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ScienceDirect
Journal of Approximation Theory 146 (2007) 87–90
Journal of Approximation Theory
www.elsevier.com/locate/jat

An extremal property of Bernstein operators ☆

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Received 1 August 2006; accepted 17 October 2006

Communicated by Zeev Ditzian
Available online 18 December 2006

Abstract

We establish a strong version of a known extremal property of Bernstein operators, as well as several characterizations of a related specific class of positive polynomial operators.

MSC: 41A36; 26D15

Keywords: Bernstein operators; Positive linear operators; Convex functions

For $n = 1, 2, \ldots$, let $\mathcal{L}_n^*$ be the class of all positive polynomial operators L acting on $\mathcal{C}$ (the space of all real continuous functions on $I := [0, 1]$) and having the form

$$Lf(x) = \sum_{k=0}^{n} p_{n,k}(x) \int_I f \, d\mu_{n,k}, \qquad f \in \mathcal{C}, \quad x \in I, \tag{1}$$

where for $0 \leqslant k \leqslant n$, $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$, and $\mu_{n,k}$ is a positive Borel measure on I, and let $\mathcal{L}_n$ be the set of all $L \in \mathcal{L}_n^*$ that preserve the affine functions, that is, such that, for $0 \leqslant k \leqslant n$, $\mu_{n,k}$ is a probability measure fulfilling the condition

$$\int_I x \, d\mu_{n,k}(x) = k/n. \tag{2}$$

☆ Research supported by the Spanish MCYT, Proyecto MTM2005-08376-C02-02.

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2006.10.003

It is clear that, for $L \in \mathcal{L}_n$, $\mu_{n,0}$ and $\mu_{n,n}$ always are the unit masses at 0 and 1, respectively. If, in addition, $\mu_{n,k} = \delta_{k/n} :=$ unit mass at k/n $(1 \leqslant k \leqslant n-1)$, we have the Bernstein operator

$$B_n f(x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n),$$

which can also be represented in the form

$$B_n f(x) = \int_{[0,n]} f(t/n)\, d\nu_{n,x}(t),$$

where $\nu_{n,x}$ is the binomial probability distribution with parameters n, x. Obviously, B_n is well-defined as a positive polynomial operator on $\mathbb{R}^I$.

In a recent paper [2], Bustamante and Quesada obtained the following theorem which improves an earlier result of Berens and DeVore [1].

Theorem 1. *Let* $n \geqslant 2$. *If* $L \in \mathcal{L}_n$, *then*

$$e_2 \leqslant B_n e_2 \leqslant L e_2,$$

where $e_2(x) := x^2$. *Moreover, if* $B_n e_2(x) = L e_2(x)$, *for some* $x \in (0,1)$, *then* $B_n = L$.

Here, we establish a result stronger than Theorem 1 using a similar idea for its proof. We denote by $\mathcal{C}_{cx}$ ($\mathcal{C}_{cx}^*$) the set of all convex (strictly convex) functions in $\mathcal{C}$. Thus, $f \in \mathcal{C}_{cx}^*$ means that f is continuous and fulfills the condition

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y), \quad 0 \leqslant x < y \leqslant 1,\ \lambda \in (0,1),$$

which amounts to saying that the right derivative of f is strictly increasing in (0, 1).

Theorem 2. *Let* $n \geqslant 2$. *If* $L \in \mathcal{L}_n$, *then*

$$f \leqslant B_n f \leqslant L f, \quad f \in \mathcal{C}_{cx}. \tag{3}$$

Moreover, if $B_n f(x) = L f(x)$, *for some* $f \in \mathcal{C}_{cx}^*$ *and some* $x \in (0,1)$, *then* $B_n = L$.

