Combinatorics, transvectants and superalgebras. An elementary constructive approach to Hilbert’s finiteness theorem

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Dedicated to Amitai Regev on the occasion of his 65th birthday

Abstract

We describe a constructive method to produce a minimal set of generators for the algebra of $SL(n)$-invariants of an $n$-ary form. The main feature of this approach is that it provides a “running bound” for the degrees of the generating invariants. The techniques are based on a superalgebraic description of transvectants and a systematic use of the symbolic method.

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1. Introduction

The main problem of classical invariant theory was that of finding finite systems of generators for the algebras of invariants/covariants of a system of $n$-ary forms (in modern language, homogeneous symmetric tensors over a space of dimension $n$); the study of this problem was one of the crucial themes of the Algebra of the second half of the 19th century and involved quite distin-
guished Mathematicians, such as Cayley, Sylvester, Hermite, Clebsch, Gordan, Klein, Brioschi, Faà di Bruno, Capelli, to name but a few (see, e.g. [22,26,31]).

Classical invariant theory culminated in Hilbert’s two landmark papers of 1890 [20] and 1893 [21]. In these papers Hilbert proved that the algebra of invariants of $n$-ary forms is a finitely generated algebra, $n$ an arbitrary positive integer; in order to do this, Hilbert introduced stunning new ideas which deeply influenced the development of modern algebra and algebraic geometry.

The first paper [20] contains a non-constructive proof of this result which relies, on the one hand, on the \textit{basis theorem} and, on the other hand, on the \textit{Cayley Omega process}. Hilbert remarked that his proof actually applies in a wider range of cases, namely in those cases in which there exists an analog of the Cayley Omega process (see, e.g., Weyl [31]).

This proof was criticized for not being constructive. In 1893 Hilbert [21] wrote his second seminal paper, and provided a further proof of the finiteness theorem in order to reply to the harsh criticism about his 1890s work. This second proof is, a posteriori, a constructive one, as shown by Sturmfels [29]; however, following Derksen [10], Hilbert’s second proof should be meant just as a “more constructive” proof, since he did not give a degree bound for the generators of the invariant ring. About a century later, Popov did obtain an explicit degree bound, by combining Hilbert’s second proof with some results of modern algebra and algebraic geometry (see, e.g. Decker–de Jong [8], Derksen [10]).

In the last twenty years, a lot of deep work has been done in order to obtain algorithms for the explicit construction of finite systems of generators of rings of invariants (clearly in the cases in which these rings are finitely generated, XIV Hilbert’s problem, see, e.g., Popov [25], Grosshans [16,17]). The substantial part of modern contributions relies on sophisticated uses of the modern theory of Groebner bases (see, e.g., Sturmfels [29], Derksen [9], Derksen–Kemper [11]). In particular, Derksen recently provided quite innovatory ideas that opened the way to surprising results: among them, we mention a refinement of Hilbert’s approach of 1890 that turns it into an algorithm and a significant improvement [10] of Popov’s degree bound.

The pre-Hilbert classical invariant theory was characterized by a clever use of combinatorial methods, essentially based on the notion of “transvectant” for binary forms (see, e.g. [7,12,15,18,22,24]). In 1868, P. Gordan [14] solved the main problem in the case of binary forms of arbitrary degree and discovered an algorithm to produce a set of generators for the algebra of invariants (see also Meyer [22] and, for a modern approach to Gordan’s method, Weyman [32]).

In this paper, we extend the combinatorial approach to the study of the ring of invariants from the case of binary forms to the case of $n$-ary forms, $n$ an arbitrary positive integer. Our strategy is based on a generalization of the so-called “electro-chemical method” of Sylvester (as admirably reformulated by P. Olver et al. [23,24]) for binary forms and, therefore, it involves a detailed study of the combinatorics of transvectants and of the “\textit{symbolic method}” in the case of $n$-ary forms. Our approach is here voluntarily limited: we examine the classical situation, namely, the study of the ring of invariants of an $n$-ary form, with respect to the action of the special linear group $SL(n, \mathbb{K})$, $\mathbb{K}$ a field of characteristic zero (even if we are confident that this approach could be extended to more general situations).

We refer to the quite elementary scheme of iterative algorithm discussed in Popov’s book (pp. 30–31). Informally speaking, at any step one constructs a finite system of linear generators for the space of homogeneous invariants of a given degree $m$ and then reduces it by eliminating those elements that are expressible as polynomials in the invariants (of lower degree) produced in the preceding steps. These basic steps can be performed in different ways, with different degrees
of efficiency. The procedure terminates by Hilbert Theorem of 1890, and, therefore, the problem is to find a termination criterion.

We submit a method that yields a “running criterion.” We define the transvectants of \( n \)-ary forms and show that one can produce a system of linear generators in degree \( m \) by forming all the transvectants of a system of linear generators in degree \( m - 1 \) (Proposition 6). Furthermore, we prove that any transvectant of an invariant factorizable into the product of \( t \) invariants, \( t \) greater than the degree \( d \) of the ground form, is still “reducible” (Proposition 5). The “running” termination criterion is obtained by combining the two results on transvectants mentioned above and is expressed by Theorem 2: if some irreducible invariants are produced in a certain step \( m \) but no irreducible invariants are produced in steps \( m + 1, \ldots, md + 1 \), \( d \) the degree of the ground form, then no irreducible invariants will be produced in further steps, and, therefore the algorithm terminates.

The paper is organized as follows. In Section 2 we state the results in the algebra \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) of polynomial functions over a space of homogeneous symmetric tensors direct sum with \( n - 1 \) copies of the ground vector space \( V \), \( \dim(V) = n \). In Section 3, we provide the proofs of these results by a systematic use of the “symbolic method.” Specifically, the algebra \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) is the epimorphic image of a symbolic algebra \( \mathbb{K}[L \cup X[P] \cup P] \) with respect to action of a \( SL(n, \mathbb{K}) \)-equivariant operator, the umbral operator \( U^d \); this fact allows the theory to be developed in the “symbolic world” \( \mathbb{K}[L \cup X[P] \cup P] \) and then transferred to the “actual world” \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) by means of the operator \( U^d \). The crucial point is that the transvectants can be lifted to the symbolic algebra where they admit a purely combinatorial description (Theorem 4); the proofs are obtained in the “symbolic world,” and rely on some deep combinatorial properties of bracket polynomials and of the action of transvectants over them.

The techniques we use in Section 3 are founded on the following ideas.

(i) The symbolic method (see, e.g. [19,27]).

(ii) The imbedding of the symbolic algebra (a letterplace algebra [13]) into a superalgebraic setting (see, e.g. [1,3,4,19,28]); this imbedding allows the full effectiveness of Capelli’s method of virtual variables to be exploited [5,6]. Specifically, this method leads to a supple use of Capelli’s special identity (Theorem 3) and of the “superstraightening formula” [19] that allows us to provide a combinatorial closed form for the action of transvectants (Theorem 4).

(iii) The “polar expansion formula” of Capelli [7], as recently restated in [5]. This expansion formula is the main tool in the proof of the “spanning theorem” (Theorem 6).

2. Position of the problem and description of the algorithm

2.1. The algebras

In the following, we set \( G = SL(n, \mathbb{K}) \), where \( \mathbb{K} \) denotes a field of characteristic 0. Given a \( \mathbb{K} \)-vector space \( V \) of \( n \), we denote by \( \text{Sym}_d(V) \) the \( d \)th homogeneous component of its symmetric algebra, that is, the space of symmetric tensors of step \( d \) on \( V \).

