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Three-dimensional subspace of $l_\infty^{(5)}$ with maximal projection constant

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Abstract

Let V be an n-dimensional real Banach space and let $\lambda(V)$ denote its absolute projection constant. For any $N \in \mathbb{N}$, $N \geqslant n$, define

$$\lambda_n^N = \sup\{\lambda(V)\colon\ \dim(V) = n,\ \ V \subset l_\infty^{(N)}\}$$

and

$$\lambda_n = \sup\{\lambda(V)\colon\ \dim(V) = n\}.$$

A well-known Grünbaum conjecture (p. 465 in [B. Grünbaum, Projection constants, Trans. Amer. Math. Soc. 95 (1960) 451–465]) says that

$$\lambda_2 = 4/3.$$

In this paper we show that

$$\lambda_3^5 = \frac{5 + 4\sqrt{2}}{7}$$

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and we determine a three-dimensional space $V \subset l_\infty^{(5)}$ satisfying $\lambda_3^5 = \lambda(V)$. In particular, this shows that Proposition 3.1 from [H. König, N. Tomczak-Jaegermann, Norms of minimal projections, J. Funct. Anal. 119 (1994) 253–280] (see p. 259) is incorrect. Hence the proof of the Grünbaum conjecture given in [H. König, N. Tomczak-Jaegermann, Norms of minimal projections, J. Funct. Anal. 119 (1994) 253–280] which is based on Proposition 3.1 is incomplete.

Keywords: Absolute projection constant; Minimal projection; Three-dimensional Hahn–Banach theorem

1. Introduction

Let X be a real Banach space and let $V \subset X$ be a finite-dimensional subspace. A linear, continuous mapping $P : X \to V$ is called *a projection* if $P|_V = id|_V$. Denote by $\mathcal{P}(X, V)$ the set of all projections from X onto V. Set

$$\lambda(V, X) = \inf\{\|P\|: P \in \mathcal{P}(X, V)\}$$

and

$$\lambda(V) = \sup\{\lambda(V, X): V \subset X\}.$$

The constant $\lambda(V, X)$ is called the *relative projection constant* and $\lambda(V)$ the *absolute projection constant*. General bounds for absolute projection constants were studied by many authors (see e.g. [2,3,9–11,13,15]). It is well known (see e.g. [16]) that if V is a finite-dimensional space then

$$\lambda(V) = \lambda(I(V), l_\infty),$$

where $I(V)$ denotes any isometric copy of V in l_∞. Denote for any $n \in \mathbb{N}$

$$\lambda_n = \sup\{\lambda(V): \dim(V) = n\}$$

and for any $N \in \mathbb{N}$, $N \geqslant n$,

$$\lambda_n^N = \sup\{\lambda(V): V \subset l_\infty^{(N)}\}.$$

By the Kadec–Snobar Theorem (see [8]) $\lambda(V) \leqslant \sqrt{n}$ for any $n \in \mathbb{N}$. However, determination of the constant λ_n seems to be difficult. In [7, p. 465] it was conjectured by B. Grünbaum that

$$\lambda_2 = 4/3.$$

In [12, Th. 1.1] an attempt has been made to prove the Grünbaum conjecture (and a more general result). The proof presented in this paper is mainly based on [12, Proposition 3.1, p. 259] and [12, Lemma 5.1, p. 273]. Unfortunately, the proof of Proposition 3.1 is incorrect. In fact the formula (3.19) from [12, p. 263] is false. This can be easily checked differentiating formula (3.12) on p. 262 with respect to the variable Z_{s1}. (I am using notation from [12].) Because of this error, the part of the proof of [12], on p. 265 is incorrect and as a result, the proof of [12, Th. 1.1] is incomplete.

In this paper we show that

$$\lambda_3^5 = \frac{5+4\sqrt{2}}{7}$$

and we determine a three-dimensional space $V \subset l_\infty^{(5)}$ satisfying $\lambda_3^5 = \lambda(V)$ (see Theorem 3.6). In particular, this shows that not only the proof of Proposition 3.1 from [12] is incorrect but also the statement of Proposition 3.1 is incorrect.

Now we briefly describe the structure of the paper.

In Section 2 we demonstrate some preliminary lemmas useful for determination of λ_3^5 as well as some general results concerning calculation of λ_n^N.

In Section 3 we determine the constant λ_3^5.

The main tools applied in our proof are the Lagrange Multiplier Theorem and the Implicit Function Theorem.

We would like to add that a proof of the Grünbaum conjecture can be found in [4].

2. Preliminary results

In this section mainly we consider the following problem. For a fixed $u_1 \in [0, 1]$ maximize a function $f_{u_1} : \mathbb{R}^{N-1} \times (\mathbb{R}^N)^n \to R$ defined by

$$f_{u_1}\big((u_2, \dots, u_N), x^1, \dots, x^n\big) = \sum_{i,j=1}^{N} u_i u_j \big|\langle x_i, x_j\rangle_n\big| \tag{1}$$

under constraints

$$\big\langle x^i, x^j\big\rangle_N = \delta_{ij}, \quad 1 \leqslant i \leqslant j \leqslant n; \tag{2}$$

$$\sum_{j=2}^{N} u_j^2 = 1 - u_1^2. \tag{3}$$

Here for $j = 1, \dots, N$, $x_j = ((x^1)_j, \dots, (x^n)_j)$, $\langle w, z\rangle_n = \sum_{j=1}^{n} w_j z_j$ for any $w = (w_1, \dots, w_n)$, $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ and $\langle p, q\rangle_N = \sum_{j=1}^{N} p_j q_j$ for any $p = (p_1, \dots, p_N)$, $q = (q_1, \dots, q_N) \in \mathbb{R}^N$. Also we will work with

$$f_{u_1,A}\big((u_2, \dots, u_N), x^1, \dots, x^n\big) = \sum_{i,j=1}^{N} u_i u_j a_{ij} \langle x_i, x_j\rangle_n, \tag{4}$$

where $A = \{a_{ij}\}$ is a fixed $N \times N$ symmetric matrix.

Lemma 2.1. *Let $C = (c_{ij})_{i,j=1,\dots,n}$ be a real $n \times n$ orthonormal matrix. Then for any $x^1, \dots, x^n$, $u \in \mathbb{R}^N$ satisfying* (2) *and* (3),

$$f_{u_1}\big((u_2, \dots, u_N), x^1, \dots, x^n\big) = f_{u_1}\big((u_2, \dots, u_N), C\big(x^1\big), \dots, C\big(x^n\big)\big),$$

and

$$f_{u_1,A}\big((u_2,\dots,u_N),x^1,\dots,x^n\big)=f_{u_1,A}\big((u_2,\dots,u_N),C\big(x^1\big),\dots,C\big(x^n\big)\big)$$

for any $N\times N$ matrix A. Here $C(x^i)=\sum_{j=1}^n c_{ij}x^j$.

Proof. It follows easily from the facts that

$$\big\langle Cx^i,Cx^j\big\rangle_N=\big\langle x^i,x^j\big\rangle_N$$

for $i,j=1,\dots,n$ and

$$\big\langle (Cx)_i,(Cx)_j\big\rangle_n=\langle x_i,x_j\rangle_n$$

for $i,j=1,\dots,N$, where $(Cx)_i=((Cx^1)_i,\dots,(Cx^n)_i)$. □

Now we recall without proof the following well-known

Lemma 2.2. *Let $(X,\langle\cdot,\cdot\rangle)$ be a finite-dimensional Hilbert space with an orthonormal basis $x^1,\dots,x^n$. Let $T:X\to X$ be a linear isometry. If C is an $n\times n$ matrix with columns $c_j=(c_{1j},\dots,c_{nj})$ defined by*

$$Tx^j=\sum_{i=1}^n c_{ji}x^i,$$

then C is an orthonormal matrix.

Lemma 2.3. *Let $x^1,\dots,x^n\in\mathbb{R}^N$ and $u\in\mathbb{R}^N$ satisfy* (2) *and* (3)*. Set $V=\operatorname{span}[x^1,\dots,x^n]$. Assume $v^1,\dots,v^n$ is an orthonormal basis of V (with respect to $\langle\cdot,\cdot\rangle_N$). Then*

$$f_{u_1}\big((u_2,\dots,u_N),x^1,\dots,x^n\big)=f_{u_1}\big((u_2,\dots,u_N),v^1,\dots,v^n\big)$$

and

$$f_{u_1,A}\big((u_2,\dots,u_N),x^1,\dots,x^n\big)=f_{u_1,A}\big((u_2,\dots,u_N),v^1,\dots,v^n\big)$$

for any $N\times N$ matrix A.

Proof. It is well known that for any $x,y\in\mathbb{R}^N$, $\langle x,x\rangle_N=\langle y,y\rangle_N=1$, there exists a linear isometry (with respect to the Euclidean norm in $\mathbb{R}^N$) $T_{x,y}:\mathbb{R}^N\to\mathbb{R}^N$ such that $Tx=y$. Applying this fact and the induction argument with respect to n we get that there exists a linear isometry $T:\mathbb{R}^N\to\mathbb{R}^N$ such that $Tx^i=v^i$ for $i=1,\dots,n$. By Lemma 2.2 there exists an orthonormal matrix C such that $Cx^i=\sum_{j=1}^n C_{ij}x^j=v^i$. By Lemma 2.1,

$$f_{u_1}\big((u_2,\dots,u_N),x^1,\dots,x^n\big)=f_{u_1}\big((u_2,\dots,u_N),v^1,\dots,v^n\big),$$

and

$$f_{u_1,A}\big((u_2,\dots,u_N),x^1,\dots,x^n\big)=f_{u_1,A}\big((u_2,\dots,u_N),v^1,\dots,v^n\big),$$

which completes the proof. $\square$

Lemma 2.4. *Let $n,N\in\mathbb{N}$, $N\geqslant n$. Fix $u=(u_1,\dots,u_N)\in\mathbb{R}^N$ with nonnegative coordinates. Let us consider a function $f:\mathbb{R}^{nN}\to\mathbb{R}$ given by*

$$f\big(x^1,\dots,x^n\big)=\sum_{i,j=1}^{N}u_iu_j\big|\langle x_i,x_j\rangle\big|_n,$$

where $x^i\in\mathbb{R}^N$ for $i=1,\dots,n$. Assume that $y^1,\dots,y^n\in\mathbb{R}^N$ are so chosen that

$$f\big(y^1,\dots,y^n\big)=\max\big\{f\big(x^1,\dots,x^n\big)\colon\ \big(x^1,\dots,x^n\big)\text{ satisfying }(2)\big\}.$$

Let $A\in\mathbb{R}^{N\times N}$ be a matrix defined by

$$a_{ij}=\operatorname{sgn}\big(\langle y_i,y_j\rangle_n\big)\tag{5}$$

for $i,j=1,\dots,N$ ($\operatorname{sgn}(0)=1$ by definition). Define $B\in\mathbb{R}^{N\times N}$ by

$$b_{ij}=u_iu_ja_{ij}\tag{6}$$

for $i,j=1,\dots,N$. Let

$$b_1\geqslant b_2\geqslant\cdots\geqslant b_N$$

denote the eigenvalues of B. (Since B is symmetric all of them are real.) Then there exist orthonormal (with respect to $\langle\cdot,\cdot\rangle_N$) eigenvectors of B, $w^1,\dots,w^n\in\mathbb{R}^N$ corresponding to $b_1,\dots,b_n$, such that

$$f\big(w^1,\dots,w^n\big)=f\big(y^1,\dots,y^n\big)=\sum_{j=1}^{n}b_j.$$

Set

$$f_1\big(x^1,\dots,x^n\big)=\sum_{i,j=1}^{N}b_{ij}\langle x_i,x_j\rangle_n.$$

If $y^1,\dots,y^n\in\mathbb{R}^N$ are such that

$$f_1\big(y^1,\dots,y^n\big)=\max\big\{f_1,\text{ under constraint }(2)\big\}=\max\big\{f,\text{ under constraint }(2)\big\}$$

and $b_n>b_{n+1}$ then $\operatorname{span}[y^i\colon\ i=1,\dots,n]=\operatorname{span}[w^i\colon\ i=1,\dots,n]$.

