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Some Problems in the Qualitative Theory of Ordinary Differential Equations*

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In my department, I have a group of persons doing research on the qualitative theory of ordinary differential equations (ODEs). We are especially interested in the global structure of the phase portrait of quadratic differential systems, stability theory, dynamical systems on two manifolds, etc. The purpose of this paper is to give a summary of some of our research work on the first topic over the past years.

The system to be considered is

$$\frac{dx}{dt} = P_2(x, y), \qquad \frac{dy}{dt} = Q_2(x, y), \tag{$*$}$$

where P_2 and Q_2 are polynomials (in general, nonhomogeneous) of second degree with real constant coefficients, and x, y, t are also considered to be real.

A quadratic differential system (QDS, for abbreviation) in two dimensions is the simplest nonlinear ODE which is, in general, nonintegrable. Just as professor M. L. Cartwright had said in [1]: "It is best to tackle a really difficult problem first in its simplest form," the study of (*) will reveal many new phenomena and raise many new problems for autonomous ODE in the plane.

As QDS appear more and more in applied mathematics and engineering, one wants to know the global stability of critical points, existence or nonexistence of limit cycles and also the global topological structure of the phase-portrait of such QDS.

Let $E(2)$ denote the maximal number of limit cycles for a QDS. Petrovskii and Landis (see Coppel [2] for reference) asserted that $E(2) = 3$, but, as we shall mention later, it is now known that $E(2) \geqslant 4$. Even if it can be shown that $E(2) = 4$, say, by usig the methods of complex variables similar to those

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of Petrovskii and Landis, this result is similar to the fundamental theorem of Gauss on the number of zeros of a polynomial. The result cannot be used to examine any given QDS with real variables and real coefficients or to determine the existence and number of limit cycles. On the other hand, in order to find the maximum number of real roots of all algebraic equations with real coefficients and of a fixed degree, we can use differential calculus or a Sturm sequence without knowing Gauss' theorem. Hence, there is no reason to say that in order to find the maximum number of limit cycles of all QDS(*), one must carry x, y from real variables into complex variables at the very beginning. It also makes the situation complicated. Nevertheless, it seems to me that at a suitable time the use of complex analysis may be helpful.

My work in 1957 and that of Tung in 1958 about possible and impossible relative positions of limit cycles of (*) can be found in detail in a survey paper written by professor W. A. Coppel [2]. This paper also contains several applications of QDS.

Since 1958, I have classified those systems (*) which may have periodic orbits into three typical types [3],

$$\frac{dx}{dt} = -y + \delta x + lx^2 + mxy + ny^2 = P_2(x, y), \qquad \frac{dy}{dt} = x, \tag{I}$$

$$\frac{dx}{dt} = P_2(x, y), \qquad \frac{dy}{dt} = x(1 + ax), \qquad a \neq 0, \tag{II}$$

$$\frac{dx}{dt} = P_2(x, y), \qquad \frac{dy}{dt} = x(1 + ax + by), \qquad b \neq 0, \tag{III}$$

and began to examine each of these systems in detail. Up to 1966, I, and some of my students, solved the problem for system (I) completely, at least from the qualitative point of view (see [4–7]). A part of our results appeared in my book entitled "The Theory of Limit Cycles" (1965) (see *MR* **34** (1967), #3007). But this book is now completely out of date, since many new and important results have been discovered in the last 15 years. I am rewriting this book and hope that the revised edition will be published at the end of 1982.

Our results about system (I) are as follows:

THEOREM A. (i) *If* $\delta = 0$, $m(l+n) = 0$, *then* (I) *has a center at* (0, 0). *If* $\delta = 0$, $m(l+n) \neq 0$, *then* (I) *has no periodic orbit* (1962).

(ii) *If* $\delta m(l+n) > 0$, *then* (I) *has no periodic orbit.*

(iii) *If* $\delta m(l+n) < 0$ *and* $0 < |\delta| < \delta^* = f(l, m, n)$, *where* f *is a definite function of* l, m, n, *then* (I) *has one and only one limit cycle, which* *grows as* $|\delta|$ *increases and becomes a singular cycle* (*homoclinic orbit*) *when* $|\delta| = \delta^*$. *If* $|\delta| > \delta^*$, *then* (I) *has no periodic orbit* (1962–1966).[1]

From Theorem A, three problems arise:

(a) How can one determine the function $\delta^* = f(l, m, n)$? Is it algebraic or transcendental?

