On two affine-like dynamical systems in a local field

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\textbf{Article Info} \\
Article history: \\
Received 1 June 2011 \\
Revised 10 February 2012 \\
Accepted 1 May 2012 \\
Available online 9 August 2012 \\
Communicated by David Goss

\textbf{Abstract} \\
Let $K$ be a local field with valuation $v$ and residue field $k$. We study two dynamical systems defined on $K$ that can be considered as affine. The first one is the dynamical system $(K,\varphi)$ where $\varphi(x) = x^{p^h} + a$ ($h \in \mathbb{N}$, $a \in K$, $v(a) \geq 0$) and $p$ is the characteristic of $k$. We prove that the minimal subsets of $(K,\varphi)$ are cycles. For $K$ of finite characteristic, the action of the Carlitz module on $K$ gives a dynamical system that is similar to an affine system in characteristic 0.

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1. Introduction

A dynamical system is a pair $(X,G)$ where $X$ is a set and $G$ is a group (or a semi-group) acting on $X$. We assume that $X$ is a topological space and $G$ acts continuously on $X$. Classically, a dynamical system is obtained by iterations of a continuous self map $\varphi$ on $X$.

\textbf{Notation.} In a simpler way, we will denote the dynamical system induced by iterations of the application $\varphi$ on $X$ by $(X,\varphi)$.

First, recall some general definitions.

\textbf{Definitions 1.} \\
(1) Let $(X,G)$ be a dynamical system and $x \in X$. The orbit of $x$ is the set $O(x) = \{g.x \mid g \in G\}$. 

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http://dx.doi.org/10.1016/j.jnt.2012.05.011
Let $x \in X$ and $f$ be a self map. The point $x$ is a periodic point of the system $(X, f)$ if there exists $r \geq 1$ such that $f^r(x) = x$. The cardinality of the orbit of $x$ is the period of $x$.

A cycle is the orbit of a periodic point.

Recently, some interest has been given to dynamical systems in a non-archimedean context (see [1] and [2] and their references). For instance, in the case of monomial systems, Khrennikov showed the following result (here $\mathbb{Q}_p$ denotes the field of $p$-adic numbers):

**Theorem 2.** (See [6]) Let $p > 2$, $n \geq 1$ and for $j \geq 1$ let $m_j = \gcd(n^j - 1, p - 1)$. The dynamical system $(\mathbb{Q}_p, x^a)$ has a cycle of period $r$ if and only if $m_r$ is not divisible by any $m_j$ for $1 \leq j \leq r - 1$.

In [7, Chapter 4], small perturbations of $(\mathbb{Q}_p, x^a)$ are studied. In particular, the theory is applied on dynamical systems $(\mathbb{Q}_p, x^a + a)$ where $a$ is of small valuation. In [3], the authors describe the dynamical system induced by an affine self map on $\mathbb{Q}_p$ and more generally of a local field of characteristic 0.

**Notations.** In what follows, $K$ will denote a local field endowed with a discrete valuation $v$, and $V = \{x \in K \mid v(x) \geq 0\}$ will be its ring of integers and $\mathfrak{M}$ its maximal ideal. The prime $p$ will denote the characteristic of the residue field $k = V/\mathfrak{M}$.

In this article, we are interested in two dynamical systems that are induced by maps that can be considered to be affine applications. First, in Section 2 we consider the dynamical system $(K, \varphi)$ where $h \in \mathbb{N}$, $a \in V$ and $\varphi(x) = x^{p^h} + a$. This dynamical system can been seen as a generalization of the dynamical system $(K, x + a)$ studied in [3]. We will determine the dynamic of the system $(K, \varphi)$ and in particular, we will describe the cycles of $(K, \varphi)$.

In Section 3, denoting by $q$ a power of $p$, we will assume that $K$ is the completion of $\mathbb{F}_q[T]$ for a $p$-adic valuation, where $P$ is a monic irreducible polynomial of $\mathbb{F}_q[T]$. Using the Carlitz module, we will define a continuous action of $\mathbb{F}_q[T]$ on $K$. We will show that the dynamics of $(K, \mathbb{F}_q[T])$ are quite similar to those of the dynamical system $(\mathbb{Q}_p, f)$, where $f$ is an affine map of $\mathbb{Q}_p$ in itself.

### 2. The dynamical system $(K, x^{p^h} + a)$

Let $a \in V$ and $h \in \mathbb{N}$ and $\varphi(x) = x^{p^h} + a$. In this section, we describe the dynamic of the system $(K, \varphi)$. Recall some definitions.

**Definitions 3.**

1. The point $x \in X$ is a recurrent point of $(X, G)$ if $x$ is an accumulation point of its orbit.
2. The system $(X, G)$ is minimal if, for all $x \in X$, the orbit of $x$ is dense in $X$.
3. A subset $E$ of $X$ is minimal if $E$ is invariant by $G$ and the subsystem $(E, G)$ is minimal.

The existence of minimal subset is given by the following theorem which is a consequence of the Zorn lemma.

**Theorem 4 (Birkhoff).** Let $X$ be a compact topological space and let $G$ be a group acting continuously on $X$. The system $(X, G)$ admits a minimal subset.

**Notation.** In this section, $q$ will denote the cardinality of the residue field $k$.

#### 2.1. Reduction to the system $(V, \varphi)$

Obviously, if $x \in K \setminus V$, $v(\varphi^j(x))$ tends to $-\infty$. Therefore, every element of a minimal subset of $K$ belongs to $V$. Thus, every minimal subset of the system $(K, \varphi)$ is included in $V$. In particular, in the sequel, we will be interested in the system $(V, \varphi)$. 
It is obvious that every cycle of length \( r \) in \( V \) induces a cycle of length \( r' \leq r \) with \( r' \mid r \) in the residue field \( k \). In fact, we will show that, for every cycle of \((V, \varphi)\), there exists a cycle of the same length in the residue field and conversely.

2.2. Minimal subsets of \((V, \varphi)\)

Recall the

**Definition 5.** A map \( f \) defined on a subset \( E \) of \( K \) is a 1-lipschitzian map of \( E \) if, for all \( x, y \in E \),

\[
v(f(x) - f(y)) \geq v(x - y).
\]

The following proposition deals with the density of the range of 1-lipschitzian map.

