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On the energetic Galerkin boundary element method applied to interior wave propagation problems

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ABSTRACT

We consider two-dimensional interior wave propagation problems with vanishing initial and mixed boundary conditions, reformulated as a system of two boundary integral equations with retarded potential. These latter are then set in a weak form, based on a natural energy identity satisfied by the solution of the differential problem, and discretized by the related energetic Galerkin boundary element method. Numerical results are presented and discussed.

1. Introduction

Time-dependent problems that are frequently modelled by hyperbolic partial differential equations can be dealt with by the boundary integral equations (BIEs) method. The transformation of the problem to a BIE follows the same well-known method for elliptic boundary value problems. Boundary element methods (BEMs) have been successfully applied in the discretization phase. In principle, both the frequency-domain and time-domain BEM can be used for hyperbolic boundary value problems. The consideration of the time-domain (transient) problem yields the unknown time-dependent quantities directly. In this case, the representation formula in terms of single-layer and double-layer potentials uses the fundamental solution of the hyperbolic partial differential equation and jump relations, giving rise to retarded BIEs. Usual numerical discretization procedures include collocation techniques and Laplace–Fourier methods coupled with Galerkin boundary elements in space. The convolution quadrature method for time discretization has been developed in [1]. It provides a straightforward way to obtain an efficient time stepping scheme using the Laplace transform of the kernel function, although stability and convergence are assured under strong regularity assumptions on problem data. The application of the Galerkin BEM in both space and time has been implemented by several authors but in this direction only the weak formulation due to Ha Duong [2] furnishes genuine convergence results. The only drawback of the method is that it has stability constants growing exponentially in time, as stated in [3].

Recently, in [4], we have considered two-dimensional Dirichlet or Neumann exterior problems for a temporally homogeneous (normalized) scalar wave equation in the time interval $[0, T]$, reformulated as a BIE with retarded potential. Special attention has been devoted to a natural energy identity related to the differential problem, that leads to a space–time weak formulation for the BIE, having, under suitable constraint, precise continuity and coerciveness properties. Consequently it can be discretized by unconditionally stable schemes with well-behaved stability constants even for large times.

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In the present paper we focus on two-dimensional interior wave propagation problems with vanishing initial and mixed boundary conditions. These problems are more difficult to be treated with respect to exterior ones, since the propagation of the wave in a bounded domain can give rise, at the discretization level, to significant noise or even huge unstable behavior in the numerical solution, as reported in several literature works (see e.g. [5,6]).

We reformulate the differential problem as a system of two BIEs with retarded potentials and the related energetic space–time weak formulation is introduced. The energetic Galerkin BEM used in the discretization phase, after a double analytic integration in time variables, has to deal also with weakly singular, singular and hypersingular double integrals in space variables, which are not present in the one-dimensional case and have been numerically evaluated by efficient quadrature schemes. The stability and accuracy of the energetic approach is shown by means of several numerical results.

2. Model problem and its boundary integral weak formulation

We will consider a mixed boundary value problem for the wave equation with homogeneous initial conditions, in a bounded domain $\Omega \subset \mathbb{R}^2$, with a boundary Γ referred to a Cartesian orthogonal coordinate system $\mathbf{x} = (x_1, x_2)$ and partitioned into two non-intersecting subsets Γ_u and Γ_p such that $\overline{\Gamma}_u \cup \overline{\Gamma}_p = \overline{\Gamma}$:

$$u_{tt} - \Delta u = 0, \qquad \mathbf{x} \in \Omega, t \in (0, T) \tag{2.1}$$

$$u(\mathbf{x}, 0) = u_t(\mathbf{x}, 0) = 0, \qquad \mathbf{x} \in \Omega \tag{2.2}$$

$$u(\mathbf{x}, t) = \bar{u}(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \Sigma_T^u := \Gamma_u \times [0, T] \tag{2.3}$$

$$p(\mathbf{x}, t) := \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}, t) = \bar{p}(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \Sigma_T^p := \Gamma_p \times [0, T], \tag{2.4}$$

where $\mathbf{n}$ is the unit outward normal vector of Γ, $\bar{\Gamma} = \bar{\Gamma}_u \cup \bar{\Gamma}_p$, $\Gamma_u \cap \Gamma_p = \emptyset$ and $\bar{u}, \bar{p}$ are given boundary data of Dirichlet and Neumann type, respectively. Let us consider the boundary integral representation of the solution of (2.1)–(2.4), for $\mathbf{x} \in \Omega, t \in (0, T)$:

$$u(\mathbf{x}, t) = \int_\Gamma \int_0^t \left[G(r, t-\tau) p(\boldsymbol{\xi}, \tau) - \frac{\partial G}{\partial \mathbf{n}_{\boldsymbol{\xi}}}(r, t-\tau) u(\boldsymbol{\xi}, \tau) \right] \mathrm{d}\tau \, \mathrm{d}\gamma_{\boldsymbol{\xi}}, \tag{2.5}$$

where $r = \|\mathbf{r}\|_2 = \|\mathbf{x} - \boldsymbol{\xi}\|_2$ and

$$G(r, t-\tau) = \frac{1}{2\pi} \frac{H[t-\tau-r]}{[(t-\tau)^2 - r^2]^{\frac{1}{2}}} \tag{2.6}$$

is the forward fundamental solution of the two-dimensional wave operator, with $H[\cdot]$ the Heaviside function. With a limiting process for $\mathbf{x}$ tending to Γ we obtain the space–time BIE (see [3])

$$\frac{1}{2} u(\mathbf{x}, t) = \int_\Gamma \int_0^t G(r, t-\tau) p(\boldsymbol{\xi}, \tau) \, \mathrm{d}\tau \, \mathrm{d}\gamma_{\boldsymbol{\xi}} - \int_\Gamma \int_0^t \frac{\partial G}{\partial \mathbf{n}_{\boldsymbol{\xi}}}(r, t-\tau) u(\boldsymbol{\xi}, \tau) \, \mathrm{d}\tau \, \mathrm{d}\gamma_{\boldsymbol{\xi}},$$

which can be written, with obvious meaning of notation, in the compact form

$$\frac{1}{2} u(\mathbf{x}, t) = (Vp)(\mathbf{x}, t) - (Ku)(\mathbf{x}, t). \tag{2.7}$$