(b) In the proof of (ii) and (iii) of Theorem A, we used not only the generalized theory of rotated vector fields of Duff [8, 29] but also the uniqueness theorem of limit cycles of Chang Chi-fen [9]. This is due to the fact that, in Duff's theory, a semi-stable limit cycle may appear suddenly and then split into two. Hence, if we can prove that for this special family (I) with δ as a parameter, no such semi-stable limit cycle can occur when δ varies, then there will be no need to use the theorem of Chang. The answer is, of course, in the affirmative, since the uniqueness of limit cycles had already been proved in Theorem A. It remains only to find a proof.

Therefore, in my opinion, concerning the theory of rotated vector fields, a two-sided problem exists. On the one hand, we need to strengthen Duff's condition in some manner so that semi-stable cycles will not appear. On the other hand, we must enlarge the object, but not strengthen the condition, and begin to investigate the theory of half plane rotated vector fields to see if a periodic orbit has points in both half planes, and what will happen when the parameter varies. By half plane rotated vector fields, we mean that there exists an unbounded curve, which divides the plane into two infinite parts, such that, when the parameter varies, vectors in one half plane all rotate clockwise while vectors in another half plane all rotate counterclockwise. We have frequently encountered such vector fields.

(c) In order to apply Chang's uniqueness theorem, we have made several changes of variables, both linear and nonlinear, until system (I) becomes a system of the Liénard type. It is seen that the ultimate system is much more complicated than the original system (I). Hence, it is interesting to ask: How can we prove the uniqueness of limit cycles of (I) without making these changes of variables?

Before discussing system (II), let us first examine a special case of (III), i.e.,

$$\frac{dx}{dt} = -y + \delta x + lx^2 + mxy + ny^2, \qquad \frac{dy}{dt} = x(1 + by). \qquad (\mathrm{III})_{a=0}$$

This system differs from (I) in two respects. It has an integral line

[1] The uniqueness of limit cycles of (I) has been proved independently also by two Russian mathematicians; see [30, 31].

$1 + by = 0$, and, aside from $(0, 0)$, it may have a second focus $(0, 1/n)$ for certain values of the parameter. Nevertheless, Theorem A applies to $(\mathrm{III})_{a=0}$ as well.

In proving Theorem A(i) for $(\mathrm{I})_{\delta=0}$ or $(\mathrm{III})_{a=\delta=0}$, we found a very interesting fact: integrating factors[2] for the center case $m(l+n) = 0$ can be used to prove the nonexistence of periodic orbits for the case $m(l+n) \neq 0$ through the well-known Bendixson–Dulac theorem. Thus for $(\mathrm{I})_{\delta=0}$ we can take a Dulac function[3]

$$B_1(x, y) = e^{(mn\alpha - 2l)y}(x - n\alpha y + \alpha)^{\alpha m},$$

where α is a positive root of $n^2\alpha^2 - m\alpha - 1 = 0$, and the line $x - n\alpha y + \alpha = 0$ is a tangent line of a separatrix at the saddle point $(0, 1/n)$. Then it is easily seen that

$$\frac{\partial}{\partial x}(B_1 P_2) + \frac{\partial}{\partial y}(B_1 Q_2) = \alpha m(l+n)\, x^2 (x - n\alpha y + \alpha)^{m\alpha - 1}\, e^{(mn\alpha - 2l)y}$$

from which we obtain immediately Theorem A(i) for $(\mathrm{I})_{\delta=0}$.

As for $(\mathrm{III})_{a=\delta=0}$, in 1964, N. D. Chu (now deceased) of the Mathematics Institute found another Dulac function

$$B_2(x, y) = (1 + by)^{(1/b)(mn\theta - b - 2l)}(n\theta y - x - \theta)^{m\theta},$$

where θ is a real root of the equation $n(n+b)\,\theta^2 - m\theta - 1 = 0$. However, B_2 can be used to prove Theorem A(i) only under the condition $m^2 + 4n(n+b) \geqslant 0$.