The diagonal action of \( G \) over \( \text{Sym}_d(V) \) induces, in a standard way, a (contravariant) action of \( G \) over

\[ \mathbb{K}[\text{Sym}_d(V)] , \]

the algebra of polynomial functions over \( \text{Sym}_d(V) \). We will regard the algebra \( \mathbb{K}[\text{Sym}_d(V)] \) as a subalgebra of
\[
\mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] = \mathbb{K}[\text{Sym}_d(V)] \otimes \mathbb{K}[V]^{\oplus(n-1)},
\]
and, therefore, the canonical embedding
\[
\mathbb{K}[\text{Sym}_d(V)] \hookrightarrow \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]
\]
is a \(G\)-equivariant injection.

Let \( E = \{e_i; i = 1, \ldots, n\} \) be a linearly ordered basis of \( V \), and let \( \Phi = \{\varphi_j \in V^*; j = 1, \ldots, n\} \) be its dual basis, namely, the basis of \( V^* \) such that
\[
\langle e_i | \varphi_j \rangle = \delta_{ij},
\]
for every \( i, j = 1, \ldots, n \). Let \( (r_1, \ldots, r_n) \in \mathbb{N}^n \) be a multi-index such that \( r_1 + \cdots + r_n = d \). We denote by \( a_{r_1,\ldots,r_n} \) the element of the basis of \((\text{Sym}_d(V))^*\) defined by setting
\[
(a_{r_1,\ldots,r_n} | e_1^{r_1} \cdots e_n^{r_n}) = \delta_{(r_1,\ldots,r_n),(s_1,\ldots,s_n)} \binom{d}{r_1,\ldots,r_n}^{-1}.
\]
We recall that the classical mapping
\[
(\text{Sym}_d(V))^* \rightarrow \text{Sym}_d(V^*)
\]
defined by setting
\[
a_{r_1,\ldots,r_n} \rightarrow \varphi_1^{r_1} \cdots \varphi_n^{r_n} (d!)^{-1}
\]
is a \(G\)-equivariant isomorphism; in modern terms, this is the starting point of the symbolic method for symmetric tensors (see Section 3).

Given a symmetric tensor \( f \in \text{Sym}_d(V) \), we write
\[
f = \sum_{r_1+\cdots+r_n=d} \binom{d}{r_1,\ldots,r_n} (a_{r_1,\ldots,r_n} | f) e_1^{r_1} \cdots e_n^{r_n}.
\]
Let
\[
\{(x_i | \varphi_j); i = 1, \ldots, n - 1, \ j = 1, \ldots, n\}
\]
be the dual basis of \((V^{\oplus(n-1)})^*\); in plain words, the symbol \((x_i | \varphi_j)\) denotes the \(j\)th coordinate function (with respect to the basis \( E \)) of the \(i\)th fold of the direct sum \( V^{\oplus(n-1)} \). Sometimes we will write \( x_{ij} \) in place of \((x_i | \varphi_j)\). Since \( \mathbb{K} \) is an infinite field, the \( \mathbb{K} \)-algebra
\[
\mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]
\]
is identified with the \( \mathbb{K} \)-algebra
\[
\mathbb{K}[a_{r_1,\ldots,r_n}, (x_i | \varphi_j)]
\]
of formal polynomials in the “variables” \( a_{r_1,\ldots,r_n} \)'s and \((x_i | \varphi_j)\)'s, and its subalgebra \( \mathbb{K}[\text{Sym}_d(V)] \) is identified with the subalgebra \( \mathbb{K}[a_{r_1,\ldots,r_n}] \) of formal polynomials in the “variables” \( a_{r_1,\ldots,r_n} \)'s.
2.2. \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) as a graded algebra

Since the algebra of polynomial functions \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) is identified with the polynomial algebra

\[
\mathbb{K}[a_{r_1,\ldots,r_n}, (x_i|\varphi_j)] = \mathbb{K}[(x_i|\varphi_j)][a_{r_1,\ldots,r_n}],
\]

we will regard \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) as a \( \mathbb{Z} \)-graded \( \mathbb{K}[(x_i|\varphi_j)] \)-algebra; in plain words, the \( \mathbb{Z} \)-homogeneous component \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]_h = \mathbb{K}[a_{r_1,\ldots,r_n}, (x_i|\varphi_j)]_h \) is the \( \mathbb{K}[(x_i|\varphi_j)] \)-subspace spanned by the homogeneous monomials of total degree \( h \in \mathbb{Z} \) in the “variables” \( a_{r_1,\ldots,r_n} \).

Notice that the action of \( G \) over \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) and \( \mathbb{K}[\text{Sym}_d(V)] \) is a homogeneous action.

2.3. Invariants and joint invariants

An element \( \varphi \in \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) is called a joint invariant (of a generic symmetric tensor of step \( d \) and \( n-1 \) vectors in \( V \)) whenever

\[
g \cdot \varphi = \varphi,
\]

for every \( g \in G \).

A joint invariant \( \varphi \) that belongs to \( \mathbb{K}[\text{Sym}_d(V)] \) is called an invariant.

The sets

\[
\mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]^G
\]

of joint invariants and

\[
\mathbb{K}[\text{Sym}_d(V)]^G
\]

of invariants are subalgebras of \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \) and \( \mathbb{K}[\text{Sym}_d(V)] \), respectively.

Since the action of \( G \) is a homogeneous action, the algebra \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]^G \) is a graded subalgebra of \( \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}] \), that is,

\[
\mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]^G = \bigoplus_{h \in \mathbb{Z}} (\mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]^G \cap \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]_h),
\]

and the algebra of invariants \( \mathbb{K}[\text{Sym}_d(V)]^G \) is a graded subalgebra of \( \mathbb{K}[\text{Sym}_d(V)] \).

In the following, we set

\[
C = \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]^G,
\]

\[
C_h = \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]^G \cap \mathbb{K}[\text{Sym}_d(V) \oplus V^{\oplus(n-1)}]_h.
\]

**Proposition 1.** \( C_h \) is a finite-dimensional vector space, for every \( h \in \mathbb{Z} \).
Theorem 1 (Hilbert’s 1890 Finiteness Theorem). The \( \mathbb{K} \)-algebra

\[
C = \mathbb{K}[\text{Sym}_d(V) \oplus V^\oplus(n-1)]^G = \mathbb{K}[a_{r_1, \ldots, r_n}, (x_i|\varphi_j)]^G
\]

is finitely generated.

A minimal finite set of homogeneous algebra generators of \( C \) is called a fundamental system of joint invariants. We recall that the number of the elements, and their degrees, in a fundamental system of joint invariants is uniquely determined.

2.4. \( C_1 \) is one-dimensional

Proposition 2. The space \( C_1 \) is generated by the element

\[
F = \sum_{d = r_1 + \ldots + r_n} \binom{d}{r_1, \ldots, r_n} a_{r_1, \ldots, r_n} x_1^{r_1} \cdots x_n^{r_n}
= F(a_{r_1, \ldots, r_n}; x_{1i}; x_{2i}; \ldots; x_{n-1,i}),
\]

where \( X_i \) is the signed maximal \( i \)th minor of the matrix

\[
\begin{pmatrix}
x_{11} & x_{12} & \ldots & x_{1n} \\
x_{21} & x_{22} & \ldots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n-1,1} & x_{n-1,2} & \ldots & x_{n-1,n}
\end{pmatrix},
\]

that is, the minor obtained by deleting the \( i \)th column, taken with sign \((-1)^{i+1}\).

The fact that \( F \) is a joint invariant is a reformulation of the classical statement “the form is a covariant of itself.”