**Proposition 6.** (See [4].) Let \( E \) be a precompact subset of a valued field \( K \) and let \( f \) be a 1-lipschitzian map from \( E \) to \( E \). Then \( f(E) = E \) if and only if \( f \) is an isometry of \( E \), that is,

\[
v(f(x) - f(y)) = v(x - y),
\]

for all \( x, y \in E \).

We deduce the

**Corollary 7.** Let \( K \) be a valued field and \( f \) be a 1-lipschitzian map from \( K \) to \( K \). The map \( f \) is an isometry of every precompact minimal subset of \((K, f)\).

Since \( V \) is a compact subset of \( K \) and is invariant with respect to \( \varphi \), the Birkhoff Theorem implies that the system \((V, \varphi)\) admits minimal subsets. The following theorem shows that all minimal subsets of the system \((K, \varphi)\) are cycles.

**Theorem 8.** Let \( f \in V[X] \) be such that \( v(f'(x)) \geq 1 \) for every \( x \in V \). The elements of a minimal system of \((V, f)\) are pairwise uncongruent modulo \( M \). In particular, a minimal subset of \((V, f)\) is a cycle of length \( \leq q \).

**Proof.** Let \( E \subseteq V \) be a minimal subsystem of \((V, f)\). Since the map \( f \) is 1-lipschitzian, by Proposition 6, \( f \) is an isometry of \( E \) and \( v(f(x) - f(y)) = v(x - y) \) for all \( x, y \in E \). By Taylor’s formula, for every \( x, y \in E \), one has

\[
f(x) - f(y) = (x - y)(f'(x) + f_2(x)(x - y) + \cdots + f_n(x)(x - y)^{n-1})
\]

where \( f_2, \ldots, f_{n-1} \in V[X] \) and \( \deg f = n \). In particular, since \( f \) is an isometry, for every \( x \neq y \in E \), \( v(x - y) = 0 \). Hence, the subset \( E \) is finite of cardinality \( \leq q \). \( \square \)

**Lemma 9.** For all \( x, y \in V \) such that \( v(x - y) \geq 1 \) and for every \( m \in \mathbb{N} \), one has

\[
v(\varphi^m(x) - \varphi^m(y)) \geq v(x - y) + m.
\]

**Proof.** By Taylor’s formula, we can write

\[
\varphi(x) - \varphi(y) = (x - y)(\varphi'(x) + \varphi_1(x)(x - y) + \cdots + \varphi_{p-1}(x)(x - y)^{p-1}),
\]

where \( \varphi_1, \ldots, \varphi_{p-1} \in V[X] \). Since \( v(x - y) > 0 \) and \( v(\varphi'(x)) \geq 1 \),
By the previous lemma, if \( m > 1 \), then
\[
\nu\left(\varphi(x) - \varphi(y)\right) \geq \nu(x - y) + 1.
\]

One concludes by induction on \( m \). \( \square \)

The following lemma is an immediate consequence.

**Lemma 10.** Let \( E \) and \( F \) be two distincts minimal subsets of \((V, \varphi)\). For all \((x, y) \in E \times F\), one has \( \nu(x - y) = 0 \).

**Proof.** By the previous lemma, if \( \nu(x - y) \geq 1 \), the sequence \((\varphi^n(x) - \varphi^n(y))_{n \in \mathbb{N}}\) converges to 0. \( \square \)

The following theorem gives a relationship between the lengths of minimal subsets of \((K, \varphi)\).

**Theorem 11.** Let \( K \) be a local field with valuation domain \( V \) and maximal ideal \( \mathfrak{M} \). Let \( q \) be the cardinality of its residue field \( k \), \( a \in V \) and \( \varphi(x) = x^q + a \). There are only finitely many minimal subsets of the dynamical system \((K, \varphi)\). They are cycles in \( V \) of lengths \( r_1, r_2, \ldots, r_k \) and one has
\[
r_1 + r_2 + \cdots + r_k = q.
\]

**Proof.** The number of elements of each of the minimal subsets of \((V, \varphi)\) is less than \( q \). Let \( E_1, E_2, \ldots, E_k \) be distinct minimal subsets of \((K, \varphi)\). By Theorem 8, they are cycles in \( V \) of lengths \( r_1, r_2, \ldots, r_k \). According to Lemma 9, \( r_1 + r_2 + \cdots + r_k \leq q \). Assume that \( r_1 + r_2 + \cdots + r_k < q \), then
\[
E' := \left\{ x \in V \mid \nu(x - y) = 0, \ \forall y \in \bigcup_{i=1}^{s} E_i \right\} \neq \emptyset.
\]

Since \( \varphi(x) - \varphi(y) \equiv (x - y)^q \mod \mathfrak{M} \) for every \( x, y \in V \), \( \varphi \) induces a bijection on \( k \), then \( E' \) is an invariant compact of \( V \). Thanks to Theorem 4, the system \((E', \varphi)\) admits a minimal subset \( E_{s+1} \) of cardinality \( r_{s+1} \). By iteration, we obtain a finite number of cycles with lengths \( r_1, r_2, \ldots, r_k \) and \( r_1 + r_2 + \cdots + r_k = q \). \( \square \)

**Remarks 12.**

1. The union of all minimal subsets \( E_1, E_2, \ldots, E_k \) forms a complete system of cosets of \( k \).
2. The set of all recurrent points of the system \((V, \varphi)\) is the union \( E_1 \cup E_2 \cup \cdots \cup E_k \).
3. If \( \nu(a) > 0 \), then \( \mathfrak{M} \) is invariant by \( \varphi \). By Birkhoff's Theorem, it contains a minimal subset \( E \) and, by Theorem 8, \( E \) is a singleton.

**Lemma 13.** Let \( E = \{\varphi^i(x_0) \mid i \in \mathbb{N}\} \) be a minimal subset of \((K, \varphi)\) with cardinality \( r \) and \( x \in V \) be such that \( \nu(x - x_0) \geq 1 \). For every \( 0 \leq j < r \), the sequence \((\varphi^{nr+j}(x))^n \in \mathbb{N}\) converges to \( \varphi^j(x_0) \).

