Available online at www.sciencedirect.com
SCIENCE DIRECT®
Theoretical Computer Science 319 (2004) 333–356
Theoretical Computer Science
www.elsevier.com/locate/tcs

A q-enumeration of convex polyominoes by the festoon approach

Svjetlan Feretić*
Šetalište Joakima Rakovca 17, 51000 Rijeka, Croatia

Abstract

In 1938, Pólya stated an identity involving the perimeter and area generating function for parallelogram polyominoes. To obtain that identity, Pólya presumably considered *festoons*. A festoon (so named by Flajolet) is a closed path w which can be written as $w = uv$, where each step of u is either $(1,0)$ or $(0,1)$, and each step of v is either $(-1,0)$ or $(0,-1)$.

In this paper, we introduce four new festoon-like objects. As a result, we obtain explicit expressions (and not just identities) for the generating functions of parallelogram polyominoes, directed convex polyominoes, and convex polyominoes.

Keywords: Closed lattice path; Factorization; Enclosed region; Convex polyomino; q-Enumeration

1. Introduction

In 1938, George Pólya [14] made a study of lattice polygons (also known as polyominoes).[1] One of Pólya's results concerns $p(x,q)$, the perimeter and area generating function for parallelogram polyominoes.[2] Namely, $p(x,q)$ satisfies the identity

$$2x^2 + p(x,q) + p(x,q^{-1}) = 1 - \frac{1}{\sum_{n=0}^{\infty} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^2 x^{2n} q^{-i(n-i)}}, \tag{1}$$

* Corresponding author. Fax: +385-51-332-816.
E-mail address: svjetlan.feretic@gradri.hr (S. Feretić).

[1] In the above-mentioned year, Pólya just summarized that study in his diary. He wrote the related paper [14] only in the late 1960s.

[2] What is a parallelogram polyomino? See Section 2 for the definition and/or Fig. 1 for an example.

doi:10.1016/j.tcs.2004.02.011

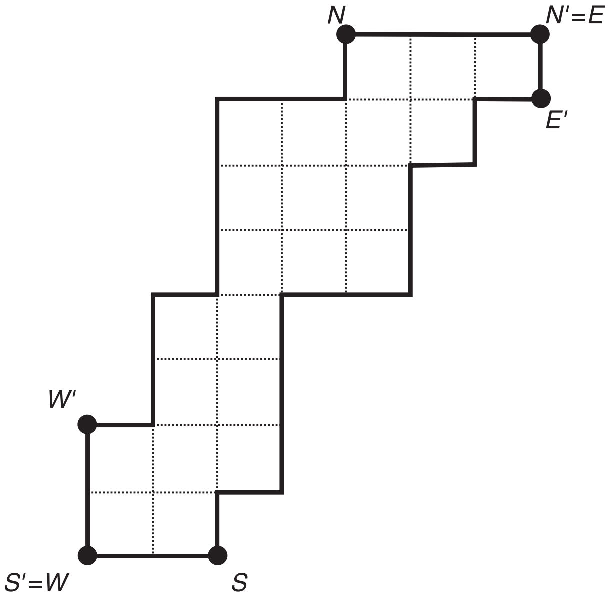

Fig. 1. A parallelogram polyomino.

where $\left[{n \atop i}\right]$ is the q-binomial coefficient, or Gaussian polynomial

$$\left[\begin{matrix} n \\ i \end{matrix}\right] = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-i+1})}{(1-q)(1-q^2)\cdots(1-q^i)}. \tag{2}$$

(In (2), all empty products are taken to be one.)

Pólya himself published no proof of (1). However, by now this identity has been proved at least twice. In [11], a paper containing a q-Lagrange inversion formula, one of the examples is right the derivation of (1). On the other hand, in [10], an attempt is made to reconstruct Pólya's original argument. (Incidentally, both papers [10,11] distinguish between horizontal and vertical perimeters.)

According to Flajolet [10], Pólya obtained identity (1) by examining *festoons*. What is a festoon? It is a closed path w which can be written in the form $w = uv$, where u is a path on the step-set $\{(1,0),(0,1)\}$, and v is a path on the step-set $\{(-1,0),(0,-1)\}$. See Fig. 2. The paths u and v can have any number $k \in \{0,1,2,\ldots\}$ of internal vertices in common. These common vertices split a festoon into, so to speak, atoms. Some atoms involve just one horizontal step of u and one horizontal step of v. Similarly, there are also small vertical atoms. These midgets excepted, any atom bounds a parallelogram polyomino. This parallelogram polyomino is either topped by u and bottomed by v, or topped by v and bottomed by u.

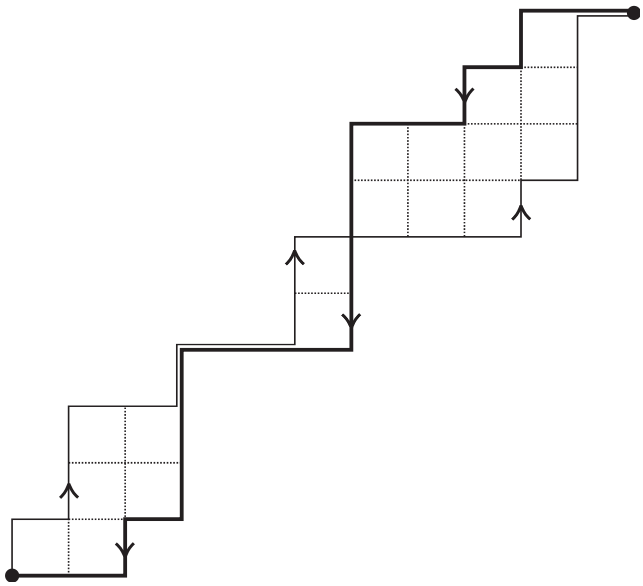

Fig. 2. A Pólya festoon (herein also called a delta festoon).

Altogether, with the variable x marking the number of steps, and the variable q marking

(the sum of the ordinates of the rightward steps)
minus (the sum of the ordinates of the leftward steps),

the generating function for festoons is equal to

$$1+\sum_{k=0}^{\infty}[2x^2+p(x,q)+p(x,q^{-1})]^{k+1}=\frac{1}{1-[2x^2+p(x,q)+p(x,q^{-1})]}. \tag{3}$$

On the other hand, this generating function is easily seen to be

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}^2 x^{2n}q^{-i(n-i)}. \tag{4}$$

(The proof of (4) uses some familiar facts about q-binomial coefficients. See Section 3 for the details.)

Equating expressions (3) and (4) at once gives identity (1).

So, this was the purported Pólya's argument. In the present paper, this argument is developed further (hopefully without a drastic loss of elegance). In addition to Pólya's festoons, we introduce four other kinds of festoons. As a result, we get explicit expressions (and not just identities) for the generating functions of parallelogram polyominoes, directed convex polyominoes, and general convex polyominoes. In the cases of parallelogram polyominoes and directed convex polyominoes, our expressions can be found in the literature (proved in a different way). In the case of general convex polyominoes, our expression is new.

In a Pólya festoon, there is no guarantee that (the sum of the ordinates of the rightward steps) is greater or equal than (the sum of the ordinates of the leftward steps). Therefore, the generating function for the Pólya festoons is a Laurent series (i.e., a series where positive, zero and negative powers of q are all involved). The same is true of our new kinds of festoons and their respective generating functions.

Before our q-enumeration (i.e., enumeration by perimeter and area), convex polyominoes have been q-enumerated about five times [2–4,6,13]. Four of those five times, the q-enumeration bears the signature of Bousquet-Mélou. In [2–4,6,13], the generating function for convex polyominoes is expressed by means of ordinary Taylor series. Not that we pride ourselves on having obtained Laurent series in place of Taylor series. However, considering the appeal of the proofs, we believe that this paper does have a raison d'être.

To continue with, now (in Section 2) we state the necessary definitions. Then we proceed to the enumerations. The agenda is the following. Section 3: Pólya festoons. Section 4: directed convex polyominoes which are not parallelogram polyominoes. Section 5: all directed convex polyominoes. Section 6: convex polyominoes which are not directed and remain not directed under the 180 degree rotation. Section 7: all convex polyominoes.

Remark 1. Our method is able to derive both the generating function (gf) for convex polyominoes and the gf for parallelogram polyominoes. Now, those two derivations have a common beginning (the enumeration of Pólya festoons), but are otherwise independent from each other. Since our main object here is the gf for convex polyominoes, we relegate the gf for parallelogram polyominoes to a remark in Section 4.

2. Definitions, conventions and notations

2.1. Convex polyominoes

If c is a closed unit square in the Cartesian plane, and if the vertices of c have integer coordinates, then c is called a cell. A *polyomino* is a union of cells which is finite and possesses connected interior. A *convex polyomino* is a polyomino having a convex intersection both with every horizontal straight line and with every vertical straight line. For certain prominent points of a convex polyomino, we use the name *poles* and we write W, W', N, $N', \dots$. See Fig. 3.