2.5. Cayley operators and transvectants

Let us consider the algebra

\[
\mathbb{K}[V^\oplus(n-1) \oplus V^\oplus(n-1)] = \mathbb{K}[V^\oplus(n-1)] \otimes \mathbb{K}[V^\oplus(n-1)]
\]

\[
= \mathbb{K}[(x_i|\varphi_j)] \otimes \mathbb{K}[(y_i|\varphi_j)] = \mathbb{K}[(x_i|\varphi_j), (y_i|\varphi_j)]
\]

where \( i = 1, \ldots, n-1 \) and \( j = 1, \ldots, n \), and where the symbols \((x_i|\varphi_j)\) and \((y_i|\varphi_j)\) denote the \( j \)th coordinate function of the \( i \)th fold of the first and the second summand in \( V^\oplus(n-1) \oplus V^\oplus(n-1) \), respectively. Sometimes we will write \( x_{ij} \) in place of \((x_i|\varphi_j)\), and \( y_{ij} \) in place of \((y_i|\varphi_j)\).
2.5.1. Cayley operators

We define a family of $K$-linear endomorphism of $K[[x_i | \varphi_j), (y_i | \varphi_j)]$ by setting

$$
\Omega_s = \Omega[x_s, y_1, \ldots, y_{n-1}] = \det \left[ \begin{array}{cccc}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \\
\frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \cdots & \frac{\partial}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial}{\partial y_{n-1}} & \frac{\partial}{\partial y_{n-2}} & \cdots & \frac{\partial}{\partial y_{n-1}}
\end{array} \right],
$$

for every $s = 1, \ldots, n - 1$.

Since

$$K[a_{r_1, \ldots, r_n}, (x_i | \varphi_j), (y_i | \varphi_j)] = K[a_{r_1, \ldots, r_n}] \otimes K[(x_i | \varphi_j), (y_i | \varphi_j)],$$

we can extend the operator $\Omega[x_s, y_1, \ldots, y_{n-1}]$ to the algebra

$$K[a_{r_1, \ldots, r_n}, (x_i | \varphi_j), (y_i | \varphi_j)]$$

by tensorizing with the identity map on $K[a_{r_1, \ldots, r_n}]$ and, since no confusion could arise, we keep the same symbol for this $K$-linear map.

The operators $\Omega_s = \Omega[x_s, y_1, \ldots, y_{n-1}]$ are called the Cayley operators.

From Capelli’s special identity (see, e.g. Theorem 3 below), it follows

**Proposition 3.** *The Cayley operators*

$$\Omega_s = \Omega[x_s, y_1, \ldots, y_{n-1}]$$

are $G$-equivariant $K$-linear endomorphism of $K[a_{r_1, \ldots, r_n}, (x_i | \varphi_j), (y_i | \varphi_j)]$.

2.5.2. The specialization map $[\ ]_{y \mapsto x}$

We define a $K$-algebra endomorphism of $K[(x_i | \varphi_j), (y_i | \varphi_j)]$ by setting

$$(x_i | \varphi_j) \mapsto (x_i | \varphi_j), \quad (y_i | \varphi_j) \mapsto (x_i | \varphi_j),$$

for every $i = 1, \ldots, n - 1$ and $j = 1, \ldots, n$. Following the notation of the classical Authors of the 19th century, we denote this operator by the symbol

$$[\ ]_{y \mapsto x}.$$

Again, we can extend the operator $[\ ]_{y \mapsto x}$ to the algebra

$$K[a_{r_1, \ldots, r_n}, (x_i | \varphi_j), (y_i | \varphi_j)]$$

by tensorizing with the identity map on $K[a_{r_1, \ldots, r_n}]$ and, since no confusion could arise, we keep the same symbol for this $K$-algebra homomorphism.
2.5.3. Transvectants

Let \( F = F(a_{r_1}, \ldots, a_{r_n}; (x_i|\varphi_j)) \) be the generator of the one-dimensional space \( C_1 \) (cf. Proposition 2), and consider the element \( F_y = F(a_{r_1}, \ldots, a_{r_n}; (y_i|\varphi_j)) \), that is the polynomial obtained from \( F \) by mapping \((x_i|\varphi_j)\) to \((y_i|\varphi_j)\) for every \( i, j \).

For every multi-index \( j = (j_1, \ldots, j_{n-1}) \) of weight \(|j| = j_1 + \cdots + j_{n-1} \leq d\), we consider the \( K \)-linear endomorphism \( S^j = S^{j_1, \ldots, j_{n-1}} \) of \( \mathbb{K}[a_{r_1}, \ldots, a_{r_n}; (x_i|\varphi_j)] \) defined as follows:

\[
S^{j_1, \ldots, j_{n-1}}(g) = \left[ \prod_{i=1}^{j_1} \Omega^{j_1}_{1} \cdots \prod_{n-1}^{j_{n-1}} \Omega^{j_{n-1}}_{n-1} (g F_y) \right]_{y \mapsto x},
\]

for every \( g = g(a_{r_1}, \ldots, a_{r_n}; (x_i|\varphi_j)) \). Note that, for \( n = 2 \), the evaluation \( S^j(g) \) is essentially the classical \( j \)th transvectant evaluated on \( g \) and \( F \); for this reason, we will call transvectants the equivariant operators \( S^j \).

**Proposition 4.** If \( g \) is a joint invariant in \( C_h \), then \( S^{j_1, \ldots, j_{n-1}}(g) \) is a joint invariant in \( C_{h+1} \).

2.6. Transvectants and factorizability

**Proposition 5.** Let \( g = g_1 \cdots g_t \in C_{m-1} \), with \( g_i \in C_{m_i} \), \( 0 < m_i < m - 1 \). If \( t > d \), then

\[
S^j(g) = \sum_{i=1}^{t} h_i g_i, \quad h_i \in C_{s_i}, \quad s_i \leq m - 1,
\]

for every composition \( j \) of weight \(|j| \leq d\).

In plain words, if an invariant \( g \) of degree \( m - 1 \) is factorizable into the product of \( t > d \) invariants, then any of its transvectants is expressible as a “polynomial” into invariants of degrees \( \leq m - 1 \).

2.7. Transvectants on \( C_m \) span \( C_{m+1} \)

**Proposition 6.** Let \( L_m \) be a (finite, minimal) set of \( K \)-linear generators for the \( K \)-space \( C_m \). The set

\[
\bigcup_{j \in \mathcal{J}} S^j[L_m]
\]

is a (finite) set of \( K \)-linear generators of \( C_{m+1} \).

2.8. Main result

In the following, we will denote by

\[
\mathbb{K}\left[ \bigoplus_{i=1}^{h} C_i \right]
\]
the \( \mathbb{K} \)-subalgebra of \( C = \mathbb{K}[a_{r_1, \ldots, r_n}, (x_i | j)]^G \) generated by the finite dimensional subspace \( \bigoplus_{i=1}^h C_i \), for every positive integer \( h \).

Informally speaking, the meaning of the next result is the following: if no “irreducible” joint invariants are found in degrees \( h + 1, \ldots, hd + 1 \), then no “irreducible” invariants will be found in higher degrees (\( d \) the degree of the ground form).

**Theorem 2.** If, for some \( h \in \mathbb{Z}^+ \),

\[
\mathbb{K}\left[ \bigoplus_{i=1}^h C_i \right] = \mathbb{K}\left[ \bigoplus_{i=1}^{h+1} C_i \right] = \cdots = \mathbb{K}\left[ \bigoplus_{i=1}^{hd+1} C_i \right] \tag{1}
\]

then

\[
\mathbb{K}\left[ \bigoplus_{i=1}^h C_i \right] = \mathbb{K}\left[ \bigoplus_{i=1}^{hd+k} C_i \right]
\]

for every \( k \geq 1 \).