The BIE (2.7) is generally used to solve Dirichlet problems but it can be employed for mixed problems too. However, in the latter case, one can consider a second space–time BIE, obtainable from (2.5), performing the normal derivative with respect to $\mathbf{n}_{\mathbf{x}}$ and operating a limiting process for $\mathbf{x}$ tending to Γ:

$$\frac{1}{2} p(\mathbf{x}, t) = \int_\Gamma \int_0^t \frac{\partial G}{\partial \mathbf{n}_{\mathbf{x}}}(r, t-\tau) p(\boldsymbol{\xi}, \tau) \mathrm{d}\tau \mathrm{d}\gamma_{\boldsymbol{\xi}} - \int_\Gamma \int_0^t \frac{\partial^2 G}{\partial \mathbf{n}_{\mathbf{x}} \partial \mathbf{n}_{\boldsymbol{\xi}}}(r, t-\tau) u(\boldsymbol{\xi}, \tau) \mathrm{d}\tau \mathrm{d}\gamma_{\boldsymbol{\xi}},$$

which can be written in the compact form

$$\frac{1}{2} p(\mathbf{x}, t) = (K'p)(\mathbf{x}, t) - (Du)(\mathbf{x}, t). \tag{2.8}$$

Note that the operator K' is the adjoint of the Cauchy singular operator K, which can be expressed as

$$Ku(\mathbf{x}, t) = -\int_\Gamma \frac{\partial r}{\partial \mathbf{n}_{\boldsymbol{\xi}}} \int_0^t G(r, t-\tau) \left[u_t(\boldsymbol{\xi}, \tau) + \frac{u(\boldsymbol{\xi}, \tau)}{(t-\tau+r)} \right] \mathrm{d}\tau \, \mathrm{d}\gamma_{\boldsymbol{\xi}}. \tag{2.9}$$

Expression (2.9) can be obtained starting from the definition of the double-layer operator K and observing that

$$\frac{\partial}{\partial r} \frac{H[t-\tau-r]}{\sqrt{t-\tau-r}} = \frac{\partial}{\partial \tau} \frac{H[t-\tau-r]}{\sqrt{t-\tau-r}}. \tag{2.10}$$

In fact, using (2.10), we have

$$\frac{\partial G}{\partial \mathbf{n}_{\xi}}(r, t-\tau) = \frac{\partial G}{\partial r}(r, t-\tau)\frac{\partial r}{\partial \mathbf{n}_{\xi}} = \frac{1}{2\pi}\frac{\partial}{\partial r}\left[\frac{1}{\sqrt{t-\tau+r}}\frac{H[t-\tau-r]}{\sqrt{t-\tau-r}}\right]\frac{\partial r}{\partial \mathbf{n}_{\xi}}$$
$$= \frac{1}{2\pi}\left[-\frac{1}{2}\frac{1}{t-\tau+r}\frac{H[t-\tau-r]}{\sqrt{(t-\tau)^2-r^2}} + \frac{1}{\sqrt{t-\tau+r}}\frac{\partial}{\partial \tau}\frac{H[t-\tau-r]}{\sqrt{t-\tau-r}}\right]\frac{\partial r}{\partial \mathbf{n}_{\xi}}. \tag{2.11}$$

Now, inserting (2.11) in the definition of K, integrating in the sense of distributions the term containing the derivative with respect to τ, one gets, up to the factor $-\frac{1}{2\pi}$,

$$\int_{\Gamma}\frac{\partial r}{\partial \mathbf{n}_{\xi}}\int_0^t\left\{\frac{1}{2}\frac{H[t-\tau-r]}{\sqrt{(t-\tau)^2-r^2}}\frac{u(\boldsymbol{\xi},\tau)}{t-\tau+r} + \frac{H[t-\tau-r]}{\sqrt{t-\tau-r}}\frac{\partial}{\partial\tau}\left[\frac{u(\boldsymbol{\xi},\tau)}{\sqrt{t-\tau+r}}\right]\right\}\mathrm{d}\tau\,\mathrm{d}\gamma_{\xi},;$$

by expressing the time derivative of the second term in the integrand function explicitly, one finally deduces (2.9).