It is interesting to note that three Russian mathematicians published two papers [10, 11] in 1967 (five years later than our's), using another method to prove Theorem A(i) for $(\mathrm{I})_{\delta=0}$. And then in 1970, another Russian mathematician, L. A. Cherkas, used their method to prove Theorem A(i) for $(\mathrm{III})_{a=\delta=0}$ without any condition. Nevertheless, recently, another student of mine, L. S. Chen, now working in the Mathematics Institute in Beijing, found a third Dulac function

$$B_3 = (1 + by)^{-2l/b - 1} e^{(2m/\sigma)tg^{-1}[-(2n(n+b)/\sigma(1-ny))x - m/\sigma]},$$

where $\sigma = [-m - 4n(n+b)]^{1/2}$, quite different from B_2. But B_3 can be used to prove Theorem A(i) for $(\mathrm{III})_{a=\delta=0}$ just under the condition

[2] Strictly speaking, the functions B_i below are not definite; they vary with the coefficients of the QDS considered.

[3] By a Dulac function we mean the function used by H. Dulac in his extension of the famous Bendixson criterion on the nonexistence of closed orbits of a plane autonomous system; see [32, Chap. V, Sect. 9].

$m^2 + 4n(n+b) < 0$. Furthermore, B_3 can be transformed into a B_2^* through the well-known formula

$$tg^{-1} z = \frac{1}{2i} \ln \frac{i - z}{i + z}$$

and B_2^* can be used instead of B_2, to get the same result for the case $m^2 + 4n(n+b) \geqslant 0$. In other words, in order to prove Theorem A(i) for $(\mathrm{III})_{a=\delta=0}$ the Dulac function method is available as well. And it is just because of this fact that we hope that perhaps the theory of complex variables can be used in such a typical question of real analysis.

About $(\mathrm{III})_{a=0}$, I should like to mention at last that, in China, Theorem A(iii) was proved only for some special cases before 1966, and it was proved completely after 1970 in two papers of Russian mathematicians [12, 13] by using a more complicated theorem on the uniqueness of limit cycles. The theorem of Chang does not seem to be applicable to this case.

Now, let us turn to system (II). In 1962 we analysed a very special case, namely,[4]

$$\frac{dx}{dt} = -y - y^2 + mxy, \qquad \frac{dy}{dt} = x(1 + ax), \qquad m \neq 0, \quad (\mathrm{II})_{\delta = l = 0}$$

which can be proved easily to have no periodic orbit. But, in general, this system has two foci and two saddles and, therefore, there are eight separatrices, four of them go into the saddles, and another four go out. Here arises the question as to the relative positions of certain pairs of separatrices passing through different saddles, say, L_1 and L_2, L_3 and L_4, L_5 and L_6, L_5 and L_4 in Fig. 1.

In the (m, a) parameter plane, we obtained six bifurcation curves (Fig. 2):

1. $m = -a$ $(R = N)$,
2. $27a = 4m^3$ (two infinite critical points coincide),
3. $m = 1/a - a$ ($L_1 = L_2$ and become a straight line),
4. C_4 $(L_3 = L_4)$,
5. C_5 $(L_5 = L_6)$,
6. C_6 $(L_5 = L_4)$.

The existence of the latter three, C_4, C_5, C_6, was only proved qualitatively (see [28]), and hence each of them may not consist of only one branch, but may have also an odd number of branches, or even contain interior points. Therefore, the question arises as to the equations and number of branches of

[4] Similar work has been done in [14] concerning the system $(\mathrm{III})_{\delta=m=0, n=-1}$, but the situation is much more complicated.