**Proof.** According to Lemma 10, \( \nu(x - x_0) \geq 1 \) implies that for every \( n \in \mathbb{N} \), one has
\[
\nu\left(\varphi^{nr+j}(x) - \varphi^{nr+j}(x_0)\right) \geq \nu(x - x_0) + nr + j.
\]

Since \( \varphi^{nr+j}(x_0) = \varphi^j(x_0) \), we have
\[
\nu\left(\varphi^{nr+j}(x) - \varphi^j(x_0)\right) \geq \nu(x - x_0) + nr + j. \quad \square
\]
In particular, we have:

**Corollary 14.** The lengths of the cycles of $\varphi$ in $V$ and of the induced map in $k$ are the same.

**Proof.** Every cycle of $\varphi$ in $V$ induces a cycle in $k$. The converse is a consequence of the previous lemma. □

### 2.3. Lengths of cycles

We have seen that all minimal subsets of the system $(K, \varphi)$ are cycles whose periods $r_1, r_2, \ldots, r_k$ are such that $r_1 + r_2 + \cdots + r_k = q$. Let $\text{Per}(a, h) = \{r_1, r_2, \ldots, r_k\}$ be the set of all periods of $\varphi$. Obviously, a positive integer $r$ is a period of $\varphi$ if and only if $r$ is multiple of some $r_i$ $(1 \leq i \leq k)$. The aim of this section is to determine the set $\text{Per}(a, h)$.

From now on, we will use the same notation for $\varphi$ and its induced map on $k$.

By Corollary 14, to determine the possible values of $r_i$’s, we just need to determine the values of $\text{period so fc y c l e so ft h e m a p} \varphi$ in the residue field $k$.

By Theorem 8, one has $r_i \leq q = pf$. We begin with a simple case.

**Proposition 15.** Suppose that $f$ divides $h$.

1. If $v(a) = 0$, every minimal subset of the system $(K, \varphi)$ is a cycle of length $p$.
2. If $v(a) \geq 1$, the system $(K, \varphi)$ admits exactly $q$ fixed points.

(In particular, if $v(a) \geq 1$ every minimal subset of the system $(K, \varphi)$ is of the type $E = \{x_0\}$ where $x_0$ is a fixed point of $\varphi$.)

**Proof.** Since $f \mid h$, for every $n \in \mathbb{N}$ and every $x$ in the residue field we have $\varphi^n(x) = x + na$. If $v(a) = 0$, the length of any cycle in the residue field is $p$. If $v(a) \neq 0$ then for every element $x$ of the residue field, $\varphi(x) = x$. □

From now on, we will suppose that $f$ does not divide $h$. Denote $\delta = \text{GCD}(f, h)$ and $m = f / \delta$. Let $\sigma$ be the automorphism of $k$ defined by $\sigma(x) = x^{ph}$ for all $x \in k$ and let $G$ be the cyclic group generated by $\sigma$. It follows that $k$ is a Galoisian extension of $k^G$ of dimension $m = f / \delta$. By the normal basis theorem, there exists $w \in k$ such that $(w, \sigma(w), \ldots, \sigma^{m-1}(w))$ is a basis of $k$ over $k^G$. Thus, every element $x \in k$ can be written

$$x = \sum_{j=0}^{m-1} x_j w^{bj} \quad (x_j \in k^G).$$

The trace of $x \in k$ relative to $k^G$ will be denoted $\text{Tr}(x)$.

**Lemma 16.** Let $x \in k$.

1. Writing $x = \sum_{i=0}^{m-1} x_j w^{bj}$ with $x_j \in k^G$ $(0 \leq j < m)$, one has $\text{Tr}(x) = s(x) \text{Tr}(w)$ where

$$s(x) = \sum_{i=0}^{m-1} x_i.$$

2. The equality $\text{Tr}(x) = 0$ holds if and only if $s(x) = 0$. 

Proof. (1) We have

\[
\text{Tr}(x) = \sum_{j=0}^{m-1} \sigma_j(x) = \sum_{j=0}^{m-1} \sum_{l=0}^{m-1} x_l w_{p^{bij}} = \sum_{j=0}^{m-1} \sum_{l=0}^{m-1} w^{p^j} = s(x) \text{Tr}(w).
\]

(2) This is an immediate consequence of \(\text{Tr}(w) \neq 0\). □

The simplicity of the function \(\varphi\) allows us to compute its iterates explicitly. The next formula is fundamental. Let \(n \in \mathbb{N}\). For every \(x \in k\), one has

\[
\varphi^n(x) = x^{p^{nh}} + ap_{n-1}h + \cdots + ap_{1}h + a,
\]

\[
= \sigma^n(x) + \sum_{j=0}^{n-1} \sigma^j(a).
\]

We have the

Lemma 17. For every \(x \in k\) and \(r \in \mathbb{N}\) \((r = \alpha m + r_0, \alpha, r_0 \in \mathbb{N}, 0 \leq r_0 < m)\), the \(r\)-th iterate of \(\varphi\) at \(x\) is:

\[
\varphi^r(x) = \varphi^{r_0}(x) + \alpha \text{Tr}(a).
\]

Proof. By equality (2) above, we have

\[
\varphi^r(x) = \sigma^r(x) + \sum_{j=0}^{r-1} \sigma^j(a) = \sigma^r(x) + \sum_{j=0}^{\alpha m + r_0 - 1} \sigma^j(a)
\]

\[
= x^{r_0} + \alpha \sum_{j=0}^{m-1} \sigma^j(a) + \sum_{j=\alpha m}^{\alpha m + r_0 - 1} \sigma^j(a) \quad \text{ (since } \sigma \text{ is of order } m) \]

\[
= \sigma^{r_0} + \alpha \text{Tr}(a) + \sum_{j=0}^{r_0-1} \sigma^j(a). \quad \Box
\]

Remark 18. By the previous lemma, for every \(r \in \mathbb{N}\) \((r = \alpha m + r_0, \alpha, r \in \mathbb{N}, r_0 < m)\) and every \(x \in k\), we have

\[
\varphi^r(x) = \varphi^{r_0}(x) + \alpha s(a) \sum_{i=0}^{m-1} w^{p^{bij}}.
\]

Lemma 19. If \(m = f/\delta\) and \(r \in \text{Per}(a, h)\) then \(r\) divides \(pm\).

Proof. Since \(r\) is a length of a cycle of \(\varphi\) in \(k\), it suffices to show that for every \(x \in k\), \(\varphi^{pm}(x) = x\). Thanks to the previous lemma, we have

\[
\varphi^{pm}(x) = \varphi^{0}(x) + p \text{Tr}(a) = x. \quad \Box
\]
We will prove that the periods of the system \((K, \varphi)\) depend on \(\text{Tr}(a)\) only. Since the next lemma is trivial and well-known, we skip its proof.