**Proof.** We proceed by induction on parameter \( k \). For \( k = 1 \), the statement is true by hypothesis; we suppose the statement true for \( k \), and prove it for \( k + 1 \). By Proposition 6, we have

\[
C_{hd+k+1} = \sum S^j[C_{hd+k}];
\]

by definitions and induction hypothesis, we have

\[
\sum S^j[C_{hd+k}] = \sum S^j[\mathbb{K}\left[ \bigoplus_{i \leq hd+k} C_i \right] \cap C_{hd+k}]
\]

\[
= \sum S^j[\mathbb{K}\left[ \bigoplus_{i \leq h} C_i \right] \cap C_{hd+k}] .
\]

By Proposition 5, we have

\[
\sum S^j[\mathbb{K}\left[ \bigoplus_{i \leq h} C_i \right] \cap C_{hd+k}] \subseteq \mathbb{K}\left[ \bigoplus_{i \leq hd+k} C_i \right] = \mathbb{K}\left[ \bigoplus_{i \leq h} C_i \right] . \quad \square
\]

**Remark 1.** By Hilbert’s theorem,

\[
H = \left\{ h \in \mathbb{Z} : \mathbb{K}\left[ \bigoplus_{i \leq h} C_i \right] = \mathbb{K}\left[ \bigoplus_{i \leq h+1} C_i \right] = \cdots = \mathbb{K}\left[ \bigoplus_{i \leq hd+1} C_i \right] \right\} \neq \emptyset.
\]
Remark 2. Let \( h_0 = \text{Min } H \); by Theorem 2, it follows
\[
\mathbb{K} \left[ \bigoplus_{i \leq h_0} C_i \right] = \mathbb{K} \left[ \text{Sym}_d(V) \oplus V^\oplus(n-1) \right]^G.
\]

2.9. The algorithm

For every \( n \) and \( d \), the algorithm computes, degree by degree, a fundamental system of joint invariants of a symmetric tensor of step \( d \) and \( n - 1 \) vectors over the \( n \)-dimensional \( \mathbb{K} \)-vector space \( V \).

In plain words, the algorithm works as follows.

- **step 1** First set \( A_1 = \{ F \} \), where \( F \) is “the form covariant of itself” which generates the one-dimensional space \( C_1 \) (cf. Proposition 2).
- **step 2** Construct the set of all transvectants \( S^j(F) \), which is a set of linear generators for \( C_2 \); reduce this set (in the linear sense) to a subset \( A_2 \) modulo the subspace generated by \( F^2 \); set \( L_2 = A_2 \cup \{ F^2 \} \), which is a linear basis of \( C_2 \).
- **step 3** Construct the set of all transvectants \( S^j[L_2] \), which is a set of linear generators for \( C_3 \); reduce this set (in the linear sense) to a subset \( A_3 \) modulo the subspace generated by all the polynomials of degree 3 which are products of the elements of the set \( A_1 \cup A_2 \); set \( L_3 \) as linear basis of \( C_3 \).
- **step m** Continue this process until a sufficiently long sequence (cf. hypothesis of Theorem 2) of consecutive \( A_i \)’s, \( A_i = \emptyset \), is found.

By Remark 1, such a sequence will be indeed found after a finite number \( j_0 \) of steps, and, therefore, the algorithm stops. By Remark 2, the set \( A_1 \cup A_2 \cup \cdots \cup A_{h_0} \) is a set of generators for the algebra \( C \), and by construction it is a fundamental system of joint invariants.

Algorithm

**Input:** the dimension \( n \) of the ground space \( V \), and the step \( d \) of the symmetric tensor.

**Output:** a finite sequence \( A_1, A_2, \ldots, A_q \), with \( A_i \subset C_i \), such that \( A_1 \cup A_2 \cup \cdots \cup A_q \) is a fundamental system of joint invariants of a symmetric tensor of step \( d \) and \( n - 1 \) vectors over the \( n \)-dimensional \( \mathbb{K} \)-vector space \( V \).

**Procedure:**
- build the set \( J \) of all the multi-indexes \( j = (j_1, j_2, \ldots, j_{n-1}) \) of weight \( |j| \leq d \);
- set \( L_1 := \{ F \} \), \( A_1 := \{ F \} \), where \( F \) is “the form covariant of itself” (see Proposition 2);
- set \( p := 1 \);
- set \( m := 2 \);
- while \( m < p \times d + 1 \)
  - build \( S := \bigcup_{j \in J} S^j[L_{m-1}] \);
  - build \( M := \) the set of all the polynomials of degree \( m \) which are products of the elements of the set \( A_1 \cup A_2 \cup \cdots \cup A_{m-1} \);
  - build (e.g. via the Gauss algorithm) two subsets \( M_m \subset M \) and \( A_m \subset S \) minimal such that \( \langle M_m \rangle = \langle M \rangle \) and \( \langle M_m \cup A_m \rangle = C_m \);
  - set \( L_m := M_m \cup A_m \).
- if \( A_m \neq \emptyset \) then \( p := m \) end if
set \( m := m + 1; \)
end while
return: \( A_1, A_2, \ldots, A_p. \)
end procedure

**Termination:** the variable \( p \) records the last \( m \) for which \( A_m \neq \emptyset \); the algorithm, due to Remark 1, stops; by Remark 2, this happens at the step \( m = p \times d + 1. \)

**Remark 3.** To obtain a fundamental system of invariants, one simply takes the sets \( A_1, A_2, \ldots, A_{j_0} \) and keeps just the polynomials in which only the variables \( a_{r_1}, \ldots, a_{r_n} \) appear.

3. **Proofs**

3.1. **The symbolic method**

3.1.1. **Commutative letterplace algebras**

Let \( L = \{ \alpha_i; \ i \in \mathbb{Z}^+ \} \) be a countable linearly ordered set, called the alphabet of *umbral letters*, that will be associated to a generic symmetric tensor in \( \text{Sym}_d(V) \). Let \( X = \{ x_h; \ h \in n - 1 \} \) be a linearly ordered set of \( n - 1 \) letters, that will be associated to \( n - 1 \) generic vectors in \( V \); let also \( Y = \{ y_h; \ h \in n - 1 \} \) be a copy of \( X \). Let \( P = \{1, \ldots, n\} = n; \ j \in P \) will be called the \( j \)th place, and will be associated to the \( j \)th coordinate function \( \varphi_j \in V^* \). The set

\[
[L \cup X \cup Y|P] = \{ (\alpha_i|j), (x_h|j), (y_h|j); \ i \in \mathbb{Z}^+, \ h \in n - 1, \ j \in n \}
\]

will be called the set of letterplace variables.

In the following, we will denote by the symbol

\[
\mathbb{K}[L \cup X \cup Y|P]
\]

the free commutative algebra, with unity, generated by the set of letterplace variables over the field \( \mathbb{K} \).