Proof. Since u_j are nonnegative,

$$f_1(x^1, \dots, x^n) \leqslant f(x^1, \dots, x^n)$$

for any $x^1, \dots, x^n \in \mathbb{R}^N$. Moreover,

$$f_1(y^1, \dots, y^n) = f(y^1, \dots, y^n).$$

Hence f_1 attains its maximum under constraints (2) at $(y^1, \dots, y^n)$. We now apply the Lagrange Multiplier Theorem to the function f_1. This is possible since f_1 is a C^∞ function. Notice that by [12, p. 261] $\operatorname{rank}(G'(y^1, \dots, y^n)) = n(n+1)/2$ where G is the $n(n+1)/2 \times nN$ matrix associated with conditions (2). Consequently there exist Lagrange multipliers k_{ij}, $1 \leqslant i \leqslant j \leqslant n$, such that

$$\frac{\partial(f_1 - \sum_{1\leqslant i\leqslant j\leqslant n} k_{ij} G_i)}{\partial(x^i)_j}(y^1, \dots, y^n) = 0 \tag{7}$$

for $i = 1, \dots, n$, $j = 1, \dots, N$, where $G_i(x^1, \dots, x^n) = \langle x^i, x^j \rangle_N$. Let us define for $i, j \in \{1, \dots, n\}$, $\gamma_{ij} = k_{ij}/2$ if $i < j$, $\gamma_{ij} = k_{ji}/2$, if $j < i$ and $\gamma_{ii} = k_{ii}$. Hence the system (7) can be rewritten (compare with [12, p. 262, formula (3.14)]) as:

$$B(y^m) = \sum_{i=1}^n \gamma_{mi} y^i \tag{8}$$

for $m = 1, \dots, n$. Let $\Gamma = \{\gamma_{ij}, i, j = 1, \dots, n\}$. Observe that Γ is a symmetric $n \times n$ matrix. Hence it has real eigenvalues $a_1, \dots, a_n$. Without loss of generality we can assume that

$$a_1 \geqslant a_2 \geqslant \dots \geqslant a_n. \tag{9}$$

Let $V = [v_{ij}]$ be the $n \times n$ orthonormal matrix consisting of eigenvectors of Γ. Then

$$V^T \Gamma V = D, \tag{10}$$

where D is a diagonal matrix with $d_{ii} = a_i$ for $i = 1, \dots, n$. Now we show that

$$a_i = b_i \tag{11}$$

for $i = 1, \dots, n$. First we prove that a_m, $m = 1, \dots, n$, are also eigenvalues of B. To do this, fix $m \in \{1, \dots, n\}$. Define

$$w^m = \sum_{j=1}^n v_{jm} y^j. \tag{12}$$

We show that $Bw^m = a_m w^m$. Note that

$$Bw^m = B\left(\sum_{j=1}^n v_{jm} y^j\right) = \sum_{j=1}^n v_{jm} B(y^j) = \sum_{j=1}^n v_{jm}\left(\sum_{i=1}^n \gamma_{ji} y^i\right)$$
$$= \sum_{i=1}^n \left(\sum_{j=1}^n v_{jm}\gamma_{ji}\right) y^i = \sum_{i=1}^n \left(\sum_{j=1}^n v_{jm}\gamma_{ij}\right) y^i = \sum_{i=1}^n (\Gamma V)_{im} y^i$$

(by (10))

$$= \sum_{i=1}^n (VD)_{im} y^i = \sum_{i=1}^n v_{im} a_m y^i = a_m\left(\sum_{i=1}^n v_{im} y^i\right) = a_m w^m.$$

Hence for $m = 1, \dots, n$, a_m are eigenvalues of B with the corresponding vectors w^m. By Lemma 2.3, $\langle w^i, w^j\rangle_N = \delta_{ij}$. Notice that by (12) and Lemma 2.3

$$f_1(y^1, \dots, y^n) = f_1(w^1, \dots, w^n).$$

Since for any $m = 1, \dots, n$ and $i = 1, \dots, N$,

$$(Bw^m)_i = a_m (w^m)_i,$$

multiplying each of the above equations by $(w^m)_i$ and summing them up we get that

$$\sum_{j=1}^n a_m = f_1(w^1, \dots, w^n) = f_1(y^1, \dots, y^n) = f(y^1, \dots, y^n).$$

If $a_i \neq b_i$ for some $i \in \{1, \dots, n\}$, let $v^1, \dots, v^n$ be the orthonormal eigenvectors of B corresponding to $b_1, \dots, b_n$. Reasoning as above, we get

$$f(v^1, \dots, v^n) \geqslant \sum_{i,j=1}^N u_i u_j \operatorname{sgn}(\langle y_i, y_j\rangle_n)\langle v_i, v_j\rangle_n$$
$$= \sum_{i=1}^n b_i > \sum_{i=1}^n a_i = f(y^1, \dots, y^n);$$

a contradiction. The fact that $\operatorname{span}[y^i\colon i = 1, \dots, n] = \operatorname{span}[w^i\colon i = 1, \dots, n]$ follows from (12) and invertibility of the matrix V. □

Reasoning as in the proof of Lemma 2.4 we can show

Theorem 2.1. *Let $\mathcal{A}$ denote the set of all $N \times N$ symmetric matrices (a_{ij}) such that $a_{ij} = \pm 1$ and $a_{ii} = 1$ for $i, j = 1, \dots, N$. Let f_{u_1} be given by* (1). *Then*

$$\max\bigl\{f_{u_1}\colon \bigl((u_2,\dots,u_N),x^1,\dots,x^n\bigr) \text{ satisfying (2), (3)}\bigr\}$$
$$=\max\left\{\sum_{i=1}^n b_i(v,A)\colon\ A\in\mathcal{A},\ v=(v_1,\dots,v_n)\in\mathbb{R}^N,\ \sum_{i=1}^N v_i^2=1,\ v_1=u_1\right\},$$

where $b_1(v,A)\geqslant b_2(v,A)\geqslant\cdots\geqslant b_n(v,A)$ *denote the biggest eigenvalues of an* $N\times N$ *matrix* $(v_iv_ja_{ij})_{ij=1}^N$. *Analogously for any* $A=(a_{ij})\in\mathcal{A}$,

$$\max\left\{\sum_{i,j=1}^N u_iu_ja_{ij}\langle x_i,x_j\rangle_n\colon\ \bigl(x^1,\dots,x^n\bigr) \text{ satisfying (2)},\right.$$
$$\left. u_j=\sqrt{\bigl(1-u_1^2\bigr)/(N-1)},\ j=2,\dots,N\right\}$$
$$=\max\left\{\sum_{i=1}^n b_i(v,A)\colon\ A\in\mathcal{A},\ v=\bigl(u_1,c(u_1),\dots,c(u_1)\bigr)\right\},$$

where $c(u_1)=\sqrt{(1-u_1^2)/(N-1)}$. *Also*

$$\max\left\{\sum_{i,j=1}^N u_iu_j\bigl|\langle x_i,x_j\rangle\bigr|_n\colon\ \bigl(x^1,\dots,x^n\bigr) \text{ satisfying (2)},\ \sum_{j=1}^N u_j^2=1\right\}$$
$$=\max\left\{\sum_{i=1}^n b_i(v,A)\colon\ A\in\mathcal{A},\ v=(v_1,\dots,v_n)\in\mathbb{R}^N,\ \sum_{i=1}^N v_i^2=1\right\}.$$

Now for $n,N\in\mathbb{N}$, $N\geqslant n$ define

$$\lambda_n^N=\sup\bigl\{\lambda\bigl(V,l_\infty^{(N)}\bigr)\colon\ V\subset l_\infty^{(N)},\ \dim(V)=n\bigr\}. \tag{13}$$

Lemma 2.5. *For any* $n,N\in\mathbb{N}$, $2\leqslant n\leqslant N$,

$$\lambda_{n-1}^{N-1}\leqslant\lambda_n^N.$$

Proof. Let $V\subset l_\infty^{(N-1)}$ be an $(n-1)$-dimensional subspace with a basis $w^1,\dots,w^{n-1}$. Define

$$V_1=\operatorname{span}\bigl[e_1,\bigl(0,w^j\bigr)\colon\ j=1,\dots,n-1\bigr]\subset l_\infty^N.$$

Let $P\in\mathcal{P}(l_\infty^{(N)},V_1)$ be such that

$$\|P\|=\lambda\bigl(V_1,l_\infty^{(N)}\bigr).$$

(Since V_1 is finite-dimensional such a projection exists.) Define $Q\in\mathcal{L}(l_\infty^{(N-1)},V)$ by

$$Qx=\bigl(P(0,x)_2,\dots,P(0,x)_n\bigr).$$

It is clear that $Q(l_\infty^{(N-1)}) \subset V$ and $Qw^j = w^j$ for $j = 1, \dots, n-1$. Hence $Q \in \mathcal{P}(l_\infty^{(N-1)}, V)$. Moreover, $\|Q\| \leqslant \|P\|$. Taking supremum over V we get that

$$\lambda_{n-1}^{N-1} \leqslant \lambda_n^N,$$

as required. □

Theorem 2.2. *Let $n, N \in \mathbb{N}$, $N \geqslant n$. Then*

$$\lambda_n^N = \max\left\{ \sum_{i,j=1}^N u_i u_j \left|\langle x_i, x_j\rangle\right|_n \colon \left(x^1, \dots, x^n\right) \textit{ satisfying } (2),\ \sum_{j=1}^N u_j^2 = 1 \right\}.$$

Proof. By [12, Prop. 2.2 and (3.7), p. 260],

$$\lambda_n^N \leqslant \max\left\{ \sum_{i,j=1}^N u_i u_j \left|\langle x_i, x_j\rangle\right|_n \colon \left(x^1, \dots, x^n\right) \text{ satisfying } (2),\ \sum_{j=1}^N u_j^2 = 1 \right\}.$$

To prove a converse assume that there exist $n, N \in \mathbb{N}$, $N \geqslant n$, such that

$$\lambda_n^N < \phi_n^N = \max\left\{ \sum_{i,j=1}^N u_i u_j \left|\langle x_i, x_j\rangle\right|_n \colon \left(x^1, \dots, x^n\right) \text{ satisfying } (2),\ \sum_{j=1}^N u_j^2 = 1 \right\}.$$

Without loss of generality we can assume that

$$n = \min\left\{m \in \mathbb{N}\colon\ \lambda_m^M < \phi_m^M \text{ for some } M \geqslant m\right\}$$

and

$$N = \min\left\{M \in \mathbb{N},\ M \geqslant n\colon\ \lambda_n^M < \phi_n^M\right\}.$$

Let us define

$$f\left(u, x^1, \dots, x^n\right) = \sum_{i,j=1}^N u_i u_j \left|\langle x_i, x_j\rangle\right|_n.$$

Let $y^1, \dots, y^n \in \mathbb{R}^N$ satisfying (2) and $u^o \in \mathbb{R}^N$ with $\sum_{j=1}^N (u_j^o)^2 = 1$, be such that

$$f\left(u^o, y^1, \dots, y^n\right) = \phi_n^N.$$

Define as in Lemma 2.4

$$a_{ij} = \operatorname{sgn}\left(\langle y_i, y_j\rangle_n\right) \tag{14}$$

for $i, j = 1, \dots, N$. Also let $B \in \mathbb{R}^{N\times N}$ be given by

$$b_{ij} = u_i^o u_j^o a_{ij} \tag{15}$$

for $i = 1, \dots, N$. By Lemma 2.4 and Theorem 2.1 we can get that

$$f(u^o, y^1, \dots, y^n) = \sum_{i=1}^{n} b_i(u^o, A)$$

where $b_1(u^o, A) \geqslant b_2(u^o, A) \geqslant \dots \geqslant b_n(u^o, A)$ denote the biggest eigenvalues of the above defined matrix B. First suppose that $u^o_j = 0$ for some $j \in \{1, \dots, N\}$. Without loss of generality we can assume that $u^o_1 = 0$. Let B_1 be an $(N-1) \times (N-1)$ matrix given by

$$B_1 = \{b_{ij}\}_{i,j=2,\dots,N}$$

(the part of B without the first row and the first column). Let $d_1 \geqslant \dots \geqslant d_{N-1}$ be the eigenvalues of B_1 and $z^1, \dots, z^{N-1}$ the corresponding orthonormal eigenvectors. Since $u^o_1 = 0$, $v^j = (0, z^j)$, $j = 1, \dots, N-1$, are the orthonormal eigenvectors of B corresponding to d_j. Also $d_o = 0$ is an eigenvalue of B with e_1 as an eigenvector. Consequently

$$b_j(u^o, A) \in \{0, d_k,\ k = 1, \dots, N-1\}$$

for $j = 1, \dots, n$. If $b_j(u^o, A) > 0$ for $j = 1, \dots, n$, then $b_j(u^o, A)$ are also the eigenvalues of B_1. By Theorem 2.1,

$$\sum_{i=1}^{n} b_i(u^o, A) = \phi^N_n = \phi^{N-1}_n = \lambda^{N-1}_n \leqslant \lambda^N_n;$$

a contradiction with the definition of N. If $b_j(u^o, A) = 0$ for some $j \in \{1, \dots, n\}$, then again by Theorem 2.1

$$\phi^N_n \leqslant \sum_{i \neq j} b_i(u^o, A) \leqslant \phi^{N-1}_{n-1} = \lambda^{N-1}_{n-1}.$$

Consequently by Lemma 2.5,

$$\lambda^N_n \geqslant \lambda^{N-1}_{n-1} = \phi^{N-1}_{n-1} \geqslant \phi^N_n,$$

which again leads to a contradiction. Now assume that $u^o_j > 0$ for $j = 1, \dots, N$. Let $w^1, \dots, w^n$ be the orthonormal eigenvectors corresponding to $b_i(u^o, A)$ for $i = 1, \dots, n$. By the proof of Lemma 2.4

$$f_1(u^o, w^1, \dots, w^n) = \phi^N_n.$$

Define, for $j = 1, \dots, n$,

$$z^j = (w^j_1/u^o_1, \dots, w^j_N/u^o_N)$$

and let

$$V = \operatorname{span}[z^j:\ j = 1, \dots, n] \subset l^{(N)}_\infty.$$

We show that $\lambda(V, l_\infty^{(N)}) = \sum_{j=1}^n b_j(u^o, A) = \phi_n^N$. Define, for $j = 1, \dots, n$,

$$f^j = \left(w_1^j u_1^o, \dots, w_N^j u_N^o\right)$$

and let $P \in \mathcal{L}(l_\infty^{(N)}, V)$ be given by

$$Px = \sum_{j=1}^n \langle f^j, x\rangle_N z^j.$$

Since the vectors w^j are orthogonal with respect to $\langle\cdot,\cdot\rangle_N$, $P \in \mathcal{P}(l_\infty^{(N)}, V)$. Now we show that

$$\|P\| = \lambda\left(V, l_\infty^{(N)}\right) = \phi_n^N.$$

Since the function f_1 attains its conditional maximum at $u^o, w^1, \dots, w^n$ (compare with the proof of Lemma 2.4) by the Lagrange Multiplier Theorem there exist $k_{ij} \in \mathbb{R}$, $1 \leqslant i \leqslant j \leqslant n$, and $d \in \mathbb{R}$ such that

$$\frac{\partial(f_1 - \sum_{1\leqslant i\leqslant j\leqslant n} k_{ij} G_i - d(\sum_{j=1}^N u_j^2 - 1))}{\partial(u_j)}\left(u^o, w^1, \dots, w^n\right) = 0. \tag{16}$$

