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Two-sided cells in type B (asymptotic case)

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Abstract

We compute two-sided cells of Weyl groups of type B for the "asymptotic" choice of parameters. We also obtain some partial results concerning Lusztig's conjectures in this particular case.

Keywords: Weyl group; Kazhdan–Lusztig; Unequal parameters; Lusztig's conjectures

Let W_n be a Weyl group of type B_n. The present paper is a continuation of the work done by L. Iancu and the author [3] concerning Kazhdan–Lusztig theory of W_n for the asymptotic choice of parameters [3, §6]. To each element $w \in W_n$ is associated a pair of standard bi-tableaux $(P(w), Q(w))$ (see [18] or [3, §3]): this can be viewed as a Robinson–Schensted type correspondence. Our main result [3, Theorem 7.7] was the complete determination of the left cells: two elements w and w' are in the same left cell if and only if $Q(w) = Q(w')$. For the corresponding result for the symmetric group, see [12] and [1]. We have also computed the character afforded by a left cell representation [3, Proposition 7.11] (this character is irreducible).

In this paper, we are concerned with the computation of the two-sided cells. Let us state the result here. If $w \in W_n$, write $Q(w) = (Q^+(w), Q^-(w))$ and denote by $\lambda^+(w)$ and $\lambda^-(w)$ the shape of $Q^+(w)$ and $Q^-(w)$, respectively. Note that $(\lambda^+(w), \lambda^-(w))$ is a bipartition of n.

Theorem. (See 3.9.) *For the choice of parameters as in* [3, §6]*, two elements w and w' are in the same two-sided cell if and only if* $(\lambda^+(w), \lambda^-(w)) = (\lambda^+(w'), \lambda^-(w'))$.

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Lusztig [16, Chapter 14] has proposed fifteen conjectures on Kazhdan–Lusztig theory of Hecke algebras with unequal parameters. In the asymptotic case, Geck and Iancu [9] use some of our results, namely some informations on the preorder $\leqslant_{\mathcal{LR}}$ (see Theorem 3.5 and Proposition 4.2), to compute the function **a** and to prove Lusztig's conjectures P_i, for $i \in \{1,2,3,4,5,6,7,8,11,12,13,14\}$. On the other hand, Geck [8] has shown that Lusztig's conjectures P_9 and P_{10} hold. More precisely, he proved that the Kazhdan–Lusztig basis is *cellular* (in the sense of [11]). He also proved a slightly weaker version of P_{15} (but his version is sufficient for constructing the homomorphism from the Hecke algebra to the asymptotic algebra J).

The present paper is organized as follows. In Section 1, we study some consequences of Lusztig's conjectures on the multiplication by T_{w_0}, where w_0 is the longest element of a finite Weyl group. From Section 2 to the end of the paper, we assume that the Weyl group is of type B_n and that the choice of parameters is done as in [3, §6]. In Section 2, we establish some preliminary results concerning the Kazhdan–Lusztig basis. In Section 3, we prove the above theorem by introducing a new basis of the Hecke algebra: this was inspired by the work of Geck on the induction of Kazhdan–Lusztig cells [7]. Section 4 contains some results related to Lusztig's conjectures. In Section 5, we determine which specializations of the parameters preserve the Kazhdan–Lusztig basis.

1. Generalities

1.1. Notation

We slightly modify the notation used in [3, §5]. Let (W,S) be a Coxeter group with $|S|<\infty$. We denote by $\ell : W \to \mathbb{N} = \{0,1,2,\dots\}$ the length function relative to S. If W is finite, w_0 denotes its longest element. Let $\leqslant$ denote the Bruhat ordering on W. If $I \subset S$, we denote by W_I the standard parabolic subgroup of W generated by I.

Let Γ be a totally ordered abelian group which will be denoted additively. The order on Γ will be denoted by $\leqslant$. If $\gamma_0 \in \Gamma$, we set

$$\Gamma_{<\gamma_0} = \{\gamma \in \Gamma \mid \gamma < \gamma_0\}, \qquad \Gamma_{\leqslant\gamma_0} = \{\gamma \in \Gamma \mid \gamma \leqslant \gamma_0\},$$
$$\Gamma_{>\gamma_0} = \{\gamma \in \Gamma \mid \gamma > \gamma_0\} \quad \text{and} \quad \Gamma_{\geqslant\gamma_0} = \{\gamma \in \Gamma \mid \gamma \geqslant \gamma_0\}.$$

Let A be the group algebra of Γ over $\mathbb{Z}$. It will be denoted exponentially: as a $\mathbb{Z}$-module, it is free with basis $(v^\gamma)_{\gamma\in\Gamma}$ and the multiplication rule is given by $v^\gamma v^{\gamma'} = v^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. If $a \in A$, we denote by a_γ the coefficient of a on v^γ, so that $a = \sum_{\gamma\in\Gamma} a_\gamma v^\gamma$. If $a \neq 0$, we define the *degree* and the *valuation* of a (which we denote respectively by $\deg a$ and $\operatorname{val} a$) as the elements of Γ equal to

$$\deg a = \max\{\gamma \mid a_\gamma \neq 0\}$$

and

$$\operatorname{val} a = \min\{\gamma \mid a_\gamma \neq 0\}.$$

By convention, we set $\deg 0 = -\infty$ and $\operatorname{val} 0 = +\infty$. So $\deg : A \to \Gamma \cup \{-\infty\}$ and $\operatorname{val} : A \to \Gamma \cup \{+\infty\}$ satisfy $\deg ab = \deg a + \deg b$ and $\operatorname{val} ab = \operatorname{val} a + \operatorname{val} b$ for all $a, b \in A$. We denote by $A \to A$, $a \mapsto \bar{a}$ the automorphism of A induced by the automorphism of Γ sending γ to $-\gamma$. Note that $\deg a = -\operatorname{val} \bar{a}$. If $\gamma_0 \in \Gamma$, we set