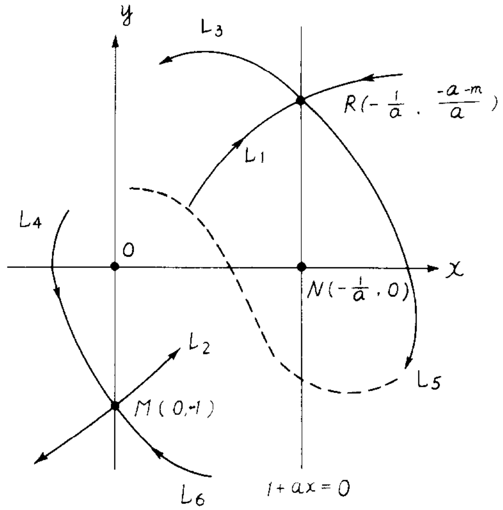


FIGURE 1

each of the curves C_4, C_5 and C_6. Of course, such questions will happen frequently in the analysis of other QDS; and, if limit cycles can appear, we will have other kinds of bifurcation curves.

We may transform the above question into the nonlinear eigenvalue problem

$$\frac{dy}{dx}=\frac{x(1+ax)}{-y-y^2+mxy}, \qquad y(0)=-1, \qquad y\left(-\frac{1}{a}\right)=\frac{-a-m}{a}.$$

I do not know whether this latter problem is easier to discuss.

After my work on $(\mathrm{II})_{\delta=l=0}$, in 1964, two of my students investigated the more complicated system,

$$\frac{dx}{dt}=-y+\delta x-y^2+mxy, \quad \frac{dy}{dt}=x(1+ax), \quad a<0, \quad m\neq 0, \quad (\mathrm{II})_{l=0}$$

for the existence of limit cycles [15]. This system can be proved to have no periodic orbit if $m\delta \leqslant 0$ or $m(m-\delta)\leqslant 0$. But it certainly will have a limit cycle around $(0,0)$ if $m\delta>0$ and $0<|\delta|\ll 1$, or, in some cases, around another focus if $m\delta>0$ and $0<|m-\delta|\ll 1$. Hence, the problem arises of the co-existence of limit cycles around two different foci. This is again a new type of problem for autonomous differential systems in the plane. We have here

THEOREM B. *For* $(\mathrm{II})_{l=0}$ $(a<0,\ m\neq 0)$:

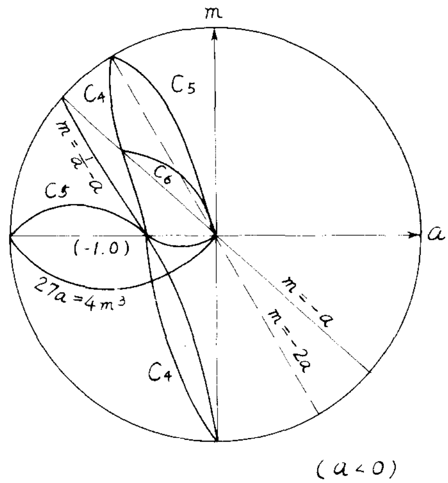


FIGURE 2

(i) *If $m > -a > 0$, then co-existence of cycles appears.*

(ii) *If $0 < m < 1/a - a$, then, for δ in a certain interval I_1, there can appear at least* 2 *cycles around* (0, 0); *and, for δ in another interval I_2, there can appear at least* 2 *cycles around another focus. But we do not know the relationship between I_1 and I_2, nor the possibility of co-existence of cycles.*

(iii) *If $a < m < 0$, then co-existence is impossible.*

(iv) *If $m \leqslant a$, then, around the focus on $1 + ax = 0$, no cycles can appear.*

Later, in 1975–1978, other methods have been discovered by three Chinese mathematicians in order to solve the co-existence problem of cycles for the general QDS, see, e.g., [16]. But even the problem in Theorem B(ii) remains open.

As to the problem of the existence of two and just two limit cycles around the same critical point, very few results can be found in the literature. Only in recent years, Ričkov [17], Chang Chi-fen [18] and Wang Kezhen [19] have obtained some results in this direction. At the end of my book, I conjectured that the system

$$\frac{dx}{dt} = -y + ny^2 + \delta x + lx^2, \qquad \frac{dy}{dt} = x(1 + ax) \qquad (\text{II})_{m=0}$$

has at most two cycles around $(0, 0)$. Notice that here $\phi(y) = -y + ny^2$, $g(x) = x(1 + ax)$ are not monotone, different from the usual assumption for a system of Liénard type. But this problem still remains open, too.