It is easy to see that (16) reduces to (compare with [12, (3.12), p. 262])

$$\sum_{j=1}^N u_j^o a_{ij} \langle w_i, w_j\rangle_n = d u_i^o$$

for $i = 1, \dots, N$. Multiplying the above equalities by u_i^o and summing them up, we get that

$$d = f_1\left(u^o, w^1, \dots, w^n\right) = \phi_n^N.$$

Also since $u_i^o > 0$ for $i = 1, \dots, N$, (16) reduces to

$$\left(\sum_{j=1}^N u_j^o a_{ij} \langle w_i, w_j\rangle_n\right) / u_i^o = d.$$

Consequently, by definition of $\langle\cdot,\cdot\rangle_n$, we get, for $i = 1, \dots, N$,

$$d = \sum_{k=1}^n \left(\sum_{j=1}^N a_{ij} u_j^o w_j^k\right) w_i^k / u_i^o = \sum_{k=1}^n \left(\sum_{j=1}^N a_{ij} f_j^k\right) z_i^k = \left(P(a_{i1}, \dots, a_{iN})\right)_i. \tag{17}$$

Since $\|(a_{i1}, \dots, a_{iN})\|_\infty = 1$, $\|P\| \geqslant d$. On the other hand, for any $x = (x_1, \dots, x_N) \in l_\infty^{(N)}$, $\|x\|_\infty = 1$ and $i \in \{1, \dots, N\}$,

$$\begin{aligned}\left|(Px)_i\right| &= \left|\sum_{j=1}^{n}\langle f^j, x\rangle_N z_i^j\right| = \left|\sum_{k=1}^{N} x_k\left(\sum_{j=1}^{n} f_k^j z_i^j\right)\right| \\ &= \left|\sum_{k=1}^{N} x_k\left(\sum_{j=1}^{n} w_k^j u_k^o w_i^j/u_i^o\right)\right| \leqslant \left(\sum_{k=1}^{N} u_k^o\left|\langle w_k, w_i\rangle_n\right|\right)/u_i^o \\ &= \left(\sum_{k=1}^{N} u_k^o a_{ik}\langle w_k, w_i\rangle_n\right)/u_i^o = d,\end{aligned}$$

since $a_{ij} = \operatorname{sgn}(\langle y_j, y_i\rangle_n) = \operatorname{sgn}(\langle w_j, w_i\rangle_n)$ for $i, j = 1, \dots, N$. Hence

$$\|P\| = d = \phi_n^N.$$

Now we show that

$$\|P\| = \lambda\big(V, l_\infty^{(N)}\big).$$

To do this set for $i = 1, \dots, N$, $a^i = (a_{i1}, \dots, a_{iN})$ and define an operator $E_p : l_\infty^{(N)} \to l_\infty^{(N)}$ by

$$E_p(x) = \sum_{i=1}^{N}\big(u_i^o\big)^2 x_i a^i.$$

We show that $E_p(V) \subset V$. Note that for any $k = 1, \dots, N$, and $j = 1, \dots, n$

$$\begin{aligned}\big(E_p\big(z^j\big)\big)_k &= \sum_{i=1}^{N}\big(u_i^o\big)^2\big(w_i^j/u_i^o\big)\big(a^i\big)_k = \sum_{i=1}^{N} u_i^o w_i^j a_{ki} \\ &= b_j\big(u^o, A\big)w_k^j/u_k^o = b_j\big(u^o, A\big)z_k^j,\end{aligned}$$

since w^j is an eigenvector associated to $b_j(u^o, A)$. Observe that by (17)

$$\big(Pa^i\big)_i = d = \|P\|$$

for $i = 1, \dots, N$ and $\sum_{i=1}^{N}(u_i^o)^2 = 1$. By [5] (see also [14, Th. 1.3]), P is a minimal projection in $\mathcal{P}(l_\infty^{(N)}, V)$. Finally

$$\lambda_n^N \geqslant \lambda\big(V, l_\infty^{(N)}\big) = \|P\| = d = \phi_n^N,$$

which leads to a contradiction. The proof is complete. □

Lemma 2.6. *For any $n \geqslant 2$,*

$$\lambda_n^{n+1} = 2 - 2/(n+1).$$

Moreover, $\lambda_n^{n+1} = \lambda(\ker(f), l_\infty^{(n+1)})$ if and only if $f = c(\pm 1, \dots, \pm 1)$, where c is a positive constant.

Proof. It is clear that

$$\lambda_n^{n+1} = \max\left\{\lambda\left(\ker(f), l_\infty^{(n+1)}\right):\ f \in l_1^{(n+1)} \setminus \{0\},\ \|f\|_1 = 1\right\}.$$

By [1], if $f = (f_1, \ldots, f_{n+1}) \in l_1^{(n+1)}$, $\|f\|_1 = 1$ is so chosen that $\lambda(\ker(f), l_\infty^{(n+1)}) > 1$, then $|f_j| < 1/2$ for any $j = 1, \ldots, n+1$ and

$$\lambda\left(\ker(f), l_\infty^{(n+1)}\right) = 1 + \left(\sum_{i=1}^{n+1} \frac{|f_j|}{(1 - 2|f_j|)}\right)^{-1}.$$

Hence it is easy to see that

$$\lambda_n^{n+1} = \max\left\{1 + \left(\sum_{i=1}^{n+1} \frac{f_j}{(1 - 2f_j)}\right)^{-1}\right\}$$

under constraints

$$\left\{\sum_{j=1}^{n+1} f_j = 1,\ 1/2 \geqslant f_j \geqslant 0,\ j = 1, \ldots, n+1\right\}. \tag{18}$$

Now we show by induction argument that

$$\lambda_n^{n+1} = 2 - 2/(n+1).$$

If $n = 2$, by the Lagrange Multiplier Theorem the only functional $f = (f_1, f_2, f_3)$ which can maximize the function $\phi_2(f) = 1 + (\sum_{i=1}^{3} \frac{f_j}{(1-2f_j)})^{-1}$ under constraint (18) is $f = (1/3, 1/3, 1/3)$ and $\phi_2(1/3, 1/3, 1/3) = 4/3$. Now assume that $\lambda_k^{k+1} = 2 - 2/(k+1)$ for any $k \leqslant n$. Then by the Lagrange Multiplier Theorem the only functional $f = (f_1, \ldots, f_{n+1})$ which can maximize the function $\phi_n(f) = 1 + (\sum_{i=1}^{n+1} \frac{f_j}{(1-2f_j)})^{-1}$ under constraint (18) is $f = (1/(n+1), \ldots, 1/(n+1))$ and $\phi_n(1/(n+1), \ldots, 1/(n+1)) = 2 - 2/(n+1)$. Notice that $\phi_{n+1}(1/(n+2), \ldots, 1/(n+2)) = 2 - 2/(n+2)$, where $\phi_{n+1}(f) = 1 + (\sum_{i=1}^{n+1} \frac{f_j}{(1-2f_j)})^{-1}$. Consequently, by the induction hypothesis,

$$\lambda_{n+1}^{n+2} = \max\left\{1 + \left(\sum_{i=1}^{n+2} \frac{f_j}{(1 - 2f_j)}\right)^{-1}\right\}$$

under constraints

$$\left\{\sum_{j=1}^{n+2} f_j = 1,\ 1/2 > f_j > 0,\ j = 1, \ldots, n+2\right\}. \tag{19}$$

Again by the Lagrange Multiplier Theorem the only $f = (f_1, \ldots, f_{n+2})$ which can maximize ϕ_{n+1} under constraints (19) is $f = (1/(n+2), \ldots, 1/(n+2))$. Hence $\lambda_{n+1}^{n+2} = 2 - 2/(n+2)$, as required. By the above proof, any functional f satisfying $\lambda(\ker(f), l_\infty^{(n+1)}) = \lambda_n^{n+1}$ is of the form $c(\pm 1/(n+1), \dots, \pm 1/(n+1))$. The proof is complete. □

Lemma 2.7. *Let us consider problem* (1) *with $u_1 = 0$ and fixed $N \geqslant n+2$. Assume that $\lambda_n^{N-1} > \lambda_{n-1}^{N-1}$. Then the maximum of f_{u_1} under constraints* (2) *and* (3) *is equal to λ_n^{N-1}.*

Proof. By [12, Th. 1.2] and Theorem 2.2 for any $n, N \in \mathbb{N}$, $N \geqslant n+1$,

$$\lambda_n^N = \max\left\{ \sum_{i,j=1}^N u_i u_j \left|\langle x_i, x_j\rangle_n\right| \right\} \tag{20}$$

under constraints

$$\left\langle x^i, x^j \right\rangle_N = \delta_{ij}, \quad 1 \leqslant i \leqslant j \leqslant n; \tag{21}$$

$$\sum_{j=1}^N u_j^2 = 1. \tag{22}$$

Moreover, if $u, y^1, \dots, y^n \in \mathbb{R}^N$ satisfying (21) and (22) are such that

$$\sum_{i,j=1}^N u_i u_j \left|\langle y_i, y_j\rangle_n\right| = \lambda_n^N,$$

then by Lemma 2.4 and Theorem 2.1,

$$\lambda_n^N = \sum_{j=1}^n b_j, \tag{23}$$

where $b_1 \geqslant b_2 \geqslant \dots \geqslant b_n$ are the biggest eigenvalues of the $N \times N$ matrix $B = (b_{ij})_{i,j=1,\dots,N}$ defined by $b_{ij} = u_i u_j \operatorname{sgn}(\langle y_i, y_j\rangle_n)$.

Now, assume

$$\begin{aligned} &f_{u_1}\big((v_2, \dots, v_n), y^1, \dots, y^n\big) \\ &\quad = \max\big\{ f_{u_1}\big((u_2, \dots, u_n), x^1, \dots, x^n\big)\colon (u_1, \dots, u_n), \big(x^1, \dots, x^n\big) \text{ satisfying (2), (3)} \big\}. \end{aligned}$$

Since $u_1 = 0$, by (20), and Theorem 2.2,

$$f_{u_1}\big((v_2, \dots, v_n), y^1, \dots, y^n\big) \geqslant \phi_n^{N-1} = \lambda_n^{N-1}.$$

To prove the opposite inequality, let B be an $N \times N$ matrix defined by

$$b_{ij} = v_i v_j \operatorname{sgn}\big(\langle y_i, y_j\rangle_n\big).$$

Let $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_N$ be the eigenvalues of B (with multiplicities). By Lemma 2.4,

$$f_{u_1}\big((v_2, \ldots, v_n), y^1, \ldots, y^n\big) = \sum_{j=1}^{n} b_j.$$

Let $C = \{b_{ij}\}_{i,j=2,\ldots,N}$ and let $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{N-1}$ be the eigenvalues of C. Since $u_1 = 0$,

$$\{c_1, \ldots, c_{N-1}\} \cup \{0\} = \{b_1, \ldots, b_N\}.$$

If $b_{j_o} = 0$ for some $j_o \in \{1, \ldots, n\}$, then again by [12, Th. 1.2], (20), Theorem 2.1 and Lemma 2.6

$$\begin{aligned}\lambda_n^{N-1} &\leqslant f_{u_1}\big((v_2, \ldots, v_n), y^1, \ldots, y^n\big)\\ &= \sum_{j=1}^{n} b_j \leqslant \sum_{j<j_o} b_j \leqslant \lambda_{n-1}^{N-1};\end{aligned}$$

a contradiction with our assumptions. Hence $b_i = c_i$ for $i = 1, \ldots, n$. Now let $z^1, \ldots, z^n \in \mathbb{R}^{N-1}$ be the corresponding to $b_1, \ldots, b_n$ orthonormal eigenvectors of C. Hence for any $j = 1, \ldots, n$ and $i = 1, \ldots, N-1$

$$\big(Cz^j\big)_i = c_j\big(z^j\big)_i.$$

Multiplying each of the above equations by $(z^j)_i$ and summing them up we get

$$\begin{aligned}\max\{f_{u_1}\} &= \sum_{j=1}^{n} c_j = \sum_{i,j=2}^{N} b_{ij}\langle z_{i-1}, z_{j-1}\rangle_n\\ &= \sum_{i,j=2}^{N} v_i v_j \operatorname{sgn}\big(\langle y_i, y_j\rangle_n\big)\langle z_{i-1}, z_{j-1}\rangle_n \leqslant \lambda_n^{N-1}.\end{aligned}$$

The proof is complete. □

Lemma 2.8. *Let $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$ and let $z = (z_2, \ldots, z_n) \in \{-1, 1\}^{N-1}$. Let A_z be $N \times N$ matrix defined by $z_j = a_{1j} \in \{\pm 1\}$ for $j = 2, \ldots, N$, $a_{ij} = -1$ for $i, j = 2, \ldots, N$, $i \neq j$ and $a_{ii} = 1$ for $i = 1, \ldots, N$. Let $B_z = \{(b_z)_{ij},\ i, j = 1, \ldots, N\}$ where $(b_z)_{ij} = u_i u_j (A_z)_{ij}$. Hence*

$$B_z = \begin{pmatrix} u_1^2 & z_2u_1u_2 & z_3u_1u_3 & \ldots & z_Nu_1u_N\\ z_2u_1u_2 & u_2^2 & -u_2u_3 & \ldots & -u_2u_N\\ z_3u_1u_2 & -u_2u_3 & u_3^2 & \ldots & -u_2u_N\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ z_Nu_1u_N & -u_2u_N & \ldots & \ldots & u_N^2 \end{pmatrix}. \tag{24}$$

Let σ be a permutation of $\{1, \ldots, N\}$ such that $\sigma(1) = 1$ and let for any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $x_- = (x_1, -x_2, \ldots, -x_N)$. Then the matrices

$$B_{\sigma(z)} = \{u_{\sigma(i)}u_{\sigma(j)}(A\sigma(z))_{ij},\ i,j=1,\dots,N\},$$
$$B_{z_-} = \{(u_i u_j (A_{z_-})_{ij}),\ i,j=1,\dots,N\}$$

and B_z have the same eigenvalues.