\( \mathbb{K}[L \cup X \cup Y|P] \) is a \( G \)-module under the contravariant “place action” of \( G = SL(n, \mathbb{K}) \) obtained by extending diagonally the action on the letterplace variables

\[
A \cdot (z|h) = \sum_{j=1}^{n} \bar{a}_{hj}(z|j), \quad \text{where } [\bar{a}_{hj}] = A^{-1}
\]

for all matrices \( A \in SL(n, \mathbb{K}), \) letters \( z \in L \cup X \cup Y, \) and places \( h \in P. \)

We denote by

\[
\mathbb{K}^d_m[L|P]
\]

the \( G \)-invariant subspace of \( \mathbb{K}[L|P] \) generated by the monomials in which only the umbral letters \( \alpha_i, i = 1, \ldots, m \) appear, each with multiplicity \( d. \)
3.1.2. The umbral operator $U^d$

The umbral operator $U^d$ is the $\mathbb{K}$-linear map

$$U^d : \mathbb{K}[L|P] \to \mathbb{K}[\text{Sym}_d(V)]$$

defined by the following conditions

$$U^d((\alpha_1|1)^{r_1}\cdots(\alpha_1|n)^{r_n}(\alpha_2|1)^{p_1}\cdots(\alpha_2|n)^{p_n}\cdots)$$

$$= U^d((\alpha_1|1)^{r_1}\cdots(\alpha_1|n)^{r_n})U^d((\alpha_2|1)^{p_1}\cdots(\alpha_2|n)^{p_n})\cdots;$$

$$U^d((\alpha_i|1)^{r_1}\cdots(\alpha_i|n)^{r_n}) = \begin{cases} \frac{d^{r_1+\cdots+r_n}}{r_1!\cdots r_n!}, & \text{if } r_1 + \cdots + r_n = d, \\ 1, & \text{if } r_1 = \cdots = r_n = 0, \\ 0, & \text{otherwise}. \end{cases} \forall i \in \mathbb{Z}^+,$$

Proposition 7. Let $m$ be a positive integer. Then:

- $U^d[\mathbb{K}^d_m[L|P]] = \mathbb{K}[\text{Sym}_d(V)]_m$;
- $U^d$ is a $G$-equivariant $\mathbb{K}$-linear map;
- $U^d[\mathbb{K}^d_m[L|P]^G] = \mathbb{K}[\text{Sym}_d(V)]^G_m$.

Since

$$\mathbb{K}[L \cup X \cup Y|P] = \mathbb{K}[L|P] \otimes \mathbb{K}[(x_1|j), (y_i|j)]$$

and

$$\mathbb{K}[\text{Sym}_d(V) \oplus V^\oplus(n-1) \oplus V^\oplus(n-1)] = \mathbb{K}[\text{Sym}_d(V)] \otimes \mathbb{K}[(x_i|\varphi_j), (y_i|\varphi_j)],$$

the operator $U^d$ naturally extends to a $G$-equivariant $\mathbb{K}$-linear map from $\mathbb{K}[L \cup X \cup Y|P]$ to $\mathbb{K}[\text{Sym}_d(V) \oplus V^\oplus(n-1) \oplus V^\oplus(n-1)]$ just by tensorizing with the “identity map”

$$(x_i|j) \to (x_i|\varphi_j), \quad (y_i|j) \to (y_i|\varphi_j).$$

This operator will be denoted again by the symbol $U^d$. Proposition 7 extends accordingly.

Let

$$\mathbb{K}^d_m[L \cup X|P]$$

denote the $G$-invariant subspace of $\mathbb{K}[L \cup X|P]$ generated by the monomials in which only the umbral letters $\alpha_i, i = 1, \ldots, m$ occur, each with multiplicity $d$, and the letters $x_1, \ldots, x_{n-1}$ occur with any multiplicity. We denote by the symbol

$$C_m = \mathbb{K}^d_m[L \cup X|P]^G$$

the space of invariants in $\mathbb{K}^d_m[L \cup X|P]$.

By the first fundamental theorem of vector invariant theory, the space $C_m$ is generated by the rectangular bitableaux of length $n$ (products of brackets of the form $[u_1u_2\ldots u_n]$, with $u_i \in$
L ∪ X) with “content” d in each umbral letter α_i, i = 1, ..., m, and any content in the symbols x_1, ..., x_{n-1}.

Hence, by skew-symmetry of the brackets, non-zero rectangular bitableaux in C_m have at most md rows, and their number is finite. Therefore, C_m = U^d[C_m] is a finite-dimensional vector space, and this proves Proposition 1. The same argument shows that C_1 is generated by the element

F_x^1 = [α_1 x_1 ... x_{n-1}]^d,

furthermore U^d(F_x^1) = F, and this proves Proposition 2.

3.2. The Cayley operator and umbral transvectants

In Section 2.5, for every s = 1, ..., n − 1, we defined the Cayley operators

Ω_s = Ω[x_s, y_1, ..., y_{n-1}]: \mathbb{K}[x_j, (y_i)|j)] \to \mathbb{K}[x_j, (y_i)|j)];

we recall that they are G-equivariant maps.

Since

\mathbb{K}[L ∪ X ∪ Y | P] = \mathbb{K}[L | P] \otimes \mathbb{K}[x_j, (y_i)|j)],

we can extend the operator Ω_s to the algebra \mathbb{K}[L | P] \otimes \mathbb{K}[x_j, (y_i)|j)] by tensorizing with the identity map on \mathbb{K}[L | P], modulo the identification (x_i|j) → (x_i|φj), (y_i|j) → (y_i|φj). Since no confusion could arise, we keep the same symbol for this \mathbb{K}-linear map.

Notice that the umbral map commutes with the Cayley operators:

Ω_s ◦ U^d = U^d ◦ Ω_s.

We have also defined the specialization map

[ ]_{y→x}: \mathbb{K}[x_j, (y_i)|j)] \to \mathbb{K}[x_j, (y_i)|j)].

Again, we can extend the operator [ ]_{y→x} to the algebra \mathbb{K}[L | P] \otimes \mathbb{K}[x_j, (y_i)|j)] by tensorizing with the identity map on \mathbb{K}[L | P], modulo the identification (x_i|j) → (x_i|φj), (y_i|j) → (y_i|φj). Since no confusion could arise, we keep the same symbol for this \mathbb{K}-algebra homomorphism.

Notice that the umbral map commutes with the specialization map.

3.2.1. Umbral transvectants

Let

F_y^m = [α_m y_1 ... y_{n-1}]^d;

notice that F_x^1 = [α_1 x_1 ... x_{n-1}]^d is a generator of the one-dimensional space C_1.
In this subsection we define the umbral transvectants, namely the following class of \( K \)-linear endomorphisms

\[
S_j^j = S_{j_1} \cdots S_{j_{n-1}}
\]

de of

\[
C = \bigoplus_h C_h,
\]

where \( j = (j_1, \ldots, j_{n-1}) \) ranges over the set of the compositions of weight \( |j| = j_1 + \cdots + j_{n-1} \leq d \). For every \( m > 1 \) and \( G \in C_{m-1} \) we set

\[
S_{j_1} \cdots S_{j_{n-1}}(G) = [\Omega_{j_1} \cdots \Omega_{j_{n-1}}(G F^m_y)]_{y \mapsto x}.
\]

We explicitly note that \( S_{j_1} \cdots S_{j_{n-1}}[C_{m-1}] \subseteq C_m \).

**Proposition 8.** The umbral map commutes with transvectants:

\[
S_j^j \circ U^d = U^d \circ S_j^j.
\]

### 3.3. Superalgebras, biproducts, and Capelli operators

In the following, we will need some superalgebraic techniques; in this subsection we recall a few basic notions and facts pertaining to the theory of letterplace superalgebras. A systematic exposition of this theory can be found in [5,6]. Let \( Z = Z^- \cup Z^+ \), \( Z^- \) and \( Z^+ \) countable sets, and \( P = P^- \cup P^+ \) be signed (i.e., endowed with a \( \mathbb{Z}_2 \)-grading \( |: Z \rightarrow \mathbb{Z}_2 \), \( | : P \rightarrow \mathbb{Z}_2 \) ) alphabets, called the letter alphabet and the place alphabet, respectively. The letterplace alphabet \([Z|P] = \{(a|b); \ a \in Z, \ b \in P \}\) inherits a signature (i.e., \( \mathbb{Z}_2 \)-grading) by setting \(|(a|b)| = |a| + |b| \in \mathbb{Z}_2 \).

