Journal of Multivariate Analysis 102 (2011) 896–907

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Bounds for mixtures of order statistics from exponentials and applications

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ARTICLE INFO

Article history:
Received 2 February 2010
Available online 22 January 2011

MSC2010:
60E15
62G30
62N05

Keywords:
Exponential distribution
Order statistics
Usual stochastic order
Hazard rate order
Mixture of distributions

ABSTRACT

This paper deals with the stochastic comparison of order statistics and their mixtures. For a random sample of size n from an exponential distribution with hazard rate λ, and for $1 \le k \le n$, let us denote by $F_{k:n}^{(\lambda)}$ the distribution function of the corresponding kth order statistic. Let us consider m random samples of same size n from exponential distributions having respective hazard rates $\lambda_1, \ldots, \lambda_m$. Assume that $p_1, \ldots, p_m > 0$, such that $\sum_{i=1}^m p_i = 1$, and let U and V be two random variables with the distribution functions $F_{k:n}^{(\lambda)}$ and $\sum_{i=1}^m p_i F_{k:n}^{(\lambda_i)}$, respectively. Then, V is greater in the hazard rate order (or the usual stochastic order) than U if and only if $\lambda \ge \sqrt[k]{\sum_{i=1}^m p_i \lambda_i^k}$, and V is smaller in the hazard rate order (or the usual stochastic order) than U if and only if $\lambda \le \min_{1\le i\le m} \lambda_i$, for all $k = 1, \ldots, n$.

These properties are used to find the best bounds for the survival functions of order statistics from independent heterogeneous exponential random variables. For the proof, we will use a mixture type representation for the distribution functions of order statistics.

1. Introduction

The paper is motivated by the importance of the stochastic comparison of some basic Markov models in reliability theory. Recall that a k-*out-of*-n system is a structure composed of n independent components which works if and only if at least k of the n components work. This model and its extensions find various applications in industrial, economic or biological fields. In particular, the Markov models in reliability deals with exponential distribution, which enjoys the phenomenon of loss of memory. A detailed description of coherent systems in reliability and basic properties of the exponential distribution can be found in the books [1–3,26]. Recall that the lifetime of a $(n+1-k)$-*out-of*-n system is the kth order statistic from the lifetimes of their n independent components. A rich literature treats the stochastic comparisons of order statistics, especially in the exponential case. We mention here the papers [2,7,8,24,29,31]. A series of recent papers, such as [9,16–18,25,33,34,36], deals with the characterization of the stochastic comparison between a k-*out-of*-n Markov system having non-identical components with a similar system having identical components. For further results in this direction, we will point out some fine ordering properties for the mixtures of some distributions of order statistics. Mixture representation for order statistics, monotony properties of the mixtures and related characteristics have been studied in many works, among which we mention the papers [3–6,12,13,15,17,18,20–23,27,28,32,35]. In particular, the paper [21] includes bounds for mixtures of order statistics on the type $\sum_{i=1}^n p_i F_{i:n}$.

Let $X_{k:n}^{(\lambda)}$ be the kth order statistic of a random sample $X_1, \ldots, X_n$ from an exponential distribution with hazard rate λ. The distribution function of $X_{k:n}^{(\lambda)}$ will be denoted by $F_{k:n}^{(\lambda)}(t),\ t \ge 0$.

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doi:10.1016/j.jmva.2011.01.006

Consider now a random variable U with the distribution function $F_{k:n}^{(\lambda)}$ and a random variable V whose distribution function F_V is a mixture of distribution functions of kth order statistics from exponentials:

$$F_V(t) = \sum_{i=1}^{m} p_i F_{k:n}^{(\lambda_i)}(t), \quad t \geq 0,$$

where $\lambda_1, \ldots, \lambda_m$ and $p_1, \ldots, p_m$ are $2m$ arbitrary positive numbers, such that $\sum_{i=1}^{m} p_i = 1$.

We show that

$$V \geq_{hr} U \Leftrightarrow V \geq_{st} U \Leftrightarrow \lambda \geq \left(\sum_{i=1}^{m} p_i \lambda_i^k\right)^{\frac{1}{k}}, \tag{1}$$

and

$$V \leq_{hr} U \Leftrightarrow V \leq_{st} U \Leftrightarrow \lambda \leq \min_{i=1,\ldots,m} \lambda_i, \tag{2}$$

where $\geq_{st}$ designates the usual stochastic order and $\geq_{hr}$ designates the hazard rate order. Notice that, for $k = 1$, these relations can be derived from Lemma 2.1 in [25].

Further, we establish that, for $k > 1$, the distribution of the random variable $X_{k:n}$ is the convolution between the distribution of the first order statistic $X_{1:n}$ and a mixture involving the distributions of $(k-1)$th order statistics corresponding to all subsets with $n-1$ elements of the set $\{X_1, \ldots, X_n\}$. We will apply these results to find the best bounds for the survival functions of order statistics from independent heterogeneous exponential random variables. Thus, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$, and common hazard rate λ, respectively. Firstly, we find an alternative proof for the following equivalence given in [9]:

$$X_{k:n} \geq_{st} Y_{k:n} \Leftrightarrow \lambda \geq \left(\binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}\right)^{\frac{1}{k}}, \quad k = 1, 2, \ldots, n. \tag{3}$$

Secondly, we obtain the following extension:

$$X_{k:n} \leq_{st} Y_{k:n} \Leftrightarrow \lambda \leq \frac{\sum_{i=1}^{n+1-k} \lambda_{(i)}}{n+1-k}, \quad k = 1, 2, \ldots, n, \tag{4}$$

where $\lambda_{(1)} \leq \lambda_{(2)} \leq \cdots \leq \lambda_{(n)}$ is the increasing arrangement of $\lambda_1, \ldots, \lambda_n$. The relations are evident for $k = 1$. For $k = 2$, the equivalences (3) and (4) also hold in the case of the hazard rate order (see [25]). In the same case $k = 2$, similar characterizations have been recently formulated for the likelihood ratio order (see [36]), for the mean residual life order (see [33]), and for the dispersive ordering (see [34]). For $k = n$, relation (3) is valid for the hazard rate order (see [16]).

2. Preliminaries

Let us recall some notions and elementary results which are useful in what follows. The comparison of positive random variables (lifetimes) is a developed concern of applied probabilities, particularly of reliability theory. The most important ordering for lifetimes is the usual stochastic order. On the other hand, the hazard rate order lies in a central position among the strong stochastic orders.

Definition 2.1. Let X and Y be two positive absolute continuous random variables with the distribution functions $F_X(t) = P(X \leq t)$ and $F_Y(t) = P(Y \leq t)$, survival functions $\overline{F}_X(t) = P(X > t)$ and $\overline{F}_Y(t) = P(Y > t)$, density functions $f_X(t)$ and $f_Y(t)$, and hazard rate functions $r_X(t) = f_X(t)/\overline{F}_X(t)$ and $r_Y(t) = f_Y(t)/\overline{F}_Y(t)$, respectively.

1. X is said to be greater than Y in the usual stochastic order, denoted by $X \geq_{st} Y$, if $\overline{F}_X(t) \geq \overline{F}_Y(t)$, for all $t \geq 0$.
2. X is said to be greater than Y in the hazard rate order, denoted by $X \geq_{hr} Y$, if the function $\overline{F}_X(t)/\overline{F}_Y(t)$ is monotone non-decreasing in $t \geq 0$ (an equivalent definition: $r_X(t) \leq r_Y(t)$, for all $t \geq 0$).