Proof. If $f \in \mathcal{C}_{cx}$, we have by Jensen's inequality

$$\int_I f(x)\, d\mu_{n,k}(x) \geqslant f\left(\int_I x\, d\mu_{n,k}(x)\right) = f(k/n), \quad 0 \leqslant k \leqslant n, \tag{4}$$

as well as

$$B_n f(x) \geqslant f\left(\int_{[0,n]} (t/n)\, d\nu_{n,x}(t)\right) = f(x), \quad x \in I,$$

and (3) follows from the nonnegativity of the polynomials $p_{n,k}(\cdot)$. To get the conclusion in the second part, we only need to show that $\mu_{n,k} = \delta_{k/n}$, for $1 \leqslant k \leqslant n-1$. Fix $1 \leqslant k \leqslant n-1$, let F and G be the respective distribution functions of $\mu_{n,k}$ and $\delta_{n/k}$, and denote by f' the right derivative of the function $f \in \mathcal{C}_{cx}^*$ in the hypothesis of the second part of the theorem. By (4), the assumption on f and Fubini's theorem (or integration by parts), we have

$$\begin{aligned} 0 &= \int_I f(x)\, d(F(x) - G(x)) = \int_0^1 f'(t)(G(t) - F(t))\, dt \\ &= \int_0^1 (f'(t) - f'(k/n))(G(t) - F(t))\, dt + f'(k/n) \int_0^1 (G(t) - F(t))\, dt \\ &= \int_0^1 (f'(t) - f'(k/n))(G(t) - F(t))\, dt, \end{aligned}$$

the last equality by (2). Since f' is strictly increasing in (0, 1), and $G(t) - F(t)$ is nonpositive (nonnegative) for $0 < t < k/n$ ($k/n < t < 1$), we conclude that $F = G$ a.e., and this entails $F = G$, by the right-continuity of F and G. This finishes the proof of the theorem. □

Remark 1. Property (3) actually characterizes the elements of $\mathcal{L}_n$ within the class $\mathcal{L}_n^*$, since B_n preserves the affine functions on I, and we have $Lf = B_n f$, for each affine function f (because f is both convex and concave). Other characterizations giving further insights on the size of $\mathcal{L}_n$ are provided in the following theorem. We denote by $L \circ M$ the composition of the operators L and M.

Theorem 3. *Let $n \geqslant 1$ and let $L : \mathcal{C} \longrightarrow \mathbb{R}^I$ be a positive linear operator. Then, the following assertions are equivalent*:

(a) $L \in \mathcal{L}_n$.
(b) $L = B_n \circ L^*$, *for some positive linear operator $L^* : \mathcal{C} \longrightarrow \mathcal{C}$ preserving the affine functions on I.*
(c) $L = B_n \circ L^*$, *for some positive linear operator $L^* : \mathcal{C} \longrightarrow \mathbb{R}^I$ preserving the affine functions on I.*

Proof. Let $L \in \mathcal{L}_n$ be given as in (1), and define L^* in the following way:

$$L^* f(x) := \int_I f\, d\mu_x, \quad f \in \mathcal{C},\ x \in I,$$

where $\mu_x := (k + 1 - nx)\mu_{n,k} + (nx - k)\mu_{n,k+1}$, if $k/n \leqslant x \leqslant (k + 1)/n$ $(0 \leqslant k \leqslant n - 1)$. It is readily checked that L^* fulfills the requirements in (b), and this shows that (a) implies (b). It is trivially true that (b) implies (c). Finally, assume that (c) holds true. From the assumptions on L^*, we have, by the Riesz representation theorem,

$$L^* f(x) = \int_I f\, d\nu_x, \quad f \in \mathcal{C},\ x \in I,$$

where for each $x \in I$, ν_x is a probability measure on I having mean x, and we therefore have that L is given by (1), with $\mu_{n,k} = \nu_{k/n}$. This means that (a) also holds true, and the proof of the theorem is complete. □

Remark 2. It is clear from the proof that the preceding theorem remains true if $\mathcal{L}_n$ is replaced by $\mathcal{L}_n^*$ in (a) and the condition of preserving affine functions is dropped in (b) and (c).

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