**Lemma 20.** Let \(L\) be a field, \(n\) a positive integer and \(a_0, a_1, \ldots, a_n\) elements of \(L\). The system

\[
\begin{align*}
    x_1 &= x_2 + a_1, \\
    x_2 &= x_3 + a_2, \\
    &\vdots \\
    x_n &= x_1 + a_n
\end{align*}
\]

admits a solution in \(L^n\) if and only if \(\sum_{i=1}^{n} a_i = 0\).

Furthermore, if \(\sum_{i=1}^{n} a_i = 0\) then the set of the solutions is an affine space of \(L^n\) of dimension 1.

Recall that \(m = f/\delta\) where \(\delta = \text{GCD}(f, h)\).

**Notation.**

1. For every \(n \in \mathbb{Z}\), we denote by \(\theta(n)\) the unique non-negative integer such that \(\theta(n) = n \pmod{m}\) and \(0 \leq \theta(n) < m\).
2. For every \(r \in \mathbb{N}\), we denote by \(o(r)\) its order in \((\mathbb{Z}/m\mathbb{Z}, +)\) and \(d(r)\) the non-negative integer such that \(o(r) = d(r)m\).

**Proposition 21.** Let \(r \in \mathbb{N}\). The equation \(\varphi^r(x) = x\) admits a solution if and only if \(d(r)s(a) = 0\). In which case, the equation \(\varphi^r(x) = x\) admits \(p^{\frac{f}{\delta}}\) solutions.

**Proof.** Write \(r = \alpha m + r_0\) (\(\alpha, r_0 \in \mathbb{N}\) and \(r_0 < m\)) and \(x = \sum_{i=0}^{m-1} x_i w^{hi}\) (\(x_i \in kG\) for every \(0 \leq i < m\)). By equality (3), one has

\[
\varphi^r(x) = \sum_{i=0}^{m-1} x_{\theta(i-r_0)} + \sum_{l=0}^{r_0-1} \sum_{i=0}^{r_0-1} a_{\theta(i-l)} + \alpha s(a) w^{hi}.
\]

Hence,

\[
\varphi^r(x) = x \iff x_{\theta(i-r_0)} + \sum_{l=0}^{r_0-1} a_{\theta(i-l)} + \alpha s(a) = x_i \quad \text{for all } 0 \leq i < m
\]

\[
\iff x_{\theta(i-jr_0)} + \sum_{l=0}^{r_0-1} a_{\theta(i-(j-1)r_0-l)} + \alpha s(a) = x_{\theta(i-(j-1)r_0)}
\]

for all \(0 \leq i < m/o(r_0)\) and \(0 \leq j < o(r_0)\).

By Lemma 20, for every \(0 \leq i < m/o(r_0)\), the system

\[
S_i: \quad x_{\theta(i-jr_0)} + \sum_{l=0}^{r_0-1} a_{\theta(i-(j-1)r_0-l)} + \alpha s(a) = x_{\theta(i-(j-1)r_0)} \quad (0 \leq j < o(r_0))
\]

admits a solution if and only if
\[
\sum_{j=0}^{o(r_0)-1} \left( \sum_{l=0}^{r_0-1} a_\delta(i-(j-1)r_0-l) + \alpha s(a) \right) = 0,
\]
i.e. if and only if
\[
\sum_{i=0}^{o(r_0)-1} \sum_{l=0}^{r_0-1} a_\delta(i-(j-1)r_0-l) + o(r_0)\alpha s(a) = 0. \tag{4}
\]
Since
\[
\{ \theta(i - jr_0 - l) \mid 0 \leq j < o(r_0), 0 \leq l < r_0 - 1 \} = \left[ 0, o(r_0)r_0 \right],
\]
we obtain
\[
\sum_{i=0}^{o(r_0)-1} \sum_{l=0}^{r_0-1} a_\delta(i-(j-1)r_0-l) = \sum_{j=0}^{o(r_0)r_0-1} a_\delta(j) = d(r_0)s(a). \tag{5}
\]
Noticing that \( o(r) = o(r_0) \), we deduce that \( d(r) = d(r_0) + \alpha o(r_0) \). By condition (4) and equality (5), the system \( S_i \) has solutions if and only if \( d(r)s(a) = 0 \), that is, if and only if \( d(r) Tr(a) = 0 \) by Lemma 16(2).

As the systems \( S_i \)’s \( (0 \leq i < m/o(r)) \) are independent, the equation \( \varphi^r(x) = x \) has solutions if and only if \( d(r) Tr(a) = 0 \). Again, by Proposition 21, when the condition \( d(r) Tr(a) = 0 \) is satisfied, each system \( S_i \) admits \( p^\delta \) solutions. Hence, the equation \( \varphi^r(x) = x \) has \( (p^\delta)^{m/o(r)} = p^{j/o(r)} \) solutions. \( \square \)

We are now able to give the periods of \((K, \varphi)\). Recall that \( Per(a, h) \) is the set of all periods of the system \((V, \varphi)\). We distinguish two cases:

2.3.1. The case \( Tr(a) = 0 \)

We assume that \( Tr(a) = 0 \).

**Theorem 22.** If \( r \in \mathbb{N} \), then \( r \in Per(a, h) \) if and only if \( r \) divides \( m \).

In other words, the set of the periods of \( \varphi \) is the set of divisors of \( m \).

**Proof.** To show the “if” part, it is sufficient to prove that \( \varphi^m(x) = x \) for every \( x \in K \). This is an immediate consequence of Proposition 21. Conversely, let \( r \in \mathbb{N} \) be a divisor of \( m \) and \( r' \) be a strict divisor of \( r \). The order of \( r \) (resp. \( r' \)) in \((\mathbb{Z}/m\mathbb{Z}, +)\) is \( m/r \) (resp. \( m/r' \)). By Proposition 21,

\[
\text{Card} \bigcup_{r'|r} \{ x \in K \mid \varphi^r(x) = x \} \leq \sum_{r'|r} p^{j/r'} \leq \sum_{r'=1}^{[r/2]} p^{j/r'} \leq \frac{p^{j/r}(r/2+1)}{p^j - 1} < p^{j/r}.
\]

Hence, there exists cycle with period \( r \). \( \square \)

**Corollary 23.** For every \( a \in V \) such that \( Tr(a) = 0 \), the integers 1 and \( m \) are elements of \( Per(a, h) \). In particular, the equation \( x^p + a = x \) admits \( p \) solutions in \( K \).
2.3.2. The case $\text{Tr}(a) \neq 0$
In this subsection, we assume $\text{Tr}(a) \neq 0$. We can write $m = p^nm_0$ with $n, m_0 \in \mathbb{N}$ and $m_0$ and $p$ are coprime.