The following implication holds: $X \geq_{hr} Y \Rightarrow X \geq_{st} Y$. A comprehensive treatment of the properties of the stochastic orders can be found in [30].

Let us now present some elementary properties of order statistics. In what follows, the set $\{1, \ldots, n\}$ will be denoted by N_n and the cardinality of some finite set I will be denoted by $|I|$. Let $X_1, \ldots, X_n$ be independent positive random variables. Denote by $X_{1:n} \leq \cdots \leq X_{n:n}$ the associated order statistics. For all $k \in N_n$, the random variable $X_{k:n}$ has the following survival function:

$$\overline{F}_{X_{k:n}}(t) = \sum_{s=0}^{k-1} \sum_{\substack{I \subset N_n; \\ |I|=s}} \left[\prod_{i \in I} F_{X_i}(t) \prod_{j \in N_n \setminus I} \overline{F}_{X_j}(t)\right], \quad t \geq 0.$$

In what follows our attention will be focused on the particular case of exponential distributions. For n independent exponential random variables X_i, with respective hazard rates λ_i, we have:

$$\overline{F}_{X_{k:n}}(t) = \sum_{s=0}^{k-1} \sum_{\substack{I \subset N_n; \\ |I|=s}} \left[e^{-(\sum_{j \in N_n \setminus I} \lambda_j)t} \prod_{i \in I} (1 - e^{-\lambda_i t}) \right], \quad t \geq 0. \tag{5}$$

The density function of $X_{k:n}$ is:

$$f_{X_{k:n}}(t) = -\overline{F}'_{X_{k:n}}(t) = \sum_{\substack{I \subset N_n; \\ |I|=k-1}} \left[\left(\sum_{j \in N_n \setminus I} \lambda_j \right) e^{-(\sum_{j \in N_n \setminus I} \lambda_j)t} \prod_{i \in I} (1 - e^{-\lambda_i t}) \right], \quad t \geq 0. \tag{6}$$

Consider now the particular case of identical parameters. Assume that $\lambda_i = \lambda$, for all $i \in N_n$, and denote by $X_{k:n}^{(\lambda)}$ the kth order statistics from X_i. Also use the following notations $F_{k:n}^{(\lambda)}, \overline{F}_{k:n}^{(\lambda)}, f_{k:n}^{(\lambda)}$ and $r_{k:n}^{(\lambda)}$ for the distribution function, the survival function, the density function and the hazard rate function of $X_{k:n}^{(\lambda)}$, respectively. From (5) and (6) we get

$$\overline{F}_{k:n}^{(\lambda)}(t) = \sum_{s=0}^{k-1} \binom{n}{s} e^{-\lambda t(n-s)} (1 - e^{-\lambda t})^s, \quad t \geq 0; \tag{7}$$

$$f_{k:n}^{(\lambda)}(t) = \lambda(n+1-k) \binom{n}{k-1} e^{-\lambda t(n+1-k)} (1 - e^{-\lambda t})^{k-1}, \quad t \geq 0; \tag{8}$$

$$\begin{aligned} r_{k:n}^{(\lambda)}(t) &= \frac{\lambda(n+1-k) \binom{n}{k-1} e^{-\lambda t(n+1-k)} (1 - e^{-\lambda t})^{k-1}}{\sum_{s=0}^{k-1} \binom{n}{s} e^{-\lambda t(n-s)} (1 - e^{-\lambda t})^s} \\ &= \frac{\lambda(n+1-k) \binom{n}{k-1} (e^{\lambda t} - 1)^{k-1}}{\sum_{s=0}^{k-1} \binom{n}{s} (e^{\lambda t} - 1)^s}, \quad t \geq 0. \end{aligned} \tag{9}$$

Relation (9) can be rewritten in a well-known integral form:

$$r_{k:n}^{(\lambda)}(t) = \frac{\lambda (e^{\lambda t} - 1)^{k-1}}{\int_0^1 y^{n-k} (e^{\lambda t} - y)^{k-1} dy}, \quad t \geq 0. \tag{10}$$

Note that this integral formula can be obtained by using the properties of Euler's function *Beta*. A consequence of this formula is the fact that

$$r_{k:n}^{(\lambda)}(t) = \frac{\lambda}{\int_0^1 y^{n-k} \left(1 + \frac{1-y}{e^{\lambda t}-1}\right)^{k-1} dy},$$

increases in λ, for all $t > 0$ and all $k \in N_n$. Then:

$$\lambda \leq \lambda' \Rightarrow r_{k:n}^{(\lambda)}(t) \leq r_{k:n}^{(\lambda')}(t), \quad \forall t \geq 0. \tag{11}$$

Actually, this conclusion can be directly derived by using Theorem 1.B.34 of [30].

3. Mixtures of distributions of order statistics

Our purpose is to compare a mixture of distributions of kth order statistics from exponential samples of size n with a single distribution of the same type. More precisely, we will describe sharp bounds on the survival and hazard rate functions of order statistics from randomly chosen i.i.d. exponential populations with various scale parameters by means of respective functions of order statistics from fixed i.i.d. exponential populations. The interest in considering this kind of problem is revealed in the last section. Thus, these results will be used to obtain (by induction) precise lower and upper uniform evaluations of survival functions of arbitrarily fixed order statistics from independent non-identically distributed exponential samples by means of survival functions of the order statistics for the i.i.d. exponential samples.

We need some technical lemmas.

Lemma 3.1. *Let n and k be two positive integers with $k \le n$. The function $g_{k:n} : [0, \infty) \to [0, \infty)$, defined by*

$$g_{k:n}(x) = \frac{x^{1/k}(e^{x^{1/k}} - 1)^{k-1}}{\int_0^1 y^{n-k}(e^{x^{1/k}} - y)^{k-1}dy}, \quad x \ge 0,$$

is monotone increasing and concave on $[0, \infty)$.

Proof. Assume two fixed positive integers k and n, such that $k \le n$. For $k = 1$, $g_{1:n}(x) = nx,\ x \ge 0$, which is a concave increasing function. For $k \ge 2$, let us consider the auxiliary functions $z, h : [0, \infty) \to [0, \infty)$, defined by $z(x) = x^{1/k}$, for all $x \ge 0$, and $h(z) = \frac{(e^z-1)^{k-1}}{\int_0^1 y^{n-k}(e^z-y)^{k-1}dy}$, for all $z \ge 0$. We have $g_{k:n}(x) = z(x)h(z(x))$, for all $x \ge 0$. Since $g'_{k:n}(x) = \frac{1}{k}z^{1-k}(x)[h(z(x)) + z(x)h'(z(x))],\ x > 0$ and

$$h'(z) = \frac{(k-1)e^z(e^z-1)^{k-2}\int_0^1 y^{n-k}(1-y)(e^z-y)^{k-2}dy}{\left(\int_0^1 y^{n-k}(e^z-y)^{k-1}dy\right)^2} > 0, \quad \forall z > 0, \tag{12}$$

we find $g'_{k:n}(x) > 0,\ \forall x > 0$. Therefore $g_{k:n}$ is monotone increasing on $[0, \infty)$.

