}, \mathcal{R}) = NF(\leadsto)$.

PROOF. "$\subseteq$" Trivial.
"$\supseteq$" Let $t \in NF(\leadsto)$ and suppose $t \notin NF(\mathcal{F}, \mathcal{R}_j)$ for some $j \in \{1, \ldots, n\}$. According to Theorem 6.14, $(\mathcal{F}, \mathcal{R}_j)$ is normalizing. Hence there is a term $t' \in NF(\mathcal{F}, \mathcal{R}_j)$ such that $t \rightarrow^{+}_{\mathcal{R}_j} t'$. It follows $t \leadsto_{\mathcal{R}_j} t'$ which contradicts the assumption $t \in NF(\leadsto)$. We conclude $t \in NF(\mathcal{F}, \mathcal{R}_j)$ for all $j \in \{1, \ldots, n\}$. Finally, it is a consequence of Lemma 6.28 that $t \in NF(\mathcal{F}, \mathcal{R})$. □

THEOREM 6.30. If $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ are semi-complete pairwise constructor-sharing CTRSs, then $\leadsto$ is complete.

PROOF. According to Theorem 6.27, $\leadsto$ is terminating. Thus it suffices to show that $\leadsto$ has unique normal forms w.r.t. reduction. Consider $t_1 \ {}^{*}\!\!\leftsquigarrow t \leadsto^{*} t_2$, where $t_1, t_2 \in NF(\leadsto)$. By the preceding proposition, $t_1, t_2 \in NF(\mathcal{F}, \mathcal{R})$. Now since $\leadsto \subseteq \rightarrow^{*}_{\mathcal{R}}$, we obtain a conversion $t_1 \ {}^{*}_{\mathcal{R}}\!\!\leftarrow t \rightarrow^{*}_{\mathcal{R}} t_2$ between the two normal forms t_1 and t_2. It follows $t_1 = t_2$ because $(\mathcal{F}, \mathcal{R})$ is confluent (indeed semi-complete) by Theorem 6.14. □

COROLLARY 6.31. Let $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ be semi-complete pairwise constructor-sharing CTRSs and let $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. The unique normal form $t\!\downarrow$ of t w.r.t. $(\mathcal{F}, \mathcal{R})$ coincides with the unique normal form $t\wr$ of t w.r.t. $\leadsto$.

PROOF. Clearly, $t \leadsto^{*} t\wr$ implies $t \rightarrow^{*}_{\mathcal{R}} t\wr$. Furthermore, $t\wr \in NF(\mathcal{F}, \mathcal{R})$ by Proposition 6.29. It follows from the semi-completeness of $(\mathcal{F}, \mathcal{R})$ that the normal form $t\!\downarrow$ of t w.r.t. $(\mathcal{F}, \mathcal{R})$ is unique. Thus $t\!\downarrow = t\wr$. □

In order to prove the principal theorem of this subsection, we have to show that decreasingness is conserved under signature extensions. This is by no means trivial. Gramlich (1994*c*) showed that several properties (like normalization, for instance) are not preserved under signature extensions.

From now on, we assume that the CTRS $(\mathcal{F}_1, \mathcal{R}_1)$ is decreasing w.r.t. the partial ordering $>_1 \subseteq \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_1, \mathcal{V})$. It will be shown that the CTRS $(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{R}_1)$ is also decreasing for any signature $\mathcal{F}'$ with $\mathcal{F}_1 \cap \mathcal{F}' = \emptyset$. First we show that $>_1$ can be extended to a partial ordering $>_2$ on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$ which has almost all the properties necessary for showing that $(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{R}_1)$ is decreasing. Hereby, $(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{R}_1)$ is considered to be the disjoint union of the CTRSs $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}', \emptyset)$. In particular, function symbols from $\mathcal{F}_1$ are black and those from $\mathcal{F}'$ are white. In the disjoint union case, it is convenient to use the following notation. Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

$$S_2(t) = \begin{cases} S^{w}_{P}(t) & \text{if } t \text{ is top black.} \\ S^{b}_{P}(t) & \text{if } t \text{ is top white.} \end{cases}$$

DEFINITION 6.32. We define $>_2$ on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$ by: $s >_2 t$ if either

(1) $rank(s) > rank(t)$, or
(2) $rank(s) = rank(t)$ and either

 (a) $top^{b}(s) >_1 top^{b}(t)$, or

(b) $top^b(s) = top^b(t)$ and $S_2(s) >_2^{mul} S_2(t)$.