Further, considering at this stage the derivative with respect to $\mathbf{n}_{\mathbf{x}}$ of (2.9) and operating with the same arguments as before, after a cumbersome but easy calculation the hypersingular integral operator D in (2.8) can be equivalently expressed in the following way:

$$Du(\mathbf{x},t) = -\int_{\Gamma}\frac{\partial^2 r}{\partial \mathbf{n}_{\mathbf{x}}\partial \mathbf{n}_{\xi}}\int_0^t G(r,t-\tau)\left[u_t(\boldsymbol{\xi},\tau) + \frac{u(\boldsymbol{\xi},\tau)}{(t-\tau+r)}\right]\mathrm{d}\tau\,\mathrm{d}\gamma_{\xi}$$
$$+ \int_{\Gamma}\frac{\partial r}{\partial \mathbf{n}_{\mathbf{x}}}\frac{\partial r}{\partial \mathbf{n}_{\xi}}\int_0^t G(r,t-\tau)\left[u_{tt}(\boldsymbol{\xi},\tau) + \frac{2u_t(\boldsymbol{\xi},\tau)}{(t-\tau+r)} + \frac{3u(\boldsymbol{\xi},\tau)}{(t-\tau+r)^2}\right]\mathrm{d}\tau\,\mathrm{d}\gamma_{\xi}.$$

Hence, using the boundary conditions (2.3)–(2.4), a mixed boundary value wave propagation problem can be rewritten as a system of two BIEs in the boundary unknowns the functions $p(\mathbf{x},t)$ and $u(\mathbf{x},t)$ on Γ_u, Γ_p, respectively:

$$\begin{bmatrix} V_u & -K_p \\ -K'_u & D_p \end{bmatrix}\begin{bmatrix} p \\ u \end{bmatrix} = \begin{bmatrix} -V_p & \frac{1}{2}I + K_u \\ -\frac{1}{2}I + K'_p & -D_u \end{bmatrix}\begin{bmatrix} \bar{p} \\ \bar{u} \end{bmatrix}, \qquad \begin{matrix} (\mathbf{x},t)\in\Sigma_T^u \\ (\mathbf{x},t)\in\Sigma_T^p, \end{matrix} \tag{2.12}$$

where the boundary integral operator subscripts $\beta = u, p$ define their restriction to Σ_T^{β}. Then, starting from the observation that the solution of (2.1)–(2.4) satisfies the following energy identity:

$$\mathcal{E}(u,T) := \frac{1}{2}\int_{\Omega}\int_0^T\left[u_t^2(\mathbf{x},t) + |\nabla u(\mathbf{x},t)|^2\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{\Gamma}\int_0^T u_t(\mathbf{x},t)\frac{\partial u}{\partial \mathbf{n}_{\mathbf{x}}}(\mathbf{x},t)\,\mathrm{d}t\,\mathrm{d}\gamma_{\mathbf{x}},$$

obtainable by multiplying Eq. (2.1) by u_t and integrating by parts over $\Omega\times[0,T]$, and remembering (2.7) and (2.8), the energetic weak formulation of the system (2.12) is defined as

$$\begin{cases} \langle (V_u p)_t, \psi\rangle_{L^2(\Sigma_T^u)} - \langle (K_p u)_t, \psi\rangle_{L^2(\Sigma_T^u)} = \langle f_t^u, \psi\rangle_{L^2(\Sigma_T^u)} \\ -\langle K'_u p, \eta_t\rangle_{L^2(\Sigma_T^p)} + \langle D_p u, \eta_t\rangle_{L^2(\Sigma_T^p)} = \langle f^p, \eta_t\rangle_{L^2(\Sigma_T^p)}, \end{cases} \tag{2.13}$$

where $f_t^u = -(V_p\bar{p})_t + \left(\left(\frac{1}{2}I + K_u\right)\bar{u}\right)_t$, $f^p = \left(-\frac{1}{2}I + K'_p\right)\bar{p} - D_u\bar{u}$ and $\psi(\mathbf{x},t)$, $\eta(\mathbf{x},t)$ are suitable test functions, belonging to the same functional space of $p(\mathbf{x},t)$, $u(\mathbf{x},t)$, respectively (see also [7], where the energetic weak formulation was introduced for the problem (2.1)–(2.4), but with $\Omega\subset\mathbb{R}$, i.e., for a spatial domain coincident with a bounded interval of the real line). The first equation in (2.12) has been derived with respect to time and projected with the $L^2(\Sigma_T^u)$ scalar product onto the space of functions approximating $p(\mathbf{x},t)$, while the second equation in (2.12) has been projected with $L^2(\Sigma_T^p)$ onto the space of the time derivative of functions approximating $u(\mathbf{x},t)$. Note that the scalar products involved are represented by a space–time integral; hence, taking into account the space–time integral nature of operators V, K, K', D, in (2.13) we will have to deal with quadruple integrals, double in space and double in time.

2.1. Galerkin BEM discretization

For time discretization we consider a uniform decomposition of the time interval $[0,T]$ with time step $\Delta t = T/N_{\Delta t}$, $N_{\Delta t}\in\mathbb{N}^+$, generated by the $N_{\Delta t}+1$ instants

$$t_k = k\Delta t, \quad k = 0,\ldots,N_{\Delta t},$$

and we choose temporally piecewise constant shape functions for the approximation of p and piecewise linear shape functions for the approximation of u, although higher-degree shape functions can be used. Note that, for this particular choice, temporal shape functions, for $k = 0,\ldots,N_{\Delta t}-1$, will be defined as

$$v_k^p(t) = H[t-t_k] - H[t-t_{k+1}]$$

for the approximation of p, or as

$$v_k^u(t) = R(t - t_k) - R(t - t_{k+1}),$$

for the approximation of u, where $R(t - t_k) = \frac{t-t_k}{\Delta t} H[t - t_k]$ is the ramp function.