Then $g''_{k:n}(x) = -\frac{z^{1-2k}(x)\Delta(z(x))}{k^2},\ x > 0$, where the function $\Delta : (0, \infty) \to \mathbb{R}$ is defined by

$$\Delta(z) = (k-1)h(z) + (k-3)zh'(z) - z^2h''(z), \quad z > 0. \tag{13}$$

To prove the concavity of the function $g_{k:n}$ it is enough to show that $\Delta(z) > 0,\ \forall z > 0$. Observe that, from the elementary inequalities $e^z - 1 > z$ and $e^z - y \ge e^z(1-y)$, for all $z > 0$ and $y \in [0, 1]$, we find $(k-1)h(z) > zh'(z),\ \forall z > 0$. Therefore, $\Delta(z) > z\Delta_1(z)$, where $\Delta_1(z) = (k-2)h'(z) - zh''(z)$, for $z \in (0, \infty)$.

For $k = 2$ we have $\Delta_1(z) = -zh''(z) = \frac{n(n-1)ze^z(ne^z+n-1)}{(ne^z-n+1)^3} > 0,\ \forall z > 0$, so Δ is positive.

For $k \ge 3$, from (12) and the well-known inequality $e^z - 1 > z,\ \forall z > 0$, we obtain:

$$\Delta_1(z) > \frac{z}{e^z-1}(k-2)h'(z) - zh''(z) = \frac{(k-1)ze^z(e^z-1)^{k-3}A(z)}{\left(\int_0^1 y^{n-k}(e^z-y)^{k-1}dy\right)^3}, \quad \forall z > 0, \tag{14}$$

where

$$\begin{aligned} A(z) = {} & (k-2)\int_0^1 y^{n-k}(1-y)(e^z-y)^{k-2}dy\int_0^1 y^{n-k}(e^z-y)^{k-1}dy \\ & - (e^z-1)\int_0^1 y^{n-k}(1-y)(e^z-y)^{k-2}dy\int_0^1 y^{n-k}(e^z-y)^{k-1}dy \\ & - (k-2)e^z\int_0^1 y^{n-k}(1-y)(e^z-y)^{k-2}dy\int_0^1 y^{n-k}(e^z-y)^{k-1}dy \\ & - (k-2)e^z(e^z-1)\int_0^1 y^{n-k}(1-y)(e^z-y)^{k-3}dy\int_0^1 y^{n-k}(e^z-y)^{k-1}dy \\ & + 2(k-1)e^z(e^z-1)\int_0^1 y^{n-k}(1-y)(e^z-y)^{k-2}dy\int_0^1 y^{n-k}(e^z-y)^{k-2}dy. \end{aligned}$$

By grouping and reducing the terms of $A(z)$, we find

$$A(z) = (e^z-1)\left\{\int_0^1 y^{n-k}(1-y)(e^z-y)^{k-2}dy\int_0^1 y^{n-k}(e^z-y)^{k-2}[e^z+(k-1)y]dy + (k-2)e^z\Delta_2(z)\right\},$$

where

$$\begin{aligned} \Delta_2(z) = {} & \int_0^1 y^{n-k}(1-y)(e^z-y)^{k-2}dy\int_0^1 y^{n-k}(e^z-y)^{k-2}dy \\ & - \int_0^1 y^{n-k}(1-y)(e^z-y)^{k-3}dy\int_0^1 y^{n-k}(e^z-y)^{k-1}dy. \end{aligned}$$

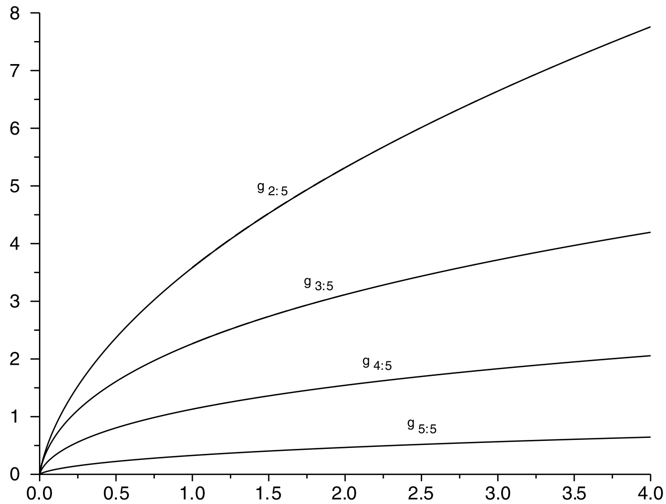

Fig. 1. Graphs of the functions $g_{k:5}$ for $k = 2, 3, 4, 5$.

Let us denote

$$U_i(z) = \int_0^1 y^{n-k}(1-y)^i(e^z - y)^{k-3}dy, \quad i = 0, 1, 2.$$

We have

$$\int_0^1 y^{n-k}(1-y)(e^z - y)^{k-2}dy = (e^z - 1)U_1(z) + U_2(z);$$
$$\int_0^1 y^{n-k}(e^z - y)^{k-2}dy = (e^z - 1)U_0(z) + U_1(z);$$
$$\int_0^1 y^{n-k}(1-y)(e^z - y)^{k-3}dy = U_1(z);$$
$$\int_0^1 y^{n-k}(e^z - y)^{k-1}dy = (e^z - 1)^2U_0(z) + 2(e^z - 1)U_1(z) + U_2(z).$$

Then we get the following useful relation

$$\Delta_2(z) = (e^z - 1)[U_0(z)U_2(z) - U_1^2(z)].$$

Assume that $z > 0$. Based on the Cauchy–Bunyakovsky–Schwarz inequality, we have $U_1^2(z) \le U_0(z)U_2(z)$. So, $\Delta_2(z) > 0$. Therefore $A(z) > 0$. Then, from inequality (14), we obtain $\Delta_1(z) > 0$. As follows, Δ is a positive function on the interval $(0, \infty)$. This ends the proof. □

The increasing monotony and the concavity of the functions $g_{k:n}$ are illustrated by Fig. 1, for $n = 5$.

Note that

$$r_{k:n}^{(\lambda)}(t) = \frac{1}{t}g_{k:n}(\lambda^k t^k), \quad \forall t > 0.$$

In what follows we need a "weighted" version of Chebyshev's sum inequality (see e.g. [19]), which is an immediate consequence of the Binet–Cauchy identity.

Lemma 3.2. *Let $p_i,\ i = 1, \ldots, m$ be m positive numbers, such that $\sum_{i=1}^m p_i = 1$. If*

$$x_1 \le x_2 \le \cdots \le x_m \quad \text{and} \quad y_1 \ge y_2 \ge \cdots \ge y_m,$$

then

$$\sum_{i=1}^m p_i x_i y_i \le \left(\sum_{i=1}^m p_i x_i\right)\left(\sum_{i=1}^m p_i y_i\right).$$

The next lemma establishes sufficient conditions to ensure a concavity property for a ratio of positive functions.