$$A_{<\gamma_0} = \bigoplus_{\gamma<\gamma_0} \mathbb{Z}v^\gamma, \qquad A_{\leqslant\gamma_0} = \bigoplus_{\gamma\leqslant\gamma_0} \mathbb{Z}v^\gamma,$$
$$A_{>\gamma_0} = \bigoplus_{\gamma>\gamma_0} \mathbb{Z}v^\gamma \quad \text{and} \quad A_{\geqslant\gamma_0} = \bigoplus_{\gamma\geqslant\gamma_0} \mathbb{Z}v^\gamma.$$

We fix a *weight function* $L : W \to \Gamma$, that is a function satisfying $L(ww') = L(w) + L(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$. We also assume that $L(s) > 0$ for every $s \in S$. We denote by $\mathcal{H} = \mathcal{H}(W, S, L)$ the *Hecke algebra* of W associated to the weight function L. It is the associative A-algebra with A-basis $(T_w)_{w\in W}$ indexed by W and whose multiplication is determined by the following two conditions:

$$\begin{array}{lll} \text{(a)} & T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ \text{(b)} & T_s^2 = 1 + \big(v^{L(s)} - v^{-L(s)}\big)T_s, & \text{if } s \in S. \end{array}$$

It is easily seen from the above relations that $(T_s)_{s\in S}$ generates the A-algebra $\mathcal{H}$ and that T_w is invertible for every $w \in W$. If $h = \sum_{w\in W} a_w T_w \in \mathcal{H}$, we set $\bar{h} = \sum_{w\in W} \bar{a}_w T_{w^{-1}}^{-1}$. Then the map $\mathcal{H} \to \mathcal{H}$, $h \mapsto \bar{h}$ is a semi-linear involutive automorphism of $\mathcal{H}$. If $I \subset S$, we denote by $\mathcal{H}(W_I)$ the sub-A-algebra of $\mathcal{H}$ generated by $(T_s)_{s\in I}$.

Let $w \in W$. By [16, Theorem 5.2], there exists a unique element $C_w \in \mathcal{H}$ such that

$$\begin{array}{ll} \text{(a)} & C_w = \overline{C}_w, \\ \text{(b)} & C_w \in T_w + \Big(\bigoplus_{y\in W} A_{<0} T_y\Big). \end{array}$$

Write $C_w = \sum_{y\in W} p^*_{y,w} T_y$ with $p^*_{y,w} \in A$. Then [16, 5.3]

$$\begin{aligned} &p^*_{w,w} = 1, \\ &p^*_{y,w} = 0, \quad \text{if } y \nleqslant w. \end{aligned}$$

In particular, $(C_w)_{w\in W}$ is an A-basis of $\mathcal{H}$: it is called the *Kazhdan–Lusztig basis* of $\mathcal{H}$. Write now $p_{y,w} = v^{L(w)-L(y)} p^*_{y,w}$. Then

$$p_{y,w} \in A_{\geqslant 0}$$

and the coefficient of $p_{y,w}$ on v^0 is equal to 1 (see [16, Proposition 5.4(a)]).

We define the relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$, $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$ as in [16, §8].

1.2. The function **a**

Let $x, y \in W$. Write

$$C_x C_y = \sum_{z\in W} h_{x,y,z} C_z,$$

where $h_{x,y,z} \in A$ for $z \in W$. Of course, we have

$$\overline{h_{x,y,z}} = h_{x,y,z}. \tag{1.1}$$

The following lemma is well known [16, Lemma 10.4(c) and formulas 13.1(a) and (b)].

Lemma 1.2. *Let x, y and z be three elements of W. Then*

$$\deg h_{x,y,z} \leqslant \min\bigl(L(x), L(y)\bigr).$$

Conjecture P_0. (Lusztig) *There exists $N \in \Gamma$ such that $\deg h_{x,y,z} \leqslant N$ for all x, y and z in W.*

If W is finite, then W satisfies obviously P_0. If W is an affine Weyl group, then it also satisfies P_0 [15, 7.2]. From now on, we assume that W satisfies P_0, so that the next definition is valid. If $z \in W$, we set

$$\mathbf{a}(z) = \max_{x,y\in W} \deg h_{x,y,z}.$$

Since $h_{1,z,z} = 1$, we have $\mathbf{a}(z) \in \Gamma_{\geqslant 0}$. If necessary, we will write $\mathbf{a}_W(z)$ for $\mathbf{a}(z)$. We denote by $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ the coefficient of $v^{\mathbf{a}(z)}$ in $h_{x,y,z}$. The next proposition shows how the function $\mathbf{a}$ can be calculated by using different bases.

Proposition 1.3. *Let $(X_w)_{w\in W}$ and $(Y_w)_{w\in W}$ be two families of elements of $\mathcal{H}$ such that, for every $w \in W$, $X_w - T_w$ and $Y_w - T_w$ belong to $\bigoplus_{y<w} A_{<0}T_y$. For all x and y in W, write*

$$X_x Y_y = \sum_{z\in W} \xi_{x,y,z} C_z.$$

Then, if $x, y, z \in W$, we have:

(a) $\deg \xi_{x,y,z} \leqslant \min\{L(x), L(y)\}$.
(b) $\xi_{x,y,z} \in \gamma_{x,y,z^{-1}} v^{\mathbf{a}(z)} + A_{<\mathbf{a}(z)}$.