We distinguish between *true* and *untrue* convex polyominoes. A convex polyomino is true when $S' \neq W$ and $N' \neq E$. Therefore, a convex polyomino is untrue when $S' = W$ or $N' = E$.

A convex polyomino is *directed* if $S' = W$. See Fig. 4.

We consider a directed convex polyomino to be *true* if $N' \neq E$, and to be *untrue* if $N' = E$. Untrue directed convex polyominoes are commonly known as *parallelogram*

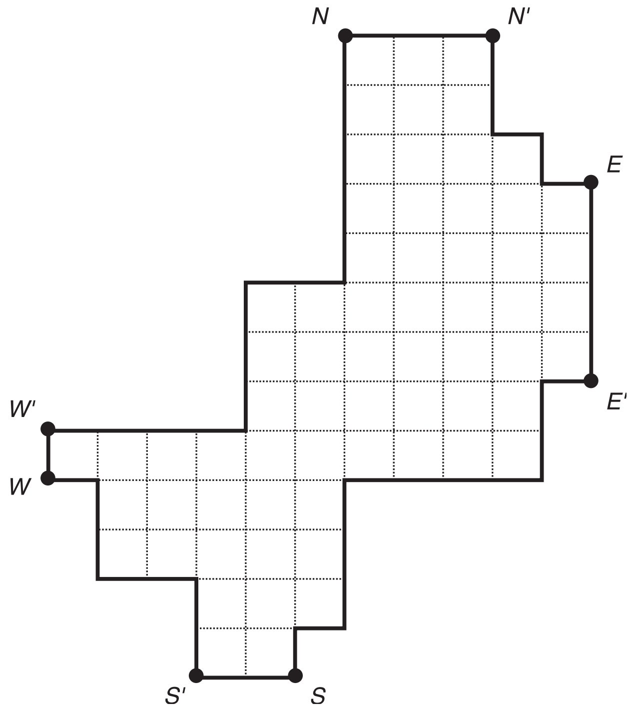

Fig. 3. A convex polyomino.

polyominoes (Fig. 1). A parallelogram polyomino with $E' = S$ is a *Ferrers diagram*. See Fig. 5.

We count polyominoes in the usual way—that is, up to translations.

Let P be a convex polyomino. Suppose that the horizontal perimeter of P is $2i$, that the vertical perimeter of P is $2j$, and that the area of P is n. Then we write $h(P) = 2i$, $v(P) = 2j$ and $a(P) = n$.

Let Θ be a subset of the set of all convex polyominoes. We define the generating function of Θ to be the formal sum

$$gf(\Theta) = \sum_{P \in \Theta} x^{h(P)} y^{v(P)} q^{a(P)}.$$

We denote the set of all convex polyominoes by C, the set of true convex polyominoes by Tc, the set of untrue convex polyominoes by Uc, the set of directed convex polyominoes by Dc, the set of true directed convex polyominoes by Tdc, the set of parallelogram polyominoes by P, and the set of Ferrers diagrams by F. For $gf(C)$, $gf(Tc)$, $gf(Uc), \ldots$, we use the abbreviated notations c, tc, $uc, \ldots$. That is, the set names begin with upper-case letters, while the generating function names begin with lower-case letters.

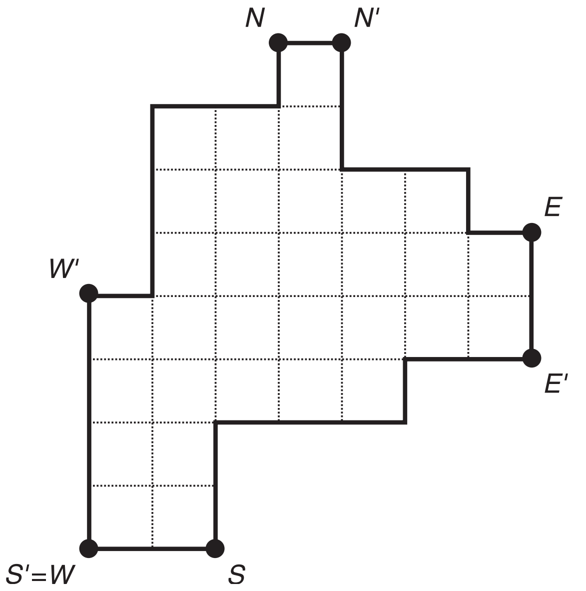

Fig. 4. A directed convex polyomino.

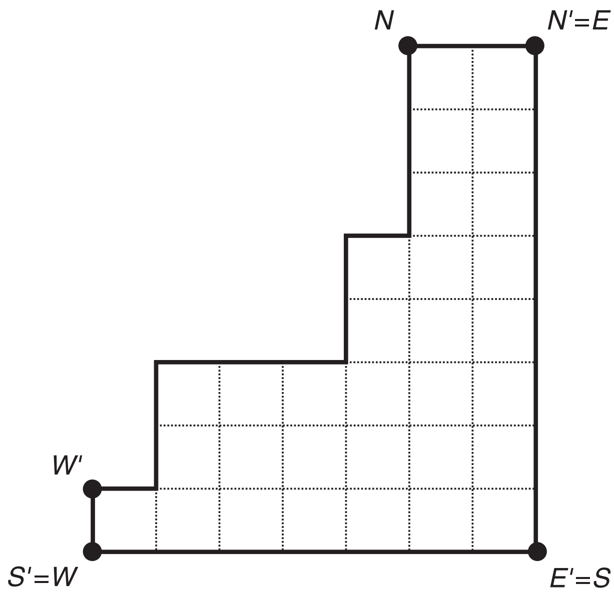

Fig. 5. A Ferrers diagram.

2.2. Paths

In this paper, our attention is confined to paths with vertices at lattice points, and with steps lying in the set $\{(1,0),(0,1),(-1,0),(0,-1)\}$. We denote the steps as follows: x is a $(1,0)$-step, y is a $(0,1)$-step, $\bar{x}$ is a $(-1,0)$-step, and $\bar{y}$ is a $(0,-1)$-step.

For $A \subseteq \{x,y,\bar{x},\bar{y}\}$, the symbol A^* stands for the set of all paths on step-set A. Let $u,v \in A^*$. The product uv is then understood as the concatenation of u and v. It is assumed that A^* contains an *empty path* 1 such that $u1 = 1u = u$, for every $u \in A^*$.

Let $u \in \{x,y,\bar{x},\bar{y}\}^*$. The symbol $|u|_x$ means the number of x-steps in u. The symbols $|u|_y$, $|u|_{\bar{x}}$ and $|u|_{\bar{y}}$ are interpreted similarly. Let $h(u) = |u|_x + |u|_{\bar{x}}$ and $v(u) = |u|_y + |u|_{\bar{y}}$. By definition, the *length* of u is the number $|u| = h(u) + v(u)$.

Let $u \in \{x,y,\bar{x},\bar{y}\}^*$ and let e be a horizontal step of u. Let v and z be the paths such that $u = vez$. Then we put $lev(e) = |v|_y - |v|_{\bar{y}}$. We say that $lev(e)$ is the *level* of e with respect to the path u.

For $u \in \{x,y,\bar{x},\bar{y}\}^*$, we define the *integral* of u to be

$$int(u) = \sum_{\substack{e \text{ a rightward} \\ \text{step of } u}} lev(e) - \sum_{\substack{e \text{ a leftward} \\ \text{step of } u}} lev(e).$$

For example, $int(y \cdot x \cdot y \cdot \bar{x} \cdot \overline{yyy} \cdot x \cdot \overline{yy} \cdot \bar{x} \cdot y) = 1 - 2 - 1 + 3 = 1$.

Let Θ be a subset of $\{x,y,\bar{x},\bar{y}\}^*$. We define the generating function of Θ to be the formal sum

$$gf(\Theta) = \sum_{u \in \Theta} x^{h(u)} y^{v(u)} q^{int(u)}.$$

If the set Θ is finite, we also define the *small generating function* of Θ

$$sgf(\Theta) = \sum_{u \in \Theta} q^{int(u)}.$$

2.3. Tailed Ferrers diagrams

For $u \in \{x,y\}^*$, let $\hat{u}$ be the closed path obtained by continuing u with $|u|_y$ downward steps, followed by $|u|_x$ leftward steps. We define a *tailed Ferrers diagram* to be a plane figure obtainable from some $u \in \{x,y\}^*$ by forming the union of $\hat{u}$ with the cells "trapped" inside $\hat{u}$. (Those "trapped" cells are $int(u)$ in number.) True to its name, a tailed Ferrers diagram looks like a Ferrers diagram with two (possibly empty) tails. See Fig. 6.