LEMMA 6.33. The relation $>_2$ is well-founded on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$.

PROOF. We show by induction on $rank(s_1)$ the impossibility of an infinite sequence

$$D: \ s_1 >_2 s_2 >_2 s_3 >_2 \dots$$

If $rank(s_1) = 0$, then s_1 is a variable and there is nothing to show. If $rank(s_1) = 1$, then $rank(s_j) = 1$ for any $j \in \mathbb{N}$, and either $s_j \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ or $s_j \in \mathcal{T}(\mathcal{F}', \mathcal{V})$. In the former case, we derive

$$top^b(s_1) >_1 top^b(s_2) >_1 top^b(s_2) >_1 \dots,$$

which contradicts the well-foundedness of $>_1$, and the latter case is obviously impossible. Therefore, let $rank(s_1) = k > 1$. The induction hypothesis states that $>_2$ is well-founded on $\mathcal{T}^{<k}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$. Hence we have $rank(s_j) = k$ and further that $root(s_j)$ has the same color as $root(s_1)$ for each $j \in \mathbb{N}$. Furthermore, the multiset extension $>_2^{mul}$ of $>_2$ is well-founded on $\mathcal{M}(\mathcal{T}^{<k}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V}))$. If s_1 is top black, then there can only be a finite number of $top^b(s_j) >_1 top^b(s_{j+1})$ steps in D (due to the well-foundedness of $>_1$). If s_1 is top white, then it follows that $top^b(s_j) = \Box = top^b(s_{j+1})$ for every $j \in \mathbb{N}$. Hence there must be an index $m \in \mathbb{N}$ such that

$$S_2(s_m) >_2^{mul} S_2(s_{m+1}) >_2^{mul} S_2(s_{m+2}) >_2^{mul} \dots$$

is infinite. This contradicts the well-foundedness of $>_2^{mul}$ on $\mathcal{M}(\mathcal{T}^{<k}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V}))$ because $S_2(s_j) \in \mathcal{M}(\mathcal{T}^{<k}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V}))$ for all $j \geq m$. □

LEMMA 6.34. Let $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$. If $s \to_{\mathcal{R}_1} t$, then $s >_2 t$.

PROOF. The lemma will be established by induction on $rank(s)$. If $rank(s) = 0$, then $s \in \mathcal{V}$ and the lemma holds vacuously. So let $rank(s) = k \geq 1$. If $rank(t) < k$, then there is nothing to show. Thus assume $rank(t) = k$. We distinguish the cases:

Case (i): s is top black.
It follows from Proposition 6.17 that $s \to_{\mathcal{R}_1}^{o} t$ implies $top^b(s) \Rightarrow_{\mathcal{R}_1} top^b(t)$. Therefore, $top^b(s) >_1 top^b(t)$ and further $s >_2 t$. If on the other hand $s \to_{\mathcal{R}_1}^{i} t$, then we may write $s = C^b[\![s_1, \dots, s_j, \dots, s_n]\!] \to_{\mathcal{R}_1} C^b[\![s_1, \dots, s_j', \dots, s_n]\!] = t$. The induction hypothesis yields $s_j >_2 s_j'$ from which we immediately get $S_2(s) >_2^{mul} S_2(t)$. Now $s >_2 t$ follows because $top^b(s) = top^b(t)$.

Case (ii): s is top white.
Here $s = C^w[\![s_1, \dots, s_j, \dots, s_n]\!] \to_{\mathcal{R}_1} C^w[s_1, \dots, s_j', \dots, s_n] = t$. If the rewrite step is non-destructive, then t is indeed equal to $C^w[\![s_1, \dots, s_j', \dots, s_n]\!]$ and the assertion follows as above. Otherwise $s_j' = \hat{C}^w\{\!\{t_1, \dots, t_m\}\!\}$. Since $rank(s_j) > rank(t_i)$ for every $i \in \{1, \dots, m\}$, we infer that $[s_j] >_2^{mul} [t_1, \dots, t_m]$. Again, we conclude $S_2(s) >_2^{mul} S_2(t)$. Now $s >_2 t$ follows from $top^b(s) = \Box = top^b(t)$. □

LEMMA 6.35. If $l \to r \Leftarrow s_1 \downarrow t_1, \dots, s_n \downarrow t_n$ is a rule from $\mathcal{R}_1$ and $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$ is a substitution, then $l\sigma >_2 s_j\sigma$ and $l\sigma >_2 t_j\sigma$ for all $j \in \{1, \dots, n\}$.