The letterplace \( K \)-superalgebra \( \text{Super}[Z|P] \) is the quotient algebra of the free associative \( K \)-algebra with \( 1 \) generated by the letterplace alphabet \([Z|P]\) modulo the bilateral ideal generated by the elements of the form:

\[
(a|b)(c|d) - (-1)^{|a|+|b|}|(c|d)(a|b), \quad a, c \in Z; \ b, d \in P.
\]

Let \( z', z \in Z \). The superpolarization \( D_{z',z} \) of the letter \( z \) to the letter \( z' \) is the unique linear operator \( D_{z',z} : \text{Super}[Z|P] \rightarrow \text{Super}[Z|P] \) such that

- \( D_{z',z}(A B) = D_{z',z}(A) B + (-1)^{|z'|+|z|} |A| A D_{z',z}(B) \), for all monomials \( A, B \in \text{Super}[Z|P] \), that is, \( D_{z',z} \) is left superderivation of \( \mathbb{Z}_2 \)-grade \( |z'| + |z| \);
- \( D_{z',z}(s|t) = \delta_{z,s} (z'|t) \), for every \( (s|t) \in [Z|P] \).

Note that the following identity holds
\[ D_{z',z} D_{w',w} - (-1)^{(|z'|+|z|)(|w'|+|w|)} D_{w',w} D_{z',z} \]
\[ = \delta_{z,w} D_{z',w} - (-1)^{(|z'|+|z|)(|w'|+|w|)} \delta_{w,z} D_{w',z}. \]

Let \( u_1, \ldots, u_r \in Z \) and \( j_1, \ldots, j_r \in P \), not necessarily distinct. For any \( \beta \neq u_i \), with \( |\beta| = 0 \), all the expressions

\[ D_{u_1 \beta} D_{u_2 \beta} \cdots D_{u_r \beta} \left( (\beta| j_1)(\beta| j_2) \cdots (\beta| j_r) \right) \]

yield the same element of \( \text{Super}[Z|P] \), which is called the biproduct of \( u_1, \ldots, u_r \) and \( j_1, \ldots, j_r \), (see, e.g., [4,19]) and is denoted by

\[ (u_1 u_2 \ldots u_r | j_1 j_2 \ldots j_r). \quad (3.1) \]

In the following, we will consider the letterplace superalgebras

\[ \text{Super}[L \cup X \cup Y|P] \hookrightarrow \text{Super}[L \cup X \cup Y \cup \{ \beta_1, \beta_2, \ldots \}|P], \]

where \( L = L^-, X = X^-, Y = Y^-, P = P^- = \{ 1, \ldots, n \} \), and \( \beta_h \) are letters of grade \( |\beta_h| = 0 \).

The positive letters \( \beta_h \) are called virtual letters.

We hardly need to recall that \( \text{Super}[L \cup X \cup Y|P] \) and \( \text{Super}[L \cup X \cup Y \cup \{ \beta_1, \beta_2, \ldots \}|P] \) are \( G \)-modules, \( G = SL(n, \mathbb{K}) \), with respect to the place action defined in Section 3.1. Furthermore, we claim that any letter polarization operator is a \( G \)-equivariant endomorphism.

Given \( n \) (not necessarily distinct) letters \( u_1, u_2, \ldots, u_n \) in \( L \cup X \cup Y \cup \{ \beta_1, \beta_2, \ldots \} \), the bracket

\[ [u_1 u_2 \ldots u_n] \]

is defined to be the biproduct \( (u_1 u_2 \ldots u_n|12 \ldots n) \).

### 3.3.1. A superalgebraic version of Capelli’s special identity

In the following, the symbols \( z_1, z_2, \ldots, z_n \) will denote negative distinct elements of the alphabet \( L \cup X \cup Y \).

**Remark.** Note that, for any positive virtual letter \( \beta \),

\[ D_{z_i \beta} \cdots D_{z_2 \beta} D_{z_1 \beta} \left( (\beta|1)(\beta|2) \cdots (\beta|n) \right) = \text{det} \left[ (z_h|k) \right]_{h,k=1,2,\ldots,n} \]
\[ = [z_1 z_2 \ldots z_n], \]

which is the usual bracket, in the sense of Cayley.

For any positive virtual letter \( \beta \), all the products

\[ D_{z_1 \beta} D_{z_2 \beta} \cdots D_{z_n \beta} \cdot D_{\beta z_n} \cdots D_{\beta z_2} D_{\beta z_1} \quad (3.2) \]

yield the same operator, when restricted to \( \text{Super}[L \cup X \cup Y|P] \); this operator is called a Capelli operator, and will be denoted by the symbol

\[ H[z_1, z_2, \ldots, z_n]. \]
Proposition 9. The Capelli operator $H[z_1, z_2, \ldots, z_n]$ leaves invariant the subspace $\text{Super}[L \cup X \cup Y | P]$.

Theorem 3 (Capelli’s Special Identity [6]). Under the above assumptions, we have the following identity between operators over $\text{Super}[L \cup X \cup Y | P]$: 

$$H[z_1, z_2, \ldots, z_n] = [z_1 z_2 \ldots z_n] \Omega[z_1, z_2, \ldots, z_n],$$

where $\Omega[z_1, z_2, \ldots, z_n]$ denotes the Cayley operator.

The operator $H$, since defined as a product of letter superpolarization operators, is a $G$-equivariant map with respect to the contravariant place $G$-action over $\text{Super}[L \cup X \cup Y | P]$. Furthermore, we submit that the action of this operator can be proved [6] to be equal to the action of the classical Capelli operator in the sense of Weyl [31].

We claim that Capelli’s special identity implies that the Cayley operators are, in turn, $G$-equivariant maps.

3.4. Combinatorial description of umbral transvectants

We open this subsection by stating and proving a formal lemma that should be regarded as a superalgebraic virtual generalization of the classical Cayley identity (see, e.g. [30]).

We recall that the symbols $z_1, z_2, \ldots, z_n$ denote negative distinct elements of the alphabet $L \cup X \cup Y$.

Lemma 1. Let $\beta$ any positive virtual letter. We have:

1. $D_{\beta z_p} [\beta^{p-1} z_p \ldots z_n][z_1 \ldots z_n] = \frac{p+1}{p} [\beta^p z_{p+1} \ldots z_n][z_1 \ldots z_n],$

2. $D_{\beta z_p} [\beta^{p-1} z_p \ldots z_n][z_1 \ldots z_n]^{k-1} = \frac{p+k-1}{p} [\beta^p z_{p+1} \ldots z_n][z_1 \ldots z_n]^{k-1},$

3. $D_{\beta z_p} \cdots D_{\beta z_1} [z_1 \ldots z_n]^{k} = \frac{k(k+1)\ldots(k+p-1)}{p!} [\beta^p z_{p+1} \ldots z_n][z_1 \ldots z_n]^{k-1}.$

Proof. (Sketch) We have

$$D_{\beta z_p} [\beta^{p-1} z_p \ldots z_n][z_1 \ldots z_n]$$

$$= [\beta^p z_p \ldots z_n][z_1 \ldots z_n] + [\beta^{p-1} z_p \ldots z_n][\beta z_1 \ldots z_p \ldots z_n].$$

By applying the straightening formula of Grosshans, Rota and Stein [19] (for an elementary proof see [2]), the second r.h.s. summand equals

$$\frac{1}{p} [\beta^p z_p \ldots z_n][z_1 \ldots z_n],$$

therefore, assertion 1 is proved.

Assertion 2 follows from assertion 1 just by noting that the polarization is a superderivation and the bracket $[z_1 \ldots z_n]$ is a $\mathbb{Z}_2$-homogeneous element of grade 0.