If a tailed Ferrers diagram L has area n, we write $a(L) = n$.

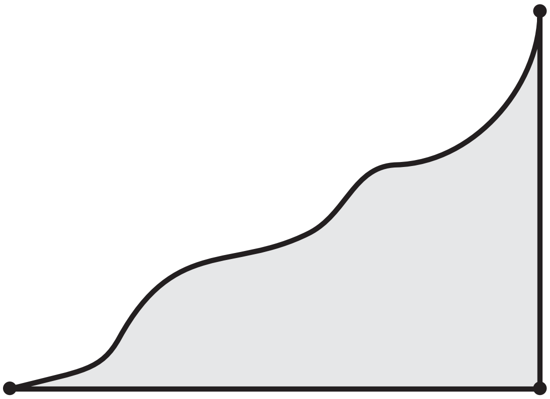

Fig. 6. Tailed Ferrers diagrams have this form.

3. Delta festoons

To begin with, we state two well-known facts about the q-binomial coefficients.

Fact 1. *Let $s_{ij} = s_{ij}(q)$ be the small generating function for S_{ij}, the set of all paths $u \in \{x, y\}^*$ such that $|u|_x = i$ and $|u|_y = j$. Let $ft_{ij} = ft_{ij}(q)$ be the area generating function for Ft_{ij}, the set of tailed Ferrers diagrams of width i and height j. We have*

$$ft_{ij} = s_{ij} = \begin{bmatrix} i+j \\ i \end{bmatrix}.$$

Let h be a formal Laurent series in the variable q. Suppose that, in h, we substitute q by q^{-1}. To denote the result (which is again a formal Laurent series), we write h_{-1}.

Fact 2. *For $i,\ j \in \mathbb{N} \cup \{0\}$ we have*

$$\begin{bmatrix} i+j \\ i \end{bmatrix}_{-1} = \begin{bmatrix} i+j \\ i \end{bmatrix} \cdot q^{-ij}.$$

The proofs of Facts 1 and 2 can be found, for example, in Andrews' book [1].

Now, a *delta festoon* is a closed path w which can be written in the form $w = uv$, where $u \in \{x, y\}^*$ and $v \in \{\bar{x}, \bar{y}\}^*$. (By stating this definition, we just provide another name for the Pólya festoons.) See Fig. 2 again. Observe that, if w is a delta festoon, then the just-mentioned factorization $w = uv$ is unique.

Denoting the set of all delta festoons by Δ, we let $\delta = gf(\Delta)$.

For $i,\ j \in \mathbb{N} \cup \{0\}$, let Δ_{ij} be the set of those $w \in \Delta$ which are made up of $2i$ horizontal steps and $2j$ vertical steps. Consider a festoon $w \in \Delta_{ij}$. Let $u \in \{x,\ y\}^*$ and $v \in \{\bar{x}, \bar{y}\}^*$ be the paths such that $w = uv$. Then $u \in S_{ij}$. Define V to be the union of $vx^i y^j$ with the cells living inside $vx^i y^j$. Clearly, V is a tailed Ferrers diagram. That is, V lies in Ft_{ij}. See Fig. 7. The correspondence $w \mapsto (u, V)$ is a bijection from Δ_{ij} to the

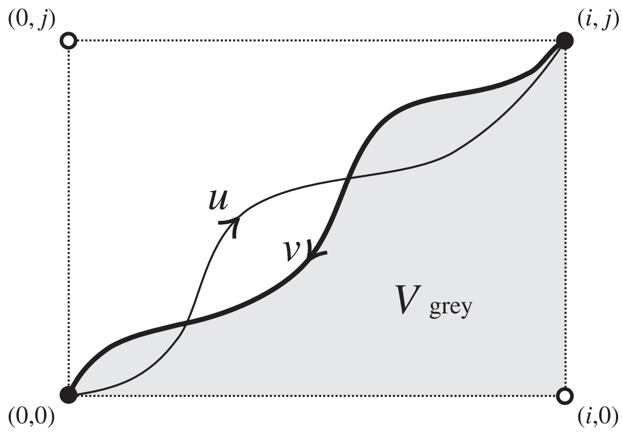

Fig. 7. The elements of Δ_{ij} have this form.

Cartesian product $S_{ij} \times Ft_{ij}$. Furthermore, we have $int(w) = int(u) - a(V)$. Altogether, this means that

$$sgf(\Delta_{ij}) = \sum_{w \in \Delta_{ij}} q^{int(w)} = \sum_{u \in S_{ij}} \sum_{V \in Ft_{ij}} q^{int(u)-a(V)}$$

$$= \left[\sum_{u \in S_{ij}} q^{int(u)}\right] \left[\sum_{V \in Ft_{ij}} q^{-a(V)}\right] = sgf(S_{ij})(ft_{ij})_{-1}$$

$$= \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ i \end{bmatrix}_{-1} = \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ i \end{bmatrix} q^{-ij}.$$

Proposition 1. *The generating function for delta festoons is given by*

$$\delta = \sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}. \qquad (5)$$

Proof. The set Δ being a disjoint union of the Δ_{ij}'s, we easily find

$$\delta = gf(\Delta) = \sum_{i,j=0}^{\infty} gf(\Delta_{ij}) = \sum_{i,j=0}^{\infty} sgf(\Delta_{ij}) x^{2i} y^{2j}$$

$$= \sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}. \qquad \square$$

4. Gamma festoons and the gf for true directed convex polyominoes

Similarly as in Section 3, we begin with stating one well-known fact about the q-binomial coefficients.

Fact 3. *Let $f_{ij} = f_{ij}(q)$ be the area generating function for F_{ij}, the set of all Ferrers diagrams of width i and height j. We have*

$$f_{ij} = \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix} q^{i+j-1}.$$

Now, a *gamma festoon* is a closed path w which can be written in the form $w = w_1 w_2 w_3$, where
($\gamma 1$) The path w_1 lies in $\{x, y\}^*$ and ends with an x-step.
($\gamma 2$) The path w_2 lies in $\{x, \bar{y}\}^*$, begins with a $\bar{y}$-step and ends with an x-step.
($\gamma 3$) The path w_3 lies in $\{\bar{x}, \bar{y}\}^*$ and begins with a $\bar{y}$-step.
See Fig. 8.

If w is a gamma festoon, then the just-mentioned factorization $w = w_1 w_2 w_3$ is unique. Indeed, the first step of w_2 is the first $\bar{y}$-step of w; the last step of w_2 is

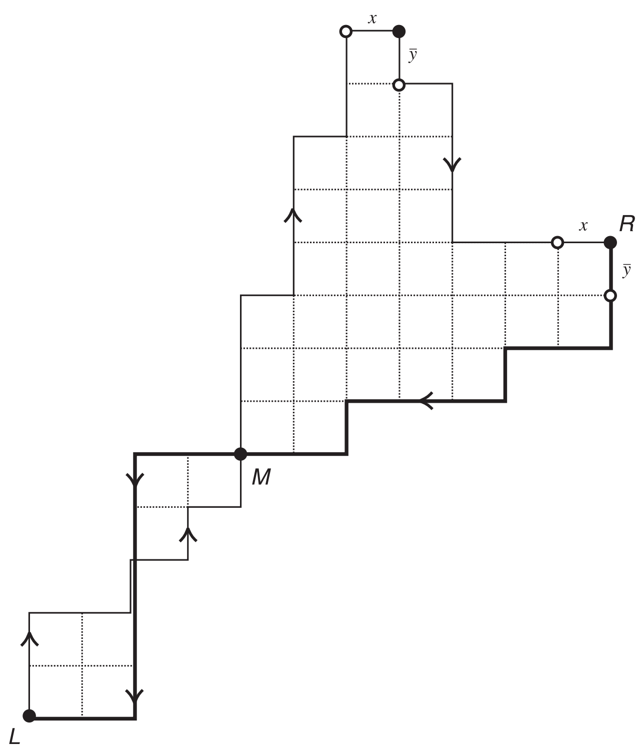

Fig. 8. A gamma festoon.

the last x-step of w. Needless to say, w_1 (resp. w_3) is the portion of w coming before (resp. after) w_2.

Denoting the set of all gamma festoons by Γ, we let $\gamma = gf(\Gamma)$.