Lemma 3.3. *Let u and v be two positive functions defined on a real interval I. Assume that $g = u/v$ is concave and monotone increasing on I, and v is monotone decreasing on I. Then, for all $x_1, \ldots, x_m \in I$ and for all $p_1, \ldots, p_m > 0$, such that $\sum_{i=1}^m p_i = 1$,*

$$\frac{\sum_{i=1}^m p_i u(x_i)}{\sum_{i=1}^m p_i v(x_i)} \le g\left(\sum_{i=1}^m p_i x_i\right). \tag{15}$$

Proof. Assume that $p_1, \ldots, p_m \in (0, 1)$, with $\sum_{i=1}^m p_i = 1$, and $x_1, \ldots, x_m \in I$. Without loss of generality, we can suppose that $x_1 \le x_2 \le \cdots \le x_m$. Denote $y_i = v(x_i),\ i = 1, \ldots, m$. Since v is positive decreasing, we have $y_1 \ge \cdots \ge y_m > 0$. From the assumption that $g = u/v$ is concave, we obtain

$$\frac{\sum_{i=1}^m p_i u(x_i)}{\sum_{i=1}^m p_i v(x_i)} = \sum_{i=1}^m \frac{p_i y_i}{\sum_{j=1}^m p_j y_j} g(x_i) \le g\left(\sum_{i=1}^m \frac{p_i x_i y_i}{\sum_{j=1}^m p_j y_j}\right). \tag{16}$$

Then, Lemma 3.2 and the increasing monotony of g assure the inequality

$$g\left(\sum_{i=1}^m \frac{p_i x_i y_i}{\sum_{j=1}^m p_j y_j}\right) \le g\left(\sum_{i=1}^m p_i x_i\right). \tag{17}$$

Thus, the conclusion follows from relations (16) and (17). □

We now present the main result of this section.

Theorem 3.1. *Denote by $F_{k:n}^{(a)}$ the distribution function of kth order statistic of a random sample of size n from an exponential distribution with common hazard rate a. Let U be a random variable with the distribution function $F_U = F_{k:n}^{(\lambda)}$ and let V be another random variable with the distribution function $F_V = \sum_{i=1}^m p_i F_{k:n}^{(\lambda_i)}$, where $\lambda, \lambda_i, p_i > 0$ and $\sum_{i=1}^m p_i = 1$. Then*

$$V \ge_{hr} U \Leftrightarrow V \ge_{st} U \Leftrightarrow \lambda \ge \left(\sum_{i=1}^m p_i \lambda_i^k\right)^{\frac{1}{k}}, \tag{18}$$

and

$$V \le_{hr} U \Leftrightarrow V \le_{st} U \Leftrightarrow \lambda \le \min_{i=1,\ldots,m} \lambda_i. \tag{19}$$

Proof. Denote $\widetilde{\lambda} = (\sum_{i=1}^m p_i \lambda_i^k)^{\frac{1}{k}}$. Consider the functions $u_{k:n}, v_{k:n}, g_{k:n} : [0, \infty) \to [0, \infty)$, defined by

$$u_{k:n}(x) = (n + 1 - k)\binom{n}{k-1} x^{1/k} \mathrm{e}^{-nx^{1/k}} (\mathrm{e}^{x^{1/k}} - 1)^{k-1};$$

$$v_{k:n}(x) = \mathrm{e}^{-nx^{1/k}} \sum_{s=0}^{k-1} \binom{n}{s} (\mathrm{e}^{x^{1/k}} - 1)^s = (n + 1 - k)\binom{n}{k-1} \mathrm{e}^{-nx^{1/k}} \int_0^1 y^{n-k} (\mathrm{e}^{x^{1/k}} - y)^{k-1} \mathrm{d}y;$$

$$g_{k:n}(x) = \frac{u_{k:n}}{v_{k:n}}(x) = \frac{x^{1/k}(\mathrm{e}^{x^{1/k}} - 1)^{k-1}}{\int_0^1 y^{n-k}(\mathrm{e}^{x^{1/k}} - y)^{k-1} \mathrm{d}y}.$$

From relations (7)–(10), we find that V has the following hazard rate function

$$r_V(t) = \frac{\sum_{i=1}^m p_i f_{k:n}^{(\lambda_i)}(t)}{\sum_{i=1}^m p_i \overline{F}_{k:n}^{(\lambda_i)}(t)} = \frac{1}{t} \frac{\sum_{i=1}^m p_i u_{k:n}(\lambda_i^k t^k)}{\sum_{i=1}^m p_i v_{k:n}(\lambda_i^k t^k)}, \quad \forall t > 0. \tag{20}$$

Note that $v_{k:n}(x) = \overline{F}^{(1)}_{k:n}(x^{1/k}),\ x \geq 0$. Then $v_{k:n}$ is decreasing on $[0, \infty)$. Also, from Lemma 3.1, $g_{k:n}$ is concave and monotone increasing on $[0, \infty)$. Therefore, applying Lemma 3.3, we obtain

$$\frac{1}{t}\frac{\sum_{i=1}^{m} p_i u_{k:n}(\lambda_i^k t^k)}{\sum_{i=1}^{m} p_i v_{k:n}(\lambda_i^k t^k)} \leq \frac{1}{t} g_{k:n}\left(\sum_{i=1}^{m} p_i \lambda_i^k t^k\right) = r^{(\widetilde{\lambda})}_{k:n}(t), \quad \forall t > 0. \tag{21}$$

On the other hand (see (11)) the hazard rate function $r^{(\lambda)}_{k:n}(t)$ is monotone increasing in λ, for all positive t. Then, from inequalities (20) and (21), we obtain $\lambda \geq \widetilde{\lambda} \Rightarrow r_V(t) \leq r_U(t),\ \forall t > 0$. Thus, we have proved the statement $\lambda \geq \widetilde{\lambda} \Rightarrow V \geq_{hr} U$. Further, $V \geq_{hr} U \Rightarrow V \geq_{st} U$. Finally, assume that $V \geq_{st} U$, or $F_V(t) \leq F_U(t)$, for all $t \geq 0$. Therefore $\limsup_{t\downarrow 0} \frac{F_V(t)}{F_U(t)} \leq 1$. But, from (8), we obtain $\lim_{t\downarrow 0}(t^{1-k} f^{(\lambda)}_{k:n}(t)) = (n+1-k)\binom{n}{n-k}\lambda^k$. Then, using l'Hôpital's rule, we find

$$\lim_{t\downarrow 0}\frac{F_V(t)}{F_U(t)} = \lim_{t\downarrow 0}\frac{f_V(t)}{f_U(t)} = \lim_{t\downarrow 0}\frac{t^{1-k} f_V(t)}{t^{1-k} f_U(t)} = \frac{\sum_{i=1}^{m} p_i \lambda_i^k}{\lambda^k}.$$

Hence $\lambda \geq \widetilde{\lambda}$. Thus, (18) is proved.