In particular,

$$\mathbf{a}(z) = \max_{x,y\in W} \deg \xi_{x,y,z}.$$

Proof. Clear. □

1.3. Lusztig's conjectures

Let $\tau : \mathcal{H} \to A$ be the A-linear map such that $\tau(T_w) = \delta_{1,w}$ if $w \in W$. It is the canonical symmetrizing form on $\mathcal{H}$ (recall that $\tau(T_x T_y) = \delta_{xy,1}$). If $z \in W$, let

$$\Delta(z) = -\deg p^*_{1,z} = -\deg \tau(C_z).$$

Let n_z be the coefficient of $p^*_{1,z}$ on $v^{-\Delta(z)}$. Finally, let

$$\mathcal{D} = \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}.$$

Conjectures. (Lusztig) *With the above notation, we have*:

P_1. *If $z \in W$, then $\mathbf{a}(z) \leqslant \Delta(z)$.*
P_2. *If $d \in \mathcal{D}$ and if $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.*
P_3. *If $y \in W$, then there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.*
P_4. *If $z' \leqslant_{\mathcal{LR}} z$, then $\mathbf{a}(z) \leqslant \mathbf{a}(z')$. Therefore, if $z \sim_{\mathcal{LR}} z'$, then $\mathbf{a}(z) = \mathbf{a}(z')$.*
P_5. *If $d \in \mathcal{D}$ and $y \in W$ satisfy $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.*
P_6. *If $d \in \mathcal{D}$, then $d^2 = 1$.*
P_7. *If $x, y, z \in W$, then $\gamma_{x,y,z} = \gamma_{y,z,x}$.*
P_8. *If $x, y, z \in W$ satisfy $\gamma_{x,y,z} \neq 0$, then $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$ and $z \sim_{\mathcal{L}} x^{-1}$.*
P_9. *If $z' \leqslant_{\mathcal{L}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{L}} z$.*
P_{10}. *If $z' \leqslant_{\mathcal{R}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{R}} z$.*
P_{11}. *If $z' \leqslant_{\mathcal{LR}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{LR}} z$.*
P_{12}. *If $I \subset S$ and $z \in W_I$, then $\mathbf{a}_{W_I}(z) = \mathbf{a}_W(z)$.*
P_{13}. *Every left cell $\mathcal{C}$ of W contains a unique element $d \in \mathcal{D}$. If $y \in \mathcal{C}$, then $\gamma_{y^{-1},y,d} \neq 0$.*
P_{14}. *If $z \in W$, then $z \sim_{\mathcal{LR}} z^{-1}$.*
P_{15}. *If $x, x', y, w \in W$ are such that $\mathbf{a}(y) = \mathbf{a}(w)$, then*

$$\sum_{y' \in W} h_{w,x',y'} \otimes_{\mathbb{Z}} h_{x,y',y} = \sum_{y' \in W} h_{y',x',y} \otimes_{\mathbb{Z}} h_{x,w,y'}$$

in $A \otimes_{\mathbb{Z}} A$.

Lusztig has shown that these conjectures hold if W is a finite or affine Weyl group and $L = \ell$ [16, §15], if W is dihedral and L is any weight function [16, §17] and if (W, L) is quasi-split [16, §16].

1.4. Lusztig's conjectures and multiplication by T_{w_0}

We assume in this subsection that W is finite. We are interested here in certain properties of the multiplication by $T_{w_0}^n$ for $n \in \mathbb{Z}$. Some of them are partially known [14, Lemma 1.11 and Remark 1.12]. If $y \in W$ and $n \in \mathbb{Z}$, we set

$$T_{w_0}^n C_y = \sum_{x \in W} \lambda_{x,y}^{(n)} C_x.$$

Note that $\lambda_{x,y}^{(n)} = 0$ if $x \not\leqslant_{\mathcal{L}} y$.

Proposition 1.4. *Assume that W is finite and satisfies Lusztig's conjectures P_1, P_4 and P_8. Let $n \in \mathbb{Z}$ and let x and y be two elements of W such that $x \leqslant_{\mathcal{L}} y$. Then*:

(a) *If $n \geqslant 0$, then $\deg \lambda_{x,y}^{(n)} \leqslant n(\mathbf{a}(x) - \mathbf{a}(w_0 x))$. If moreover $x <_{\mathcal{L}} y$, then $\deg \lambda_{x,y}^{(n)} < n(\mathbf{a}(x) - \mathbf{a}(w_0 x))$.*

(b) *If* $n \leqslant 0$, *then* $\deg \lambda_{x,y}^{(n)} \leqslant n(\mathbf{a}(y) - \mathbf{a}(w_0 y))$. *If moreover* $x <_{\mathcal{L}} y$, *then* $\deg \lambda_{x,y}^{(n)} < n(\mathbf{a}(y) - \mathbf{a}(w_0 y))$.

(c) *If* n *is even and if* $x \sim_{\mathcal{L}} y$, *then* $\lambda_{x,y}^{(n)} = \delta_{x,y} v^{n(\mathbf{a}(x)-\mathbf{a}(w_0 x))}$.