Let w be a gamma festoon. Let $w = w_1w_2w_3$, where w_1, w_2, and w_3 have properties (γ1), (γ2), and (γ3). Imagine that we travel along the path w_1w_2 from its origin up to its *next to last* vertex. After leaving the origin of w_1w_2 (= the terminus of w_3), we may or may not meet other vertices of w_3. In either case, let M be the vertex of w_3 which we meet latest. Let $w_1w_2 = u_1u_2$ and $w_3 = z_1z_2$, where u_1 and z_1 both terminate at M. It is fairly easy to see that u_1z_2 is a delta festoon, and that u_2z_1 is the boundary of a true directed convex polyomino. Denoting this delta festoon by s and the true directed convex polyomino by J, we have

$$h(w) = h(s) + h(J), \quad v(w) = v(s) + v(J)$$

and

$$int(w) = int(s) + a(J).$$

Furthermore, the correspondence $w \mapsto (s, J)$ is a bijection from Γ to the Cartesian product $\Delta \times Tdc$. So, the generating functions of these sets are related by $\gamma = \delta \cdot tdc$, or equivalently, by

$$tdc = \frac{\gamma}{\delta}. \tag{6}$$

With the formula for δ already in hand, we now embark on computing γ. Let $i, j \in \{2, 3, \ldots\}$, $k \in \{1, \ldots, i-1\}$ and $\ell \in \{1, \ldots, j-1\}$. We define $\Gamma_{ijk\ell}$ to be the set of all gamma festoons w for which, in the factorization $w = w_1w_2w_3$ satisfying (γ1)–(γ3), it happens that

(γ4) $|w_1|_x + |w_2|_x = |w_3|_{\bar{x}} = i$,
(γ5) $|w_1|_y = |w_2|_{\bar{y}} + |w_3|_{\bar{y}} = j$,
(γ6) $|w_1|_x = k$,
(γ7) $|w_3|_{\bar{y}} = \ell$.

See Fig. 9.

Consider a festoon $w \in \Gamma_{ijk\ell}$. Let w_1, w_2, and w_3 be the paths satisfying the equation $w = w_1w_2w_3$ and conditions (γ1)–(γ7). Then we can write w_1 as $w_1 = ux$, with $u \in \{x, y\}^*$. Likewise, $w_3 = \bar{y}z$, with $z \in \{\bar{x}, \bar{y}\}^*$. Define V to be the figure bounded by the path $w_2 y^{j-\ell} \bar{x}^{i-k}$. (If we reflect V about the x-axis, the resulting figure, say V', is a Ferrers diagram.) Also, define Z to be the union of $zx^i y^{\ell-1}$ with the cells living inside $zx^i y^{\ell-1}$. The figure Z is a tailed Ferrers diagram.

It is clear that $u \in S_{k-1,j}$, $V' \in F_{i-k,j-\ell}$ and $Z \in Ft_{i,\ell-1}$. Moreover, the correspondence $w \mapsto (u, V', Z)$ is a bijection from $\Gamma_{ijk\ell}$ to the Cartesian product $S_{k-1,j} \times F_{i-k,j-\ell} \times Ft_{i,\ell-1}$. And we have

$$\begin{aligned} int(w) &= int(u) + j + (i-k)j - a(V') - a(Z) \\ &= int(u) - a(V') - a(Z) + (i-k+1)j, \end{aligned}$$

with $(i-k)j - a(V') = (i-k)j - a(V)$ coming from the path w_2.

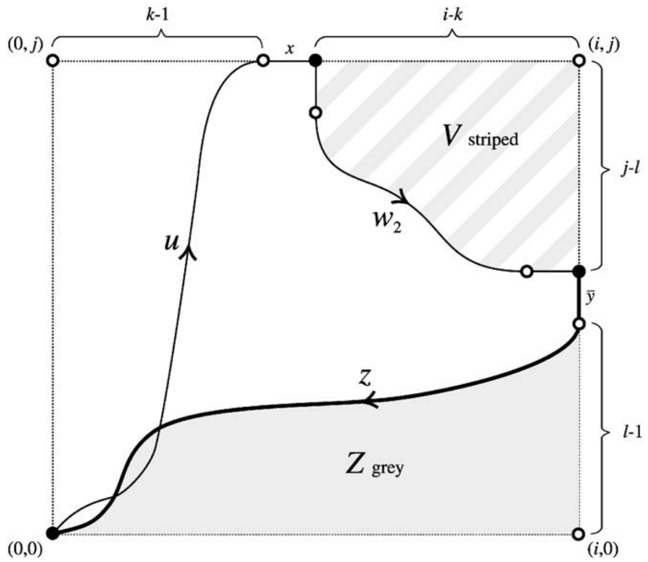

Fig. 9. The elements of $\Gamma_{ijk\ell}$ have this form.

At this point, things begin to fall into place. We see that

$$\begin{aligned} sgf(\Gamma_{ijk\ell}) &= s_{k-1,j}(f_{i-k,j-\ell})_{-1}(ft_{i,\ell-1})_{-1}q^{(i-k+1)j} \\ &= \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix} \left(\begin{bmatrix} i+j-k-\ell-2 \\ i-k-1 \end{bmatrix} q^{i+j-k-\ell-1} \right)_{-1} \\ &\quad \times \begin{bmatrix} i+\ell-1 \\ i \end{bmatrix}_{-1} q^{(i-k+1)j} \\ &= \begin{bmatrix} i+\ell-1 \\ i \end{bmatrix} \begin{bmatrix} j+k-1 \\ j \end{bmatrix} \begin{bmatrix} i+j-k-\ell-2 \\ i-k-1 \end{bmatrix} q^{i+j-k\ell}. \end{aligned}$$

Proposition 2. *The generating function for gamma festoons is given by*

$$\gamma = \sum_{i,j=2}^{\infty} \sum_{k=1}^{i-1} \sum_{\ell=1}^{j-1} \begin{bmatrix} i+\ell-1 \\ i \end{bmatrix} \begin{bmatrix} j+k-1 \\ j \end{bmatrix} \begin{bmatrix} i+j-k-\ell-2 \\ i-k-1 \end{bmatrix} x^{2i} y^{2j} q^{i+j-k\ell}. \tag{7}$$

Proof. The set Γ being a disjoint union of the $\Gamma_{ijk\ell}$'s, we obtain at once

$$\gamma = gf(\Gamma) = \sum_{i,j=2}^{\infty} \sum_{k=1}^{i-1} \sum_{\ell=1}^{j-1} gf(\Gamma_{ijk\ell})$$

$$= \sum_{i,j=2}^{\infty} \sum_{k=1}^{i-1} \sum_{\ell=1}^{j-1} sgf(\Gamma_{ijk\ell}) x^{2i} y^{2j}$$

$$= \sum_{i,j=2}^{\infty} \sum_{k=1}^{i-1} \sum_{\ell=1}^{j-1} \begin{bmatrix} i+\ell-1 \\ i \end{bmatrix} \begin{bmatrix} j+k-1 \\ j \end{bmatrix} \begin{bmatrix} i+j-k-\ell-2 \\ i-k-1 \end{bmatrix} x^{2i} y^{2j} q^{i+j-k\ell}. \quad \square$$

Now we have all the ingredients for the following theorem.

Theorem 1. *The generating function for true directed convex polyominoes is given by*

$$tdc = \frac{\sum_{i,j=2}^{\infty} \sum_{k=1}^{i-1} \sum_{\ell=1}^{j-1} \begin{bmatrix} i+\ell-1 \\ i \end{bmatrix} \begin{bmatrix} j+k-1 \\ j \end{bmatrix} \begin{bmatrix} i+j-k-\ell-2 \\ i-k-1 \end{bmatrix} x^{2i} y^{2j} q^{i+j-k\ell}}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}}.$$

Proof. Substitute (5) and (7) into (6). $\square$

Remark 2. *Parallelogram polyominoes* dwell in *epsilon festoons*, which are practically as easy as delta festoons. In fact, epsilon festoons are a subset of delta festoons: an epsilon festoon is a closed path w which can be written in the form $w = w_1 w_2$, where w_1 lies in $\{x, y\}^*$ and ends with an x-step, while w_2 lies in $\{\bar{x}, \bar{y}\}^*$ and begins with a $\bar{y}$-step. See Fig. 10.

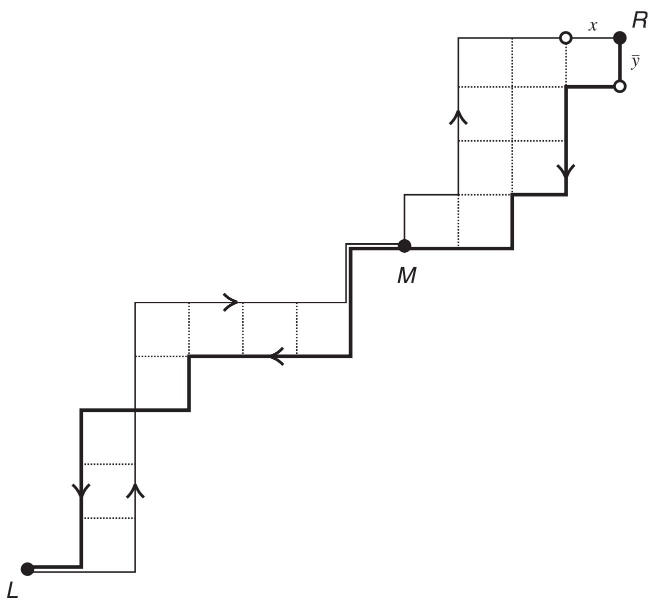

Fig. 10. An epsilon festoon.