By iterating assertion 2 one gets assertion 3.  \[\square\]
Notice that, by setting \( p = n \) in assertion 3 of the previous lemma, the special Capelli identity immediately leads to the **Cayley identity**:

\[
\Omega[z_1, \ldots, z_n][z_1 \ldots z_n]^k = k(k + 1) \cdots (k + n - 1)[z_1 \ldots z_n]^{k-1}.
\]

**Proposition 10.** Let \( A \in \mathcal{C}_{m-1} \) and \( \beta \) any positive virtual letter; for every natural number \( k \), we have

\[
D_{\beta x_i} D_{\beta y_n} \cdots D_{\beta y_1} ([y_1 \ldots y_{n-1} \alpha_m]^k A) = c[\beta \beta \ldots \beta] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} D_{\alpha_m x_i} A,
\]

where \( c \) is a rational coefficient different from zero.

**Proof.** (Sketch) First note that

\[
D_{\beta x_i} D_{\beta y_n} \cdots D_{\beta y_1} ([y_1 \ldots y_{n-1} \alpha_m]^k A) = D_{\beta x_i} \left( (D_{\beta y_n} \cdots D_{\beta y_1} [y_1 \ldots y_{n-1} \alpha_m]^k A) \right).
\]

By the previous lemma, the second expression equals

\[
c_1 D_{\beta x_i} ([\beta \ldots \beta \alpha_m] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} A) = \frac{c_1}{n} D_{\beta x_i} D_{\alpha_m \beta} ([\beta \ldots \beta \beta] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} A),
\]

where the coefficient \( c_1 \) is nonzero, namely

\[
c_1 = \frac{k(k + 1) \cdots (k + n - 2)}{(n - 1)!}.
\]

Using the commutation identity between polarizations, we can write

\[
D_{\beta x_i} D_{\alpha_m \beta} = -D_{\alpha_m \beta} D_{\beta x_i} + D_{\alpha_m x_i}.
\]

Notice that

\[
D_{\alpha_m \beta} D_{\beta x_i} ([\beta \ldots \beta \beta] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} A) = 0,
\]

since

\[
D_{\beta x_i} ([\beta \ldots \beta \beta] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} A)
\]

is a linear combination of tableaux, each containing \( n + 1 \) occurrences of the positive symbol \( \beta \), and there are no standard tableaux with such content [19].
Therefore, we get
\[
\frac{c_1}{n} D_{\alpha m x_1} ([\beta \ldots \beta] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} A) = \frac{k(k+1) \cdots (k+n-2)}{n!} [\beta \ldots \beta] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} D_{\alpha m x_1} A. \quad \square
\]

**Lemma 2.** Let \( A \in C_{m-1} \). We have:

1. \( H[y_1, \ldots, y_{n-1}, x_i]([y_1 \ldots y_{n-1} \alpha_m]^k A) = k(k+1) \cdots (k+n-2) [y_1 \ldots y_{n-1} x_i] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} D_{\alpha m x_1} A, \)
2. \( \Omega[y_1, \ldots, y_{n-1}, x_i]([y_1 \ldots y_{n-1} \alpha_m]^k A) = k(k+1) \cdots (k+n-2) [y_1 \ldots y_{n-1} \alpha_m]^{k-1} D_{\alpha m x_1} A, \)
3. \( \Omega^j[y_1, \ldots, y_{n-1}, x_i]([y_1 \ldots y_{n-1} \alpha_m]^k A) = d \cdot [y_1 \ldots y_{n-1} \alpha_m]^{k-j} D_{\alpha m x_1} A, \)

where \( d \) is a nonzero rational coefficient.

**Proof.** We need essentially to prove the first statement.

\[
H[y_1, \ldots, y_{n-1}, x_i]([y_1 \ldots y_{n-1} \alpha_m]^k A) = D_{y_1 \beta} \cdots D_{y_{n-1} \beta} D_{x_i \beta} D_{\beta y_1} \cdots D_{\beta y_{n-1}} ([y_1 \ldots y_{n-1} \alpha_m]^k A) = k(k+1) \cdots (k+n-2) \frac{1}{n!} D_{y_1 \beta} \cdots D_{y_{n-1} \beta} D_{x_i \beta} ([\beta \ldots \beta] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} D_{\alpha m x_1} A) = k(k+1) \cdots (k+n-2) [y_1 \ldots y_{n-1} x_i] [y_1 \ldots y_{n-1} \alpha_m]^{k-1} D_{\alpha m x_1} A. \quad \square
\]

**Theorem 4.** Let \( A \in C_{m-1} \). Then

\[
S_j^\perp(A) = c_j \cdot [\alpha_m x_1 \ldots x_{n-1}]^j D_{\alpha m x_1} \cdots D_{\alpha m x_{n-1}} A,
\]

where \( c_j \) is a nonzero rational coefficient and \( j_0 + j_1 + \cdots + j_{n-1} = d \).

### 3.5. Transvectants and factorizability

**Proof of Proposition 5.** Let \( g = g_1 \cdots g_t \in C_{m-1} \), with \( g_i \in C_{m_i} \), \( 0 < m_i < m - 1 \), \( m_1 + \cdots + m_t = m - 1 \), and assume that \( t > d \).

Since \( U^d \) is a \( G \)-equivariant surjective map, then there exist \( t \) invariants \( G_1, \ldots, G_t \in \mathbb{K}[L \cup X[P]]^G \) such that:

- \( G_i \) involves precisely the \( m_i \) umbral letters
  \[
  \alpha_{m_1+\cdots+m_i-1+1}, \ldots, \alpha_{m_1+\cdots+m_{i-1}+m_i},
  \]
  each of them with multiplicity \( d \), for every \( i = 1, 2, \ldots, t \);
- \( U^d(G_i) = g_i \), for every \( i = 1, 2, \ldots, t \).

Recall that

\[
U^d(G_1 \cdots G_t) = U^d(G_1) \cdots U^d(G_t) = g_1 \cdots g_t = g \in C_{m-1}
\]
and
\[ S^L(g) = U^d \left( S^L(G_1 \cdots G_t) \right). \]

By Theorem 4,
\[ S^L(G_1 \cdots G_t) = c_j \cdot \left[ \alpha_m x_1 \cdots x_{n-1} \right]^{j_0} D^{j_1}_{\alpha_{m} x_1} \cdots D^{j_{n-1}}_{\alpha_{m} x_{n-1}} (G_1 \cdots G_t). \]

(2)

Since the polarization operators are derivations, by the Leibniz rule the action of the polarization monomial \(D^{j_1}_{\alpha_{m} x_1} \cdots D^{j_{n-1}}_{\alpha_{m} x_{n-1}}\) on \(G_1 \cdots G_t\) produces a sum of terms obtained by splitting the polarization monomial in all possible ways on the \(t\) factors. Due to the fact that the number \(t\) of the factors is greater than \(d \geq j_1 + \cdots + j_{n-1}\), in each term there is at least one factor on which no polarization acts.

Therefore, the element (2) can be rewritten as a linear combination of elements of the form \(H_i G_i\), where each \(H_i\) is a linear combination obtained from the product \(G_1 \cdots G_{i-1} G_{i+1} \cdots G_t\) by suitable distributions the \(| j_1 | < t\) occurrences of the polarization operators \(D^{j_1}_{\alpha_{m} x_1}\) on the factors, and then by multiplying by \([\alpha_m x_1 \cdots x_{n-1}]^{j_0}\).