Proof. If $n = 0$, then (a), (b) and (c) are easily checked. Let us now prove (a) and (b). By [16, Proposition 11.4],

$$T_{w_0} = \sum_{u \in W} (-1)^{\ell(w_0 u)} p_{1,w_0 u}^* C_u.$$

Consequently,

$$\lambda_{x,y}^{(1)} = \sum_{\substack{u \in W \\ x \leqslant_{\mathcal{R}} u}} (-1)^{\ell(w_0 u)} p_{1,w_0 u}^* h_{u,y,x}.$$

But, by P_1, we have $\deg p_{1,w_0 u}^* \leqslant -\mathbf{a}(w_0 u)$. If moreover $x \leqslant_{\mathcal{R}} u$, then $w_0 u \leqslant_{\mathcal{R}} w_0 x$ and so $-\mathbf{a}(w_0 x) \geqslant -\mathbf{a}(w_0 u)$ by P_4. Therefore,

$$\deg \lambda_{x,y}^{(1)} \leqslant \mathbf{a}(x) - \mathbf{a}(w_0 x).$$

On the other hand, if $\deg \lambda_{x,y}^{(1)} = \mathbf{a}(x) - \mathbf{a}(w_0 x)$, then there exists $u \in W$ such that $x \leqslant_{\mathcal{R}} u$ and $\deg h_{u,y,x} = \mathbf{a}(x)$. So, by P_8, we get that $x \sim_{\mathcal{L}} y$. This shows (a) for $n = 1$.

Now, let $\nu : \mathcal{H} \to \mathcal{H}$ denote the A-linear map such that $\nu(C_w) = v^{\mathbf{a}(w_0 w) - \mathbf{a}(w)} C_w$ for all $w \in W$ and let $\mu : \mathcal{H} \to \mathcal{H}$, $h \mapsto T_{w_0} h$. Then, if $w \in W$, we have

$$\nu\mu(C_y) = \sum_{u \leqslant_{\mathcal{L}} y} v^{\mathbf{a}(w_0 u) - \mathbf{a}(u)} \lambda_{u,y}^{(1)} C_u.$$

So, by the previous discussion, we have $v^{\mathbf{a}(w_0 u) - \mathbf{a}(u)} \lambda_{u,y}^{(1)} \in A_{\leqslant 0}$. Moreover, if $u <_{\mathcal{L}} y$, then $v^{\mathbf{a}(w_0 u) - \mathbf{a}(u)} \lambda_{u,y}^{(1)} \in A_{<0}$. On the other hand, $\det \mu = \pm 1$ and $\det \nu = 1$. Therefore, if we write

$$\mu^{-1} \nu^{-1}(C_y) = \sum_{u \leqslant_{\mathcal{L}} y} \beta_{u,y} C_u,$$

then $\beta_{u,y} \in A_{\leqslant 0}$ and, if $u <_{\mathcal{L}} y$, then $\beta_{u,y} \in A_{<0}$. Finally,

$$\begin{aligned} T_{w_0}^{-1} C_y &= \mu^{-1}(C_y) \\ &= \mu^{-1} \nu^{-1} \nu(C_y) \\ &= v^{\mathbf{a}(w_0 y) - \mathbf{a}(y)} \mu^{-1} \nu^{-1}(C_y) \\ &= \sum_{u \leqslant_{\mathcal{L}} y} v^{\mathbf{a}(w_0 y) - \mathbf{a}(y)} \beta_{u,y} C_u. \end{aligned}$$

In other words, $\lambda_{x,y}^{(-1)} = v^{\mathbf{a}(w_0 y) - \mathbf{a}(y)} \beta_{x,y}$. This shows that (b) holds if $n = -1$. An elementary induction argument using P_4 shows that (a) and (b) hold in full generality.

Let us now prove (c). Let K be the field of fraction of A. Let C be a left cell of W and let $c \in C$. We set

$$\mathcal{H}^{\leqslant_{\mathcal{L}} C} = \bigoplus_{w \leqslant_{\mathcal{L}} c} AC_w \quad \text{and} \quad \mathcal{H}^{<_{\mathcal{L}} C} = \bigoplus_{w <_{\mathcal{L}} c} AC_w.$$

Then $\mathcal{H}^{\leqslant_{\mathcal{L}} C}$ and $\mathcal{H}^{<_{\mathcal{L}} C}$ are left ideals of $\mathcal{H}$. The algebra $K\mathcal{H} = K \otimes_A \mathcal{H}$ being semi-simple, there exists a left ideal I_C of $K\mathcal{H}$ such that $K\mathcal{H}^{\leqslant_{\mathcal{L}} C} = K\mathcal{H}^{<_{\mathcal{L}} C} \oplus I_C$.

We need to prove that, for all $h \in I_C$,

$$T_{w_0}^n h = v^{n(\mathbf{a}(c) - \mathbf{a}(w_0 c))} h.$$

For this, we may, and we will, assume that $n > 0$. Let $V_1^C, V_2^C, \dots, V_{n_C}^C$ be irreducible sub-$K\mathcal{H} \otimes_A K$-modules of I_C such that

$$I_C = V_1^C \oplus \cdots \oplus V_{n_C}^C.$$

Let $j \in \{1, 2, \dots, n_C\}$. Since $T_{w_0}^n$ is central and invertible in $\mathcal{H}$, there exists $\varepsilon \in \{1, -1\}$ and $i_j^C \in \Gamma$ such that

$$T_{w_0}^n h = \varepsilon v^{i_j^C} h$$

for every $h \in V_j^C$. By specializing $v^\gamma \mapsto 1$, we get that $\varepsilon = 1$. Moreover, by (a) and (b), $i_j^C \leqslant n(\mathbf{a}(c) - \mathbf{a}(w_0 c))$. On the other hand, since $\det \mu = \pm 1$, we have $\det \mu^n = 1$. But $\det \mu^n = v^r$, where

$$\begin{aligned}
r &= \sum_{C \in \mathcal{LC}(W)} \sum_{j=1}^{n_C} i_j^C \dim V_j^C \\
&\leqslant n \sum_{C \in \mathcal{LC}(W)} \big(\mathbf{a}(C) - \mathbf{a}(w_0 C)\big) \sum_{j=1}^{n_C} \dim V_j^C \\
&= n \sum_{C \in \mathcal{LC}(W)} \big(\mathbf{a}(C) - \mathbf{a}(w_0 C)\big)|C| \\
&= n \sum_{w \in W} \big(\mathbf{a}(w) - \mathbf{a}(w_0 w)\big) \\
&= 0.
\end{aligned}$$