Proof. Let b be an eigenvalue of B_z with an eigenvector $x=(x_1,\dots,x_N)$. Define $x_\sigma = (x_1, x_{\sigma(2)},\dots,x_{\sigma(N)})$ and $x_- = (x_1,-x_2,\dots,-x_N)$. Notice that

$$(B_{\sigma(z)}x_{\sigma(z)})_1 = u_1^2 x_1 + \sum_{j=2}^{n} x_{\sigma(j)}u_{\sigma(j)}u_1 = u_1^2 x_1 + \sum_{j=2}^{n} x_j u_j u_1 = bx_1.$$

Analogously, for $i=2,\dots,N$,

$$\begin{aligned}(B_{\sigma(z)}x_\sigma)_i &= u_1 u_{\sigma(i)} x_{\sigma(i)} x_1 + \sum_{j=2, j\neq i}^{n} -u_{\sigma(j)}u_{\sigma(i)}x_{\sigma(j)} + u_{\sigma(i)}^2 x_{\sigma(i)} \\ &= bx_{\sigma(i)} = b(x_\sigma)_i.\end{aligned}$$

Also notice that

$$(B_{z_-}x_-)_1 = u_1^2 x_1 + \sum_{j=2}^{N}(-x_j)u_1 u_j(-x_j) = bx_1 = b(x_-)_1$$

and for $i=2,\dots,N$

$$(B_{z_-}x_-)_i = u_1 u_i(-x_i)x_1 + \sum_{j=2}^{n} a_{ij}u_1 u_j(-x_j) = -bx_i = b(x_-)_i.$$

This shows that any eigenvalue of B is an eigenvalue of B_{z_-} and $B_{\sigma(z)}$ with the same multiplicity. By the same reasoning, any eigenvalue of B_{z_-} and $B_{\sigma(z)}$ is also an eigenvalue of B, which completes the proof. □

Theorem 2.3. *Let $n=3$ and $N=5$. Let $z=(z_2,z_3,z_4,z_5)$ be such that $z_i=\pm 1$, for $i=2,\dots,5$ and $z_j=-1$ for exactly one $j\in\{2,3,4,5\}$. Assume that $A_z=(a_{ij}(z))$ is a 5×5 matrix defined by*

$$A_z = \begin{pmatrix} 1 & z_2 & z_3 & z_4 & z_5 \\ z_2 & 1 & -1 & -1 & -1 \\ z_3 & -1 & 1 & -1 & -1 \\ z_4 & -1 & -1 & 1 & -1 \\ z_5 & -1 & -1 & -1 & 1 \end{pmatrix}. \tag{25}$$

Let

$$M_A = \max\left\{ \sum_{i,j=1}^{5} u_i u_j a_{ij}(z)\langle x_i, x_j\rangle_3 \colon \left(x^1, x^2, x^3\right) \in \left(\mathbb{R}^5\right)^3 \textit{ satisfying } (2),\ \sum_{i=1}^{5} u_i^2 = 1 \right\}.$$

Then $M_A = 3/2$.

Proof. By Lemma 2.8, we can assume that $z_2 = -1$. Fix $u \in \mathbb{R}^5$, $\sum_{i=1}^5 u_i^2 = 1$. Let B_u denote the 5×5 matrix defined by

$$(b_u)_{ij} = u_i u_j a_{ij}(z)$$

for $i, j = 1, \dots, 5$. By Lemma 2.4,

$$M_A = \max\left\{ \sum_{j=1}^{3} b_j(u, A) \colon u \in \mathbb{R}^5,\ \sum_{i=1}^{5} u_i^2 = 1 \right\},$$

where $b_1(u, A) \geqslant b_2(u, A) \geqslant b_3(u, A)$ denote the three biggest eigenvalues of B_u. Put for $i = 1, \dots, 5$, $v_i = u_i^2$. After elementary but tedious calculations (we advise to check them by the symbolic Mathematica program) we get that

$$\det(B_u - t\,Id) = -t^5 + t^4\left(\sum_{i=1}^{5} v_i\right) + 16 t v_3 v_4 v_5 (v_1 + v_2)$$
$$- 4t^2\big(v_3 v_4 v_5 + (v_1 + v_2)\big(v_4 v_5 + v_3(v_4 + v_5)\big)\big).$$

Define $w = (w_1, \dots, w_5)$ by $w_1 = 0$, $w_2 = \sqrt{u_1^2 + u_2^2}$, $w_i = u_i$ for $i = 3, 4, 5$. Observe that by the above formula B_u and B_w have the same eigenvalues. Since $w_1 = 0$, by Lemma 2.7, Theorem 2.1, Theorem 2.2 and Lemma 2.6 applied to $n = 3$ and $N = 5$ we get

$$\sum_{j=1}^{3} b_j(u, A) \leqslant \lambda_3^4 = 3/2,$$

which completes the proof. □

Lemma 2.9. *Let B be a 5×5 matrix defined by*

$$B = \begin{pmatrix} u_{o1}^2 & z_2 u_{o1} c & z_3 u_{o1} c & z_4 u_{o1} c & z_5 u_{o1} c \\ z_2 u_{o1} c & c^2 & -c^2 & -c^2 & -c^2 \\ z_3 u_{o1} c & -c^2 & c^2 & -c^2 & -c^2 \\ z_4 u_{o1} c & -c^2 & -c^2 & c^2 & -c^2 \\ z_5 u_{o1} c & -c^2 & -c^2 & -c^2 & c^2 \end{pmatrix}, \tag{26}$$

where $z_j \in \{\pm 1\}$ for $j = 2, 3, 4, 5$. Then $2c^2$ is an eigenvalue of B with multiplicity at least 2.

Proof. Let C be defined by

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & -c^2 & -c^2 & -c^2 \\ 0 & -c^2 & c^2 & -c^2 & -c^2 \\ 0 & -c^2 & -c^2 & c^2 & -c^2 \\ 0 & -c^2 & -c^2 & -c^2 & c^2 \end{pmatrix}. \tag{27}$$

Since $2c^2$ is the eigenvalue of C with the multiplicity 3 with the eigenvectors v^j, $j = 2, 3, 4$, given by (30), there exist 2 orthonormal vectors w^1, w^2 in span$[v^2, v^3, v^4]$ which are orthogonal to the first row of B, which completes the proof. □

Theorem 2.4. *Let $n = 3$ and $N = 5$. Fix $u_{o1} \in [0, 1]$. Assume $A = (a_{ij})$ is a 5×5 matrix defined by*

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}. \tag{28}$$

Let

$$M_A(u_1) = \max\left\{ \sum_{i,j=1}^{5} u_i u_j a_{ij} \langle x_i, x_j \rangle_3 \colon \left(x^1, x^2, x^3\right) \in \left(\mathbb{R}^5\right)^3 \textit{ satisfying } (2), \right.$$
$$\left. u_1 = u_{o1},\ u_i = \sqrt{1 - u_1^2}/2,\ i = 2, 3, 4, 5 \right\}.$$

Then

$$M_A(u_1) = \frac{1 + 6c^2 + \sqrt{(6c^2 - 1)^2 + 16(1 - 4c^2)c^2}}{2}$$

where $c = \sqrt{1 - u_{o1}^2}/2$. Moreover,

$$M_A = \max\left\{M_A(u)\colon\ u \in [0, 1]\right\} = \frac{5 + 4\sqrt{2}}{7} = M_A\left(\sqrt{(5 - 3\sqrt{2})/7}\right).$$

Proof. Notice that by Theorem 2.1,

$$M_A = \sum_{j=1}^{3} b_j(B),$$

where $b_1(B) \geqslant b_2(B) \geqslant \cdots \geqslant b_5(B)$ denote the eigenvalues of the matrix B given by

$$B = \begin{pmatrix} u_{o1}^2 & u_{o1}c & u_{o1}c & -u_{o1}c & -u_{o1}c \\ u_{o1}c & c^2 & -c^2 & -c^2 & -c^2 \\ u_{o1}c & -c^2 & c^2 & -c^2 & -c^2 \\ -u_{o1}c & -c^2 & -c^2 & c^2 & -c^2 \\ -u_{o1}c & -c^2 & -c^2 & -c^2 & c^2 \end{pmatrix}, \tag{29}$$

where $c = \sqrt{1 - u_{o1}^2}/2$. Hence we should calculate the eigenvalues of B. To do this, let C be given by (27). It is easy to see that the eigenvalues of C are: 0 (with the eigenvector $v^1 = (1, 0, 0, 0, 0)$), $2c^2$ (with the orthonormal eigenvectors

$$v^2 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0), \qquad v^3 = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}),$$
$$v^4 = (0, 1/2, 1/2, -1/2, -1/2)) \tag{30}$$

and $-2c^2$ (with the eigenvector $v^5 = (0, 1/2, 1/2, 1/2, 1/2)$). Hence our theorem is proved for $u_{o1} = 0$ (in this case $c = 1/2$). If $u_{o1} > 0$, since the vectors v^2, v^3 and v^5 are orthogonal to the first row of B, by Lemma 2.9, $2c^2$ (with multiplicity 2) and $-2c^2$ (with multiplicity 1) are also eigenvalues of B. Now we find the other 2 eigenvalues of B. To do this, we show that an element $(a, 1/2, 1/2, -1/2, -1/2)$ for a properly chosen a is an eigenvector of B. Let us consider a system of equations:

$$u_{o1}^2 a + 2u_{o1}c = \lambda a \tag{31}$$

and

$$u_{o1}ca + c^2 = \lambda/2 \tag{32}$$

with unknown variables a and λ. Hence we easily get that

$$u_{o1}^2 a + 2u_{o1}c = 2\big(u_{o1}ca + c^2\big)a.$$

The last equation has two solutions. Namely:

$$a_1 = \frac{u_{o1}^2 - 2c^2 + \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{4u_{o1}c}$$

and

$$a_2 = \frac{u_{o1}^2 - 2c^2 - \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{4u_{o1}c}.$$

Since a_1, λ_1 and a_2, λ_2 are the solutions of (31) and (32), it is easy to check that $(a_1, 1/2, 1/2, -1/2, -1/2)$ is an eigenvector of B corresponding to the eigenvalue

$$\lambda_1 = 2u_{o1}ca_1 + 2c^2 = 2c^2 + \frac{u_{o1}^2 - 2c^2 + \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{2}$$

and $(a_2, 1/2, 1/2, -1/2, -1/2)$ is an eigenvector of B corresponding to the eigenvalue

$$\lambda_2 = 2u_{o1}ca_2 + 2c^2 = 2c^2 + \frac{u_{o1}^2 - 2c^2 - \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{2}.$$

It is clear that $\lambda_1 > 2c^2$ and $\lambda_2 < 2c^2$. Hence by Theorem 2.1,

$$M_A = \lambda_1 + 2c^2 + 2c^2.$$

Since $u_{o1}^2 = 1 - 4c^2$,

$$\lambda_1 + 4c^2 = \frac{1 + 6c^2 + \sqrt{(6c^2 - 1)^2 + 16(1 - 4c^2)c^2}}{2},$$

which completes this part of the proof.

Now define for $c \in [0, 1/2]$,

$$h(c) = \frac{1 + 6c^2 + \sqrt{(6c^2 - 1)^2 + 16(1 - 4c^2)c^2}}{2}.$$

Notice that $h(0) = 1$ and $h(1/2) = 3/2$. After elementary calculations (substituting c^2 by x), we get that

$$c_o = \frac{\sqrt{(2 + 3\sqrt{2})/7}}{2}$$

is the only point in $[0, 1/2]$ such that $h'(c_o) = 0$. Since

$$h(c_o) = \frac{5 + 4\sqrt{2}}{7} > 3/2,$$

$$M_A = h(c_o) = \frac{5 + 4\sqrt{2}}{7}.$$

Note $u_1 = \sqrt{(5 - 3\sqrt{2})/7}$ satisfies

$$u_1^2 + 4c_o^2 = 1.$$

The proof is complete. □

Lemma 2.10. *Let B be defined by* (26). *Assume that $c \in (0, 1/2)$ is so chosen that there exist $b_4(B) \geqslant b_5(B)$ eigenvalues of B satisfying $b_4(B) < 2c^2$. Let w^1, w^2, w^3 be the orthonormal eigenvectors corresponding to the three biggest eigenvalues of B. Assume that*

$$\sum_{i,j=1}^{5} b_{ij}\langle w_i, w_j\rangle_3 = M = \max\left\{\sum_{i,j=1}^{5} u_i u_j \left|\langle z_i, z_j\rangle\right|_3\colon z^1, z^2, z^3 \in \mathbb{R}^5\right\}, \tag{33}$$

under constraint (2) *with* $u_1 = \sqrt{1-4c^2}$ *and* $u_j = c$ *for* $j = 2, 3, 4, 5$. *Then the matrix* B_o *determined by* $1 = z_2 = z_3 = -z_4 = -z_5$ *satisfies* (33).