For the space discretization, we employ a Galerkin boundary element method. We consider Ω (suitably approximated by a domain) of polygonal type and a boundary mesh on Γ_u constituted by M_u straight elements $\{e_1^u, \ldots, e_{M_u}^u\}$, with $2l_i^u := \text{length}(e_i^u)$, $l^u = \max_i\{2l_i^u\}$, $e_i^u \cap e_j^u = \emptyset$ if $i \neq j$ and such that $\bigcup_{i=1}^{M_u} \overline{e}_i^u = \overline{\Gamma}_u$. The same is done for the Neumann part of the boundary Γ_p, with obvious change of notation. Let $\Delta x = \max\{l^u, l^p\}$. The functional background compels one to choose spatial shape functions belonging to $L^2(\Gamma_u)$ for the approximation of p and to $H^1(\Gamma_p)$ for the approximation of u. Hence, having defined $\mathcal{P}_{d_i}$, the space of algebraic polynomials of degree d_i, we consider, respectively, the space of piecewise polynomial functions

$$X_{-1,\Delta x} := \{w^p(\mathbf{x}) \in L^2(\Gamma_u) : w^p_{|e_i^u} \in \mathcal{P}_{d_i}, \forall e_i^u \subset \Gamma_u\}; \tag{2.14}$$

and the space of continuous piecewise polynomial functions

$$X_{0,\Delta x} := \{w^u(\mathbf{x}) \in C^0(\Gamma_p) : w^u_{|e_j^p} \in \mathcal{P}_{d_j}, \forall e_j^p \subset \Gamma_p\}. \tag{2.15}$$

Hence, denoting with $M_{\Delta x}^p$, $M_{\Delta x}^u$ the number of unknowns on Γ_u and Γ_p, respectively, and having introduced the standard piecewise polynomial boundary element basis functions $w_j^p(\mathbf{x})$, $j = 1, \ldots, M_{\Delta x}^p$, in $X_{-1,\Delta x}$ and $w_j^u(\mathbf{x})$, $j = 1, \ldots, M_{\Delta x}^u$ in $X_{0,\Delta x}$, the approximate solutions of the problem at hand will be expressed as

$$\tilde{p}(\mathbf{x}, t) := \sum_{k=0}^{N_{\Delta t}-1} \sum_{j=1}^{M_{\Delta x}^p} \alpha_{pj}^{(k)} w_j^p(\mathbf{x}) v_k^p(t), \qquad \tilde{u}(\mathbf{x}, t) := \sum_{k=0}^{N_{\Delta t}-1} \sum_{j=1}^{M_{\Delta x}^u} \alpha_{uj}^{(k)} w_j^u(\mathbf{x}) v_k^u(t).$$

The Galerkin BEM discretization coming from energetic weak formulation (2.13) produces the linear system

$$\mathbb{E}\boldsymbol{\alpha} = \mathbf{b}, \tag{2.16}$$

where matrix $\mathbb{E}$ has a block lower triangular Toeplitz structure, since its elements depend on the difference $t_h - t_k$, and in particular they vanish if $t_h \leq t_k$. Each block has dimension $M_{\Delta x} := M_{\Delta x}^p + M_{\Delta x}^u$. If we indicate with $\mathbb{E}^{(\ell)}$ the block obtained when $t_h - t_k = (\ell + 1)\Delta t$, $\ell = 0, \ldots, N_{\Delta t} - 1$, the linear system can be written as

$$\begin{pmatrix} \mathbb{E}^{(0)} & 0 & 0 & \cdots & 0 \\ \mathbb{E}^{(1)} & \mathbb{E}^{(0)} & 0 & \cdots & 0 \\ \mathbb{E}^{(2)} & \mathbb{E}^{(1)} & \mathbb{E}^{(0)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \mathbb{E}^{(N_{\Delta t}-1)} & \mathbb{E}^{(N_{\Delta t}-2)} & \cdots & \mathbb{E}^{(1)} & \mathbb{E}^{(0)} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^{(0)} \\ \boldsymbol{\alpha}^{(1)} \\ \boldsymbol{\alpha}^{(2)} \\ \vdots \\ \boldsymbol{\alpha}^{(N_{\Delta t}-1)} \end{pmatrix} = \begin{pmatrix} \mathbf{b}^{(0)} \\ \mathbf{b}^{(1)} \\ \mathbf{b}^{(2)} \\ \vdots \\ \mathbf{b}^{(N_{\Delta t}-1)} \end{pmatrix} \tag{2.17}$$

where

$$\boldsymbol{\alpha}^{(\ell)} = \left(\alpha_j^{(\ell)}\right) \quad \text{and} \quad \mathbf{b}^{(\ell)} = \left(b_j^{(\ell)}\right) \quad \text{with } \ell = 0, \ldots, N_{\Delta t} - 1;\ j = 1, \ldots, M_l.$$

Note that each block has a 2×2 block substructure of the type

$$\mathbb{E}^{(\ell)} = \begin{bmatrix} \mathbb{E}_{uu}^{(\ell)} & \mathbb{E}_{up}^{(\ell)} \\ \mathbb{E}_{pu}^{(\ell)} & \mathbb{E}_{pp}^{(\ell)} \end{bmatrix} \tag{2.18}$$

where diagonal subblocks have dimensions $M_{\Delta x}^p$, $M_{\Delta x}^u$, respectively, $\mathbb{E}_{pu}^{(\ell)} = (\mathbb{E}_{up}^{(\ell)})^\top$ and the unknowns are organized as follows:

$$\boldsymbol{\alpha}^{(\ell)} = \left(\alpha_{p1}^{(\ell)}, \ldots, \alpha_{pM_{\Delta x}^p}^{(\ell)}, \alpha_{u1}^{(\ell)}, \ldots, \alpha_{uM_{\Delta x}^u}^{(\ell)}\right)^\top.$$