PROOF. According to Proposition 4.18, σ can be decomposed into $\sigma_2 \circ \sigma_1$, where σ_1 is black and σ_2 is top white. Since $(\mathcal{F}_1, \mathcal{R}_1)$ is decreasing w.r.t. $>_1$, we have $l\sigma_1 >_1 s_j\sigma_1$ and $l\sigma_1 >_1 t_j\sigma_1$ for all $j \in \{1, \ldots, n\}$. Since $\mathcal{V}ar(s_j) \subseteq \mathcal{V}ar(l)$ and $\mathcal{V}ar(t_j) \subseteq \mathcal{V}ar(l)$ for any $j \in \{1, \ldots, n\}$, it follows $rank(l\sigma) \geq rank(s_j\sigma)$ and $rank(l\sigma) \geq rank(t_j\sigma)$. Now it follows from $top^b(l\sigma) = l\sigma_1 >_1 s_j\sigma_1 = top^b(s_j\sigma)$ and $top^b(l\sigma) >_1 top^b(t_j\sigma)$ that also $l\sigma >_2 s_j\sigma$ and $l\sigma >_2 t_j\sigma$ for all $j \in \{1, \ldots, n\}$. □

The preceding lemmata show that $>_2$ has three out of the four properties required for showing that the CTRS $(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{R}_1)$ is decreasing. Unfortunately, it lacks the subterm property. If, for example, $g, a \in \mathcal{F}'$, then $g(a) \not>_2 a$. However, we can extend $>_2$ with the subterm property. To be exact, we define $>_3 = (>_2 \cup \rhd)^+$. $>_3$ is a relation which has the subterm property and obviously Lemmata 6.34 and 6.35 also hold when $>_2$ is replaced with $>_3$. But it is not obvious that $>_3$ is a well-founded partial ordering since $>_2$ is not closed under contexts. In order to prove this, it suffices to prove its well-foundedness because $>_3$ is transitive by definition.

LEMMA 6.36. The relation $>_3 = (>_2 \cup \rhd)^+$ is well-founded on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$.

PROOF. Let $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$. We first show:

(1) If s is top black and $s \rhd t$, then $s >_2 t$.
(2) If s is top white and $s \rhd t$, then $s \geq_2 t$.

(1) If $rank(s) > rank(t)$, then the claim is true. Otherwise $rank(s) = rank(t)$ and it is clear that $s = C^b\{\!\{s_1, \ldots, s_n\}\!\}$ and $t = \hat{C}^b\{\!\{s_i, s_{i+1}, \ldots, s_{m-1}, s_m\}\!\}$, where $C^b\{, \ldots, \} \rhd \hat{C}^b\{, \ldots, \}$ and $1 \leq i \leq m \leq n$. Therefore, $top^b(s) = C^b\{, \ldots, \} \rhd \hat{C}^b\{, \ldots, \} = top^b(t)$. Since $>_1$ has the subterm property, it follows $top^b(s) >_1 top^b(t)$. Thus $s >_2 t$.
(2) As in case (1), we may assume $rank(s) = rank(t)$ and $s = C^w\{\!\{s_1, \ldots, s_n\}\!\}$ as well as $t = \hat{C}^w\{\!\{s_i, s_{i+1}, \ldots, s_{m-1}, s_m\}\!\}$, where $C^w\{, \ldots, \} \rhd \hat{C}^w\{, \ldots, \}$ and $1 \leq i \leq m \leq n$. Clearly, $top^b(s) = \Box = top^b(t)$ and $S_2(t) \subseteq S_2(s)$. Now $s \geq_2 t$ is a direct consequence.

Now suppose that there is an infinite sequence

$$D: \ s_1 >_3 s_2 >_3 s_3 >_3 \ldots$$

It follows immediately from the above claim that

$$D: \ s_1 \geq_2 s_2 \geq_2 s_3 \geq_2 \ldots$$

If there were only finitely many $s_j >_2 s_{j+1}$ steps in D, then there would be an infinite subsequence

$$D': \ s_l \rhd s_{l+1} \rhd s_{l+2} \rhd \ldots$$

in contrast to the well-foundedness of $\rhd$. Hence there are infinitely many $s_j >_2 s_{j+1}$ steps in D which contradicts the well-foundedness of $>_2$. □

PROPOSITION 6.37. If the CTRS $(\mathcal{F}_1, \mathcal{R}_1)$ is decreasing, then $(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{R}_1)$ is decreasing for any $\mathcal{F}'$ with $\mathcal{F}_1 \cap \mathcal{F}' = \emptyset$.