Then in 1976, L. S. Chen made a conjecture as follows:

Limit cycles of a QDS having at least a fine focus must be nested around one focus.[5]

The general form of such QDS can be written as

$$\frac{dx}{dt} = -y + lx^2 + mxy + ny^2, \qquad \frac{dy}{dt} = x(1 + ax + by) \qquad (\mathrm{III})_{\delta=0}$$

for which (0, 0) is a fine focus.

Another student of mine, Wang Minshu, helped him to solve this conjecture [21]. But the answer is as follows:

1. If, in addition to the two foci already existing, there exists at least a third critical point, then Chen's conjecture is true. This is again proved by using the Dulac function method. We see now the importance of Dulac functions in the qualitative investigation of QDS, just as that of Liapunov functions in stability theory. Is there some interrelation between them?

2. If $(\mathrm{III})_{\delta=0}$ has the two foci as its only critical points, then counterexamples can be given. For example, the system

$$\frac{dx}{dt} = -y - 3x^2 + xy + y^2, \qquad \frac{dy}{dt} = x(1 + \tfrac{1}{6}x - 3y)$$

has at least two cycles, one round (0, 0), the other around (0, 1). This can be proved easily by using the well-known Poincaré–Bendixson theorem on the upper or lower Poincaré sphere.

After that, Wang tried to make (0, 0) more fine and obtained the system[6]

$$\frac{dx}{dt} = -y - 3x^2 + xy + y^2, \qquad \frac{dy}{dt} = x(1 + \tfrac{2}{9}x - 3y). \qquad (\Delta)$$

Here (0, 0) is already a fine focus of the second order, and, as before, around each focus there is a cycle, say, Γ_1 and Γ_2, respectively.

[5] If, for a focus of a differential system, the linear part of the system has a pair of purely imaginary eigenvalues, then we call it a fine focus. Similarly, if for a saddle point, the linear part of the system has two real eigenvalues $\lambda_1 \neq 0$ and $\lambda_2 = -\lambda_1$, then we call it a fine saddle.

[6] For the focus (0, 0) of a QDS, we may define its order of fineness by the following formula of Bautin [20]:

$$\rho - \rho_0 = \rho_0[2\pi\lambda_1\psi_1 + \bar{v}_3\psi_3\rho_0^2 + \bar{v}_5\psi_5\rho_0^4 + \bar{v}_7\psi_7\rho_0^6].$$

If $\lambda_1 \neq 0$, (0, 0) is not fine; if $\lambda_1 = 0$, $\bar{v}_3 \neq 0$, it is called a fine focus of order 1; if $\lambda_1 = \bar{v}_3 = 0$, $\bar{v}_5 \neq 0$, it is of order 2; if $\lambda_1 = \bar{v}_3 = \bar{v}_5 = 0$, $\bar{v}_7 \neq 0$, it is of order 3. Thus, any focus of a QDS is of order $\leqslant 3$.

Then it is easily seen from the technique of Bautin [20], that the perturbed system

$$\frac{dx}{dt} = -y - \delta_2 x - 3x^2 + (1-\delta_1)\,xy + y^2, \qquad \frac{dy}{dt} = x(1 + \tfrac{2}{9}x - 3y),$$

where $0 < \delta_2 \ll \delta_1 \ll 1$, will have two more cycles $\Gamma_4 \subset \Gamma_3$ ($\subset \Gamma_1$) in the small neighborhood of $(0, 0)$, since $(0, 0)$ has changed its stability twice after adding to system (Δ) first the perturbed term $-\delta_1 xy$ and then the perturbed term $-\delta_2 x$. It is just in this way that Wang got a counterexample to the incorrect conclusion of Petrovskii and Landis, which is a (3, 1) distribution of cycles, three cycles around one of the foci and one around the other. For the details, see [21].

Independent of Wang's result, Shi Son-ling, a graduate student of professor Chin Yuan-xun, produced another counterexample,

$$\frac{dx}{dt} = \lambda x - y - 10x^2 + (5+\delta)\,xy + y^2, \; \frac{dy}{dt} = x + x^2 + (-25 + 8\varepsilon - 9\delta)\,xy,$$

where $0 < -\lambda \ll -\varepsilon \ll -\delta \ll 1$. The cycles of this system are also a (3, 1) distribution, but it has $(0, 0)$ as a fine focus of the third order when $\lambda = \varepsilon = \delta = 0$. For the proof, see [22].