Here, $\mathcal{LC}(W)$ denotes the set of left cells in W and, if $C \in \mathcal{LC}(W)$, $\mathbf{a}(C)$ denotes the value of $\mathbf{a}$ on C (according to P_4). The fact that $r = 0$ forces the equality $i_j^C = n(\mathbf{a}(C) - \mathbf{a}(w_0 C))$ for every left cell C and every $j \in \{1, 2, \dots, n_C\}$. $\square$

Remark 1.5. Assume here that w_0 is central in W and keep the notation of the proof of Proposition 1.4(c). Let $j \in \{1, 2, \dots, n_C\}$. Then there exists $\varepsilon_j(C) \in \{1, -1\}$ et $e_j(C) \in \Gamma$ such that $T_{w_0} h = \varepsilon_j(C) v^{e_j(C)} h$ for every $h \in V_j^C$.

Question. *Let $j, j' \in \{1, 2, \dots, n_C\}$. Does $\varepsilon_j(C) = \varepsilon_{j'}(C)$?*

A positive answer to this question would allow to generalize Proposition 1.4(c) to the case where w_0^n is central.

Corollary 1.6. *Assume that W is finite and satisfies Lusztig's conjectures P_1, P_2, P_4, P_8, P_9 and P_{13}. Let $w \in W$ and let $n \in \mathbb{N}$. Then $\deg \tau(T_{w_0}^{-n} C_w) \leqslant -\mathbf{a}(w) + n(\mathbf{a}(w_0 w) - \mathbf{a}(w))$. Moreover, $\deg \tau(T_{w_0}^{-n} C_w) = -\mathbf{a}(w) + n(\mathbf{a}(w_0 w) - \mathbf{a}(w))$ if and only if $w_0^n w^{-1} \in \mathcal{D}$.*

Proof. Assume first that n is even. In particular, $w_0^n = 1$. By Proposition 1.4, we have

$$\tau\bigl(T_{w_0}^{-n} C_w\bigr) = v^{n(\mathbf{a}(w_0 w) - \mathbf{a}(w))} \tau(C_w) + \sum_{x <_{\mathcal{L}} w} \lambda_{x,w}^{(-n)} \tau(C_x).$$

But, if $x <_{\mathcal{L}} w$, then $\deg \tau(C_x) = -\Delta(x) \leqslant -\mathbf{a}(x) \leqslant -\mathbf{a}(w)$ by P_1 and P_4. So, by Proposition 1.4(b), we have that $\deg \lambda_{x,w}^{(-n)} \tau(C_x) < -\mathbf{a}(w) + n(\mathbf{a}(w_0 w) - \mathbf{a}(w))$. Moreover, again by P_1, we have $\deg \tau(C_w) = -\Delta(w) \leqslant -\mathbf{a}(w)$. This shows that $\deg \tau(T_{w_0}^{-n} C_w) \leqslant -\mathbf{a}(w) + n(\mathbf{a}(w_0 w) - \mathbf{a}(w))$ and that equality holds if and only if $\Delta(w) = \mathbf{a}(w)$, that is, if and only if $w \in \mathcal{D}$, as desired.

Assume now that $n = 2k + 1$ for some natural number k. Recall that, by [16, Proposition 11.4], $T_{w_0} = \sum_{u \in W} (-1)^{\ell(w_0 u)} p_{1,w_0 u}^* C_u$. Therefore,

$$\begin{aligned} T_{w_0}^{-n} C_w &= \sum_{u \in W} (-1)^{\ell(w_0 u)} p_{1,w_0 u}^* T_{w_0}^{-n-1} C_u C_w \\ &= \sum_{\substack{u,x \in W \\ x \leqslant_{\mathcal{L}} w \text{ and } x \leqslant_{\mathcal{R}} u}} (-1)^{\ell(w_0 u)} p_{1,w_0 u}^* h_{u,w,x} T_{w_0}^{-n-1} C_x. \end{aligned}$$

This implies that

$$\tau\bigl(T_{w_0}^{-n} C_w\bigr) = \sum_{\substack{u,x \in W \\ x \leqslant_{\mathcal{L}} w \text{ and } x \leqslant_{\mathcal{R}} u}} (-1)^{\ell(w_0 u)} p_{1,w_0 u}^* h_{u,w,x} \tau\bigl(T_{w_0}^{-n-1} C_x\bigr),$$

so

$$\deg \tau\bigl(T_{w_0}^{-n} C_w\bigr) \leqslant \max_{\substack{u,x \in W \\ x \leqslant_{\mathcal{L}} w \text{ and } x \leqslant_{\mathcal{R}} u}} \deg\bigl(p_{1,w_0 u}^* h_{u,w,x} \tau\bigl(T_{w_0}^{-n-1} C_x\bigr)\bigr).$$

Let u and x be two elements of W such that $x \leqslant_{\mathcal{L}} w$ and $x \leqslant_{\mathcal{R}} u$. Since $n+1$ is even and by the previous discussion, we have

$$\deg \tau\bigl(T_{w_0}^{-n-1} C_x\bigr) \leqslant -\mathbf{a}(x) + (n+1)\bigl(\mathbf{a}(w_0 x) - \mathbf{a}(x)\bigr).$$