PROOF. Let $(\mathcal{F}_1, \mathcal{R}_1)$ be decreasing w.r.t. the partial ordering $>_1 \subseteq \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_1, \mathcal{V})$. Define $>_3$ on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{V})$ as above. Then $(\mathcal{F}_1 \uplus \mathcal{F}', \mathcal{R}_1)$ is decreasing w.r.t. $>_3$. □

PROPOSITION 6.38. Let $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ be pairwise constructor-sharing CTRSs such that every $\mathcal{R}_j$ is finite. If the systems are decreasing, then the function nf defined by

$$nf(s) = \{t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid s \leadsto^* t,\ t \in NF(\leadsto)\}$$

is computable.

PROOF. By Theorem 6.27, $\leadsto$ is terminating. The computability of the function nf is shown by induction on the well-founded partial ordering $(\mathcal{T}(\mathcal{F}, \mathcal{V}), \leadsto^+)$. Let $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, and let

$$n\mathit{f}_j(s) = \{t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid s \to^+_{\mathcal{R}_j} t,\ t \in NF(\mathcal{F}, \mathcal{R}_j)\} = \{t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid s \leadsto_{\mathcal{R}_j} t\}.$$

Note that the CTRSs $(\mathcal{F}, \mathcal{R}_1), \ldots, (\mathcal{F}, \mathcal{R}_n)$ are decreasing by Proposition 6.37. Thus, for any $j \in \{1, \ldots, n\}$, the finite set $n\mathit{f}_j(s)$ is computable, see Dershowitz *et al.* (1988). If $n\mathit{f}_j(s)$ is empty for all $j \in \{1, \ldots, n\}$, then $s \in NF(\mathcal{F}, \mathcal{R}_j)$ for all $j \in \{1, \ldots, n\}$ and thus $s \in NF(\leadsto)$. In this case $nf(s) = \{s\}$ and we are done. Otherwise, the finite set $R(s) = \bigcup_{j=1}^{n} n\mathit{f}_j(s)$ of all one step reducts of s w.r.t. $\leadsto$ is not empty. Let $t \in R(s)$. Since $s \leadsto t$, it follows from the induction hypothesis that the finite set $nf(t)$ is computable. Hence the finite set $nf(s) = \bigcup_{t \in R(s)} nf(t)$ is computable. □

THEOREM 6.39. Let $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ be pairwise constructor-sharing CTRSs such that every $\mathcal{R}_j$ is finite. If the systems are decreasing and confluent, then their combined system $(\mathcal{F}, \mathcal{R})$ is semi-complete and the unique normal form $s\!\downarrow$ of a term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with respect to $(\mathcal{F}, \mathcal{R})$ is computable by computing the normal form of s with respect to $\leadsto$.

PROOF. Since the CTRSs $(\mathcal{F}_1, \mathcal{R}_1), \ldots, (\mathcal{F}_n, \mathcal{R}_n)$ are decreasing, they are particularly terminating. Hence they are complete. Semi-completeness of $(\mathcal{F}, \mathcal{R})$ is a consequence of Theorem 6.14. It remains to prove the computability of the function which calculates the unique normal form $s\!\downarrow \in NF(\mathcal{F}, \mathcal{R})$ of a given term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. According to Theorem 6.30, $\leadsto$ is complete. Moreover, by Proposition 6.38, the unique normal form $s\wr$ of s with respect to $\leadsto$ is computable. By Corollary 6.31, $s\!\downarrow = s\wr$ which concludes the proof. □

6.4. THE SIMPLIFYING PROPERTY

In the preceding subsection, we have seen that decreasing CTRSs behave quite "nicely" w.r.t. combinations with shared constructors. The objective of this subsection is to prove that the related simplifying property behaves even nicer. That is to say, it is modular even for composable CTRSs. This will be proven by a straightforward reduction to Theorem 5.16.

DEFINITION 6.40. A CTRS $\mathcal{R}$ is *simplifying* (Kaplan, 1987) if there exists a simplification ordering $>$ with $l > r, l > s_j$, and $l > t_j$, for each rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ of $\mathcal{R}$ and every index $j \in \{1, \ldots, n\}$.

If a finite CTRS is simplifying, then it is also decreasing. The converse is not true; see Dershowitz *et al.* (1988).