**Lemma 24.** Let $r \in \mathbb{N}$. The equation $\varphi^r(x) = x$ has a solution in $k$ if and only if $\nu_p(r) > \nu_p(m)$ where $\nu_p(r)$ and $\nu_p(m)$ are the $p$-adic valuation of $r$ and $m$.

**Proof.** By Proposition 21, the equation $\varphi^r(x) = x$ has a solution in $k$ if and only if $d(r)\text{Tr}(a) = 0$. Since $\text{Tr}(a)$ is nonzero, this is equivalent to $p$ divides $d(r)$. As $o(r) r = d(r)m$, the equation $\varphi^r(x) = x$ has a solution in $k$ if and only if $\nu_p(r) > \nu_p(m)$. \quad \square

**Theorem 25.** Let $m = p^nm_0$ where $m_0$ and $p$ are coprime and let $r \in \mathbb{N}$. Then $r \in \text{Per}(a, h)$ if and only if $r = p^nm + d$ where $d$ is a divisor of $m_0$.

**Proof.** Let $r \in \text{Per}(a, h)$. Since $m$ divides $l_0 = p^{n+1}m_0$, $o(l_0) = 1$ and by Lemma 21, there are $p^j$ solutions of the equation $\varphi^j(x) = x$. Consequently, every $x \in k$ is solution of the equation $\varphi^j(x) = x$. Hence, $r$ divides $l_0$. By Lemma 24, $\nu_p(r) > \nu_p(m)$. We deduce that $r = p^nm + d$ where $d$ is a divisor of $m_0$. Conversely, let $r = p^nm + d$ be with $d$ a divisor of $m_0$. In the same way as in the proof of Theorem 22, one shows that there exist elements of $k$ belonging to a cycle of period $r$ and not belonging to a cycle of period $r' < r$. \quad \square

**Remark 26.** If $\text{Tr}(a) \neq 0$ and $\gcd(p, m) = 1$, then the set of periods of $\varphi$ is

$$\text{Per}(a, h) = \{pd \mid d|m\}.$$ 

3. Action of the Carlitz module

3.1. Definitions and minimal subsets

Let $p$ be a prime number and $q$ be a power of $p$. Recall that the Carlitz module is the injective $\mathbb{F}_q$-morphism defined by

$$C : \mathbb{F}_q[T] \rightarrow \text{End}(\mathcal{G}_a(\mathbb{F}_q(T))),$$

$$T \rightarrow TX + X^q.$$

For an irreducible polynomial $P$ of $\mathbb{F}_q[T]$, we denote by $V$ the completion of $\mathbb{F}_q[T]$ for the $P$-adic valuation $v$, by $\mathfrak{M}$ its maximal ideal and by $K$ its quotient field. From now on, we assume that $q^{\deg P} \geq 3$. For every $H \in K$, the map

$$\tau_H : \mathbb{F}_q[T] \rightarrow K,$$

$$(a, x) \mapsto C_a(H) + x,$$

defines an action of the group $(\mathbb{F}_q[T], +)$ on $K$. More explicitly, set $D_0 = 1$ and for all $i \in \mathbb{N}^*$,

$$D_i = \prod_{h \in \mathbb{F}_q[T] \text{ monic} \atop \deg h = i} h.$$

For every $i \in \mathbb{N}$, denote by $\Psi_i$ the polynomial
\[
\Psi_i(x) = \frac{1}{D_i} \prod_{\substack{h \in \mathbb{F}_q[T] \\
\deg h < i}} (X - h).
\]

For every \( a \in \mathbb{F}_q[T] \), we have (see [5])

\[
C_a(X) = \sum_{j=0}^{\deg a} \Psi_j(a) X^{q^j}.
\]

**Lemma 27.** Let \( H \in K \setminus V \). Then, \( \lim_{\deg a \to +\infty} v(C_a(H)) = -\infty \).

**Proof.** Let \( a \in \mathbb{F}_q[T] \). By hypothesis, \( v(H) < 0 \). It is well-known that for every \( i \in \mathbb{N}, \Psi_i(\mathbb{F}_q[T]) \subseteq \mathbb{F}_q[T] \) (see [5, Corollary 3.3.5]). Hence, for all \( 0 \leq i < \deg a \), one has

\[
v(\Psi_i(a) H^{q^i}) \geq q^i v(H) > q^{\deg a} v(H).
\]

We deduce that \( v(C_a(H)) = q^{\deg a} v(H) \). This proves the lemma. \( \square \)

Thanks to this lemma, we can restrict the study of the dynamics of \( \tau_H \) to the case \( H \in V \). We suppose this condition satisfied from now on. Assume that \( H \) is a point of \( a \)-division for some \( a \in \mathbb{F}_q[T] \), that is \( C_a(H) = 0 \). Then, for every \( x \in K \), the orbit of \( x \) under the action of \( \tau_H \) is finite. Indeed, let \( b \in \mathbb{F}_q[T] \). Write \( b = aR + S \) with \( R, S \in \mathbb{F}_q[T] \) and \( \deg S < \deg a \). One obtains that

\[
C_b(H) = C_{aR+S}(H) = C_R(C_a(H)) + C_S(H) = C_S(H).
\]

**Hypothesis.** From now on, we assume that \( H \in V \) and \( H \) is not a point of division.

**Lemma 28.** Let \( x \in V \).

1. If \( v(x) > 0 \), then for every \( a \in \mathbb{F}_q[T], v(C_a(x)) = v(a) + v(x) \).
2. If \( v(x) = 0 \), then \( v(C_P(x)) = 0 \).