By P_1 and P_4, $\deg p_{1,w_0 u}^* \leqslant -\mathbf{a}(w_0 u) \leqslant -\mathbf{a}(w_0 x)$. Moreover, $\deg h_{u,w,x} \leqslant \mathbf{a}(x)$. Consequently,

$$\begin{aligned}\deg\bigl(p^*_{1,w_0u}h_{u,w,x}\tau\bigl(T_{w_0}^{-n-1}C_x\bigr)\bigr) &\leqslant -\mathbf{a}(x)+n\bigl(\mathbf{a}(w_0x)-\mathbf{a}(x)\bigr)\\ &\leqslant -\mathbf{a}(w)+n\bigl(\mathbf{a}(w_0w)-\mathbf{a}(w)\bigr).\end{aligned}$$

Moreover, equality holds if and only if $w_0u \in \mathcal{D}$, $\deg h_{u,w,x} = \mathbf{a}(x) = \mathbf{a}(w)$ and $x \in \mathcal{D}$.

We first deduce that

$$\deg\tau\bigl(T_{w_0}^{-n}C_w\bigr) \leqslant -\mathbf{a}(w)+n\bigl(\mathbf{a}(w_0w)-\mathbf{a}(w)\bigr)$$

which is the first assertion of the proposition.

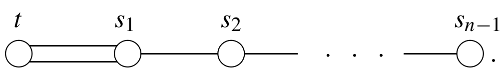
Assume now that $\deg\tau(T_{w_0}^{-n}C_w) = -\mathbf{a}(w)+n(\mathbf{a}(w_0w)-\mathbf{a}(w))$. Then there exists u and x in W such that $x \leqslant_{\mathcal{L}} w$, $x \leqslant_{\mathcal{R}} u$, $w_0u \in \mathcal{D}$, $\deg h_{u,w,x} = \mathbf{a}(x) = \mathbf{a}(w)$ and $x \in \mathcal{D}$. Since $\deg h_{u,w,x} = \mathbf{a}(x)$ and $x \in \mathcal{D}$, we deduce from P_2 that $w = u^{-1}$, which shows that $w_0w^{-1} \in \mathcal{D}$.

Conversely, assume that $w_0w^{-1} \in \mathcal{D}$. To show that $\deg\tau(T_{w_0}^{-n}C_w) = -\mathbf{a}(w)+n(\mathbf{a}(w_0w) - \mathbf{a}(w))$, it is sufficient to show that there is a unique pair (u,x) of elements of W such that $x \leqslant_{\mathcal{L}} w$, $x \leqslant_{\mathcal{R}} u$, $w_0u \in \mathcal{D}$, $\deg h_{u,w,x} = \mathbf{a}(x) = \mathbf{a}(w)$ and $x \in \mathcal{D}$. The existence follows from P_{13} (take $u = w^{-1}$ and x be the unique element of $\mathcal{D}$ belonging to the left cell containing w). Let us now show unicity. Let (u,x) be such a pair. Since $\deg h_{u,w,x} = \mathbf{a}(x)$ and $x \in \mathcal{D}$, we deduce from P_2 that $u = w^{-1}$. Moreover, since $\mathbf{a}(x) = \mathbf{a}(w)$ and $x \leqslant_{\mathcal{L}} w$, we have $x \sim_{\mathcal{L}} w$ by P_9. But, by P_{13}, x is the unique element of $\mathcal{D}$ belonging to the left cell containing w. □

2. Preliminaries on type *B* (asymptotic case)

From now on, we are working under the following hypothesis.

Hypothesis and notation. *We assume now that $W = W_n$ is of type B_n, $n \geqslant 1$. We write $S = S_n = \{t, s_1, \dots, s_{n-1}\}$ as in* [3, §2.1]: *the Dynkin diagram of W_n is given by*



We also assume that $\Gamma = \mathbb{Z}^2$ and that Γ is ordered lexicographically:

$$(a,b) \leqslant (a',b') \iff a < a' \quad \textit{or} \quad (a = a' \textit{ and } b \leqslant b').$$

We set $V = v^{(1,0)}$ and $v = v^{(0,1)}$ so that $A = \mathbb{Z}[V, V^{-1}, v, v^{-1}]$ is the Laurent polynomial ring in two algebraically independent indeterminates V and v. If $w \in W_n$, we denote by $\ell_t(w)$ the number of occurrences of t in a reduced expression of w. We set $\ell_s(w) = \ell(w) - \ell_t(w)$. Then ℓ_s and ℓ_t are weight functions and we assume that $L = L_n : W_n \to \Gamma$, $w \mapsto (\ell_t(w), \ell_s(w))$. So $\mathcal{H} = \mathcal{H}_n = \mathcal{H}(W_n, S_n, L_n)$. We denote by $\mathfrak{S}_n$ the subgroup of W generated by $\{s_1, \dots, s_{n-1}\}$: it is isomorphic to the symmetric group of degree n.