Now, let us prove the second relation (19). Without loss of generality, we can assume that $\lambda_1 \leq \cdots \leq \lambda_m$. It is well known that the hazard rate function of a mixture at a fixed point t is always between the minimum and the maximum of the hazard rate functions of the members of the mixture at this point t. Hence, as $X^{(\lambda_i)}_{k:n}$ are hr-ordered, $r^{(\lambda_1)}_{k:n}$ and $r^{(\lambda_m)}_{k:n}$ are bounds for the hazard rate function of the mixture r_V for all $p_1, \ldots, p_m$. Moreover, it is well known that the hazard rate function of a mixture goes to the hazard rate of the stronger member of the mixture when $t \to \infty$ (see e.g. [6], or [22]). Hence, $r^{(\lambda_1)}_{k:n}$ is the best lower bound for r_V, that is $V \leq_{hr} U \Leftrightarrow \lambda \leq \lambda_1$. Obviously, $\lambda \leq \lambda_1 \Rightarrow V \leq_{st} U$. Now assume that $V \leq_{st} U$, i.e., $\overline{F}_V(t) \leq \overline{F}_U(t)$, for all $t \geq 0$. From (7) and the definition of U, we get $e^{t\lambda(n+1-k)}\overline{F}_U(t) = \binom{n}{k-1} + o(1)$, for $t \to \infty$. Also, from (7) and the definition of V, we obtain without difficulty the following asymptotic property:

$$e^{t\lambda_1(n+1-k)}\overline{F}_V(t) = \binom{n}{k-1}\left(\sum_{i:\lambda_i=\lambda_1} p_i\right) + o(1), \quad \text{for} \quad t \to \infty.$$

So, if $\lambda > \lambda_1$, then $\lim_{t\to\infty}\frac{\overline{F}_V(t)}{\overline{F}_U(t)} = \infty$, in contradiction with the hypothesis. Thus, $V \leq_{st} U \Rightarrow \lambda \leq \lambda_1$. This completes the proof of the theorem. $\square$

We illustrate Theorem 3.1 for $n = 5$, $k = 3$ and $m = 4$. Consider the vectors $\boldsymbol{\lambda} = (1, 5, 7, 10)$ and $\boldsymbol{p} = (0.29, 0.23, 0.42, 0.06)$. We have $a = \min \lambda_i = 1$ and $b = \sqrt[3]{\sum_{i=1}^{4} p_i \lambda_i^3} \approx 6.187$. In Fig. 2 we plot the hazard rate functions of the random variable V and the random variable U, with the parameters $\lambda = a$ and $\lambda = b$, respectively.

Note that, from Navarro and Shaked [22], we know that $r_V/r^{(a)} \to 1$ as $t \to \infty$ (see Fig. 2). Actually, this a general property, that is, r_V and $r^{(a)}$ are asymptotically equivalent as $t \to \infty$.

Comments.

1. Let us define the function $l(t) = \log(f_U(t)/f_V(t)),\ t \in (0, \infty)$. We easily establish the equivalence $\lambda \geq (\sum_{i=1}^{m} p_i \lambda_i^k)^{\frac{1}{k}} \Leftrightarrow \lim_{t\downarrow 0} l(t) \geq 1$. Then, from Theorem 1 in the recent paper Yu [32], for the proof of the equivalence (18) it suffices to check whether l is a concave function on $(0, \infty)$. However, this theoretical version of the proof can be rather difficult.
2. The characterization given by (18) is not valid for the likelihood ratio order (lr). Suppose $k = 1$. Then U is an exponential random variable with hazard rate $n\lambda$, and V is a mixture of exponential random variables with respective hazard rates $n\lambda_i,\ i = 1, \ldots, m$. From Lemma 3.1 in [36] we find

$$V \geq_{lr} U \Leftrightarrow \lambda \geq \frac{\sum_{i=1}^{m} p_i \lambda_i^2}{\sum_{i=1}^{m} p_i \lambda_i}.$$

The inequality of Cauchy–Bunyakovsky–Schwarz provides

$$\frac{\sum_{i=1}^{m} p_i \lambda_i^2}{\sum_{i=1}^{m} p_i \lambda_i} \geq \sum_{i=1}^{m} p_i \lambda_i.$$

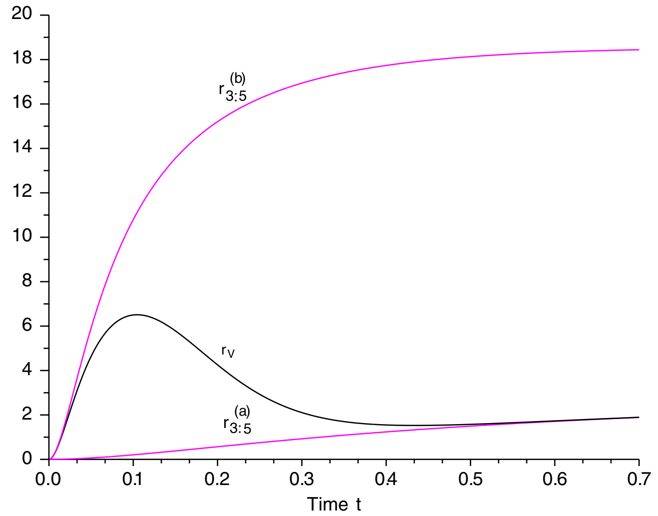

Fig. 2. The best hazard rate bounds for a mixture of distributions of third order statistics from exponential samples of size 5.

This inequality is strict for heterogeneous parameters λ_i. Therefore, if

$$\sum_{i=1}^{m} p_i\lambda_i \le \lambda < \frac{\sum_{i=1}^{m} p_i\lambda_i^2}{\sum_{i=1}^{m} p_i\lambda_i},$$

then V is not greater than U in the likelihood ratio order.

4. Bounds for the survival functions of order statistics from heterogeneous exponentials

Our goal is to indicate the best bounds for the survival function of kth order statistic from a set of heterogeneous independent exponential random variables in terms of survival functions of same kth order statistic from a set (of same size) of i.i.d. exponential random variables. Note that this problem has been partially solved in [9]. The corresponding problem for hazard rate order, has been solved in [25], but only for the second order statistic. Also, the subject was treated in [16] for the last order statistic, which corresponds to the lifetime of a parallel system in reliability. Necessary and sufficient conditions for the comparison in likelihood ratio order, mean residual life order and dispersive order of the second order statistics from independent exponential random variables were recently obtained in [36,33,34], respectively.

Essentially, to achieve the desired results, we need a convenient representation for the distributions of the order statistics from a set of independent exponential random variables.

Theorem 4.1. *Let $S = \{X_1, \ldots, X_n\}$ be a set of $n > 1$ independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$. For $i = 1, \ldots, n$, let us denote $S^{[i]} = S \setminus \{X_i\}$ and let $X^{[i]}_{j:n-1}$ be the jth order statistic from $S^{[i]}$, with the distribution function denoted by $F^{[i]}_{j:n-1}$, where $j \in \{1, \ldots, n-1\}$. Then, for all $k = 1, \ldots, n-1$, the $(k+1)$th order statistic $X_{k+1:n}$ from S is the sum of the following two independent random variables: the first order statistic $X_{1:n}$, having an exponential distribution with the hazard rate $\Lambda := \sum_{j=1}^{n} \lambda_j$, and a mixture Z_k of order statistics, whose distribution function is given by*

$$F_{Z_k}(t) = P\{Z_k \le t\} = \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F^{[i]}_{k:n-1}(t), \quad t \ge 0. \tag{22}$$

Proof. We first mention that the arguments presented below are based on some well-known properties of order statistics from exponential random variables (see e.g. [3], or [2], as general reference). Let $F_{j:n}$ be the distribution function of $X_{j:n}$, $j = 1, \ldots, n$. Assume that $k \in \{1, \ldots, n-1\}$. By conditioning on the events $\{X_{1:n} = X_i\}$, we have:

$$F_{k+1:n}(t) = \sum_{i=1}^{n} P\{X_{k+1:n} \le t | X_{1:n} = X_i\} P\{X_{1:n} = X_i\}, \quad t \ge 0, \tag{23}$$

with

$$P\{X_{1:n} = X_i\} = P\{X_i \le X_j,\ j = 1, \ldots, n\} = \frac{\lambda_i}{\Lambda}, \quad i = 1, \ldots, n. \tag{24}$$