Proof. By Theorem 2.1 we need to calculate the sum of the three biggest eigenvalues of any matrix B satisfying (26). If $z_i = z_j = 1$ for exactly two indices $i, j \in \{2, 3, 4, 5\}$ then applying Theorem 2.4 and Lemma 2.8, we can show that B has the same eigenvalues as B_o. Now assume that $z_i = -1$ for exactly one $i \in \{2, 3, 4, 5\}$. Then by Theorem 2.3,

$$b_1(B) + b_2(B) + b_3(B) \leqslant 3/2,$$

where $b_1(B) \geqslant b_2(B) \geqslant b_3(B)$ denote the three biggest eigenvalues of B. Notice that by Theorem 2.4,

$$M \geqslant M_A > 3/2.$$

By Lemma 2.8 the same conclusion holds true if $z_i = 1$ for exactly one $i \in \{2, 3, 4, 5\}$.

Now assume that $z_i = 1$ for $i = 2, 3, 4, 5$. Then, reasoning as in Theorem 2.4, we get that the eigenvalues of B are: $2c^2$ with the multiplicity 3,

$$1/2 - 3c^2 + \sqrt{1 + 12c^2 - 60c^4}/2 \quad \text{and} \quad 1/2 - 3c^2 - \sqrt{1 + 12c^2 - 60c^4}/2.$$

After elementary calculations we obtain that

$$1/2 - 3c^2 + \sqrt{1 + 12c^2 - 60c^4}/2 \geqslant 2c^2$$

if and only if $1/2 \geqslant c \geqslant 1/\sqrt{5}$. If B_o satisfies (33), by Theorem 2.1, we should have

$$b_1(B) \leqslant b_1(B_o),$$

which by the above calculations and Theorem 2.4 is equivalent to

$$\sqrt{1 + 12c^2 - 60c^4}/2 < 2c^2 + \sqrt{1 + 8c^2 - 32c^4}/2$$

or

$$2c^2 < 1/2 - c^2 + \sqrt{1 + 4c^2 - 28c^4}/2.$$

After elementary calculations we get that both inequalities are equivalent to

$$0 < c < 1/2,$$

which shows our claim. If $z_i = -1$ for $i = 2, 3, 4, 5$, by Lemma 2.8 the conclusion is the same. Finally, by Theorem 2.1, B_o satisfies (33). □

Lemma 2.11. *Let $A = \{a_{ij},\ i, j = 1, \dots, 5\}$ be a 5×5 symmetric matrix such that $a_{ij} \in \{\pm 1\}$ for $i, j = 1, \dots, 5$ and $a_{ii} = 1$ for $i = 1, \dots, 5$. Consider a function*

$$f_{u_1,A}\big((u_2, \dots, u_5), x^1, x^2, x^3\big) = \sum_{i,j=1}^{5} u_i u_j a_{ij} \langle x_i, x_j \rangle_3 \tag{34}$$

under constraints (2) *and* (3). *Then there exist $x^1, x^2, x^5 \in \mathbb{R}^5$ satisfying* (2) *and (u_2, u_3, u_4, u_5) satisfying* (3) *maximizing the function $f_{u_1,A}$ such that $x_4^3 = x_5^3 = 0$, $x_2^3 \geqslant 0$, $x_2^2 = 0$, $x_4^2 \geqslant 0$ and $x_2^1 \geqslant 0$.*

Proof. Let y^1, y^2, y^3 and (u_2, u_3, u_4, u_5) be any vectors satisfying (2) and (3) maximizing $f_{u_1,A}$. Let $V = \operatorname{span}[y^1, y^2, y^3]$. Since $\dim(V) = 3$, there exist linearly independent $f, g \in \mathbb{R}^5$ such that $V = \ker(f) \cap \ker(g)$. Hence we can find $d^3 \in V \setminus \{0\}$, which is orthogonal to e_4, e_5 such that $d_2^3 \geqslant 0$. Set $x^3 = d^3/\|d^3\|_2$. Analogously we can find $d^2 \in V \setminus \{0\}$, orthogonal to x^3 and e_2 satisfying $d_4^2 \geqslant 0$. Define $x^2 = d^2/\|d^2\|_2$. Finally we can find $d^1 \in V \setminus \{0\}$, orthogonal to x^3 and x^2 with $d_2^1 \geqslant 0$. Set $x^1 = d^1/\|d^1\|_2$. Note that $x^i \in V$ for $i = 1, 2, 3$ and they are orthonormal. By Lemma 2.3, x^1, x^2, x^3 and (u_2, u_3, u_4, u_5) maximize the function $f_{u_1,A}$, which completes the proof. □

Lemma 2.12. *Let A be a fixed 5×5 matrix given by*

$$A = \begin{pmatrix} 1 & z_2 & z_3 & z_4 & z_5 \\ z_2 & 1 & -1 & -1 & -1 \\ z_3 & -1 & 1 & -1 & -1 \\ z_4 & -1 & -1 & 1 & -1 \\ z_5 & -1 & -1 & -1 & 1 \end{pmatrix}, \tag{35}$$

where $z_i \in \{\pm 1\}$ for $i = 2, 3, 4, 5$. Let

$$g_{t,u_1,A}\big((u_2, \dots, u_5), x^1, x^2, x^3\big) = f_{u_1,A}\big((u_2, \dots, u_5), x^1, x^2, x^3\big) + t\left(\sum_{i=2}^{5} u_i + x_4^2 - x_5^2 + x_2^3 - x_3^3\right)$$

where $t > 0$ is fixed and $(u_2, \dots, u_5)$, (x^1, x^2, x^3) satisfy (2) *and* (3). *Let $u_1 = 0$ and let $(u_2, \dots, u_5)$ and $(x, y, z) \in \mathbb{R}^{15}$ satisfying* (2), (3) *maximize $g_{t,u_1,A}$. Assume that $x_2 \geqslant 0$. Then $u_i = 1/\sqrt{2}$, for $i = 2, 3, 4, 5$, $x = (0, 1/2, 1/2, 1/2, 1/2)$, $y = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2})$, and $z = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0)$.*

Proof. By Lemma 2.7, the above mentioned x, y, z and $(u_2, \dots, u_5)$ maximize $f_{0,A}$ and

$$f_{0,A}\big((u_2, \dots, u_5), x^1, \dots, x^3\big) = 3/2.$$

Since the maximum of $\sum_{i=2}^{5} u_i + x_4^2 - x_5^2 + x_2^3 - x_3^3$ under restrictions $\sum_{i=2}^{5} u_i^2 = 1 - u_1^2$, $\sum_{j=1}^{5} (x_j^i)^2 = 1$ for $i = 2, 3$ is attained only for $u_i = \sqrt{(1 - u_1^2)/2}$ for $i = 2, 3, 4, 5$, $x^2 = y$ and $x^3 = z$,

$$g_{t,0,A}\big((u_2, \ldots, u_5), x^1, \ldots, x^3\big) = 3/2 + t(4/\sqrt{2} + 2).$$

Now assume that $(v_2, \ldots, v_5)$ and (x^1, y^1, z^1) maximize the function $g_{t,0,A}$. Hence in particular, $\sum_{i=1}^{5} v_i = 4/\sqrt{2}$, which shows that $v_i = 1/\sqrt{2} = u_i$ for $i = 2, \ldots, 5$. Analogously, $y = y^1$ and $z = z^1$. By Lemma 2.4,

$$\operatorname{span}\big[x^1, y^1, z^1\big] = \operatorname{span}[x, y, z].$$

Assume that $x^1 = px + qy + rz$. Since $y = y^1$, $z = z^1$ and x^1, y^1, z^1 are orthonormal, we get $q = r = 0$. Hence $p = \pm 1$. Since $x_2^1 \geqslant 0$ and $x_2 > 0$, $x^1 = x$, which completes the proof. □

The next lemma is a simple consequence of the Implicit Function Theorem.

Lemma 2.13. *Let $U \subset \mathbb{R}^l$ be an open, non-empty set and let $f : U \times \mathbb{R}^n \to \mathbb{R}$ and $G_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, k$ be fixed C^2 functions. Let $g : U \times \mathbb{R}^{n+k} \to \mathbb{R}$ be defined by*

$$g(u, x, d) = f(u, x) - \sum_{i=1}^{k} d_i G_i(x)$$

for $u \in U$, $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^k$. Assume that $\frac{\partial g}{\partial z_j}(u^o, x^o, d^o) = 0$ for $j = 1, \ldots, n+k$ and

$$\det\left(\frac{(\partial^2 g)}{\partial z_i \partial z_j}\big(u^o, x^o, d^o\big)\right) \neq 0$$

for some $(u^o, x^o, d^o) \in U \times R^{n+k}$ and $i, j = 1, \ldots, n+k$. (We do not differentiate with respect to the coordinates of u.) Assume that $(u^m, x^m, d^m) \in U \times R^{n+k}$, and $(u^m, y^m, z^m) \in U \times R^{n+k}$, are such that $(u^m, x^m, d^m) \to (u^o, x^o, d^o)$ and $(u^m, y^m, z^m) \to (u^o, x^o, d^o)$ with respect to any norm in $\mathbb{R}^{l+n+k}$. If, for any $m \in \mathbb{N}$, $\frac{\partial g}{\partial z_j}(u^m, x^m, d^m) = 0$ and $\frac{\partial g}{\partial z_j}(u^m, y^m, z^m) = 0$ for $j = 1, \ldots, n+k$ then

$$\big(u^m, x^m, d^m\big) = \big(u^m, y^m, z^m\big)$$

for $m \geqslant m_o$.

Proof. It suffices to apply the Implicit Function Theorem to the function

$$G(u, x, d) = \left(\frac{\partial g}{\partial z_1}(u, x, d), \ldots, \frac{\partial g}{\partial z_{n+k}}(u, x, d)\right)$$

and $(u, x, d) = (u^o, x^o, d^o)$. □

Lemma 2.14. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Let $\lambda_{k_1}, \lambda_{k_2}$ be eigenvalues of A, λ_{k_i} of multiplicity j_i for $i=1,2$. Assume that for $i=1,2$ and $j=1,\dots,j_i$, v^{ij} is an orthonormal basis corresponding to the eigenspace of λ^{k_i}. Define a $(j_1+j_2)\times n$ matrix V with rows v^{ij}, $i=1,2$, $j=1,\dots,j_i$. Let*

$$C = \begin{pmatrix} A-\lambda_{k_1} id & V^T \\ V & 0 \end{pmatrix}. \tag{36}$$

Then $\det(C) \neq 0$.

Proof. Let E_j for $j=1,\dots,n+j_1+j_2$ denote the rows of the matrix C. Assume on the contrary that

$$\sum_{k=1}^{n+j_1+j_2} a_j E_j = 0 \tag{37}$$

and $\sum_{k=1}^{n+j_1+j_2} |a_j| > 0$. By (36),

$$\langle v^{ij}, (a_1,\dots,a_n)\rangle_n = 0 \tag{38}$$

for $i=1,2$ and $j=1,\dots,j_i$. If $j_1+j_2=n$, then by the orthonormality conditions $a_i=0$ for $j=1,\dots,n$. Again by the orthonormality conditions and (36) $a_j=0$ for $j=n+1,\dots,j_1+j_2+n$, a contradiction. If $j_1+j_2<n$,

$$(a_1,\dots,a_n) = \sum_{j=1}^{k} b_j w_j,$$

where $w_1,\dots,w_k$ are some orthonormal eigenvectors of A corresponding to eigenvalues $\gamma_1,\dots,\gamma_k$ of A different from λ_{k_1} and λ_{k_2}. Since A is symmetric, by (37) and (38),

$$\sum_{j=1}^{k} b_j(\gamma_j-\lambda_{k_1})w_j = \sum_{j=n+1}^{j_1+n} a_j v^{1j} + \sum_{j=n+j_1+1}^{j_1+j_2+n} a_j v^{2j}.$$

Since $\gamma_j \neq \lambda_{k_i}$ for $j=1,\dots,k$ and $i=1,2$, we have

$$\sum_{j=n+1}^{j_1+n} a_j v^{1j} + \sum_{j=n+j_1+1}^{j_1+j_2+n} a_j v^{2j} = 0$$

and consequently by the orthonormality conditions $a_j=0$ for $j=n+1,\dots,n+j_1+j_2$ and $b_j=0$ for $j=1,\dots,k$. In particular, this shows that $(a_1,\dots,a_n)=0$, a contradiction. □

Lemma 2.15. *Assume that $t\in\mathbb{R}$ and let B, E be fixed $m\times m$ matrices and let A be a fixed $n\times n$ matrix. Define*

$$C(t) = \begin{pmatrix} A & D \\ D_1 & B+tE \end{pmatrix}, \tag{39}$$

where D is a fixed $n \times m$ matrix and D_1 is a fixed $m \times n$ matrix. If $\det(C(t)) = \sum_{j=0}^{m} a_j t^j$, *then*

$$a_m = \det(A)\det(E).$$

Proof. Let, for $k \in \mathbb{N}$, Π_k denote a set of all permutations of k elements. By definition of determinant

$$\det\bigl(C(t)\bigr) = \sum_{\sigma \in W_m} \operatorname{sgn}(\sigma)\left(\prod_{j=1}^{n+m} c_{j,\sigma(j)}(t)\right) \sum_{\sigma \in W_m} \operatorname{sgn}(\sigma)\left(\prod_{j=1}^{n+m} c_{j,\sigma(j)}(t)\right) + \sum_{\sigma \notin W_m} \operatorname{sgn}(\sigma)\left(\prod_{j=1}^{n+m} c_{j,\sigma(j)}(t)\right),$$

where

$$W_m = \bigl\{\sigma \in \Pi_{n+m}\colon \sigma\bigl(\{1,\dots,n\}\bigr) \subset \{1,\dots,n\}\bigr\}.$$

Notice that to calculate the coefficient a_n it is sufficient to consider only the sum over W_n. But

$$\begin{aligned}
&\sum_{\sigma \in W_m} \operatorname{sgn}(\sigma)\left(\prod_{j=1}^{n+m} c_{j,\sigma(j)}(t)\right) \\
&= \sum_{\sigma \in \Pi_n} \operatorname{sgn}(\sigma)\left(\prod_{j=1}^{n} a_{j,\sigma(j)}\right) \sum_{\sigma \in \Pi_m} \operatorname{sgn}(\sigma)\left(\prod_{j=1}^{m} (b_{j,\sigma(j)} + t e_{j,\sigma(j)})\right) \\
&= \det(A)\det(B + tE).
\end{aligned}$$

Hence, again by definition of determinant,

$$a_m = \det(A)\det(E),$$

as required. □

3. Determination of λ_3^5

In this section we will work with functions f_{u_1} and $f_{u_1,A}$ defined by (1) and (4). The main idea of our proof is to show that the function f_{u_1} attains its maximum under conditions (2) and (3) in

$$\bigl(u_2, u_3, u_4, u_5, x^1, x^2, x^5\bigr)$$

given in Theorem 2.4. This shows that λ_3^5 has been calculated in Theorem 2.4. The main difficulty to do this, is to demonstrate that if $(u_2, u_3, u_4, u_5, x^1, x^2, x^3)$ maximize f_{u_1} under conditions (2) and (3) then $u_i = \sqrt{1 - u_1^2}/2$ for $i = 2, 3, 4, 5$. Here Lemma 2.13, Lemma 2.14 and Lemma 2.15 are applied.