We now recall some notation from [3, §2.1 and 4.1]. Let $r_1 = t_1 = t$ and, if $1 \leqslant i \leqslant n-1$, let $r_{i+1} = s_ir_i$ and $t_{i+1} = s_it_is_i$. If $0 \leqslant l \leqslant n$, let $a_l = r_1r_2\cdots r_l$. We denote by $\mathfrak{S}_l$, W_l, $\mathfrak{S}_{l,n-l}$, $W_{l,n-l}$ the standard parabolic subgroups of W_n generated by $\{s_1, s_2, \dots, s_{l-1}\}$,

$\{t, s_1, s_2, \dots, s_{l-1}\}$, $S_n \setminus \{t, s_l\}$ and $S_n \setminus \{s_l\}$, respectively. The longest element of $\mathfrak{S}_l$ is denoted by σ_l. Let

$$Y_{l,n-l} = \big\{a \in \mathfrak{S}_n \mid \forall \sigma \in \mathfrak{S}_{l,n-l},\ \ell(a\sigma) \geqslant \ell(\sigma)\big\}.$$

If $w \in W_n$ is such that $\ell_t(w) = l$, then [3, §4.6] there exist unique $a_w, b_w \in Y_{l,n-l}$, $\sigma_w \in \mathfrak{S}_{l,n-l}$ such that $w = a_w a_l \sigma_w b_w^{-1}$. Recall that $\ell(w) = \ell(a_w) + \ell(a_l) + \ell(\sigma_w) + \ell(b_w)$.

2.1. Some submodules of $\mathcal{H}$

If l is a natural number such that $0 \leqslant l \leqslant n$, we set

$$\mathcal{T}_l = \bigoplus_{\substack{w\in W_n\\ \ell_t(w)=l}} AT_w, \qquad \mathcal{T}_{\leqslant l} = \bigoplus_{\substack{w\in W_n\\ \ell_t(w)\leqslant l}} AT_w, \qquad \mathcal{T}_{\geqslant l} = \bigoplus_{\substack{w\in W_n\\ \ell_t(w)\geqslant l}} AT_w,$$
$$\mathcal{C}_l = \bigoplus_{\substack{w\in W_n\\ \ell_t(w)=l}} AC_w, \qquad \mathcal{C}_{\leqslant l} = \bigoplus_{\substack{w\in W_n\\ \ell_t(w)\leqslant l}} AC_w \quad\text{and}\quad \mathcal{C}_{\geqslant l} = \bigoplus_{\substack{w\in W_n\\ \ell_t(w)\geqslant l}} AC_w.$$

Let $\Pi_?^T : \mathcal{H}_n \to \mathcal{T}_?$ and $\Pi_?^C : \mathcal{H}_n \to \mathcal{C}_?$ be the natural projections (for $? \in \{l, \leqslant l, \geqslant l\}$).

Proposition 2.1. *Let l be a natural number such that $0 \leqslant l \leqslant n$. Then*:

(a) *$\mathcal{T}_l$ and $\mathcal{C}_l$ are sub-$\mathcal{H}(\mathfrak{S}_n)$-modules-$\mathcal{H}(\mathfrak{S}_n)$ of $\mathcal{H}_n$. The maps Π_l^T and Π_l^C are morphisms of $\mathcal{H}(\mathfrak{S}_n)$-modules-$\mathcal{H}(\mathfrak{S}_n)$.*
(b) *$\mathcal{T}_{\leqslant l} = \mathcal{C}_{\leqslant l}$.*
(c) *$\mathcal{C}_{\geqslant l}$ is a two-sided ideal of $\mathcal{H}_n$.*

Proof. (a) follows from [3, Theorem 6.3(b)]. (b) is clear. (c) follows from [3, Corollary 6.7]. $\square$

The next proposition is a useful characterization of the elements of the two-sided ideal $\mathcal{C}_n$.

Proposition 2.2. *Let $h \in \mathcal{H}_n$. The following are equivalent*:

(1) $h \in \mathcal{C}_n$.
(2) $\forall x \in \mathcal{H}_n$, $(T_t - V)xh = 0$.
(3) $\forall x \in \mathcal{H}(\mathfrak{S}_n)$, $(T_t - V)xh = 0$.

Proof. If $\ell_t(w) = n$, then $tw < w$ so $(T_t - V)C_w = 0$ by [16, Theorem 6.6(b)]. Since $\mathcal{C}_n$ is a two-sided ideal of $\mathcal{H}_n$ (see Proposition 2.1(c)), we get that (1) implies (2). It is also obvious that (2) implies (3). It remains to show that (3) implies (1).

Let $I = \{h \in \mathcal{H}_n \mid \forall x \in \mathcal{H}(\mathfrak{S}_n),\ (T_t - V)xh = 0\}$. Then I is clearly a sub-$\mathcal{H}(\mathfrak{S}_n)$-module-$\mathcal{H}_n$ of $\mathcal{H}_n$. We need to show that $I \subset \mathcal{C}_n$. In other words, since $\mathcal{C}_n \subset I$, we need to show that $I \cap \mathcal{C}_{\leqslant n-1} = 0$. Let $I' = I \cap \mathcal{C}_{\leqslant n-1} = I \cap \mathcal{T}_{\leqslant n-1}$ (see Proposition 2.1(b)). Let $X = \{w \in W_n \mid \tau(I'T_{w^{-1}}) \neq 0\}$. Showing that $I' = 0$ is equivalent to showing that $X = \emptyset$.

Assume $X \neq \emptyset$. Let w be an element of X of maximal length and let h be an element of I' such that $\tau(hT_{w^{-1}}) \neq 0$. Since $h \in \mathcal{T}_{\leqslant n-1}$, we have $\ell_t(w) \leqslant n-1$. Moreover, $T_t h = Vh$, so

$\tau(T_t h T_{w^{-1}}) = \tau(h T_{w^{-1}} T_t) \neq 0$. By the maximality of $\ell(w)$, we get that $tw < w$. So, there exists $s \in S_n$ such that $sw > w$ and $s \neq t$. Then

$$\begin{aligned}\tau(T_s h T_{(sw)^{-1}}) &= \tau(h T_{(sw)^{-1}} T_s)\\ &= \tau(h T_{w^{-1}}) + \big(v - v^{-1}\big)\tau(h T_{(sw)^{-1}})\\ &\neq 0,\end{aligned}$$

the last inequality following from the maximality of $\ell(w)$ (which implies that $\tau(h T_{(sw)^{-1}}) = 0$). But $T_s h \in I'$ and so $sw \in X$. This contradicts the maximality of $\ell(w)$. □

2.2. Some results on the Kazhdan–Lusztig basis

In this subsection, we study the elements of the Kazhdan–Lusztig basis of the form $C_{a_l\sigma}$ where $0 \leqslant l \leqslant n$ and $\sigma \in \mathfrak{S}_n$.