**Proof.** (1) For every \( 1 \leq i \leq \deg a \), one has

\[
\frac{\Psi_i(a)}{\Psi_{i-1}(a)} = \frac{D_{i-1}}{D_i} \prod_{\substack{h \in \mathbb{F}_q[T] \\
\deg h = i-1}} (a - h).
\]

By [8, proof of Lemma 11], the following equalities hold:

\[
v\left( \frac{D_i}{D_{i-1}} \right) = \sum_{e=1}^{\lfloor \deg P \rfloor} q^{i-e}\deg P - \sum_{e=1}^{\lfloor \deg P \rfloor} q^{i-1-e}\deg P
\]

\[
= \begin{cases} 
q^{i-\deg P - \beta} \left(q^{\beta+1} - 1\right) / q^{\deg P - 1} & \text{if } i = l \deg P + \beta \\
q^{i-\deg P - 1} & \text{with } l, \beta \in \mathbb{N} \ (1 \leq \beta < \deg P)
\end{cases}
\]

\text{else.}
We deduce that
\[ v\left( \frac{\Psi_i(a)x^i}{\Psi_{i-1}(a)x^{q^i-1}} \right) \geq (q^i - q^{i-1})v(x) - \frac{q^i - q^{i-1}}{q^{\deg P} - 1} \geq (q^i - q^{i-1})\left( v(x) - \frac{1}{q^{\deg P} - 1} \right) . \]

Since \( q^{\deg P} \geq 3 \) and \( v(x) \geq 1 \), we obtain
\[ v\left( \frac{\Psi_i(a)x^i}{\Psi_{i-1}(a)x^{q^i-1}} \right) > 0. \]

This proves that
\[ v(C_\alpha(x)) = v(\Psi_0(a)x^{q^0}) = v(a) + v(x). \]

(2) One has
\[ C_\alpha(x) = \sum_{i=0}^{\deg P} \Psi_i(P)x^i. \]

Since \( \Psi_{\deg P}(P) = 1 \) and for every \( 0 \leq i < \deg P \), \( v(\Psi_i(P)) \geq 1 \), we deduce that
\[ v(C_\alpha(x)) = v(x^{\deg P}) = 0. \]

\[ \text{Proposition 29. Each point } x \in K \text{ is recurrent.} \]

**Proof.** For all \( a \in \mathbb{F}_q[T] \), one has \( \tau_H(a, x) - x = C_\alpha(H) \). Hence, it is sufficient to show that there exists a sequence \( (b_n)_{n \in \mathbb{N}} \) of \( \mathbb{F}_q[T] \) such that \( \lim_{n \to +\infty} b_n(H) = 0 \). For every \( l \in \mathbb{N} \), there exist \( c_l, d_l \in \mathbb{F}_q[T] \) (\( c_l \neq d_l \)) such that \( C_{c_l}(H) - C_{d_l}(H) \in \mathfrak{M}^l \). Set \( b_l = c_l - d_l \). Thus, one has \( C_{b_l}(H) \in \mathfrak{M}^l \) for all \( l \in \mathbb{N}. \) This proves the proposition. \( \square \)

\[ \text{Remark 30. The proof of this proposition shows that there exists a monic polynomial } W \text{ with the least degree such that } C_W(H) \in \mathfrak{M}. \text{ We set } v_0 = v(C_W(H)). \]

\[ \text{Lemma 31. One has } \deg W \leq \deg P \text{ and } \gcd(W, P) = 1. \text{ Moreover, if } a \in \mathbb{F}_q[T], \text{ then } C_a(H) \in \mathfrak{M} \text{ if and only if } W \text{ divides } a. \]

**Proof.** Suppose that \( \deg W > \deg P \). There exist \( a, a' \in \mathbb{F}_q[T] \) with \( \deg a \leq \deg P \) and \( \deg a' \leq \deg P \) such that \( C_a(H) = C_{a'}(H) \mod \mathfrak{M} \). Hence, \( C_{a-a'}(H) = 0 \mod \mathfrak{M} \). By definition of \( W \), \( a - a' = 0 \). This proves that \( \deg W \leq \deg P \). If \( H \in \mathfrak{M} \), then \( C_1(H) = H \in \mathfrak{M} \) and \( \gcd(W, P) = 1 \). If \( H \notin \mathfrak{M} \), by Lemma 28(2), one has \( v(C_1(P)) = 0 \) and \( W \neq P \). Again, we conclude that \( \gcd(W, P) = 1 \). Let \( a \in \mathbb{F}_q[T] \). Write \( a = WQ + R \) (\( Q, R \in \mathbb{F}_q[T] \)) with \( \deg R < \deg W \). We have
\[ C_a(H) = C_Q(C_W(H)) + C_R(H). \]

By Lemma 28(1), \( C_Q(C_W(H)) \in \mathfrak{M} \), and \( C_a(H) \in \mathfrak{M} \) if and only if \( C_R(H) \in \mathfrak{M} \), that is if and only if \( R = 0 \). \( \square \)
Set $L_0 = 1$ and for every $i \in \mathbb{N}^*$

$$L_i = \prod_{j=1}^{n} (T^{q^j} - T).$$

Let $\exp_C$ (resp. $\log_C$) be the Carlitz exponential (resp. the Carlitz logarithm) defined on $\mathfrak{M}$ by

$$\exp_C(x) = \sum_{n \geq 0} \frac{x^{q^n}}{D_n} \left( \text{resp. } \log_C(x) = \sum_{n \geq 0} (-1)^n \frac{x^{q^n}}{L_n} \right).$$

The Carlitz exponential satisfies the following functional equation (see [5, Chapter 3]): for all $x \in \mathfrak{M}$ and for all $a \in \mathbb{F}_q[T]$

$$\exp_C(a \cdot x) = C_a(\exp_C(x)). \tag{6}$$

Moreover, the Carlitz exponential and the Carlitz logarithm are continuous on $\mathfrak{M}$ and are inverse (see [8]): for all $x \in \mathfrak{M}$

$$\exp_C(\log_C(x)) = \log_C(\exp_C(x)) = x.$$

**Proposition 32.** Let $x, y \in K$. One has $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(y)}$ if and only if there exist $G_0 \in V$ and $S_0$ in $\mathbb{F}_q[T]$ with $\deg S_0 < \deg W$ such that

$$C_{S_0}(H) = y - x \pmod{\mathfrak{M}}, \tag{7}$$

$$G_0 \log_C(C_W(H)) = \log_C(y - x - C_{S_0}(H)). \tag{8}$$

**Proof.** Assume $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(y)}$. Then, one has $y \in \overline{\mathcal{O}(x)}$, that is, there exists a sequence $(b_n)_{n \in \mathbb{N}}$ of $\mathbb{F}_q[T]$ such that $y = \lim_{n \to +\infty} C_{b_n}(H) + x$. There exists $S_0 \in \mathbb{F}_q[T]$ (deg $S_0 < deg W$) such that infinitely many $b_n$ satisfy $b_n \equiv S_0 \pmod{W \mathbb{F}_q[T]}$. We consider such a subsequence $(b_n)_{n \in \mathbb{N}}$. Write $b_n = W R_n + S_0$ ($R_n \in \mathbb{F}_q[T]$). We deduce that