The solution of (2.17) is obtained with a block forward substitution; i.e., at every time instant $t_\ell = (\ell + 1)\Delta t$, $\ell = 0, \ldots, N_{\Delta t} - 1$, one computes

$$\mathbf{z}^{(\ell)} = \mathbf{b}^{(\ell)} - \sum_{j=1}^{\ell} \mathbb{E}^{(j)} \boldsymbol{\alpha}^{(\ell-j)}$$

and then solves the reduced linear system:

$$\mathbb{E}^{(0)} \boldsymbol{\alpha}^{(\ell)} = \mathbf{z}^{(\ell)}. \tag{2.19}$$

Procedure (2.19) is a time-marching technique, where the only matrix to be inverted is the positive definite $\mathbb{E}^{(0)}$ diagonal block, while all the other blocks are used to update at every time step the right-hand side. Owing to this procedure we can construct and store only the blocks $\mathbb{E}^{(0)}, \ldots, \mathbb{E}^{(N_{\Delta t}-1)}$, with a considerable reduction of computational cost and memory requirement.

Having set $\Delta_{hk} = t_h - t_k$, the matrix elements in blocks of the type $\mathbb{E}^{(\ell)}_{uu}$, $\mathbb{E}^{(\ell)}_{up}$, $\mathbb{E}^{(\ell)}_{pp}$, after a double analytic integration in the time variables, are of the form, respectively,

$$\sum_{\alpha,\beta=0}^{1} (-1)^{\alpha+\beta} \int_{\Gamma_u} w_i^p(\mathbf{x}) \int_{\Gamma_u} H[\Delta_{h+\alpha,k+\beta} - r]\mathcal{V}(r, t_{h+\alpha}, t_{k+\beta}) w_j^p(\boldsymbol{\xi})\, d\gamma_{\boldsymbol{\xi}}\, d\gamma_{\mathbf{x}}, \tag{2.20}$$

where

$$\mathcal{V}(r, t_h, t_k) = \frac{1}{2\pi}\left[\log\left(\Delta_{hk} + \sqrt{\Delta_{hk}^2 - r^2}\right) - \log r\right]; \tag{2.21}$$

$$\sum_{\alpha,\beta=0}^{1} (-1)^{\alpha+\beta} \int_{\Gamma_u} w_i^p(\mathbf{x}) \int_{\Gamma_p} H[\Delta_{h+\alpha,k+\beta} - r]\mathcal{K}(r, t_{h+\alpha}, t_{k+\beta}) w_j^u(\boldsymbol{\xi})\, d\gamma_{\boldsymbol{\xi}}\, d\gamma_{\mathbf{x}}, \tag{2.22}$$

where

$$\mathcal{K}(r, t_h, t_k) = \frac{1}{2\pi\,\Delta t}\frac{\mathbf{r}\cdot\mathbf{n}_{\boldsymbol{\xi}}}{r^2}\sqrt{\Delta_{hk}^2 - r^2}; \tag{2.23}$$

$$\sum_{\alpha,\beta=0}^{1} (-1)^{\alpha+\beta} \int_{\Gamma_p} w_i^u(\mathbf{x}) \int_{\Gamma_p} H[\Delta_{h+\alpha,k+\beta} - r]\mathcal{D}(r, t_{h+\alpha}, t_{k+\beta}) w_j^u(\boldsymbol{\xi})\, d\gamma_{\boldsymbol{\xi}}\, d\gamma_{\mathbf{x}}, \tag{2.24}$$

where

$$\mathcal{D}(r, t_h, t_k) = \frac{1}{2\pi(\Delta t)^2}\left\{\frac{\mathbf{r}\cdot\mathbf{n}_{\mathbf{x}}\mathbf{r}\cdot\mathbf{n}_{\boldsymbol{\xi}}}{r^2}\frac{\Delta_{hk}\sqrt{\Delta_{hk}^2 - r^2}}{r^2} + \frac{(\mathbf{n}_{\mathbf{x}}\cdot\mathbf{n}_{\boldsymbol{\xi}})}{2}\left[\log\left(\Delta_{hk} + \sqrt{\Delta_{hk}^2 - r^2}\right) - \log r - \frac{\Delta_{hk}\sqrt{\Delta_{hk}^2 - r^2}}{r^2}\right]\right\}. \tag{2.25}$$

Using the standard element by element technique, the evaluation of every double integral in (2.20), (2.22) or (2.24) is reduced to the assembling of local contributions of the type

$$\int_{e_i} \widetilde{w}_i^{(d_i)}(\mathbf{x}) \int_{e_j} H[\Delta_{hk} - r]\mathcal{S}(r, t_h, t_k)\widetilde{w}_j^{(d_j)}(\boldsymbol{\xi})\, d\gamma_{\boldsymbol{\xi}}\, d\gamma_{\mathbf{x}}, \tag{2.26}$$

where $\mathcal{S}$ represents one of the kernels (2.21), (2.23) or (2.25) and $\widetilde{w}_i^{(d_i)}(\mathbf{x})$, $\widetilde{w}_j^{(d_j)}(\mathbf{x})$ indicate, respectively, one of the local Lagrangian basis functions in the space variable of degree d_i, d_j defined over the element e_i, e_j of the boundary mesh.

Looking at (2.21), (2.23) and (2.25), we observe space singularities of type $\log r$, $O(r^{-1})$ and $O(r^{-2})$ as $r \to 0$, which are typical of weakly singular, singular and hypersingular kernels related to two-dimensional elliptic problems. Hence, efficient evaluation of double integrals of type (2.26) is particularly required when $e_i \equiv e_j$ and when e_i, e_j are consecutive. Note that, when the kernel is hypersingular and $e_i \equiv e_j$, we define both the inner and the outer integrals as Hadamard finite parts, while if e_i and e_j are consecutive, only the outer integral is understood in the finite part sense: the correct interpretation of double integrals is the key point for any efficient numerical approach based on an element by element technique (see [8]).