Now, three problems arise:

1. Are (4, 0) and (2, 2) distributions of limit cycles possible for a QDS?

2. We have already proved that $E(2) \geqslant 4$. What is the exact value of $E(2)$? Is it finite?

3. We see, e.g., in Shi's system that if λ remains zero, i.e., the characteristic roots of the linear part of this system remain always purely imaginary, we can vary only the coefficients of the second degree terms, so that at least two cycles will appear in the neighborhood of $(0, 0)$. What will happen when ε and δ grow larger and larger independently? Notice that, neither ε nor δ can be considered as a parameter of a rotated vector field in a region containing $(0, 0)$ in its interior. It seems tht no such problem has been studied up to now.

As for the possibility of the (2, 2) distribution, early in 1964 I conjectured at the end of my book that system $(\mathrm{II})_{l=0}$ might have, for a certain value of δ, two semi-stable cycles, each around a focus, such that when δ increases, one of them disappears, while the other splits into two cycles, but when δ decreases, the disappearance and splitting phenomenon reverses. This still remains a conjecture; no one has either proven or disproven it. Anyhow, if, for any QDS of type (II) or (III), this phenomenon takes place, then we may take it as an example of a (2, 2) distribution, because just like a multiple root of an algebraic equation, a semi-stable cycle can be considered as a multiple cycle as well.

On the other hand, some results obtained recently in China reveal that the (4, 0) distribution might be impossible for QDS. For example, Wang Minshu and also Zai Shui-ling of Chekiang University have independently proven (not yet published) that if $(\mathrm{III})_{n=0}$ has a fine focus (0, 0) of order 3, so that, in this case, it must take the form

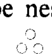
$$\frac{dx}{dt} = -y + lx^2 + xy, \qquad \frac{dy}{dt} = x(1 + \tfrac{1}{5}x + 3ly),$$

then no periodic orbit can exist around (0, 0). At this time they are attempting to prove the same conclusion for the case $n \neq 0$.

At last, let us enumerate some of the main interesting results already proven for QDS (*). Some of them are geometrical, some are topological, and some are dynamical in nature.

1. Any periodic orbit must be a strictly convex oval [23].

2. A limit cycle and a center cannot exist for the same system.

3. Periodic orbits must be nested around one or two foci or centers, the relative positions  and  being impossible [23].

4. Inside any limit cycle, there is one and only one critical point, which must be a focus [24].

5. If the system has an ellipse or a circle as its limit cycle, then it must be the unique cycle of this system, and it is a simple cycle [25].

6. If the system belongs to type (I), then limit cycle must be unique, if it exists (Theorem A).

7. If the system has a fine focus and a straight line integral, then no limit cycle can exist (Theorem A(i) for $(\mathrm{III})_{a=0}$).

8. If the system has a straight line integral, then limit cycle must be unique, if it exists (Theorem A(ii), (iii) for $(\mathrm{III})_{a=0}$).

9. If the system has two fine foci, or two fine saddles, or one fine focus and one fine saddle, then no limit cycle can exist [26].

10. If the system has two straight lines as its integral lines, then no limit cycle can exist (trivial).

11. If the system has two fine foci, then each fine focus must be of order 1 (Ye and Wang, not yet published).

12. There are also many interesting properties concerning singular cycles (homoclinic orbits) of QDS (*), discovered by Tung (see [27]).

It is doubtless that more and more interesting and important properties of the phase-portrait of QDS (*) will be found. Nevertheless, all of these facts so far are just like much data obtained by experimental physicists. There must be something more profound and not yet visible which lies behind the subject.

I am sure that in the future some excellent mathematicians, standing at a high level and having deep insight, will discover all the secrets in this fantastic field of mathematics and this will certainly propel the history of ODE forward by a long step.

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Note added in proof. The non-coexistence of limit cycles in case (ii) of Theorem B was proved recently by Mr. Z. J. Lian of Mid-China Normal University, but the same problem for the case $1/a - a \leqslant m \leqslant -a$ (the only gap in the range of m) still remains open.

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