DEFINITION 6.41. Let $(\mathcal{F},\mathcal{R})$ be a CTRS without extra variables. With $(\mathcal{F},\mathcal{R})$ we associate the unconditional TRS $(\mathcal{F},\mathcal{R}^u)$, where

$$\begin{aligned}\mathcal{R}^u = \quad & \{l \to r \mid l \to r \Leftarrow s_1 \downarrow t_1, \dots, s_n \downarrow t_n \in \mathcal{R}\} \\ \cup \quad & \{l \to s_j \mid l \to r \Leftarrow s_1 \downarrow t_1, \dots, s_n \downarrow t_n \in \mathcal{R};\ j \in \{1,\dots,n\}\} \\ \cup \quad & \{l \to t_j \mid l \to r \Leftarrow s_1 \downarrow t_1, \dots, s_n \downarrow t_n \in \mathcal{R};\ j \in \{1,\dots,n\}\}.\end{aligned}$$

We omit the simple proofs of the following two lemmata.

LEMMA 6.42. Let $(\mathcal{F},\mathcal{R})$ be a CTRS without extra variables and let $(\mathcal{F},\mathcal{R}^u)$ be its associated TRS. Then $(\mathcal{F},\mathcal{R})$ is simplifying if and only if $(\mathcal{F},\mathcal{R}^u)$ is simplifying.

LEMMA 6.43. Let $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ be composable CTRSs without extra variables. Then their associated unconditional TRSs $(\mathcal{F}_1,\mathcal{R}_1^u)$ and $(\mathcal{F}_2,\mathcal{R}_2^u)$ are composable.

THEOREM 6.44. The simplifying property is modular for composable CTRSs.

PROOF. Let $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ be composable CTRSs. We have to show that their combined system $(\mathcal{F},\mathcal{R})$ is simplifying if and only if both $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ are simplifying. The only-if case is straightforward. So let $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ be simplifying. We have to show that $(\mathcal{F},\mathcal{R})$ is simplifying, or equivalently by Lemma 6.42, that the TRS $(\mathcal{F},\mathcal{R}^u)$ associated with $(\mathcal{F},\mathcal{R})$ is simplifying. By Lemma 6.43, the TRSs $(\mathcal{F}_1,\mathcal{R}_1^u)$ and $(\mathcal{F}_2,\mathcal{R}_2^u)$ are composable. Hence their combined TRS $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1^u \cup \mathcal{R}_2^u)$ is simplifying according to Theorem 5.16. Now the equality $(\mathcal{F},\mathcal{R}^u) = (\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1^u \cup \mathcal{R}_2^u)$ concludes the proof. □

7. Related work and open problems

Another extension of combinations with shared constructors are hierarchical combinations. In a hierarchical combination one of the systems may use defined symbols of the other in the right-hand sides of its rewrite rules without importing the rules defining those symbols (a precise definition can be found in Ohlebusch, 1994*b*, for instance). The standard example of a hierarchical combination is the following one, where the base system

$$\mathcal{R}_1 = \left\{ \begin{array}{lcl} 0 + x & \to & x \\ S(x) + y & \to & S(x+y) \end{array} \right.$$

is extended with

$$\mathcal{R}_2 = \left\{ \begin{array}{lcl} 0 * x & \to & 0 \\ S(x) * y & \to & (x * y) + y. \end{array} \right.$$

Here the defined symbol $+$ occurs as a constructor in the right-hand side of the second rule of $\mathcal{R}_2$ and $*$ does not appear in $\mathcal{R}_1$. Clearly, $\mathcal{R}_1$ and $\mathcal{R}_2$ are complete constructor systems. Using standard techniques, their hierarchical combination $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ can also easily be shown to be a complete constructor system. But with our former results, we cannot infer completeness of $\mathcal{R}$ from the completeness of its constituents. On the other hand, the combination of $\mathcal{R}_1' = \{a \to b\}$ and $\mathcal{R}_2' = \{F(b) \to F(a)\}$ shows that almost all interesting properties are destroyed under hierarchical combinations. So what is the difference between the two examples, that is, why is the former so benign and the latter so malignant? In essence, this is due to the fact that in the right-hand side $(x * y) + y$ the function symbol $+$ from $\mathcal{D}_1$ occurs above the function symbol $*$ from $\mathcal{D}_2$, whereas in the right-hand side $F(a)$ the function symbol a from $\mathcal{D}'_1$ occurs below the function symbol F from $\mathcal{D}'_2$. This fact has been observed independently and contemporaneously by Dershowitz (1994) and Krishna Rao (1993). Other hierarchical combinations for which termination is modular are described in Fernández and Jouannaud (1994) (their results are based on the new notion of "alien-decreasingness"). We will not go into details here but just relate other known results to ours.