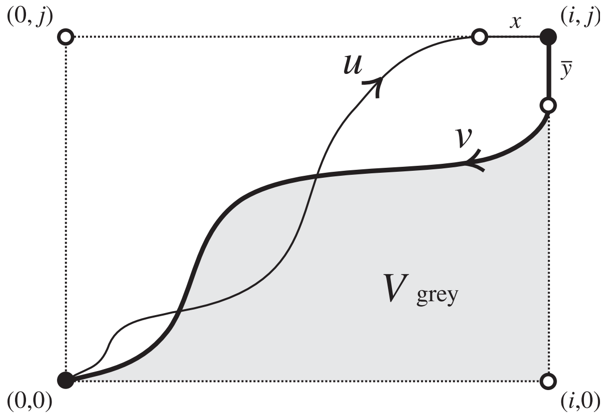

Fig. 11. The elements of E_{ij} have this form.

Denoting the set of all epsilon festoons by E, we let $\varepsilon = gf(E)$. By reapplying the argument which lead us to (6), we now find that $p = \varepsilon/\delta$.

For $i,\ j \in \mathbb{N}$, let E_{ij} be the set of those $w \in E$ which are made up of $2i$ horizontal steps and $2j$ vertical steps (Fig. 11). Consider a festoon $w \in E_{ij}$. Let $u \in \{x, y\}^*$ and $v \in \{\bar{x}, \bar{y}\}^*$ be the paths such that $w = ux\bar{y}v$. Then $u \in S_{i-1,j}$. Define V to be the union of $vx^i y^{j-1}$ with the cells living inside $vx^i y^{j-1}$. The figure V is a tailed Ferrers diagram. That is, $V \in Ft_{i,j-1}$.

The correspondence $w \mapsto (u, V)$ is a bijection. Therefore, from the fact that $int(w) = int(u) + j - a(V)$ it follows that

$$sgf(E_{ij}) = s_{i-1,j}(ft_{i,j-1})_{-1}q^j = \begin{bmatrix} i+j-1 \\ j \end{bmatrix} \begin{bmatrix} i+j-1 \\ i \end{bmatrix} q^{i+j-ij}.$$

Accordingly,

$$\begin{aligned} \varepsilon = gf(E) &= \sum_{i,j=1}^{\infty} gf(E_{ij}) = \sum_{i,j=1}^{\infty} sgf(E_{ij})x^{2i}y^{2j} \\ &= \sum_{i,j=1}^{\infty} \begin{bmatrix} i+j-1 \\ i \end{bmatrix} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} x^{2i}y^{2j}q^{i+j-ij}. \end{aligned} \tag{8}$$

Theorem 2. *The generating function for parallelogram polyominoes is given by*

$$p = \frac{\sum_{i,j=1}^{\infty} \begin{bmatrix} i+j-1 \\ i \end{bmatrix} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} x^{2i}y^{2j}q^{i+j-ij}}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i}y^{2j}q^{-ij}}. \tag{9}$$

Proof. Substitute (5) and (8) into the equation $p = \varepsilon/\delta$. □

The q-enumeration of parallelogram polyominoes has an eventful history. For an overall account see, for example, Bousquet-Mélou's *habilitation* [5]. However, if we concentrate on the Laurent-series results, all we have to tell is the following:

Formula (9) apparently originated in Goulden and Jackson's book [12, Exercise 5.5.2.b]. Feretić [9] derived (9) in another way. Fédou and Rouillon [8] expressed the gf p by a formula which looks rather different from (9), but still involves Laurent series.

5. Beta festoons and the gf for all directed convex polyominoes

A *beta festoon* is a closed path w which can be written in the form $w = w_1 w_2 w_3$, where:

(β1) The path w_1 lies in $\{x, y\}^*$ and ends with a y-step.
(β2) The path w_2 lies in $\{x, \bar{y}\}^*$, begins with an x-step and ends with a $\bar{y}$-step.
(β3) The path w_3 lies in $\{\bar{x}, \bar{y}\}^*$ and begins with an $\bar{x}$-step.

So to speak, beta festoons and gamma festoons have equal bones, but different joints. See Fig. 12.

Denoting the set of all beta festoons by B, we let $\beta = gf(B)$. Once again inferring as in the derivation of (6), we readily obtain

$$dc = \frac{\beta}{\delta}. \tag{10}$$

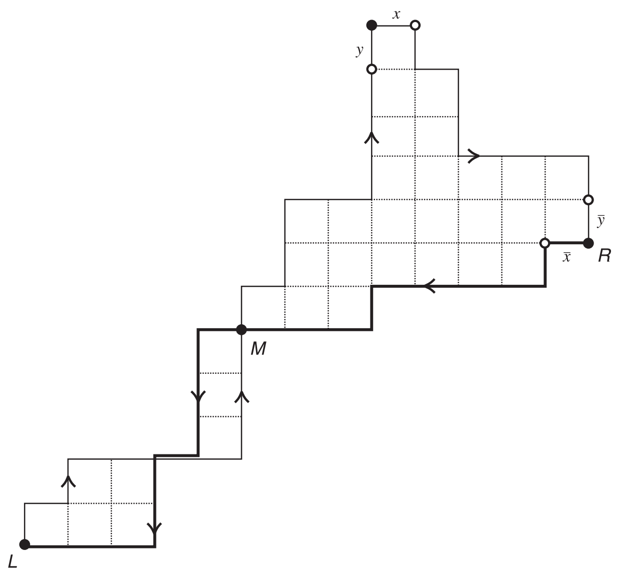

Fig. 12. A beta festoon.

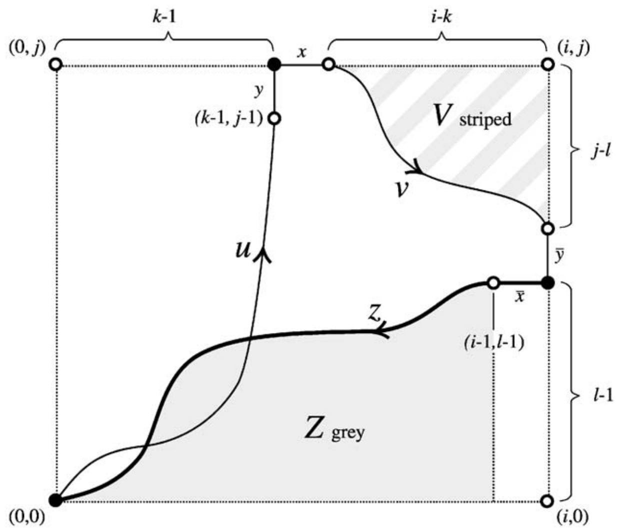

Fig. 13. The elements of $B_{ijk\ell}$ have this form.

Our next task is to compute β. Let $i,\ j \in \mathbb{N}$, $k \in \{1,\ldots,i\}$ and $\ell \in \{1,\ldots,j\}$. We define $B_{ijk\ell}$ to be the set of all beta festoons w for which, in the factorization $w = w_1w_2w_3$ satisfying (β1)–(β3), it happens that

(β4) $|w_1|_x + |w_2|_x = |w_3|_{\bar{x}} = i$,
(β5) $|w_1|_y = |w_2|_{\bar{y}} + |w_3|_{\bar{y}} = j$,
(β6) $|w_1|_x = k - 1$,
(β7) $|w_3|_{\bar{y}} = \ell - 1$.

See Fig. 13.

Consider a festoon $w \in B_{ijk\ell}$. Let w_1, w_2, and w_3 be the paths satisfying the equation $w = w_1w_2w_3$ and conditions (β1)–(β7). Then we can write w_1 as $w_1 = uy$, with $u \in \{x, y\}^*$. Likewise, $w_2 = xv\bar{y}$, with $v \in \{x, \bar{y}\}^*$, and $w_3 = \bar{x}z$, with $z \in \{\bar{x}, \bar{y}\}^*$. Define V to be the union of $vy^{j-\ell}\bar{x}^{i-k}$ with the cells living inside $vy^{j-\ell}\bar{x}^{i-k}$. (If we reflect V about the x-axis, the resulting figure, say V', is a tailed Ferrers diagram.) Also, define Z to be the union of $zx^{i-1}y^{\ell-1}$ with the cells living inside $zx^{i-1}y^{\ell-1}$. The figure Z, too, is a tailed Ferrers diagram.