It is well known in the literature that $X_{1:n}$ has an exponential distribution with the hazard rate Λ and is independent of other spacings (see [17], for the proof). Thus we find that the distribution of the random variable $X_{k+1:n}$, conditioned by $X_{1:n} = X_i$, is the convolution between the exponential distribution with parameter Λ and the distribution of the random variable $Z^{[i]} := [X_{k+1:n} - X_{1:n} | X_{1:n} = X_i]$, i.e., $[X_{k+1:n} | X_{1:n} = X_i]$ is the sum of the independent random variables $X_{1:n}$ and $Z^{[i]}$. On the other hand, we can easily see that the random variable $Z^{[i]}$ is the kth order statistic from the independent random variables $Z_j^{[i]} := [X_j - X_i | X_j > X_i]$, where $j \neq i$. But, from the memoryless property of the exponential distribution, we observe that $Z_j^{[i]}$ has the same distribution as X_j. As follows, $Z^{[i]}$ has the distribution function $F_{k:n-1}^{[i]}$. Denote by $G * H$ the convolution of two distribution functions G and H. Then, from (23) and (24), we obtain

$$F_{k+1:n} = \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} (F_{1:n} * F_{k:n-1}^{[i]}) = F_{1:n} * \left(\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F_{k:n-1}^{[i]} \right).$$

So we get the conclusion. □

Corollary 4.1. *In the particular case $\lambda_i = \lambda,\ i = 1, \ldots, n$, we have,*

$$F_{k+1:n}^{(\lambda)} = F_{1:n}^{(\lambda)} * F_{k:n-1}^{(\lambda)}.$$

Notice that the statement of Theorem 4.1 complements the results of Theorem 2.1 in [17]. More exactly, from the cited theorem, we can directly derive our conclusion in the case $k = 1$. In fact, the representation given by Theorem 4.1 is a consequence of the property of Markov which governs the corresponding stochastic process in continuous time, with transition rates, defined on a state space with 2^n elements. It is well known that there is a sequential formula for the distribution function of the time to entry in a given subset of this space. Thus, the proof of Theorem 4.1 can be transcribed in this language.

Now, we will apply the above results to indicate the best bounds for the survival function of some order statistic from heterogeneous independent exponential random variables in terms of the survival functions, of the same order statistic, from i.i.d. exponential random variables. In this sense, the best lower bound has been earlier obtained in [9] by using a different method.

Theorem 4.2. *Let $X_1, \ldots, X_n$ be a set of heterogeneous independent exponential random variables, with respective hazard rates $\lambda_1, \ldots, \lambda_n$. Let $Y_1, \ldots, Y_n$ be i.i.d. exponential random variables, with common hazard rate λ. Then, for all $k = 1, \ldots, n$,:*

$$X_{k:n} \geq_{st} Y_{k:n} \Leftrightarrow \lambda \geq \left(\binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right)^{\frac{1}{k}}, \tag{25}$$

$$X_{k:n} \leq_{st} Y_{k:n} \Leftrightarrow \lambda \leq \frac{\sum_{i=1}^{n+1-k} \lambda_{(i)}}{n+1-k}, \tag{26}$$

where $\lambda_{(1)} \leq \cdots \leq \lambda_{(n)}$ is the increasing arrangement of the parameters $\lambda_i,\ i = 1, \ldots, n$.

Proof. The proof of the implication

$$\lambda \geq \left(\binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right)^{1/k} \Rightarrow X_{k:n} \geq_{st} Y_{k:n} \tag{27}$$

of relation (25) is given by induction on the integers k. The assertion is evident for $k = 1$ and for all integers $n \geq 1$.

Assume that (27) holds for some positive integer k and for all integers $n \geq k$. Let us consider $n \geq k + 1$ and $\lambda \geq \left(\binom{n}{k+1}^{-1} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} \lambda_{i_1} \cdots \lambda_{i_{k+1}} \right)^{\frac{1}{k+1}}$. We will use the notations from Theorem 4.1. The induction assumption leads to

$$\overline{F}_{k:n-1}^{[i]}(t) \geq \overline{F}_{k:n-1}^{(\widetilde{\lambda}_i)}(t), \quad \text{for all } t \geq 0, \tag{28}$$

where

$$\widetilde{\lambda}_i = \left(\binom{n-1}{k}^{-1} \sum_{\substack{I \subset N_n \setminus \{i\} \\ |I| = k}} \prod_{j \in I} \lambda_j \right)^{1/k}, \quad i = 1, \ldots, n.$$

Let V be a positive random variable, independent of $X_{1:n}$, with the survival function $\overline{F}_V = \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} \overline{F}_{k:n-1}^{(\widetilde{\lambda}_i)}$. The inequality (28) and the definition of the random variable Z_k (given in the proof of Theorem 4.1) ensures $Z_k \geq_{st} V$. The closure property of the usual stochastic order under convolutions leads to $X_{1:n}+Z_k \geq_{st} X_{1:n}+V$. Then, from Theorem 4.1, we obtain $X_{k+1:n} \geq_{st} X_{1:n}+V$. Denote

$$\lambda^* = \left(\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} \widetilde{\lambda}^k\right)^{\frac{1}{k}}.$$

Let consider now a positive random variable U, independent on $X_{1:n}$, with the distribution function $F_U(t) = F_{k:n-1}^{(\lambda^*)}(t),\ t \geq 0$. From Theorem 3.1 we have $V \geq_{st} U$. Hence, $X_{1:n} + V \geq_{st} X_{1:n} + U$. So $X_{k+1:n} \geq X_{1:n} + U$.

From the definition of $\widetilde{\lambda}_i$, using the properties of the elementary symmetrical functions, we get

$$\lambda^* = \left(\Lambda^{-1}\binom{n-1}{k}^{-1}\sum_{i=1}^{n}\lambda_i \sum_{\substack{I\subset N_n\setminus\{i\}\\ |I|=k}} \prod_{j\in I}\lambda_j\right)^{\frac{1}{k}} = \left(n\Lambda^{-1}\binom{n}{k+1}^{-1} \sum_{\substack{I\subset N_n\\ |I|=k+1}} \prod_{j\in I}\lambda_j\right)^{\frac{1}{k}}.$$

Thus, we conclude that

$$\lambda^* = (m_{k+1})^{\frac{k+1}{k}}(m_1)^{-\frac{1}{k}},$$

where $m_j = m_j(\boldsymbol{\lambda})$ is the jth mean of the vector with positive components $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, defined by:

$$m_j = \left(\binom{n}{j}^{-1} \sum_{1\leq i_1<\cdots<i_j\leq n} \lambda_{i_1}\cdots\lambda_{i_j}\right)^{\frac{1}{j}}, \quad \text{for } j = 1, \ldots, n.$$

Recall Maclaurin's inequalities (see [14], or [19], for details):

$$m_1 \geq m_2 \geq \cdots \geq m_n, \tag{29}$$

where $m_1 = (\sum_{i=1}^{n}\lambda_i)/n$ and $m_n = \sqrt[n]{\prod_{i=1}^{n}\lambda_i}$. Therefore, we obtain $\lambda^* \leq m_{k+1}$.