We have shown that semi-completeness is modular for composable TRSs and for constructor-sharing CTRSs. Very recently, Krishna Rao (1995) provided a sufficient criterion for the modularity of semi-completeness for hierarchical combinations of TRSs. His proof technique is different from ours. Moreover, we point out that there are closely related results obtained by Middeldorp (1994*a*). He proved that semi-completeness and completeness are modular for composable conditional constructor systems without extra variables. It is yet unknown if the same is true when extra variables in conditions are allowed. As a matter of fact, Middeldorp (1994*a*) conjectured that this is the case. As far as modularity of semi-completeness of CTRSs is concerned, it is definitely worthwhile to try to extend the aforementioned result to the whole class of composable CTRSs. Note, however, that the proof presented in this paper does not carry over to composable systems. By the way, the last two statements also apply to the modularity of completeness for non-duplicating CTRSs.

Several recent papers deal with the modularity of completeness for constructor systems and the more general class of overlay systems, respectively. The investigation of constructor systems started with the work of Middeldorp and Toyama (1993). Their main result has been extended to certain classes of hierarchical combinations by Krishna Rao (1993) and Dershowitz (1994). Using the strong Theorem 5.11, Gramlich (1994*b*) proved that completeness is modular for constructor-sharing overlay systems. Corollary 5.12 shows that this is true even for composable TRSs, so the result of Middeldorp and Toyama (1993) is actually a special case thereof. Lately, Gramlich (1994*c*) showed that completeness is modular for the class of disjoint conditional overlay systems with joinable critical pairs. The question whether this result extends to more general combinations has very recently also been answered affirmatively by Gramlich (1995). Using our Theorem 6.14, he was able to extend the result to constructor-sharing CTRSs.

Finally, generalizing a result of Kurihara and Ohuchi (1992), we have shown that the simplifying property is modular for composable CTRSs. Their result has also been extended by Krishna Rao (1994) to a certain class of hierarchical TRSs. In this context, the reader is also referred to Gramlich (1994*a*), Ohlebusch (1995*a*) and Kurihara and Ohuchi (1995) for related results.

Up until now, nobody has studied hierarchical combinations of CTRSs. It goes without saying that it should also be investigated which of the known modularity results for hierarchical combinations of unconditional systems can in some way be extended to conditional systems.

In a different context, Raoult and Vuillemin (1980) showed that confluence is modular for left-linear TRSs which are orthogonal to each other. Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ are called *orthogonal to each other*, if there is no overlap between a rule from $\mathcal{R}_1$ and one of $\mathcal{R}_2$ (cf. Klop, 1992). Note that this definition does not exclude the existence of critical pairs. There may be critical pairs due to overlaps between rules of $\mathcal{R}_1$ or rules of $\mathcal{R}_2$. It is easy to see that two constructor-sharing TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ are orthogonal to each other. Two composable systems $\mathcal{R}_1$ and $\mathcal{R}_2$ are, however, in general not orthogonal to each other. Overlaps between rules from $\mathcal{R}_1$ and rules from $\mathcal{R}_2$ may occur, notwithstanding the fact that these overlaps do only create critical pairs already contained in the set $CP(\mathcal{R}_1) \cup CP(\mathcal{R}_2)$ of all critical pairs between rules from $\mathcal{R}_1$ and between rules from $\mathcal{R}_2$. The reader is invited to check that in consequence of this subtle difference the proof of Raoult and Vuillemin (1980) does not extend to composable systems. Thus it is still open whether confluence is a modular property of left-linear composable TRSs.

A somewhat different approach to modularity of TRSs has been presented in Prehofer (1994). This paper deals with a property called "modular normalization", meaning that every $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ normal form of some term s can be obtained by first reducing s to an $\mathcal{R}_1$ normal form $s{\downarrow_{\mathcal{R}_1}}$ and then reducing $s{\downarrow_{\mathcal{R}_1}}$ to an $\mathcal{R}_2$ normal form. Prehofer developed sufficient criteria for this property which also cover non-complete TRSs (the main restriction being that the system $\mathcal{R}_1$ is required to be left-linear and complete). One of the given interesting applications of modular normalization is a new modular narrowing strategy.

Acknowledgements

I am grateful to Aart Middeldorp and the referees for their comments on the paper.

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