$$\lim_{n \to +\infty} C_{W R_n}(H) = y - C_{S_0}(H) - x. \tag{9}$$

Moreover, one has for all $n \in \mathbb{N}$, $\nu(C_{b_n}(H)) = \nu(C_{b_n}(C_W(H))) \geq 1$ since $\nu(C_W(H)) \geq 1$. Hence, $y - C_{S_0}(H) - x \in \mathfrak{M}$. Therefore, for all $n \in \mathbb{N}$, $\log_C(C_{W R_n}(H))$ and $\log_C(y - C_{S_0}(H) - x)$ are well-defined. By the compactness of $V$, there exists a subsequence of $(R_n)_{n \in \mathbb{N}}$ (say $(R_{n_k})_{k \in \mathbb{N}}$) converging in $V$ (say to $G_0 \in V$). By continuity of $\log_C$ and relation (6), we obtain that

$$\lim_{k \to +\infty} \log_C(y - C_{S_0}(H) - x) = \lim_{k \to +\infty} \log_C(C_{W R_{n_k}}(H))$$

$$= \lim_{k \to +\infty} \log_C(\exp_C(R_{n_k} \log_C(C_W(H))))$$

$$= \log_C(\exp_C(G_0 \log_C(C_W(H)))) = G_0 \log(C_W(H)).$$

The converse is obvious. \qed
Theorem 33. Let $H \in V$ and $(K, \mathbb{F}_q[T])_H$ be the dynamical system induced by the Carlitz module. For $x, y \in K$, the equality $\mathcal{O}(x) = \mathcal{O}(y)$ holds if and only if $v(x - y) \geq 0$ and $y - x$ is in the subgroup $\{C_a(H) \mid a \in \mathbb{F}_q[T]\}$ of $(\mathbb{F}_q[T]/\mathbb{F}_q[T], +)$.

Proof. Assume that $\mathcal{O}(x) = \mathcal{O}(y)$. By Proposition 32, there exist $G_0 \in V$ and $S_0$ in $\mathbb{F}_q[T]$ with $\deg S_0 < \deg W$ such that equalities (7) and (8) hold. Since $v(G_0) \geq 0$, one has $v(\log_C(C_W(H))) \geq v(\log_C(y - x - C_{S_0}(H)))$, and

$$v(y - x - C_{S_0}) \geq v(C_W(H)) = v_0.$$  

That is $y - x = C_{S_0}(H) \pmod{\mathcal{M}}$. Conversely, there exists $a \in \mathbb{F}_q[T]$ such that $y - x = C_a(H) \pmod{\mathcal{M}}$. Hence

$$v(y - x - C_a(H)) \geq v_0.$$  

We deduce that there exists $G_0 \in V$ such that $\log_C(y - x - C_a(H)) = G_0 \log_C(C_W(H)).$ \hfill $\Box$

Notation. For every $r \in \mathbb{Z}$, one denotes by $S_r$ the sphere with center 0 and radius $r$, i.e.

$$S_r = \{x \in K \mid v(x) = r\}.$$  

Proposition 34. Let $H \in V$ and $(K, \mathbb{F}_q[T])_H$ be the dynamical system induced by the Carlitz module.

1. For every $r \in \mathbb{N}^*$, $S_{-r}$ is the disjointed union of its $q^{(v_0 + r - 1) \deg P - \deg W - (q^{\deg P} - 1)}$ minimal sets.
2. The ring $V$ is the disjointed union of its $q^{v_0 \deg P - \deg W}$ minimal sets.

Proof. (1) There exist $\sigma_r := q^{r \deg P - q^{(r - 1) \deg P}}$ elements $x_i$ of $K$ such that

$$S_{-r} = \bigsqcup_{1 \leq i \leq \sigma_r} x_i + V.$$  

For every $1 \leq i \leq \sigma_r$ and for every $a \in \mathbb{F}_q[T]$, one has $\tau_H(a, x_i + V) \subseteq x_i + V$. Noticing that for all $x, y \in x_i + V$, $v(x - y) \geq 0$, we deduce from Theorem 33 that $x_i + V$ is the disjointed union of

$$\#((\mathbb{F}_q[T]/\mathbb{F}_q[T], +)/\{C_a(H) \mid a \in \mathbb{F}_q[T]\}) = q^{v_0 \deg P - \deg W}$$  

minimal sets.

(2) The proof is obtained similarly to (1) by noticing that $v(x - y) \geq 0$ for all $x, y \in V$. \hfill $\Box$

We deduce immediately

Corollary 35. If $H \in V$ and $(V, \mathbb{F}_q[T])_H$ is the dynamical system induced by the Carlitz module on $V$, then $V$ is minimal if and only if $\deg W = v_0 \deg P$. 

3.2. Conjugate dynamical systems

Let \( G \) be a group acting continuously on two topological spaces \( X \) and \( Y \).

**Definition 36.** The two dynamical systems \((X, G)\) and \((Y, G)\) are conjugate if there exists a homeomorphism \( S : X \to Y \) such that, for every \( g \in G \), the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow S & & \downarrow S \\
Y & \xrightarrow{g} & Y
\end{array}
\]

We now determine when the two dynamical systems \((K, \mathbb{F}_q[T])_{H_1}\) and \((K, \mathbb{F}_q[T])_{H_2}\) are conjugate. We begin with the

**Lemma 37.** The minimal sets of \((K, \mathbb{F}_q[T])_H\) are clopen.

**Proof.** Let \( \mu \) be a minimal set of \((K, \mathbb{F}_q[T])_H\). By definition, \( \mu \) is closed. Let \( x \in \mu \). The ball \( B_{v_0}(x) \) is included in \( \mu \) and \( \mu \) is open. \( \square \)

**Lemma 38.** Let \( \mu \) be a minimal set of \((K, \mathbb{F}_q[T])_H\) and \( \mu_0 \) be the minimal set containing \( 0 \). Then, \((\mu, \mathbb{F}_q[T])_H\) and \((\mu_0, \mathbb{F}_q[T])_H\) are conjugate dynamical systems.