It is clear that $u \in S_{k-1,j-1}$, $V' \in Ft_{i-k,j-\ell}$ and $Z \in Ft_{i-1,\ell-1}$. Moreover, the correspondence $w \mapsto (u, V', Z)$ is a bijection from $B_{ijk\ell}$ to the Cartesian product $S_{k-1,j-1} \times Ft_{i-k,j-\ell} \times Ft_{i-1,\ell-1}$. Observing that

$$int(w) = int(u) + j + (i-k)j - a(V') - (\ell - 1) - a(Z),$$

we now find that

$$sgf(B_{ijk\ell}) = s_{k-1,j-1}(ft_{i-k,j-\ell})_{-1}(ft_{i-1,\ell-1})_{-1}q^{(i-k+1)j-(\ell-1)}$$

$$= \begin{bmatrix} j+k-2 \\ k-1 \end{bmatrix} \begin{bmatrix} i+j-k-\ell \\ i-k \end{bmatrix}_{-1} \begin{bmatrix} i+\ell-2 \\ i-1 \end{bmatrix}_{-1} q^{(i-k+1)j-(\ell-1)}$$
$$= \begin{bmatrix} i+\ell-2 \\ i-1 \end{bmatrix} \begin{bmatrix} j+k-2 \\ j-1 \end{bmatrix} \begin{bmatrix} i+j-k-\ell \\ i-k \end{bmatrix} q^{i+j-k\ell}.$$

Hence

$$\beta = gf(B) = \sum_{i,j=1}^{\infty} \sum_{k=1}^{i} \sum_{\ell=1}^{j} gf(B_{ijk\ell})$$
$$= \sum_{i,j=1}^{\infty} \sum_{k=1}^{i} \sum_{\ell=1}^{j} sgf(B_{ijk\ell}) x^{2i} y^{2j}$$
$$= \sum_{i,j=1}^{\infty} \sum_{k=1}^{i} \sum_{\ell=1}^{j} \begin{bmatrix} i+\ell-2 \\ i-1 \end{bmatrix} \begin{bmatrix} j+k-2 \\ j-1 \end{bmatrix} \begin{bmatrix} i+j-k-\ell \\ i-k \end{bmatrix} x^{2i} y^{2j} q^{i+j-k\ell}. \qquad (11)$$

Theorem 3. *The generating function for all directed convex polyominoes is given by*

$$dc = \frac{\sum_{i,j=1}^{\infty} \sum_{k=1}^{i} \sum_{\ell=1}^{j} \begin{bmatrix} i+\ell-2 \\ i-1 \end{bmatrix} \begin{bmatrix} j+k-2 \\ j-1 \end{bmatrix} \begin{bmatrix} i+j-k-\ell \\ i-k \end{bmatrix} x^{2i} y^{2j} q^{i+j-k\ell}}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}}. \qquad (12)$$

Proof. Substitute (5) and (11) into (10). □

The first ever q-enumeration of directed convex polyominoes was done in [7]. In that paper, the formula for dc has the Taylor-series form. Formula (12) was first proved in [9].

6. Alpha festoons and the gf for true convex polyominoes

An *alpha festoon* is a closed path w which can be written in the form $w = w_1 w_2 w_3 w_4$, where:

(α1) The path w_1 lies in $\{x, y\}^*$, begins with a y-step and ends with an x-step.
(α2) The path w_2 lies in $\{x, \bar{y}\}^*$, begins with a $\bar{y}$-step and ends with an x-step.
(α3) The path w_3 lies in $\{\bar{x}, \bar{y}\}^*$, begins with a $\bar{y}$-step and ends with an $\bar{x}$-step.
(α4) The path w_4 lies in $\{\bar{x}, y\}^*$, begins with a y-step and ends with an $\bar{x}$-step.

See Fig. 14.

If w is an alpha festoon, then the above factorization $w = w_1 w_2 w_3 w_4$ is unique. Indeed, the last step of $w_1 w_2$ is the last x-step of w; the first step of w_2 is the first $\bar{y}$-step of $w_1 w_2$; the first step of w_4 is the first y-step of $w_3 w_4$.

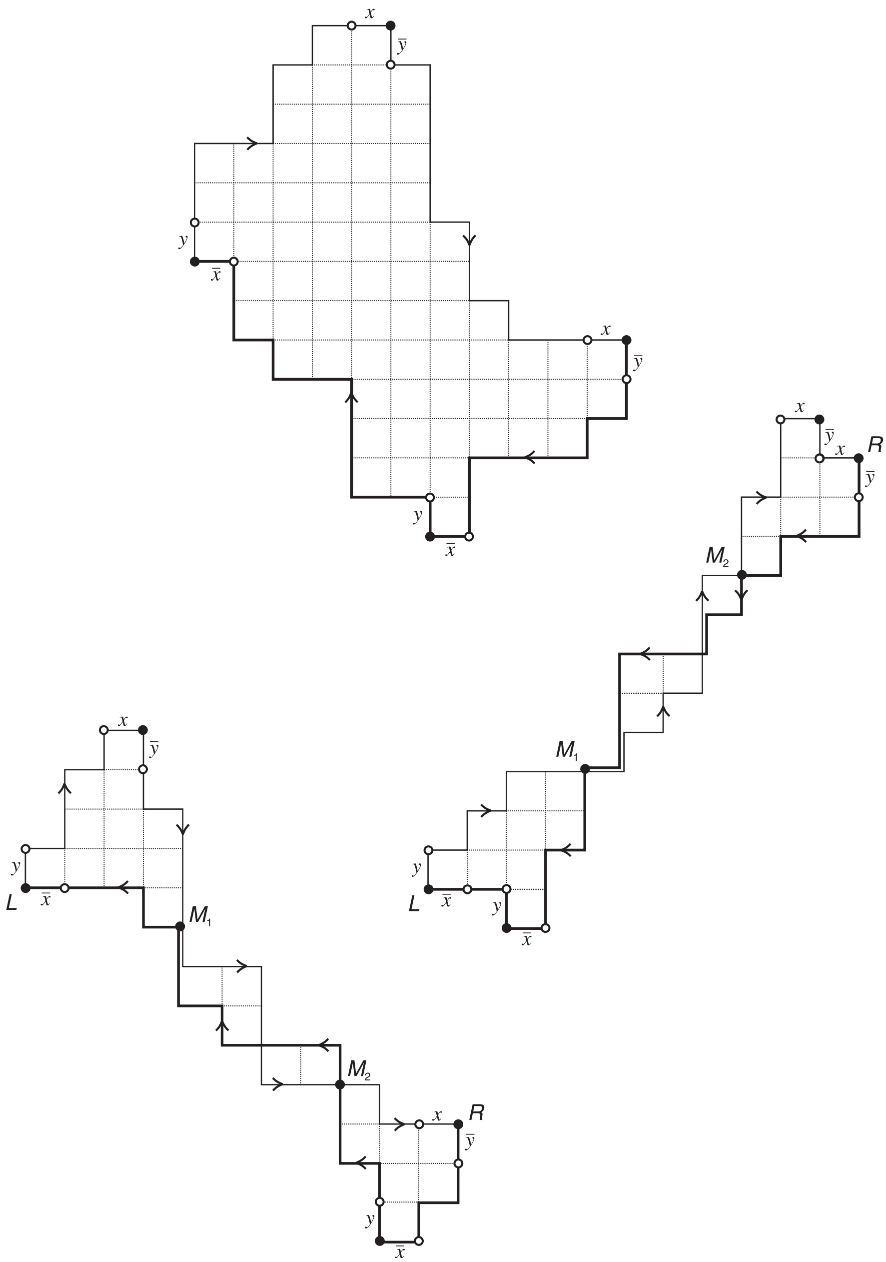

Fig. 14. Three alpha festoons.

Let w be an alpha festoon. Let $w = w_1w_2w_3w_4$, where w_1, w_2, w_3, and w_4 have properties (α1), (α2), (α3), and (α4). Naturally, w is either self-avoiding or not. If w is not self-avoiding, then there are two possibilities: that w_1 intersects w_3, and that w_2 intersects w_4. These two possibilities cannot materialize at the same time. Indeed, suppose that w_1 intersects w_3. Consider the horizontal projections of w_1 and w_3. Those projections of course overlap. But the length of the overlap is not zero—in fact, that length is at least two. Thus, the horizontal projections of w_2 and w_4 stand at least two units apart. Needless to say, it follows that w_2 and w_4 are disjoint.

