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NOTE

ON A QUESTION OF S. EILENBERG

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In [1], Eilenberg quotes the following open problem: is every $\mathbb{R}_+$-rational sequence with rational coefficients $\mathbb{Q}_+$-rational?

This question is positively answered in a paper by Soittola [4], which appeared in this journal.

We show that it is not possible to generalize this result to the case of more than one variable.

Define an $\mathbb{R}_+$-rational series S on the alphabet $X = \{x, y\}$ by:

$$S = \sum_{w \in X^*} (\alpha^{2(|w|_x - |w|_y)} + \alpha^{2(|w|_y - |w|_x)})\, w$$

where $\alpha = \frac{1}{2}(\sqrt{5} + 1)$ denotes the golden ratio and $|w|_z$ denotes the number of occurrences of the letter $z \in X$ in w.

(1) Since $S = (\alpha^2 x + \alpha^{-2} y)^* + (\alpha^{-2} x + \alpha^2 y)^*$ and α, $\alpha^{-1} \in \mathbb{R}_+$, S is an $\mathbb{R}_+$-rational series.

(2) For even $n \in \mathbb{N}$, $\alpha^n + (1/\alpha)^n = \alpha^n + \beta^n$, where $\beta = \frac{1}{2}(1 - \sqrt{5}) = -1/\alpha$, is the nth term of the Lucas sequence [3], and hence a positive integer. Thus, the coefficients of S are in $\mathbb{N}$.

(3) Suppose now that S is a $\mathbb{Q}_+$-rational series: then, according to Fliess [2], S is a $\mathbb{N}$-rational series, and, according to Eilenberg [1], for any $k \in \mathbb{N}$, $L_k = \{w \in X^* : (S, w) = k\}$ is a rational subset of X^*.

(4) Since, for any $a > 0$, $a \neq 1 : a + 1/a > 2$, one has: $L_2 = \{w \in X^* : |w|_x = |w|_y\}$ which is not a rational language.

References

[1] S. Eilenberg, *Automata, Languages and Machines*, Vol. A (Academic Press, New York, 1974).
[2] M. Fliess, Séries rationnelles positives et processus stochastiques, *Ann. Inst. H. Poincaré, Sect. B* **11** (2) (1975).
[3] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers* (Clarendon Press, Oxford, 19XX).
[4] M. Soittola, Positive rational sequences, *Theoret. Comp. Sci.* **2** (3) (1976) 317–322.