The next two theorems show how look like candidates for maximizing the function $f_{u_1,A}$.

Theorem 3.1. *Let A be defined by* (28). *Fix $t \in \mathbb{R}$ and $u_1 \in [0,1)$. Let us consider a function $h_{u_1,A,t} : \mathbb{R}^4 \times (\mathbb{R}^5)^3 \times \mathbb{R}^6 \times \mathbb{R}$ defined by*

$$\begin{aligned} &h_{u_1,A,t}\big((v_2,v_3,v_4,v_5), z^1, z^2, z^3, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big) \\ &\quad = \sum_{i,j=1}^{5} a_{ij} v_i v_j \langle z_i, z_j\rangle_3 + t\left(\sum_{i=2}^{5} v_i + z_4^2 - z_5^2 + z_2^3 - z_3^3\right) \\ &\qquad - \left(\sum_{j=1}^{3} d_j \langle z^j, z^j\rangle_5 - 1\right) - \sum_{i,j=1,i<j}^{3} d_{ij}\langle z^i, z^j\rangle_5 - d_7\big\langle (u_1,v),(u_1,v)\big\rangle_5, \end{aligned} \tag{40}$$

where $v = (v_2, v_3, v_4, v_5)$. Define for $i = 2,\ldots,5$, $u_i = \sqrt{(1-u_1^2)/2} = c$,

$$w = w(u_1) = \frac{4u_1 c}{\sqrt{(u_1^2 - 2c^2)^2 + 16c^2u_1^2} + 2c^2 - u_1^2},$$

$x_1^1 = w/\sqrt{1+w^2}$, $x_i^1 = \frac{1}{2\sqrt{1+w^2}}$, $i = 2,3$, $x_i^1 = \frac{-1}{2\sqrt{1+w^2}}$, $i = 4,5$,

$$x^2 = (0,0,0,1/\sqrt{2},-1/\sqrt{2}), \qquad x^3 = (0,1/\sqrt{2},-1/\sqrt{2},0,0),$$

$d_1 = 1/2 - c^2 + \sqrt{1+4c^2-28c^4}/2$, $d_2 = d_3 = 2c^2 + (1/\sqrt{2})t$, $d_{ij} = 0$ for $i,j = 1,2,3$, $i < j$ and

$$d_7 = 1 + t/(2c) + 2\big(x_2^1\big)^2 + \big(x_1^1 x_2^1 u_1\big)/c.$$

Then the above defined $x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7$ satisfy the system of equations:

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}\big(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big) = 0$$

for $j = 1,\ldots,26$, where

$$w_j \in \big\{v_2, v_3, v_4, v_5, z_k^i,\ k = 1,\ldots,5,\ i = 1,2,3\big\}$$

and

$$w_j \in \big\{d_{ik},\ i,k \in \{1,2,3\},\ i<k,\ d_i,\ i = 1,2,3,7\big\}.$$

(*We do not differentiate with respect to u_1.*)

Proof. Notice that the equations

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}\big(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big) = 0$$

for

$$w_j \in \{z_k^i,\ k = 1, \ldots, 5,\ i = 1, 2, 3\}$$

follow from the fact that x^i, $i = 1, 2, 3$, are the orthonormal eigenvectors of the matrix B defined by (29) corresponding to the eigenvalues d_i, $i = 1, 2, 3$, which has been established in the proof of Theorem 2.4. Also the equations

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}\big(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big) = 0$$

where

$$w_j \in \{d_1, d_2, d_3, d_{ik},\ i, k \in \{1, 2, 3\},\ i < k,\ d_7\}$$

follow immediately from the fact that $\langle x^i, x^j \rangle_5 = \delta_{ij}$ for $i, j = 1, 2, 3, i \leqslant j$ and $\langle (u_1, u), (u_1, u) \rangle_5 = 1$, where $u = (u_2, u_3, u_4, u_5)$. To end the proof, we show that

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}\big(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big) = 0$$

for

$$w_j \in \{v_2, v_3, v_4, v_5\}.$$

Notice that for $i = 2, 3, 4, 5$

$$\begin{aligned}&\frac{\partial h_{u_1,A,t}}{\partial w_i}\big(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big)\\ &\quad = 2\sum_{j=1}^{5} u_j a_{ij} \langle x_i, x_j \rangle_3 + t - 2u_i d_7.\end{aligned}$$

Since $u_1 < 1$, $u_i = \sqrt{(1 - u_1^2)/2} = c > 0$ for $i = 2, 3, 4, 5$. Hence

$$\frac{\partial h_{u_1,A,t}}{\partial w_i}\big(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big) = 0$$

if and only if

$$\left(\sum_{j=1}^{5} u_j a_{ij} \langle x_i, x_j \rangle_3\right)/c + t/(2c) = d_7.$$

Notice that for $i = 2, 3, 4, 5$,

$$\left(\sum_{j=1}^{5} a_{ij}u_j\langle x_i, x_j\rangle_3\right)/c = x_i^1\left(\sum_{j=1}^{5} a_{ij}u_jx_j^i\right)/c + x_i^2\left(\sum_{j=1}^{5} a_{ij}u_jx_j^2\right)/c + x_i^3\left(\sum_{j=1}^{5} a_{ij}u_jx_j^3\right)/c$$
$$= \left(u_1a_{i1}x_1^1x_i^1\right)/c + 2\left(x_i^1\right)^2 + 1/\sqrt{2}\left((1/\sqrt{2})c + (-1)(-1/\sqrt{2})c\right)/c$$
$$= 1 + \left(u_1a_{i1}x_1^1x_i^1\right)/c + 2\left(x_i^1\right)^2.$$

Hence for $i = 2, 3, 4, 5$,

$$d_7 = 1 + t/(2c) + 2\left(x_i^1\right)^2 + \left(x_1^1a_{i1}x_i^1u_1\right)/c.$$

Since $x_2^1 = x_3^1 = -x_4^1 = -x_5^1$, $1 = a_{21} = a_{31} = -a_{41} = -a_{51}$, and $u_i = c$ for $i = 2, 3, 4, 5$,

$$d_7 = 1 + t/(2c) + 2\left(x_2^1\right)^2 + \left(x_1^1x_2^1u_1\right)/c,$$

as required. □

Reasoning as in Theorem 3.1, we can show

Theorem 3.2. *Let A be defined by*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{pmatrix}. \tag{41}$$

Fix $t \in \mathbb{R}$ and $u_1 \in [0, 1)$. Let us consider a function $h_{u_1,A,t} : \mathbb{R}^4 \times (\mathbb{R}^5)^3 \times \mathbb{R}^6 \times \mathbb{R}$ given by (40) *with A defined as above. Define for $i = 2, \dots, 5$, $u_i = \sqrt{(1 - u_1^2)/2} = c$,*

$$x^1 = (0, 1/2, 1/2, -1/2, -1/2),$$
$$x^2 = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}), \qquad x^3 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0),$$

$d_1 = 2c^2$, $d_2 = d_3 = 2c^2 + (1/\sqrt{2})t$, $d_{ij} = 0$ for $i, j = 1, 2, 3$, $i < j$ and

$$d_7 = 3c + t/2c.$$

Then the above defined $x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7$ satisfy the system of equations:

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}\left(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\right) = 0$$

for $j = 1, \dots, 26$, where

$$w_j \in \left\{v_2, v_3, v_4, v_5, z_k^1,\ k = 1, \dots, 5,\ i = 1, 2, 3\right\}$$

and

$$w_j \in \{d_{ik},\ i, j \in \{1,2,3\},\ i < j,\ d_1, d_2, d_3, d_7\}.$$

(*We do not differentiate with respect to* u_1.)

Lemma 3.1. *Let A be defined by* (28). *For a fixed $u_1 \in (0,1)$ and $t > 0$ let $g_{u_1,A,t} : \mathbb{R}^4 \times (\mathbb{R}^5)^3 \to R$ defined by*

$$g_{u_1,A,t}\big((v_2,\dots,v_5), y^1, y^2, y^3\big) = \sum_{i,j=1}^{5} v_i v_j a_{ij} \langle y_i, y_j \rangle_3 + t\left(\sum_{i=2}^{5} v_i + y_4^2 - y_5^2 + y_2^3 - y_3^3\right).$$

Let $M_{u_1,A,t} = \max g_{u_1,A,t}$ under constraints

$$\langle y^i, y^j \rangle_5 = \delta_{ij}, \quad 1 \leqslant i \leqslant j \leqslant 3;$$

and

$$\sum_{j=2}^{5} v_j^2 = 1 - u_1^2.$$

Assume that $u_1 \in (0,1)$ is so chosen that

$$M_{u_1,A,0} = f_{u_1,A}\big((u_2,u_3,u_4,u_5), x^1, x^2, x^3, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\big),$$

where $u_2, u_3, u_4, u_5, x^1, x^2, x^3, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7$ are as in Theorem 3.1 (*for* $c = \sqrt{1-u_1^2}/2$). *Set*

$$D_{u_1} = \{(v_2, v_3, v_4, v_5, y^1, y^2, y^3)\colon\ y_4^3 = y_5^3 = y_2^2 = 0,\ y_2^1 \geqslant 0\}. \tag{42}$$

Then

$$X_{u_1} = (u_2, u_3, u_4, u_5, x^1, x^2, x^3) \tag{43}$$

is the only point maximizing $g_{u_1,A,t}$ satisfying (2) *and* (3) *belonging to* D_{u_1}.

Proof. Let

$$Y_{u_1} = (v_2, v_3, v_4, v_5, y^1, y^2, y^3) \in D_{u_1}$$

maximize $g_{u_1,A,t}$ and satisfy (2) and (3). Since $t > 0$, and the maximum of $f_{u_1,A}$ is attained at X_{u_1}, we have $v_i = u_i = \sqrt{1-u_1^2}/2$ for $i = 2,3,4,5$, $y^2 = x^2$ and $x^3 = y^3$. Since x^1, x^2, x^3 are the eigenvectors of A, by Lemma 2.4, $\operatorname{span}[y^i\colon i = 1,2,3] = \operatorname{span}[x^i\colon i = 1,2,3]$. Note that

$$\langle x^1, x^i \rangle_5 = \langle y^1, x^i \rangle_5 = 0$$

for $i = 2, 3$. Since $\operatorname{span}[y^i\colon i = 1, 2, 3] = \operatorname{span}[x^i\colon i = 1, 2, 3]$, $y^1 = dx^1$. Since $\langle y^1, y^1\rangle_5 = 1$, $y^1_2 \geqslant 0$ and $x^1_2 > 0$, $x^1 = y^1$, as required. □

Theorem 3.3. *Let A be defined by* (28). *For a fixed $u_1 \in [0, 1)$ and $t \in \mathbb{R}$ let $g_{u_1,A,t}$ and $M_{u_1,A,t}$ be as in Lemma* 3.1. *Assume that $u_1 \in [0, 1)$ is so chosen that*

$$M_{u_1,A,0} = g_{u_1,A,t}\bigl(u_2, u_3, u_4, u_5, x^1, x^2, x^3\bigr)$$

where $u_2, u_3, u_4, u_5, x^1, x^2, x^3$ are as in Theorem 3.1 (*for $c = \sqrt{1 - u_1^2}/2$*). *Let the function $h_{u_1,A,t}$ be defined by* (40). *Assume furthermore that the 23×23 matrix $D_{u,A,t}$ defined by*

$$D_{u,A,t} = \frac{\partial h_{u_1,A,t}}{\partial w_i, \partial w_j}\bigl(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\bigr), \tag{44}$$

where

$$w_i, w_j \in \bigl\{v_2, v_3, v_4, v_5, y^1_k,\ k = 1, \ldots, 5,\ y^2_1, y^2_3, y^2_4, y^2_5, y^3_1, y^3_2, y^3_3,\\ d_i,\ i = 1, 2, 3, 7,\ d_{ik},\ 1 \leqslant i < k \leqslant 3\bigr\}$$

(*we do not differentiate with respect to u_1, y^3_4, y^3_5, y^2_2*) *is such that*

$$\operatorname{Det}(D_{u,A,t}) = \sum_{j=o}^{k} a_j(u)t^j$$

and $a_j(u_1) \neq 0$ for some $j \in \{1, \ldots, k\}$. (*Here $(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7)$ are such as in Theorem* 3.1 *for $c = \sqrt{1 - u_1^2}/2$ and $t \in \mathbb{R}$.*) *Then there exists an open interval $U \subset [0, 1)$ ($U = [0, w)$ if $u_1 = 0$) such that $u_1 \in U$ and for any $u \in U$ the function $f_{u,A}$ attains its global maximum under constraints* (2) *and* (3) *at*

$$X_u = \bigl(u_2, u_3, u_4, u_5, x^1, x^2, x^3\bigr),$$

where $u_i = c_u = \sqrt{1 - u^2}/2$ for $i = 2, 3, 4, 5$ and x^1, x^2, x^3 are defined in Theorem 3.1 (*with $c = c_u$*). *The same result holds true if A will be defined by* (41). (*In this case*

$$\bigl(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7\bigr)$$

are such as in Theorem 3.2.)