Let A be the set of all alpha festoons, and let $A_{\surd}$ be the set of self-avoiding alpha festoons. For $w \in A$, let $w_1, \ldots, w_4$ again be the paths for which $w = w_1 \cdots w_4$ holds and the conditions (α1),…,(α4) are satisfied. Define

$$A_{/} = \{w \in A : w \text{ is not in } A_{\surd} \text{ because } w_1 \text{ intersects } w_3\}$$

and

$$A_{\backslash} = \{w \in A : w \text{ is not in } A_{\surd} \text{ because } w_2 \text{ intersects } w_4\}.$$

Example 1. In Fig. 14, the top festoon is in $A_{\surd}$, the middle festoon is in $A_{/}$, and the bottom festoon is in $A_{\backslash}$.

We have seen that the set A is partitioned into $A_{\surd}$, $A_{/}$, and $A_{\backslash}$. Hence, if we put $\alpha = gf(A)$, then

$$\alpha = gf(A_{\surd}) + gf(A_{/}) + gf(A_{\backslash}). \tag{13}$$

Consider a festoon $w \in A_{\surd}$. It is obvious that w bounds a true convex polyomino. Calling that polyomino J, we have $h(w) = h(J)$, $v(w) = v(J)$, and $int(w) = a(J)$. Moreover, the correspondence $w \mapsto J$ is a bijection from $A_{\surd}$ to Tc, the set of true convex polyominoes. Recasting these facts "generatingfunctionologically", we find that

$$gf(A_{\surd}) = tc. \tag{14}$$

Now consider a festoon $w \in A_{/}$. Let $w = w_1w_2w_3w_4$, where $w_1, \ldots, w_4$ have properties (α1),…,(α4). By the definition of $A_{/}$, the paths w_1 and w_3 have some vertices in common. Of those vertices, let M_1 (resp. M_2) be the one which is closest to (resp. farthest from) the origin of w_1. Let $w_1w_2 = u_1u_2u_3$ and $w_3w_4 = z_1z_2z_3$, where u_1 and z_2 terminate at M_1, and where u_2 and z_1 terminate at M_2. Now, the path $s = u_2z_2$ is a delta festoon. The path u_3z_1 bounds a true directed convex polyomino. The path u_1z_3 bounds a polyomino which, in order to become a true directed convex one, just needs to be rotated by 180°. Let I (resp. J) denote the true directed convex polyomino arising out of u_1z_3 (resp. u_3z_1). Then we have

$$h(w) = h(I) + h(s) + h(J), \quad v(w) = v(I) + v(s) + v(J)$$

and

$$int(w) = a(I) + int(s) + a(J).$$

Furthermore, the correspondence $w \mapsto (I,s,J)$ is a bijection from $A_/$ to the Cartesian product $Tdc \times \Delta \times Tdc$. Translating all of this into the language of generating functions, we obtain $gf(A_/) = tdc \cdot \delta \cdot tdc$. The series tdc being an old friend (found in Section 4), we now see that

$$gf(A_/) = \left(\frac{\gamma}{\delta}\right)^2 \delta = \frac{\gamma^2}{\delta}, \tag{15}$$

where γ and δ are given by (7) and (5), respectively.

Finally, consider a festoon $w \in A_\backslash$. Once again, let $w = w_1w_2w_3w_4$, where $w_1, \ldots, w_4$ have properties $(\alpha 1), \ldots, (\alpha 4)$. By the definition of $A_\backslash$, the paths w_2 and w_4 have some vertices in common. Of those vertices, let M_1 (resp. M_2) be the one which is closest to (resp. farthest from) the origin of w_2. Let $w_1w_2 = u_1u_2u_3$ and $w_3w_4 = z_1z_2z_3$, where u_1 and z_2 terminate at M_1, and where u_2 and z_1 terminate at M_2.

Put $s = u_2z_2$. It is good to reflect s about the x-axis and, upon that, orient the resulting path the other way round. Namely, the reflection gives us a delta festoon (say s'), but also makes the levels of horizontal steps change sign. Hence, $int(s') = -int(s)$. Changing the orientation turns s' into another delta festoon, say s''. However, where s' had rightward steps, s'' has leftward steps (and vice versa). Thus, $int(s'') = -int(s') = int(s)$.

Further, the path u_1z_3 bounds a polyomino which, in order to become a directed convex one, just needs to be reflected about the y-axis. Similarly for the path u_3z_1, except that the reflection is about the x-axis. Let I and J denote the directed convex polyominoes arising out of u_1z_3 and u_3z_1, respectively. Much as before, we have

$$h(w) = h(I) + h(s'') + h(J), \quad v(w) = v(I) + v(s'') + v(J)$$

and

$$int(w) = a(I) + int(s) + a(J) = a(I) + int(s'') + a(J).$$

The correspondence $w \mapsto (I, s'', J)$ being a bijection from $A_\backslash$ to the Cartesian product $Dc \times \Delta \times Dc$, it follows that $gf(A_\backslash) = dc \cdot \delta \cdot dc$. Clearly, this formula can be combined with the results of Section 5. This short step leads to

$$gf(A_\backslash) = \left(\frac{\beta}{\delta}\right)^2 \delta = \frac{\beta^2}{\delta}, \tag{16}$$

where β and δ are given by (11) and (5), respectively.

Now we embark on computing the series $\alpha = gf(A)$. That done, we shall solve equation (13) for the only remaining unknown—that is, for tc—and the q-enumeration of true convex polyominoes will be complete.

Let $i,\ j \in \{2,3,\ldots\}$. Let $k,\ell \in \{1,\ldots,i-1\}$ and $m,n \in \{1,\ldots,j-1\}$. We define $A_{ijk\ell mn}$ to be the set of all alpha festoons w for which, in the factorization $w = w_1w_2w_3w_4$ satisfying $(\alpha 1)$–$(\alpha 4)$, it happens that

(α5) $|w_1|_x + |w_2|_x = |w_3|_{\bar{x}} + |w_4|_{\bar{x}} = i$,
(α6) $|w_1|_y + |w_4|_y = |w_2|_{\bar{y}} + |w_3|_{\bar{y}} = j$,
(α7) $|w_1|_x = k$,

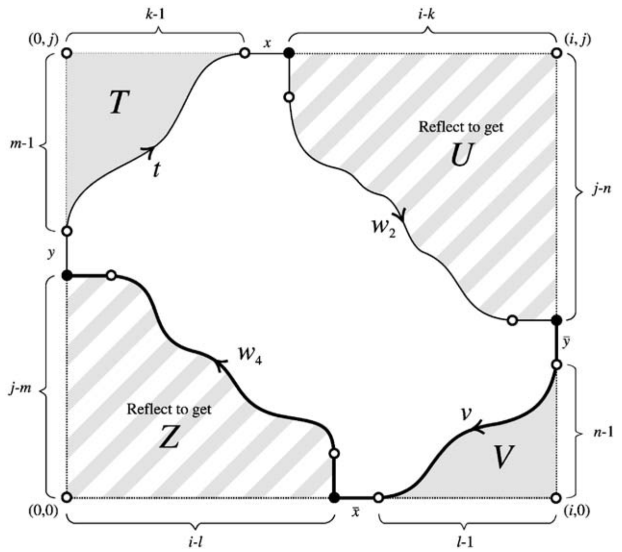

Fig. 15. The elements of $A_{ijk\ell mn}$ have this form.

(α8) $|w_3|_{\bar{x}} = \ell$,
(α9) $|w_1|_y = m$,
(α10) $|w_3|_{\bar{y}} = n$.

See Fig. 15.

Consider a festoon $w \in A_{ijk\ell mn}$. Let w_1, w_2, w_3, and w_4 be the paths satisfying the equation $w = w_1 w_2 w_3 w_4$ and the conditions (α1)–(α10). Then the path $w_2 y^{j-n} \bar{x}^{i-k}$ bounds a polyomino which, in order to become a Ferrers diagram, just needs to be reflected about the x-axis. Similarly for the path $w_4 \bar{y}^{j-m} x^{i-l}$, except that the reflection is about the y-axis. Let U and Z denote the Ferrers diagrams arising out of $w_2 y^{j-n} \bar{x}^{i-k}$ and $w_4 \bar{y}^{j-m} x^{i-l}$, respectively.

Now look at the paths w_1 and w_3. We can write w_1 as $w_1 = ytx$, with $t \in \{x, y\}^*$. Likewise, $w_3 = \bar{y} v \bar{x}$, with $v \in \{\bar{x}, \bar{y}\}^*$. Define T to be the union of $t\bar{x}^{k-1}\bar{y}^{m-1}$ with the cells living inside $t\bar{x}^{k-1}\bar{y}^{m-1}$. (If we rotate T by 180°, the resulting figure, say T', is a tailed Ferrers diagram.) Also, define V to be the union of $vx^{\ell-1}y^{n-1}$ with the cells living inside $vx^{\ell-1}y^{n-1}$. The figure V, too, is a tailed Ferrers diagram.