We have assumed that $\lambda \geq m_{k+1}$. Then clearly $Y_{k+1:n} \leq_{st} W$, where W is the $(k+1)$th order statistic of a random sample of size n from an exponential distribution with hazard rate m_{k+1}. It is well known (see e.g. [3]) that W can be represented as the sum of $k+1$ independent exponential random variables $W_1, \ldots, W_k, W_{k+1}$ with the hazard rates $b_1 = (n-k)m_{k+1}, \ldots, b_k = (n-1)m_{k+1}$, $b_{k+1} = nm_{k+1}$, respectively. Similarly, since U has the distribution of the kth order statistic of a random sample of size $n-1$ from an exponential distribution with hazard rate λ^*, we can represent U as a sum of k independent exponential random variables $U_1, U_2 \cdots, U_k$ with the hazard rates $a_1 = (n-k)\lambda^*$, $a_2 = (n-k+1)\lambda^*, \ldots, a_k = (n-1)\lambda^*$, respectively. For uniformity, let us denote $U_{k+1} = X_{1:n}$, which is an exponential random variable with the hazard rate $a_{k+1} = \Lambda = nm_1$, independent of $U_1, \ldots, U_k$. Thus, to prove (under the assumption: $\lambda \geq m_{k+1}$) the desired inequality $X_{k+1:n} \geq_{st} Y_{k+1:n}$, it suffices to show that:

$$\sum_{i=1}^{k+1} U_i \geq_{st} \sum_{i=1}^{k+1} W_i. \tag{30}$$

Clearly, $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k < b_{k+1}$. From (29), $m_{k+1} \leq m_1$. It results $a_k < a_{k+1}$. Also,

$$\frac{\prod_{i=1}^{j} a_i}{\prod_{i=1}^{j} b_i} = \left(\frac{\lambda^*}{m_{k+1}}\right)^j = \left(\frac{m_{k+1}}{m_1}\right)^{\frac{j}{k}} \leq 1, \quad \text{for } j = 1, \ldots, k,$$

and

$$\frac{\prod_{i=1}^{k+1} a_i}{\prod_{i=1}^{k+1} b_i} = \frac{m_{k+1}}{m_1}\cdot\frac{nm_1}{nm_{k+1}} = 1.$$

Then from Theorem 1 in [10] we obtain (30).

Now, we will prove in a similar way the implication

$$\lambda \leq \frac{\sum_{i=1}^{n+1-k}\lambda_{(i)}}{n+1-k} \Rightarrow X_{k:n} \leq_{st} Y_{k:n}, \tag{31}$$

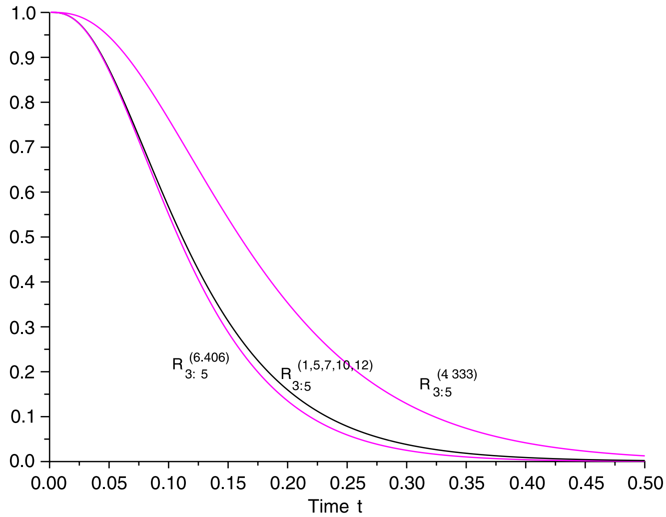

Fig. 3. The best bounds for the survival function of the order statistic 3:5 from heterogeneous independent exponentials.

where $\lambda_{(1)} \leq \cdots \lambda_{(n)}$ is the increasing arrangement of the parameters $\lambda_i,\ i = 1, \ldots, n$. Assertion (31) is clear for $k = 1$ and $n \geq k$.

Assume that (31) holds for some positive integer k and for all integers $n \geq k$. For a fixed $n \geq k + 1$, suppose $\lambda \leq \underline{\lambda}$, where $\underline{\lambda} = \frac{\sum_{i=1}^{n-k} \lambda_{(i)}}{n-k}$. Let W' be the $(k + 1)$th order statistics of a random sample of size n from an exponential distribution with hazard rate $\underline{\lambda}$. We have $Y_{k+1:n} \geq_{st} W'$ and $W' = W_1' + W_2'$, where W_1' and W_2' are independent random variables with respective distribution functions $F_{1:n}^{(\underline{\lambda})}$ and $F_{k:n-1}^{(\underline{\lambda})}$ (see Corollary 4.1). Let $\lambda_{(1)}^{[i]} \leq \cdots \leq \lambda_{(n-1)}^{[i]}$ be the increasing arrangement of the elements of the set $\{\lambda_j : j \in N_n \setminus \{i\}\}$ and denote $\underline{\lambda}_i = \frac{\sum_{j=1}^{n-k} \lambda_j^{[i]}}{n-k},\ i = 1, \ldots, n$. From the hypothesis, keeping the notations of Theorem 4.1, we have

$$\overline{F}_{k:n-1}^{[i]}(t) \leq \overline{F}_{k:n-1}^{(\underline{\lambda}_i)}(t), \quad t \geq 0,\ i = 1, \ldots, n. \tag{32}$$

Let V' be a positive random variable, independent of $X_{1:n}$, with the survival function $\overline{F}_{V'} = \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} \overline{F}_{k:n-1}^{(\underline{\lambda}_i)}$. From relation (32) we obtain $Z_k \leq_{st} V'$. Therefore, $X_{1:n} + Z_k \leq_{st} X_{1:n} + V'$ and, from Theorem 4.1, we get $X_{k+1:n} \leq_{st} X_{1:n} + V'$. We easily see that $\min_{1 \leq i \leq n} \underline{\lambda}_i = \underline{\lambda}$. Let U' be a positive random variable, independent of $X_{1:n}$, with the distribution function $F_{U'}(t) = F_{k:n-1}^{(\underline{\lambda})}(t),\ t \geq 0$. Since Theorem 3.1 provides $V' \leq_{st} U'$, we find that $X_{1:n} + V' \leq_{st} X_{1:n} + U'$. Thus $X_{k+1:n} \leq_{st} X_{1:n} + U'$. The random variables W_1' and $X_{1:n}$ are exponential with hazard rates $n\underline{\lambda}$ and Λ, respectively. Clearly, $n\underline{\lambda} \leq \Lambda$. Then $W_1' \geq_{st} X_{1:n}$. On the other hand, the random variables W_2' and U' are identically distributed. Therefore, $W_1' + W_2' \geq_{st} X_{1:n} + U'$. So, $Y_{k+1:n} \geq_{st} X_{k+1:n}$. Thus, implication (31) is proved by induction.

We now refer to the reverse implications. The proof of the implication

$$X_{k:n} \geq_{st} Y_{k:n} \Rightarrow \lambda \geq \left(\binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right)^{\frac{1}{k}}$$

can be found in [9].