**Proof.** Let \( x_0 \in \mu \). For every \( b \in \mathbb{F}_q[T] \), the following diagram is commutative.

\[
\begin{array}{ccc}
\mu & \xrightarrow{\tau_H(b,x)} & \mu \\
\downarrow x \mapsto x-x_0 & & \downarrow x \mapsto x-x_0 \\
\mu_0 & \xrightarrow{\tau_H(b,x)} & \mu_0 \quad \square
\end{array}
\]

**Theorem 39.** Let \( H_1, H_2 \in V, \ (K, \mathbb{F}_q[T])_{H_1} \) and \((K, \mathbb{F}_q[T])_{H_2}\) be the dynamical systems induced by the Carlitz module. Then, \((K, \mathbb{F}_q[T])_{H_1}\) and \((K, \mathbb{F}_q[T])_{H_2}\) are conjugate if and only if \( W_1 = W_2 \).

**Proof.** Let \( \mu_{0,H_1} \) (resp. \( \mu_{0,H_2} \)) be the minimal set of \((K, \mathbb{F}_q[T])_{H_1}\) (resp. \((K, \mathbb{F}_q[T])_{H_2}\)) containing \( 0 \). Assume that \((K, \mathbb{F}_q[T])_{H_1}\) and \((K, \mathbb{F}_q[T])_{H_2}\) are conjugate. By Lemma 38, \( \mu_{0,H_1} \) and \( \mu_{0,H_2} \) are conjugate. There exists a homeomorphism \( \varphi \) of \( K \) such that

\[
\varphi(\tau_{H_1}(b, C_a(H_1))) = \tau_{H_2}(b, \varphi(C_a(H_1))) \quad \text{for all } a, b \in \mathbb{F}_q[T].
\]

For every \( n \in \mathbb{N} \), let \( Q_n = W_1(1 + P + \cdots + P^n) \). One has for every \( n, n' \in \mathbb{N} \) (\( n' > n \)),

\[
\nu(C_{Q_{n'}}(H_1) - C_{Q_n}(H_1)) = \nu(C_{Q_{n'}} - Q_n(H_1)) = \nu(C_{W_1(1 + \cdots + P^{n'})}(H_1)) = \nu(C_{P^{n+\cdots+p^{n'}}}(W_1(H_1))) = n + \nu(W_1(H_1)) \geq n.
\]

This proves that the sequence \((C_{Q_n}(H_1))_{n \in \mathbb{N}}\) is a Cauchy sequence; therefore, it converges. Since \( \varphi \) is continuous, \((C_{Q_n}(H_2))_{n \in \mathbb{N}}\) converges. Assume that \( W_2 \) does not divide \( W_1 \). For every \( n \in \mathbb{N}^* \), we have

\[
\nu(C_{Q_{n+1}}(H_2) - C_{Q_n}(H_2)) = \nu(C_{P^{n+1}}(W_1(H_2))) = 0.
\]
since $C_{W_1}(H_2) = 0$. Consequently, $(C_{Q_a}(H_2))_{a \in \mathbb{N}}$ does not converge. Hence, $W_2$ divides $W_1$. By symmetry, $W'_2$ divides $W_2$ and $W_1 = W_2$.

Conversely, assume that $W_1 = W_2 = W$. Since

$$K = \bigsqcup_{r \in \mathbb{N}^*} S_{-r} \sqcup V,$$

by Proposition 34 and Lemma 37, it is sufficient to prove that $(\mu_{0,H_1}, \mathbb{F}_q[T])_{H_1}$ and $(\mu_{0,H_2}, \mathbb{F}_q[T])_{H_2}$ are conjugate. Let $O(0)_{H_1}$ (resp. $O(0)_{H_2}$) be the orbit of 0 under the action of $\tau_{H_1}$ (resp. $\tau_{H_2}$). Define the application $\Theta : O(0)_{H_1} \to O(0)_{H_2}$ by $\Theta(C_a(H_1)) \mapsto C_a(H_2)$ for every $a \in \mathbb{F}_q[T]$. Obviously, $\Theta$ is bijective. Let $M \in \mathbb{N}^*$, and $a, a' \in \mathbb{F}_q[T]$ such that

$$v(C_{a'}(H_1)) - v(C_a(H_1)) \geq M + v(C_W(H_1)), \quad \text{that is} \quad v(C_{a-a'}(H_1)) \geq M + v(C_W(H_1)).$$

Write $a - a' = WQ + R$ ($Q, R \in \mathbb{F}_q[T]$) with $\deg R < \deg W$. Since $v(C_{WQ}(H_1)) > 0$, necessarily $v(C_R(H_1)) > 0$. As $\deg R < \deg W$, we obtain $R = 0$ and $v(Q) \geq M$. Hence, one has

$$v(C_a(H_2) - C_{a'}(H_2)) \geq M + v(C_W(H_2)).$$

This shows that $\Theta$ is uniformly continuous on $O(0)_{H_1}$. In the same way, one proves that $\Theta^{-1}$ is uniformly continuous on $O(0)_{H_2}$. Since $\Theta$ and $\Theta^{-1}$ are bijective and uniformly continuous, $\Theta$ can be extended in a homeomorphism $\tilde{\Theta}$ of $\mu_{0,H_1}$ on $\mu_{0,H_2}$. By construction, for every $b \in \mathbb{F}_q[T]$, the following diagram is commutative:

$$\begin{array}{ccc}
\mu_{0,H_1} & \xrightarrow{\tau_{H_1}(b,x)} & \mu_{0,H_1} \\
\tilde{\Theta} \downarrow & & \downarrow \tilde{\Theta} \\
\mu_{0,H_2} & \xrightarrow{\tau_{H_2}(b,x)} & \mu_{0,H_2}
\end{array}$$

Consequently, the systems $(K, \mathbb{F}_q[T])_{H_1}$ and $(K, \mathbb{F}_q[T])_{H_2}$ are conjugate. $\square$

Remarks 40.

(1) The results obtained in this section are quite similar to those obtained in [3] for a dynamical system induced by an affine application on a local field of characteristic 0.

(2) More generally, it could be interesting to study the analogous problem where the Carlitz module is replaced by a Drinfeld module, $\mathbb{F}_q[T]$ by the ring $O_{\infty}$ of regular functions of a non-singular curve outside a prime, and $P$ by a prime of $O_{\infty}$.

References