It is clear that $T' \in Ft_{k-1,m-1}$, $U \in F_{i-k,j-n}$, $V \in Ft_{\ell-1,n-1}$, and $Z \in F_{i-\ell,j-m}$. Moreover, the correspondence $w \mapsto (T', U, V, Z)$ is a bijection from $A_{ijk\ell mn}$ to the Cartesian product $Ft_{k-1,m-1} \times F_{i-k,j-n} \times Ft_{\ell-1,n-1} \times F_{i-\ell,j-m}$. Observing that

$$int(w) = ij - a(T') - a(U) - a(V) - a(Z),$$

we now find that

$$sgf(A_{ijk\ell mn}) = q^{ij}(ft_{k-1,m-1} f_{i-k,j-n} ft_{\ell-1,n-1} f_{i-\ell,j-m})_{-1}$$

$$= q^{ij}\left(\begin{bmatrix} k+m-2 \\ k-1 \end{bmatrix}\begin{bmatrix} i+j-k-n-2 \\ i-k-1 \end{bmatrix} q^{i+j-k-n-1}\right)_{-1}$$
$$\times\left(\begin{bmatrix} \ell+n-2 \\ \ell-1 \end{bmatrix}\begin{bmatrix} i+j-\ell-m-2 \\ i-\ell-1 \end{bmatrix} q^{i+j-\ell-m-1}\right)_{-1}$$
$$= \begin{bmatrix} k+m-2 \\ k-1 \end{bmatrix}\begin{bmatrix} \ell+n-2 \\ \ell-1 \end{bmatrix}\begin{bmatrix} i+j-k-n-2 \\ i-k-1 \end{bmatrix}$$
$$\times\begin{bmatrix} i+j-\ell-m-2 \\ i-\ell-1 \end{bmatrix} q^{k+\ell+m+n-2-(i-k-\ell)(j-m-n)}.$$

Hence

$$\alpha = gf(A) = \sum_{i,j=2}^{\infty}\sum_{k,\ell=1}^{i-1}\sum_{m,n=1}^{j-1} gf(A_{ijk\ell mn})$$
$$= \sum_{i,j=2}^{\infty}\sum_{k,\ell=1}^{i-1}\sum_{m,n=1}^{j-1} sgf(A_{ijk\ell mn})x^{2i}y^{2j}$$
$$= \sum_{i,j=2}^{\infty}\sum_{k,\ell=1}^{i-1}\sum_{m,n=1}^{j-1} \begin{bmatrix} k+m-2 \\ k-1 \end{bmatrix}\begin{bmatrix} \ell+n-2 \\ \ell-1 \end{bmatrix}\begin{bmatrix} i+j-k-n-2 \\ i-k-1 \end{bmatrix}$$
$$\times\begin{bmatrix} i+j-\ell-m-2 \\ i-\ell-1 \end{bmatrix} x^{2i}y^{2j}q^{k+\ell+m+n-2-(i-k-\ell)(j-m-n)}. \qquad (17)$$

Combining (13)–(17), we obtain:

Theorem 4. *The generating function for true convex polyominoes can be written as*

$$tc = \alpha - \frac{\beta^2+\gamma^2}{\delta},$$

where α, β, γ, *and* δ *are given by* (17), (11), (7), *and* (5), *respectively*.

7. Untrue convex polyominoes and all convex polyominoes

Well, true convex polyominoes were rather a hard nut. But now we are in the final straight: it only remains to count untrue convex polyominoes, which are, so to speak, a soft nut.

In Section 2, we defined the poles $W, W', N, N', \ldots$, and then we told: "… a convex polyomino is untrue when $S' = W$ or $N' = E$". Accordingly, convex polyominoes with $S' = W$ (commonly referred to as directed convex polyominoes) are all untrue. Which convex polyominoes are untrue and not directed? Exactly those with both $S' \neq W$ and $N' = E$. Or equivalently, exactly those belonging to the image of Tdc (the set of true directed convex polyominoes) under the 180° rotation. Now, this image has the same generating function as the set Tdc itself. Therefore, the generating function for untrue convex polyominoes is

$$uc = gf(Uc) = gf(Dc) + gf(Tdc) = \frac{\beta+\gamma}{\delta},$$

where β, γ, and δ are given by (11), (7), and (5), respectively. Furthermore, the generating function for all convex polyominoes is

$$\begin{aligned} c &= gf(C) = gf(Tc) + gf(Uc) \\ &= \alpha - \frac{\beta^2+\gamma^2}{\delta} + \frac{\beta+\gamma}{\delta} = \alpha + \frac{\beta(1-\beta)+\gamma(1-\gamma)}{\delta}, \end{aligned}$$

where β, γ, and δ are as above, and α is given by (17).

Let us rewrite this last result in an easy-to-survey manner.

Theorem 5. *The generating function for all convex polyominoes is given by*

$$c = \alpha + \frac{\beta(1-\beta)+\gamma(1-\gamma)}{\delta},$$

where

$$\begin{aligned} \alpha = \sum_{i,j=2}^{\infty} \sum_{k,\ell=1}^{i-1} \sum_{m,n=1}^{j-1} & \begin{bmatrix} k+m-2 \\ k-1 \end{bmatrix} \begin{bmatrix} \ell+n-2 \\ \ell-1 \end{bmatrix} \begin{bmatrix} i+j-k-n-2 \\ i-k-1 \end{bmatrix} \\ & \times \begin{bmatrix} i+j-\ell-m-2 \\ i-\ell-1 \end{bmatrix} x^{2i} y^{2j} q^{k+\ell+m+n-2-(i-k-\ell)(j-m-n)}, \end{aligned}$$

$$\beta = \sum_{i,j=1}^{\infty} \sum_{k=1}^{i} \sum_{\ell=1}^{j} \begin{bmatrix} i+\ell-2 \\ i-1 \end{bmatrix} \begin{bmatrix} j+k-2 \\ j-1 \end{bmatrix} \begin{bmatrix} i+j-k-\ell \\ i-k \end{bmatrix} x^{2i} y^{2j} q^{i+j-k\ell},$$

$$\gamma = \sum_{i,j=2}^{\infty} \sum_{k=1}^{i-1} \sum_{\ell=1}^{j-1} \begin{bmatrix} i+\ell-1 \\ i \end{bmatrix} \begin{bmatrix} j+k-1 \\ j \end{bmatrix} \begin{bmatrix} i+j-k-\ell-2 \\ i-k-1 \end{bmatrix} x^{2i} y^{2j} q^{i+j-k\ell},$$

$$\delta = \sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}.$$

References

[1] G.E. Andrews, The theory of partitions, in: G.-C. Rota (Ed.), Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading, MA, 1976.

[2] M. Bousquet-Mélou, Convex polyominoes and algebraic languages, J. Phys. A: Math. Gen. 25 (1992) 1935–1944.

[3] M. Bousquet-Mélou, q-Énumération de polyominos convexes, J. Combin. Theory Ser. A 64 (1993) 265–288.

[4] M. Bousquet-Mélou, Polyominoes and polygons, Contemp. Math. 178 (1994) 55–70.

[5] M. Bousquet-Mélou, Rapport Scientifique pour obtenir l'habilitation à diriger des recherches, Report No. 1154-96, LaBRI, Université Bordeaux I, 1996.

[6] M. Bousquet-Mélou, J.-M. Fédou, The generating function of convex polyominoes: the resolution of a q-differential system, Discrete Math. 137 (1995) 53–75.
[7] M. Bousquet-Mélou, X.G. Viennot, Empilements de segments et q-énumération de polyominos convexes dirigés, J. Combin. Theory Ser. A 60 (1992) 196–224.
[8] J.-M. Fédou, N. Rouillon, Polyominos et q-analogues des fonctions de Bessel, une preuve combinatoire, in: B. Leclerc, J.-Y. Thibon (Eds.), Seventh FPSAC Proc., Marne-la-Vallée, 1995, pp. 203–220.
[9] S. Feretić, An alternative method for q-counting directed column-convex polyominoes, Discrete Math. 210 (2000) 55–70.
[10] P. Flajolet, Pólya festoons, Research Report No. 1507, INRIA Rocquencourt, 1991.
[11] I. Gessel, A noncommutative generalization and q-analog of the Lagrange inversion formula, Trans. Amer. Math. Soc. 257 (1980) 455–482.
[12] I.P. Goulden, D.M. Jackson, Combinatorial Enumeration, Wiley, New York, 1983.
[13] K.Y. Lin, Exact solution of the convex polygon perimeter and area generating function, J. Phys. A: Math. Gen. 24 (1991) 2411–2417.
[14] G. Pólya, On the number of certain lattice polygons, J. Combin. Theory 6 (1969) 102–105.