Further, we observe that the Heaviside function $H[\Delta_{hk} - r]$ in (2.26) and the function $\sqrt{\Delta_{hk}^2 - r^2}$ in the kernel $\mathcal{S}(r, t_h, t_k)$ give rise to other different types of trouble, which have to be properly faced, as described in [9]. Hence, the numerical treatment of (2.26) has been operated through quadrature schemes widely used in the context of the Galerkin BEM coming from elliptic problems [8], coupled with a suitable regularization technique [10], after a careful subdivision of the integration domain due to the presence of the Heaviside function. A complete illustration of the efficient numerical integration schemes we have used for the discretization of weakly, strongly and hypersingular BIEs related to wave propagation problems, and which represent a valid alternative to those proposed in [11,12], can be found in [13].

3. Numerical results

To validate the presented discretization approach, we consider a standard benchmark (see, for instance, [5]), involving a strip Ω of unit height, unbounded in the horizontal direction, fixed in the inferior part where the Dirichlet boundary datum $\bar{u} = 0$ is assigned, and subject to a uniform traction $\bar{p} = H[t]$ in its superior part, as shown in Fig. 1. A finite portion of the strip is taken into account, in such a way that the vertical dimension of the resulting rectangle is five times the other one. On the "cut" sides of the domain the equilibrium condition $\bar{p} = 0$ has been assigned. In order to apply the energetic Galerkin BEM, we have introduced on Γ at first a uniform mesh with 24 elements ($\Delta x = 0.1$) and we have used, in the spatial variable, constant shape functions for the approximation of p and linear shape functions for the approximation of u. The time

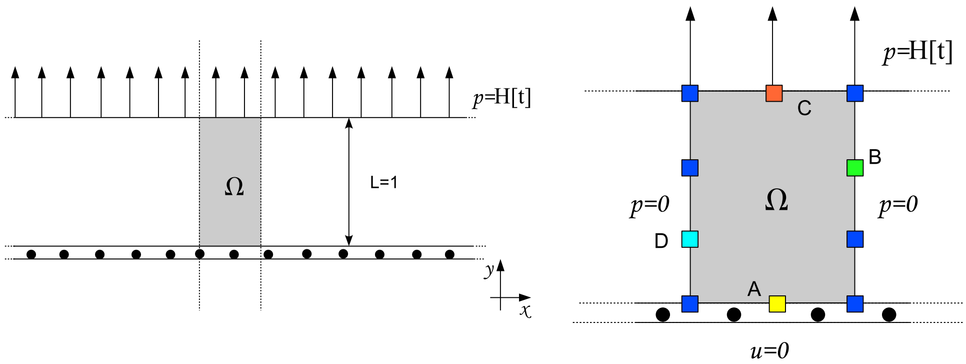

Fig. 1. Domain and mixed boundary conditions for the first test problem.

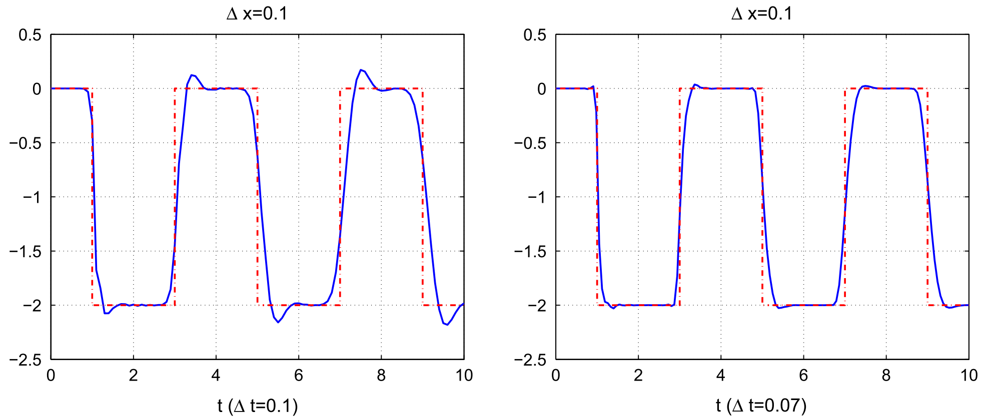

Fig. 2. Approximate solution $p(A)$ of the first test problem for $\beta = 1$ (left), $\beta = 0.7$ (right).

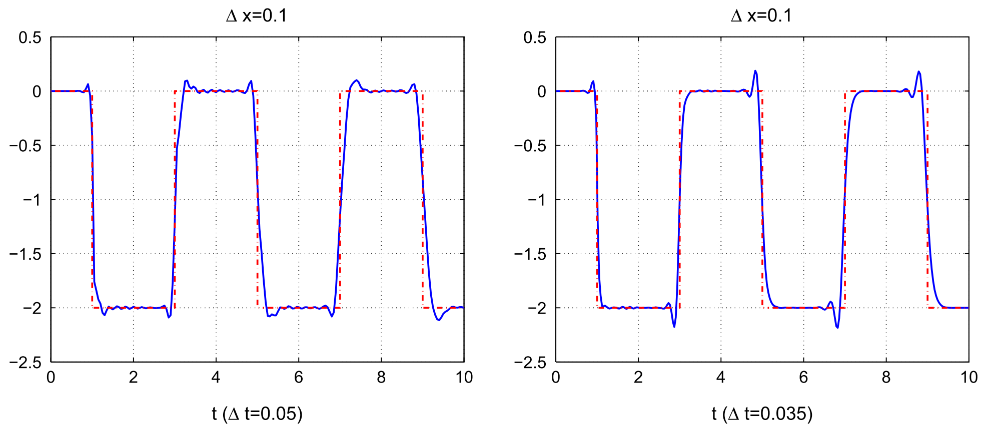

Fig. 3. Approximate solution $p(A)$ of the first test problem for $\beta = 0.5$ (left), $\beta = 0.35$ (right).