Assume that $X_{k:n} \leq_{st} Y_{k:n}$. Then

$$\limsup_{t \to \infty} \frac{\overline{F}_{X_{k:n}}(t)}{\overline{F}_{Y_{k:n}}(t)} \leq 1.$$

If $\lambda > \underline{\lambda}$ then, from (5) and (7),

$$\frac{\overline{F}_{X_{k:n}}(t)}{\overline{F}_{Y_{k:n}}(t)} = e^{t(n+1-k)(\lambda - \underline{\lambda})} \frac{\overline{F}_{X_{k:n}}(t) \cdot e^{(n+1-k)\underline{\lambda} t}}{\overline{F}_{Y_{k:n}}(t) \cdot e^{(n+1-k)\lambda t}} \to \infty, \quad \text{when } t \to \infty.$$

A contradiction. So $\lambda \leq \underline{\lambda}$. □

Let us illustrate a numerical example for Theorem 4.2. Assume that $n = 5$ and $k = 3$. For $\boldsymbol{\lambda} = (1, 5, 7, 10, 12)$, we have $\sqrt[3]{\sum_{1 \leq i_1 < i_2 < i_3 \leq 5} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} / \binom{5}{3}} \approx 6.406$and $\sum_{i=1}^{3} \lambda_i / 3 \approx 4.333$. In Fig. 3, we plot the graph of the survival function

R of the random variable $X_{3:5}$ in the following situations: for heterogeneous parameters $(1, 5, 7, 10, 12)$ and for common parameters 6.406 and 4.333, respectively.

We assume that the characterization given by Theorem 4.2 is also valid for the hazard rate order. In this regard, a previous conjecture was formulated in [25]. The key point of a proof by induction seems to be a certain closure property of the hazard rate order. Note that the preservation of the hazard rate order under mixtures of exponentials has been treated in [5]. But here we deal with mixtures of order statistics from exponential random variables.

The method outlined by this paper can serve to find similar equivalences for other stochastic orderings. Moreover, the mixture representation given by Theorem 4.1 can be considered for general coherent systems. However, the *Parrondo paradox* (see [11]) shows us that we should always be cautious on the preservation of the stochastic orders under mixtures.

Acknowledgments

I am very grateful to the three referees for the careful reading of the manuscript and for all important comments and suggestions which helped to considerably improve the presentation of this paper.

References

[1] N. Balakrishnan, A.P. Basu (Eds.), The Exponential Distribution: Theory, Methods and Applications, Gordon and Breach, Newark, New Jersey, 1995.
[2] N. Balakrishnan, C.R. Rao (Eds.), Order Statistics: Theory & Methods, in: Handbook of Statistics, vol. 16, North-Holland, Amsterdam, 1998.
[3] R.E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing, Probability Models, Holt, Rinehart and Winston, New York, 1975.
[4] J. Bartoszewicz, Mixtures of exponential distributions and stochastic orders, Statistics & Probability Letters 57 (2002) 23–31.
[5] J. Bartoszewicz, M. Skolimowska, Preservation of stochastic orders under mixtures of exponential distributions, Probability in the Engineering and Informational Sciences 20 (2006) 655–666.
[6] H.W. Block, Y. Li, T.H. Savits, Initial and final behaviour of failure rate functions for mixtures and systems, Journal of Applied Probability 40 (2003) 721–740.
[7] P.J. Boland, A reliability comparison of basic systems using hazard rate functions, Applied Stochastic Models and Data Analysis 13 (1997) 377–384.
[8] P.J. Boland, E. El-Neweihi, F. Proschan, Applications of the hazard rate ordering in reliability and order statistics, Journal of Applied Probability 31 (1994) 180–192.
[9] J.-L. Bon, E. Păltănea, Comparison of order statistics in a random sequence to the same statistics with iid variables, ESAIM: Probability and Statistics 10 (2006) 1–10.
[10] J.-L. Bon, E. Păltănea, Ordering properties of convolutions of exponential random variables, Lifetime Data Analysis 5 (1999) 185–192.
[11] A. Di Crescenzo, A Parrondo paradox in reliability theory, The Mathematical Scientist 32 (2007) 17–22.
[12] M.S. Finkelstein, Failure Rate Modeling for Reliability and Risk, Springer, London, 2008.
[13] T. Fischer, N. Balakrishnan, E. Cramer, Mixture representation for order statistics from INID progressive censoring and its applications, Journal of Multivariate Analysis 99 (2008) 1999–2015.
[14] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1934.
[15] B.W. Huff, Mixtures and order statistics, American Mathematical Monthly 99 (1992) 239–242.
[16] B.-E. Khaledi, S. Kochar, Some new results on stochastic comparisons of parallel systems, Journal of Applied Probability 37 (2000) 1123–1128.
[17] S. Kochar, R. Korwar, Stochastic orders for spacings of heterogeneous exponential random variables, Journal of Multivariate Analysis 57 (1996) 69–83.
[18] S. Kochar, J. Rojo, Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions, Journal of Multivariate Analysis 59 (1996) 272–281.
[19] D.S. Mitrinović, Analytic Inequalities, Springer, Berlin, 1970.
[20] J. Navarro, N. Balakrishnan, F.J. Samaniego, Mixture representations of residual lifetimes of used systems, Journal of Applied Probability 45 (2008) 1097–1112.
[21] J. Navarro, T. Rychlik, Reliability and expectation bounds for coherent systems with exchangeable components, Journal of Multivariate Analysis 98 (2007) 102–113.
[22] J. Navarro, M. Shaked, Hazard rate ordering of order statistics and systems, Journal of Applied Probability 43 (2006) 391–408.
[23] J. Navarro, F. Spizzichino, N. Balakrishnan, Applications of average and projected systems to the study of coherent systems, Journal of Multivariate Analysis 101 (2010) 1471–1482.
[24] P. Pledger, F. Proschan, Comparison of order statistics and of spacings from heterogeneous distributions, in: J.S. Rustagi (Ed.), Optimizing Methods in Statistics, Academic Press, New-York, 1971.
[25] E. Păltănea, On the comparison in hazard rate ordering of fail-safe systems, Journal of Statistical Planning and Inference 138 (2008) 1993–1997.
[26] M. Rausand, A. Høyland, System Reliability Theory: Models, Statistical Methods, and Applications, Wiley, Hoboken, 2004.
[27] T. Rychlik, Projecting Statistical Functionals, in: Lectures Notes in Statistics, vol. 60, Springer, New York, 2001.
[28] F.J. Samaniego, System Signatures and their Applications in Engineering Reliability, in: International Series in Operations Research & Management Science, vol. 110, Springer, New York, 2007.
[29] M. Shaked, J.G. Shanthikumar, Hazard rate ordering of k-out-of-n systems, Statistic & Probability Letters 23 (1995) 1–8.
[30] M. Shaked, J.G. Shanthikumar, Stochastic Orders, Springer, New York, 2007.
[31] J.G. Shanthikumar, D.D. Yao, Bivariate characterization of some stochastic order relations, Advances in Applied Probability 23 (1991) 642–659.
[32] Y. Yu, Stochastic ordering of exponential family distributions and their mixtures, Journal of Applied Probability 46 (2009) 244–254.
[33] P. Zhao, N. Balakrishnan, Characterization of MRL order of fail-safe systems with heterogeneous exponential components, Journal of Statistical Planning and Inference 139 (2009) 3027–3037.
[34] P. Zhao, N. Balakrishnan, Dispersive ordering of fail-safe systems with heterogeneous exponential components, Metrika, doi:10.1007/s00184-010-0297-5.
[35] P. Zhao, X. Li, Stochastic order of sample range from heterogeneous exponential random variables, Probability in the Engineering and Informational Sciences 23 (2009) 17–29.
[36] P. Zhao, X. Li, N. Balakrishnan, Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables, Journal of Multivariate Analysis 100 (2009) 952–962.