Proof. Fix $u_1 \in [0, 1)$ satisfying our assumptions and let $c_1 = \sqrt{1 - u_1^2}/2$. Let $j_o = \min\{j \in \{0, \ldots, k\}\colon a_j(u_1) \neq 0\}$. Set for $(u, t) \in [0, 1) \times \mathbb{R}$,

$$h(t, u) = \sum_{j=j_o}^{k} a_j(u)t^{j-j_o}.$$

Since $a_{j_o}(u_1) \neq 0$, and a_j are continuous there exist an open interval $U \subset [0, 1)$ and $\delta > 0$ such that $u_1 \in U$ and

$$h(t, u) \neq 0$$

for $u \in U$ and $|t| < \delta$. Fix $t_o \in (0, \delta)$. Set

$$U_{t_o} = \{u \in U\colon\ M_{u,A,t_o} \text{ is attained at } X_u\}.$$

Note that $u_1 \in U_{t_o}$. Now we show that U_{t_o} is an open set. Let $u_o \in U_{t_o}$. Assume on the contrary that there exist $\{u_n\} \in U \setminus U_{t_o}$ such that $u_n \to u_o$. Let for any $u \in U$,

$$Z_{u,t_o} = Z_u = \left(v_{2u}, v_{3u}, v_{4u}, v_{5u}, x^{1u}, x^{2u}, x^{3u}\right)$$

be a point maximizing g_{u,A,t_o} under constraints (2) and (3). Since the function

$$(f_{u,A} - g_{t_o,u,A})\left((v_2, v_3, v_4, v_5), z^1, z^2, z^3\right) = t_o\left(\sum_{i=2}^{5} v_i + z_4^2 - z_5^2 + z_2^3 - z_3^3\right)$$

is independent of z^1 and by Lemma 2.11, without loss of generality, the function g_{u,A,t_o} can be considered as a function of 16 variables

$$\left(z^1, (v_2, v_3, v_4, v_5), z_1^2, z_3^2, z_4^2, z_5^2, z_1^3, z_2^3, z_3^3\right)$$

from $\mathbb{R}^5 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^3$. (We can put $z_2^2 = z_4^3 = z_5^3 = 0$.) Consequently, we can assume that

$$Z_u \in D_u$$

(see (42)). By (2) and (3), passing to a subsequence, if necessary, we obtain that $Z_{u_n} \to Z$. By definition of D_{u_o}, $Z \in D_{u_o}$. Also by the continuity of the function

$$(v, Y) \to \left(\sum_{i,j=1}^{5} v_i v_j a_{ij} \langle y_i, y_j \rangle_3 + t_o\left(\sum_{i=2}^{5} v_i + y_4^2 - y_5^2 + y_2^3 - y_3^3\right)\right),$$

$$g_{u_o,A,t_o}(Z) = M_{u_o,A,t_o}.$$

By Lemma 2.12 and Lemma 3.1 X_{u_o} is the only point in D_{u_o} which maximizes g_{u,A,t_o} and $Z \in D_{u_o}$. Hence $Z = X_{u_o}$. Moreover, since $X_{u_o} \in \operatorname{int}(D_{u_o})$, by the Lagrange Multiplier Theorem, there exists

$$M_{u_n} = M_{u_n}(t_o) = \left(d_1^n, d_2^n, d_3^n, d_{12}^n, d_{13}^n, d_{23}^n, d_7^n\right) \in \mathbb{R}^7$$

such that

$$\frac{\partial h_{u,A,t_o}}{\partial w_i}(Z_{u_n}, M_{u_n}) = 0, \tag{45}$$

for $w_i \in X \cup DD$. Here $h_{u,A,t}$ is defined by (40) and

$$DD = \{d_i,\ i = 1, 2, 3, 7,\ d_{ij},\ 1 \leqslant i < j \leqslant 3\}.$$

Also by (2), (3), (7), (8) (see the proof of Lemma 2.4) and (45)

$$M_n \to L_{u_o} = L_{u_o}(t_o) = (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7),$$

where L_{u_o} is defined in Theorem 3.1 for $c = \sqrt{1-u_o^2}/2$ and $t = t_o$. Now we apply Lemma 2.13. Let us consider a function $G : U \times \mathbb{R}^{12} \times \mathbb{R}^4 \times \mathbb{R}^7 \to \mathbb{R}^{23}$ defined by

$$G(u, x, v, Q) = \left(\frac{\partial h_{u,A,t_o}}{\partial w_1}(u, x, v, Q), \ldots, \frac{\partial h_{t_o,u}}{\partial w_{23}}(u, x, v, Q)\right) / (t_o)^{j_o/23}$$

for $w_i \in X \cup DD$. Notice that by (45)

$$G(u_n, Z_{u_n}, M_{u_n}) = 0.$$

Also $G(u_n, X_{u_n}, L_{u_n}(t_o)) = 0$, where $(X_{u_n}, L_{u_n}(t_o))$ are defined for u_n in Theorem 3.1. Moreover,

$$(u_n, Z_{u_n}, M_{u_n}) \to (u_o, X_{u_o}, L_{u_o})$$

and

$$(u_n, X_{u_n}, L_{u_n}) \to (u_o, X_{u_o}, L_{u_o}).$$

Notice that

$$\mathrm{Det}\left(\frac{\partial G}{\partial w_j}(u_o, X_{u_o}, L_{u_o})\right) = \frac{\det(D_{u_o,A,t_o})}{(t_o^{j_o/23})^{23}} = \sum_{j=j_o}^{k} a_j(u_o) t_o^{j-j_o} = h(t_o, u_o) \neq 0,$$

by definition of j_o and t_o. By Lemma 2.13 applied to the function G, $Z_{u_n} = X_{u_n}$ and $M_{u_n} = L_{u_n}$ for $n \geqslant n_o$. Hence $u_n \in U_1$ for $n \geqslant n_o$; a contradiction. This shows that U_{t_o} is an open set. It is clear that U_{t_o} is closed. Since $u_1 \in U_{t_o}$ and U is connected, $U_{t_o} = U$. Consequently for any $n \in \mathbb{N}$, $n \geqslant n_o$ and $u \in U$, the functions $g_{u,A,1/n}$ achieve their maximum at $u_2, u_3, u_4, u_5, x^1, x^2, x^3$, where $u_i = c_u = \sqrt{1-u^2}/2$ for $i = 2, 3, 4, 5$ and x^1, x^2, x^3, are defined in Theorem 3.1 (with $c = c_u$). Since $g_{u,A,1/n}$ tends uniformly to $g_{u,A,0} = f_{u,A}$, on the set defined by (2) and (3), with $u \in U$ fixed, $f_{u,A}$ attains its maximum at $u_2, u_3, u_4, u_5, x^1, x^2, x^3$ for any $u \in U$.

By Theorem 3.2, reasoning exactly in the same way as above we can deduce our conclusion for the function $f_{u,A}$ determined by A given by (41). The proof is complete. □

Now we show that the assumptions of Theorem 3.3 concerning $D_{u,A,t}$ are satisfied. This is the most important technical result which permits us to determine the constant λ_3^5.

Theorem 3.4. *Let A be defined by* (28) *and let $D_{u,A,t}$ be given by* (44)*. Then for any $u \in [0,1)$ and $t \in \mathbb{R}$,*

$$\mathrm{Det}(D_{u,A,t}) = \sum_{j=0}^{7} a_j(u)t^j,$$

where the functions a_j are continuous for $j = 0, \dots, 7$ and $a_7(u) \neq 0$ for any $u \in [0,1)$.

Proof. Set

$$X = (x_1, b, b - b, -b, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 1/\sqrt{2}, -1/\sqrt{2}),$$
$$B = (b_1, d, d, 0, 0, 0, b_7)$$

and

$$v = (c, c, c, c).$$

Assume that we will differentiate $h_{u,A,t}$ in the following manner:

$$(w_1, \dots, w_5) = \left(x_1^1, \dots, x_5^1\right), \qquad (w_6, \dots, w_{11}) = (b_1, b_2, b_3, b_{12}, b_{13}, b_{23})$$
$$(w_{12}, \dots, w_{18}) = \left(x_1^2, x_3^2, x_4^2, x_5^2, x_1^3, x_2^3, x_3^3\right), \qquad (w_{19}, \dots, w_{23}) = (u_2, u_3, u_4, u_5, b_7).$$

(Recall that we do not differentiate with respect to u_1, x_2^2, x_4^3 and x_5^3.) Notice that by elementary but very tedious calculations (which we verified by a symbolic Mathematica program) we get that the 23×23 symmetric matrix $C = D_{u,A,t}(X, B, v)$ is given by

$$C = \begin{pmatrix} D_1 & B_1 \\ (B_1)^T & D_2 \end{pmatrix}. \tag{46}$$

Here

$$D_1 = \begin{pmatrix}
2(u^2 - b_1) & 2cu & 2cu & -2cu & -2cu & -2x_1 & 0 & 0 & 0 & 0 & 0 \\
2cu & 2(c^2 - b_1) & -2c^2 & -2c^2 & -2c^2 & -2b & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\
2cu & -2c^2 & 2(c^2 - b_1) & -2c^2 & -2c^2 & -2b & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\
-2cu & -2c^2 & -2c^2 & 2(c^2 - b_1) & -2c^2 & 2b & 0 & 0 & -1/\sqrt{2} & 0 & 0 \\
-2cu & -2c^2 & -2c^2 & -2c^2 & 2(c^2 - b_1) & 2b & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\
-2x_1 & -2b & -2b & 2b & 2b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/\sqrt{2} & 1\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}; \tag{47}$$

$D_2 = (D_{12}, D_{22})$, where

$$D_{12} = \begin{pmatrix} 2(u^2-d) & 2cu & -2cu & -2cu & 0 & 0 & 0 \\ 2cu & 2(c^2-d) & -2c^2 & -2c^2 & 0 & 0 & 0 \\ -2cu & -2c^2 & 2(c^2-d) & -2c^2 & 0 & 0 & 0 \\ -2cu & -2c^2 & -2c^2 & 2(c^2-d) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(u^2-d) & 2cu & 2cu \\ 0 & 0 & 0 & 0 & 2cu & 2(c^2-d) & -2c^2 \\ 0 & 0 & 0 & 0 & 2cu & -2c^2 & 2(c^2-d) \\ 0 & 0 & 0 & 0 & \sqrt{2}u & 3\sqrt{2}c & -\sqrt{2}c \\ 0 & 0 & 0 & 0 & -\sqrt{2}u & \sqrt{2}c & -3\sqrt{2}c \\ -\sqrt{2}u & -\sqrt{2}c & 3\sqrt{2}c & -\sqrt{2}c & 0 & 0 & 0 \\ \sqrt{2}u & \sqrt{2}c & \sqrt{2}c & -3\sqrt{2}c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{48}$$

and

$$D_{22} = \begin{pmatrix} 0 & 0 & -\sqrt{2}u & \sqrt{2}u & 0 \\ 0 & 0 & -\sqrt{2}c & \sqrt{2}c & 0 \\ 0 & 0 & 3\sqrt{2}c & \sqrt{2}c & 0 \\ 0 & 0 & -\sqrt{2}c & -3\sqrt{2}c & 0 \\ \sqrt{2}u & -\sqrt{2}u & 0 & 0 & 0 \\ 3\sqrt{2}c & \sqrt{2}c & 0 & 0 & 0 \\ -\sqrt{2}c & -3\sqrt{2}c & 0 & 0 & 0 \\ 2b^2-2b_7+1 & 1-2b^2 & 2b^2 & 2b^2 & -2c \\ 1-2b^2 & 2b^2-2b_7+1 & 2b^2 & 2b^2 & -2c \\ 2b^2 & 2b^2 & 2b^2-2b_7+1 & 1-2b^2 & -2c \\ 2b^2 & 2b^2 & 1-2b^2 & 2b^2-2b_7+1 & -2c \\ -2c & -2c & -2c & -2c & 0 \end{pmatrix}; \tag{49}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2bu & 2bu & 2bu & 2bu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6bc+2ux_1 & -2bc & 2bc & 2bc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2bc & 6bc+2ux_1 & 2bc & 2bc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2bc & -2bc & -6bc-2ux_1 & 2bc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2bc & -2bc & 2bc & -6bc-2ux_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -x_1 & -b & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & -b & -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