interval of analysis [0, 10] has been discretized with different time steps Δt, in such a way that $\beta := \frac{\Delta t}{\Delta x} = 1, 0.7, 0.5, 0.35$. The chosen discretization parameters are used in [6]. In Figs. 2 and 3, we show the time history of traction in the point A, $p(A, t)$, together with the corresponding analytical solution. Note that the oscillations in the graphs of $p(A, t)$ are due to the difficulty of approximating the jump discontinuities of the analytical solution. The best approximate result is obtained for $\beta = 0.7$, while the phenomenon of "intermittent instability", i.e., the explosive instability found in [6] for $\beta = 0.5$, is not present in our numerical results.

We have also considered a finer uniform mesh made by 48 elements ($\Delta x = 0.05$) and we have always used, in spatial variable, constant shape functions for the approximation of p and linear shape functions for the approximation of u. The time interval of analysis has been discretized with different time steps. For $\beta = 0.7, 0.5$, in Figs. 4 and 5 we show the recovered numerical solutions. In particular time history of traction in the point A, $p(A, t)$ is shown on the left, together

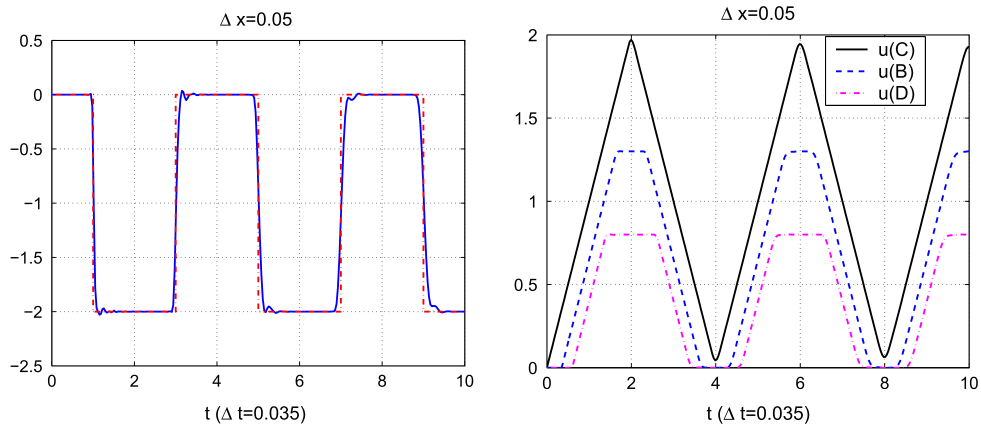

Fig. 4. Approximate solution p and u of the first test problem for $\beta = 0.7$.

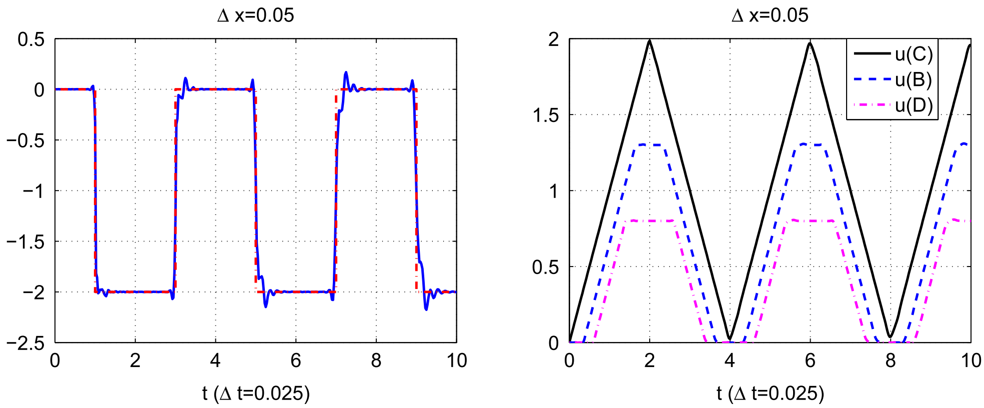

Fig. 5. Approximate solution p and u of the first test problem for $\beta = 0.5$.

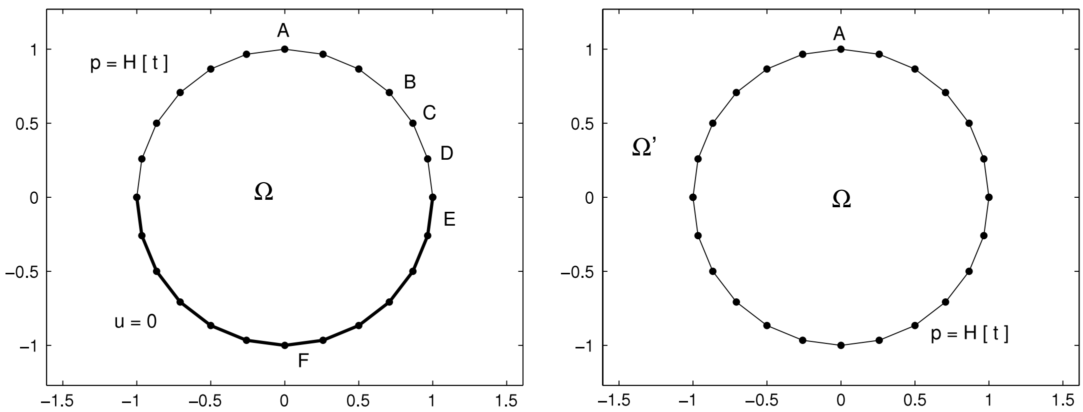

Fig. 6. Domains and boundary conditions of the second (left) and of the third (right) simulation.

with the corresponding analytical solution, while displacements in the points B, C, D, respectively $u(B, t), u(C, t), u(D, t)$, are shown on the right: here the three curves substantially overlap with their respective analytical solutions.