Therefore \(H_i\) is an invariant in \(K[L \cup X|P]^G\); since
\[ U^d (H_i G_i) = U^d (H_i) U^d (G_i) = h_i g_i, \]
where \(h_i \in C_{m-m_i}\), it follows that
\[ S^L(g) = \sum_{i=1}^{t} h_i g_i, \quad h_i \in C_{m-m_i}, \]
for every composition \(j\) of weight \(| j | \leq d\).

3.6. Transvectants, Capelli’s polar expansion formula and the spanning theorem

Let us consider the negative letter alphabet \(L_{m,X} = \{\alpha_1, \ldots, \alpha_m, x_1, \ldots, x_{n-1}\}\) and the negative place alphabet \(P = \{1, \ldots, n\} = \mathbb{N}\).

Consider the commutative letterplace algebra
\[ K[L_{m,X}|P] = K[(\alpha_1|j), \ldots, (\alpha_{m-1}|j), (\alpha_m|j), (x_h|j)] \]
generated over \(K\) by the letterplace variables
\[ (\alpha_i|j), (x_h|j), \quad i = 1, \ldots, m; \quad h = 1, \ldots, n-1; \quad j = 1, \ldots, n. \]

As a polynomial algebra, \(K[L_{m,X}|P]\) can be also regarded as the commutative letterplace algebra
\[ A[(\alpha_m|j), (x_h|j)], \]
where the coefficient ring \(A\) is the \(K\)-polynomial algebra
\[ K[(\alpha_1|j), \ldots, (\alpha_{m-1}|j)]. \]
Clearly, \( \mathbb{K}[L_m,X \mid P] \) is a graded \( \mathbb{A} \)-algebra, in the obvious way.

Let us consider the alphabet (linearly ordered)
\[
L'_{m,X}: x_1 < x_2 < \cdots < x_{n-1} < \alpha_m.
\]

It is worth to recall a couple of classical definitions (see, e.g. [4,13]).

For every \( \lambda \vdash q, \lambda_1 \leq n \), the Deruyts tableau \( X_\lambda \) is the column-constant Young tableau over the linearly ordered alphabet \( L'_{m,X}: x_1 < x_2 < \cdots < x_{n-1} < \alpha_m \) such that the \( i \)th column is filled up by the \( i \)th symbol of \( L'_{m,X} \).

Furthermore, given a pair \( (S, T) \) of tableaux over the alphabets \( L' \) and \( P \) of the same shape \( \lambda \), its bideterminant is the element \( (S \mid T) \in \mathbb{K}[L_m,X \mid P] \) defined as follows.

Let
\[
(s_{11} \ldots s_{1\lambda_1}, s_{21} \ldots s_{1\lambda_2}, \ldots)
\]
be the sequence of rows of \( S \) and
\[
(t_{11} \ldots t_{1\lambda_1}, t_{21} \ldots t_{1\lambda_2}, \ldots)
\]
be the sequence of rows of \( T \); we set
\[
(S \mid T) = (s_{11} \ldots s_{1\lambda_1} | t_{11} \ldots t_{1\lambda_1})(s_{21} \ldots s_{2\lambda_2} | t_{21} \ldots t_{2\lambda_2}) \cdots;
\]
in plain words, \( (S \mid T) \) is the “determinantal” polynomial in \( \mathbb{K}[L_m,X \mid P] \) obtained by performing the product of the biproducts (see Section 3.3) of the pairs of corresponding rows in the tableaux \( S \) and \( T \).

In the following, we will restrict our attention to homogeneous polynomials \( h \) of degree \( q \) in the \( \mathbb{A} \)-algebra \( \mathbb{K}[L_m,X \mid P] \).

Since \( \mathbb{A} \) is a ring containing the field \( \mathbb{K} \) of characteristic 0, Capelli’s polar expansion formula [5] leads to the following result.

**Theorem 5.** For every \( \lambda \vdash q, \lambda_1 \leq n \) and for every standard tableau \( S \) on the alphabet \( L' \), \( sh(S) = \lambda \), there exist a pair of \( \mathbb{A} \)-linear operators \( \mathcal{P}_{SX_\lambda}, \mathcal{P}'_{X_\lambda S} \) on \( \mathbb{K}[L_m,X \mid P] \) such that:

- The operators \( \mathcal{P}_{SX_\lambda}, \mathcal{P}'_{X_\lambda S} \) belong to the \( \mathbb{K} \)-subalgebra of \( \text{End}_{\mathbb{K}}(\mathbb{K}[L_m,X \mid P]) \) generated by the letter polarization operators involving only the letters of the subalphabet \( L' \).
- Every homogeneous polynomial \( h \) of degree \( q \) in the \( \mathbb{A} \)-algebra \( \mathbb{K}[L_m,X \mid P] \) can be written in the form:

\[
h = \sum_\lambda \sum_S \mathcal{P}_{SX_\lambda} \mathcal{P}'_{X_\lambda S}(h),
\]

where
\[
\mathcal{P}'_{X_\lambda S}(h) = \sum_T c^h_{S,T}(X_\lambda | T).
\]

The coefficients \( c^h_{S,T} \) belong to the \( \mathbb{K} \)-algebra \( \mathbb{A} \), and the sum ranges over the set of all standard tableaux \( T \) of shape \( \lambda \) on the negative place alphabet \( P = \{1, 2, \ldots, n\} \).
Remark 4 (The Reduction Principle for Umbral Letters). Assume that $h \in \mathbb{K}[L_m,X|P]$ is a $G$-invariant. Since $P'_{X_{i},S}$ is clearly a $G$-equivariant $\mathbb{K}$-linear endomorphism of the $G$-module $\mathbb{K}[L_m,X|P]$, it follows that $P'_{X_{i},S}(h)$ is a $G$-invariant.

Furthermore, the invariant $P'_{X_{i},S}(h)$ can be expressed in the form

$$[x_1 \ldots x_{n-1} \alpha_m]^{t_{\lambda}} \varphi_S(\alpha_1, \ldots, \alpha_{m-1}, x_1, \ldots, x_{n-1}),$$

where $\varphi_S$ is a $G$-invariant element of the $\mathbb{K}$-polynomial subalgebra of $\mathbb{K}[L_m,X|P]$ generated by the variables

$$(\alpha_i|j), \quad (x_h|j), \quad i = 1, \ldots, m - 1; \quad h = 1, \ldots, n - 1; \quad j = 1, \ldots, n.$$

Remark 5. Let $h \in \mathbb{K}[L_m,X|P]^G$ be such that $U^d(h)$ is a non-zero element in $C_m$. Then

$$P_{S X_{i}} P'_{X_{i},S}(h) = P_{S X_{i}} \left( [x_1 \ldots x_{n-1} \alpha_m]^{t_{\lambda}} \varphi_S \right)$$

$$= \sum_j \theta_{j S} [x_1 \ldots x_{n-1} \alpha_m]^{t_{\lambda}} D_{\alpha_m x_1}^{j_1} \cdot \cdots \cdot D_{\alpha_m x_{n-1}}^{j_{n-1}} \varphi_S$$

$$= \sum_j \theta_{j S} c^{-1}_{j S} S^j(\varphi_S),$$

where $t_{\lambda} + j_1 + \cdots + j_{n-1} = d$, and $\theta_{j S}$ are coefficients in $\mathbb{K}$.

By combining Theorem 4, Theorem 5 and Remark 5, it follows

Theorem 6 (The spanning theorem). Given a finite set $L_{m-1}$ of $\mathbb{K}$-linear generators of $C_{m-1}$, the set $\bigcup_j S^j[L_{m-1}]$ of all the transvectants of the elements of $L_{m-1}$ is a set of $\mathbb{K}$-linear generators of $C_m$.

By applying the umbral operator $U^d$, Theorem 6 yields Proposition 6.

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References