Notice that in 11th row of C the only non-zero element is $c_{11,13} = c_{13,11} = 1/\sqrt{2}$ and in 23rd row of A the only elements which could be different from 0 are $c_{23,19} = c_{23,20} = c_{23,21} = c_{23,22} = -2c$. Also the only non-zero elements in 7th row are $c_{7,14} = -\sqrt{2}$ and $c_{7,15} = \sqrt{2}$. Analogously, the only non-zero elements in 8th row are $c_{8,17} = -\sqrt{2}$ and $c_{8,18} = \sqrt{2}$. Consequently, applying the symmetry of C, subtracting 19th row from 20, 21 and 22nd row, 19th column from 20, 21 and 22nd column, adding 15th row to 14th row and 15th column to 14th column and adding 18th row to 17th row and 18th column to 17th column we get that

$$\det(C) = 8c^2 \det(A),$$

where A is a 15×15 symmetric matrix defined by

$$A = \begin{pmatrix} A_1 & F \\ F^T & A_2 \end{pmatrix}. \tag{51}$$

Here

$$A_1 = \begin{pmatrix} 2(u^2-b_1) & 2cu & 2cu & -2cu & -2cu & -2x_1 & 0 & 0 \\ 2cu & 2(c^2-b_1) & -2c^2 & -2c^2 & -2c^2 & -2b & 0 & -1/\sqrt{2} \\ 2cu & -2c^2 & 2(c^2-b_1) & -2c^2 & -2c^2 & -2b & 0 & 1/\sqrt{2} \\ -2cu & -2c^2 & -2c^2 & 2(c^2-b_1) & -2c^2 & 2b & -1/\sqrt{2} & 0 \\ -2cu & -2c^2 & -2c^2 & -2c^2 & 2(c^2-b_1) & 2b & 1/\sqrt{2} & 0 \\ -2x_1 & -2b & -2b & 2b & 2b & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & 1\sqrt{2} & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{52}$$

$A_2 = (A_{21}, A_{22})$, where

$$A_{21} = \begin{pmatrix} 2(u^2-d) & -4cu & 0 & 0 \\ -4cu & -4d & 0 & 0 \\ 0 & 0 & 2(u^2-d) & 4cu \\ 0 & 0 & 4cu & -4d \\ 0 & 0 & -2\sqrt{2}u & -4\sqrt{2}c \\ -\sqrt{2}u & 2\sqrt{2}c & -\sqrt{2}u & -2\sqrt{2}c \\ \sqrt{2}u & -2\sqrt{2}c & -\sqrt{2}u & -2\sqrt{2}c \end{pmatrix} \tag{53}$$

and

$$A_{22} = \begin{pmatrix} 0 & -\sqrt{2}u & \sqrt{2}u \\ 0 & 2\sqrt{2}c & -2\sqrt{2}c \\ -2\sqrt{2}u & -\sqrt{2}u & -\sqrt{2}u \\ -4\sqrt{2}c & -2\sqrt{2}c & -2\sqrt{2}c \\ 8b^2-4b_7 & 4b^2-2b_7 & 4b^2-2b_7 \\ 4b^2-2b_7 & 2-4b_7 & 2-4b^2-2b_7 \\ 4b^2-2b_7 & 2-4b^2-2b_7 & 2b^2-4b_7 \end{pmatrix}; \tag{54}$$

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8bc-2ux_1 & -4bc-2ux_1 & -4bc-2ux_1 \\ 0 & 0 & 0 & 0 & 8bc+2ux_1 & 4bc & 4bc \\ 0 & 0 & 0 & 0 & 0 & -4bc-2ux_1 & 4bc \\ 0 & 0 & 0 & 0 & 0 & 4bc & -4bc-2ux_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ -x_1 & 2b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_1 & -2b & 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

Now we calculate the coefficient $a_7(u)$. Notice that

$$\mathrm{Det}\big(C(t)\big) = \mathrm{Det}\big(D_{u,A,t}(X,B,v)\big) = 8c^2\,\mathrm{Det}\big(A(t)\big),$$

where $C(t)$ and $A(t)$ denote the above written matrices C and A with b_7 replaced by $b_7 + t/2c$, $d_2 = d_3 = d$ replaced by $d + (1/\sqrt{2})t$ and $x_1 = w/\sqrt{1+w^2}$, $b = \frac{1}{2\sqrt{1+w^2}}$, where w is defined in Theorem 3.1. Now we apply Lemma 2.14 and Lemma 2.15. By Lemma 2.15,

$$a_7(u) = \det(A_1)\det(E),$$

where E is a 7×7 matrix given by

$$E = \begin{pmatrix} -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2/c & -1/c & -1/c \\ 0 & 0 & 0 & 0 & -1/c & -2/c & -1/c \\ 0 & 0 & 0 & 0 & -1/c & -1/c & -2/c \end{pmatrix}. \tag{56}$$

Since $c = \sqrt{1-u_1^2}/2 > 0$ for $u_1 \in [0,1)$, E is well defined and $\det(E) \neq 0$. Also by Lemma 2.14 and Theorem 2.4, $\det(A_1) \neq 0$. Hence $a_7(u) \neq 0$ for any $u \in [0,1)$ as required. □

Now we will prove one of the main results of this section.

Theorem 3.5. *Let f_{u_1} be defined by* (1), *i.e.*

$$f_{u_1}\big(u_2, u_3, u_4, u_5, x^1, x^2, x^3\big) = \sum_{i,j=1}^{5} u_i u_j \big|\langle x_i, x_j\rangle\big|_3.$$

Let $M_u = \max(f_u)$ under constraints (2) *and* (3). *Then for any $u \in [0,1]$*

$$M_u = \frac{1 + 6c^2 + \sqrt{(6c^2-1)^2 + 16(1-4c^2)c^2}}{2},$$

where $c = c(u) = \sqrt{1-u^2}/2$.

Proof. Define

$$U = \left\{u \in [0,1):\ M_u = \frac{1 + 6c^2 + \sqrt{(6c^2-1)^2 + 16(1-4c^2)c^2}}{2}\right\}.$$

By Lemma 2.6 and Lemma 2.7, $0 \in U$, since $M_o = 3/2$. Now we show that U is an open set. Fix $u \in U$. First we consider the case $u = 0$. We apply Theorem 3.3 and Theorem 3.4. Let (X_v, L_v) where

$$X_u = \big(x^1, x^2, x^3, c(v), c(v), c(v), c(v)\big),$$

$(c(0) = 1/2)$ and

$$L_v(t) = (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7)$$

are given by Theorem 3.1 for fixed $v \in [0, 1)$ and $t \in \mathbb{R}$. Assume that $u_n \to 0$ and $u_n \notin U$. Let $(X_{u_n}, L_{u_n}(t))$ be such as in Theorem 3.3. Passing to a subsequence, if necessary, and reasoning as in Theorem 3.3, we can assume that $(X_{u_n}, L_{u_n}(t)) \to (X_o, L_o(t))$. Let

$$X_{u_n} = \big(x^{1n}, x^{2n}, x^{3n}, c(u_n), c(u_n), c(u_n), c(u_n)\big).$$

Since $X_{u_n} \to X_o$, we can assume that $\operatorname{sgn}\langle x_{in}, x_{jn}\rangle_3 = -1$ for $i, j = 2, 3, 4, 5$, $i \neq j$. Without loss of generality, again passing to a subsequence if necessary we can assume that for $n \geqslant n_o$

$$\operatorname{sgn}\langle x_{1n}, x_{jn}\rangle_3 = z_j$$

for $j = 2, 3, 4, 5$, where $z_j = \pm 1$. By Lemma 2.8 we have to consider three cases:

(a) $z_2 = -1$, $z_3 = z_4 = z_5 = 1$;
(b) $z_2 = z_3 = z_4 = z_5 = 1$;
(c) $z_2 = z_3 = -z_4 = -z_5 = 1$.

By Theorem 2.3 and Theorem 2.4, (a) can be excluded. If (b) holds true, then by Theorem 3.3, Theorem 3.2 and Theorem 3.4 applied to $u_1 = 0$ and $h_{t,A,0}$, where A is given by (41), we get that

$$M_{u_n} = 6c_u^2 \leqslant 3/2,$$

which by Theorem 2.4 leads to a contradiction. (Since $u_1 = 0$, $D_{o,A,t}$ is the same for the function $h_{o,A,t}$, determined by A given by (41). This permits us to apply Theorem 3.4 in this case.) If (c) holds true, we get a contradiction with Theorem 3.3. Consequently, there exists an interval $[0, v) \subset U$.

Now assume that $u \in U$ and $u > 0$. Assume $u_n \to u$ and $u_n \notin U$ for $n \in \mathbb{N}$. Let $(X_{u_n}, L_{u_n}(t))$ be such as in Theorem 3.3. Without loss of generality we can assume that $(X_{u_n}, L_{u_n}(t)) \to (X_u, L_u(t))$, where $(X_u, L_u(t))$ is defined in Theorem 3.3. Since $X_{u_n} \to X_u$

$$\operatorname{sgn}\langle x_{in}, x_{jn}\rangle_3 = a_{ij}$$

for $i, j = 1, 2, 3, 4, 5$ for $n \geqslant n_o$, where the matrix $\{a_{ij}\}$ is given by (28). Applying Theorem 3.3, we get that $u_n \in U$ for $n \geqslant n_o$; a contradiction. Hence the set U is open. It is easy to see that U is also closed. Since $0 \in U$ and $[0, 1)$ is connected, $U = [0, 1)$. Since $M(1, 0) = 1$ the proof is complete. □

Theorem 3.6.

$$\lambda_3^5 = \frac{5 + 4\sqrt{2}}{7}.$$

Moreover, $\lambda_3^5 = \lambda(V)$, *where* $V \subset l_\infty^{(5)}$ *is spanned by*

$$x^1 = (a/u_1, b/c_o, b/c_o, -b/c_o, -b/c_o),$$
$$x^2 = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2})/c_o$$

and

$$x^3 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0)/c_o,$$

where

$$u_1 = \sqrt{(5-3\sqrt{2})/7}, \quad c_o = \frac{\sqrt{(2+3\sqrt{2})/7}}{2}$$

and

$$a = \sqrt{(2\sqrt{2}-1)/7}, \quad b = \sqrt{1-a^2}/2.$$

Proof. Let $f_{3,5} : \mathbb{R}^5 \times (\mathbb{R}^5)^3 \to R$ be defined by

$$f_{3,5}\big((v_1, v_2, \ldots, v_5), y^1, y^2, y^3\big) = \sum_{i,j=1}^{5} v_i v_j \big|\langle y_i, y_j\rangle_3\big|.$$

Let $M_{3,5} = \max f_{3,5}$ under constraints

$$\langle y^i, y^j\rangle_5 = \delta_{ij}, \quad 1 \leqslant i \leqslant j \leqslant 3;$$

and

$$\sum_{j=1}^{5} v_j^2 = 1.$$

By Theorem 2.2,

$$\lambda_3^5 = M_{3,5}.$$

By Theorem 3.5,

$$M_{3,5} = \max\left\{h(c) = \frac{1+6c^2+\sqrt{(6c^2-1)^2+16(1-4c^2)c^2}}{2}\colon c \in [0, 1/2]\right\}.$$

By Theorem 2.4, $c_o = \frac{\sqrt{(2+3\sqrt{2})/7}}{2}$ and

$$M_{3,5} = h(c_o) = \frac{5+4\sqrt{2}}{7}.$$

By the proof of Theorem 2.2, and Theorem 2.4, the function $f_{3,5}$ attains its maximum at $z^1 = (a, b, b, -b, -b)$, $z^2 = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2})$ and $z^3 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0)$, $u = (u_1, c_o, c_o, c_o, c_o)$, where

$$u_1 = \sqrt{(5-3\sqrt{2})/7}, \quad c_o = \frac{\sqrt{(2+3\sqrt{2})/7}}{2}$$

and

$$a = \sqrt{(2\sqrt{2}-1)/7}, \quad b = \sqrt{1-a^2}/2.$$

By the proof of Theorem 2.2, x^1, x^2 and x^3, defined in the statement of our theorem, form a basis of a space V satisfying $\lambda(V) = \lambda_3^5$. $\square$

Remark 3.1. Note that (compare with [12, p. 259]) $3/2 = \lambda_3^4 < \lambda_3^5$. Also $\lambda_2^3 = 4/3$ and by the Kadec–Snobar Theorem [8] $\lambda_2^4 \leqslant \sqrt{2} < 3/2$. If x^1, x^2, x^3, u are such as in Theorem 3.6, then after elementary calculations we get

$$\|x_1\|_3 = \sqrt{\frac{2\sqrt{2}-1}{5-3\sqrt{2}}}$$

and

$$\|x_2\|_3 = \sqrt{\frac{22-2\sqrt{2}}{2+3\sqrt{2}}},$$

where $x_1 = (x_1^1, x_1^2, x_1^3)$, $x_2 = (x_2^1, x_2^2, x_2^3)$ and $\|\cdot\|_3$ is the Euclidean norm in $\mathbb{R}^3$. Hence it is easy to see that

$$\|x_1\|_3 = \|x_2\|_3$$

if and only if

$$77\sqrt{2} = 112,$$

which is false. Consequently, by the above calculations and Theorem 3.6, Proposition 3.1 from [12] is incorrect.

Remark 3.2. Notice that in [6], it has been proven that

$$\lambda(V) \leqslant 4/3$$

for any two-dimensional, real, unconditional Banach space. Recall that a two-dimensional, real Banach space V is called unconditional if there exists v^1, v^2 a basis of V such that for any $a_1, a_2 \in \mathbb{R}$ and $\epsilon_1, \epsilon_2 \in \{-1, 1\}$

$$\|a_1 v^1 + a_2 v^2\| = \|\epsilon_1 a_1 v^1 + \epsilon_2 a_2 v^2\|.$$

Moreover, the Grünbaum conjecture has been recently proved (see [4]).

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