As a second simulation, we consider a unitary disk, whose upper semi-circular boundary is subject to the Neumann boundary datum $\bar{p} = H[t]$, while its lower semi-circular boundary is fixed. The domain and mixed boundary conditions are shown in Fig. 6 on the left. For the discretization phase, we have approximated the boundary Γ introducing a uniform mesh with 24 straight elements and we have used, in the spatial variable, constant shape functions for the approximation of p and linear shape functions for the approximation of u. The time interval of analysis [0, 10] has been discretized with different time steps. In Fig. 7, the approximate solution obtained with the energetic approach, fixing $\Delta t = 0.2$, is presented. In particular, on the left the time history $p(E, t), p(F, t)$ of p on the elements E, F is shown, and one can note that while the

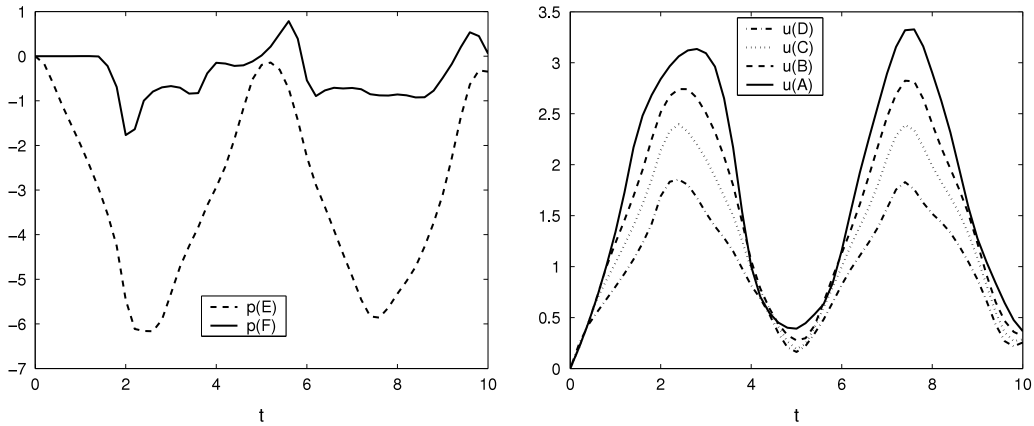

Fig. 7. Approximate solution of the second simulation.

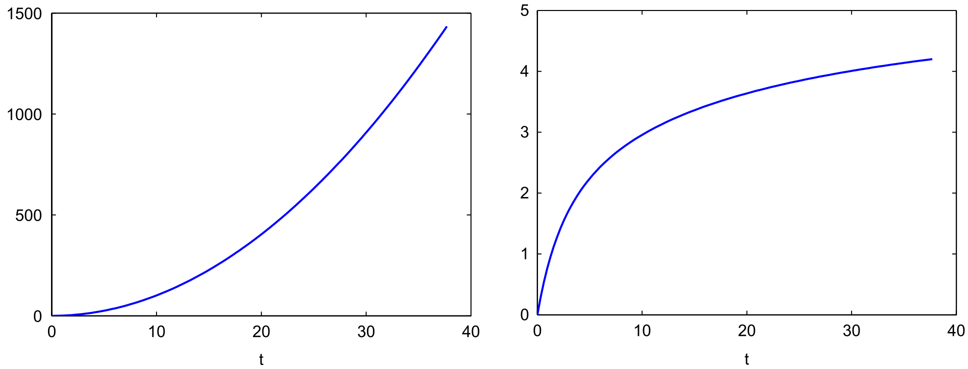

Fig. 8. Approximate solutions of the interior (left) and exterior (right) problem on the unitary circle with Neumann boundary conditions.

element E near the Neumann boundary is immediately affected by the wave, the solution on the element F is trivial till the time instant $t \simeq 1.5$. On the right the time history of the solution u at nodes A, B, C, D is shown: of course, the nearer these nodes are to Γ_u, which is fixed, the lower is the value of the corresponding solution.

For the last test problem, we consider the same domain of the previous simulation, subject to pure Neumann boundary conditions $\bar{p} = H[t]$ (see Fig. 6 on the right). For the discretization phase, the boundary Γ has been approximated introducing a decomposition in 24 straight elements, equipped with linear shape functions. The time interval of analysis $[0, 12\pi]$ has been discretized with $\Delta t = \pi/10$. In Fig. 8, on the left, the time history of the approximate solution $u(A, t)$ in the point A of the mesh, obtained with the energetic approach, is presented for this interior problem. For the sake of completeness, the approximate solution $u(A, t)$ of the exterior wave propagation problem, defined in $\Omega' = \mathbb{R}^2 \setminus \Omega$ with the same Neumann boundary conditions, obtained with the same discretization parameters, is shown on the right. This latter is in perfect agreement with that reported in [14].

Acknowledgements

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