Proposition 2.3. *Let $\sigma \in \mathfrak{S}_n$ and let $0 \leqslant l \leqslant n$. Then $C_{a_l} C_\sigma = C_{a_l\sigma}$ and $C_\sigma C_{a_l} = C_{\sigma a_l}$.*

Proof. Let $C = C_{a_l} C_\sigma$. Then $\overline{C} = C$ and

$$C - T_{a_l\sigma} = \sum_{w < a_l\sigma} \lambda_w T_w$$

with $\lambda_w \in A$ for $w < a_l\sigma$. To show that $C = C_{a_l\sigma}$, it is sufficient to show that $\lambda_w \in A_{<0}$.

But, $C_{a_l} = T_{a_l} + \sum_{x < a_l} V^{\ell_t(x) - l} \beta_x T_x$ with $\beta_x \in \mathbb{Z}[v, v^{-1}]$ (see [3, Theorem 6.3(a)]). Hence,

$$C = T_{a_l\sigma} + \sum_{\tau < \sigma} p^*_{\tau,\sigma} T_{a_l\tau} + \sum_{x < a_l} V^{\ell_t(x) - l} \beta_x T_x C_\sigma.$$

But, if $x < a_l$, then $\ell_t(x) < l$. This shows that $\lambda_w \in A_{<0}$ for every $w < a_l\sigma$. This shows the first equality. The second one is obtained by a symmetric argument. □

Proposition 2.3 shows that it can be useful to compute in different ways the elements C_{a_l} to be able to relate the Kazhdan–Lusztig basis of $\mathcal{H}_n$ to the Kazhdan–Lusztig basis of $\mathcal{H}(\mathfrak{S}_n)$. Following the work of Dipper, James and Murphy [4], Ariki and Koike [2] and Graham and Lehrer [11, §5], we set

$$\begin{aligned}P_l &= \big(T_{t_1} + V^{-1}\big)\big(T_{t_2} + V^{-1}\big)\cdots\big(T_{t_l} + V^{-1}\big)\\ &= \sum_{0 \leqslant k \leqslant l} V^{k-l}\bigg(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant l} T_{t_{i_1} \ldots t_{i_k}}\bigg).\end{aligned}$$

Lemma 2.4. *P_n is central in $\mathcal{H}_n$.*

Proof. First, P_n commutes with T_t (indeed, $tt_i = t_i t > t_i$ for $1 \leqslant i \leqslant n$). By [2, Lemma 3.3], P_n commutes with T_{s_i} for $1 \leqslant i \leqslant n - 1$. Since the notation and conventions are somewhat different, we recall here a brief proof. First, if $j \notin \{i, i+1\}$, $s_i t_j = t_j s_i > t_j$ so T_{s_i} commutes with T_{t_j}. Therefore, it is sufficient to show that T_{s_i} commutes with $(T_{t_i} + V^{-1})(T_{t_{i+1}} + V^{-1})$. This follows from a straightforward computation using the fact that $s_i t_i > t_i$, that $t_{i+1} s_i < t_{i+1}$ and that $s_i t_i = t_{i+1} s_i$. □

Proposition 2.5. *If $0 \leqslant l \leqslant n$, then $C_{a_l} = P_l T_{\sigma_l}^{-1} = T_{\sigma_l}^{-1} P_l$.*

Proof. The computation may be performed in the subalgebra of $\mathcal{H}_n$ generated by $\{T_t, T_{s_1}, \ldots, T_{s_{l-1}}\}$ so we may, and we will, assume that $l = n$. First, we have $T_t P_n = V P_n$. Since P_n is central in $\mathcal{H}_n$, it follows from the characterization of $\mathcal{C}_n$ given by Proposition 2.2 that $P_n \in \mathcal{C}_n$.

Now, let $h = C_{a_n} - P_n T_{\sigma_n}^{-1}$. Then, by Proposition 2.1(a), we have $h \in \mathcal{C}_n$. Moreover, it is easily checked that $h \in \mathcal{T}_{\leqslant n-1} = \mathcal{C}_{\leqslant n-1}$. So $h = 0$. □

Corollary 2.6. *If $0 \leqslant l \leqslant n$ and $\sigma \in \mathfrak{S}_n$, then $\Pi_0^T(C_{a_l\sigma}) = V^{-l} T_{\sigma_l}^{-1} C_\sigma$. In particular, $\tau(C_{a_l\sigma}) = V^{-l}\tau(T_{\sigma_l}^{-1} C_\sigma)$.*

Proof. Since Π_0^T is a morphism of right $\mathcal{H}(\mathfrak{S}_n)$-modules (see Proposition 2.1(a)) and since $C_{a_l\sigma} = P_l T_{\sigma_l}^{-1} C_\sigma$ (see Propositions 2.3 and 2.5), we have $\Pi_0^T(C_{a_l\sigma}) = \Pi_0^T(P_l) T_{\sigma_l}^{-1} C_\sigma$. But, $\Pi_0^T(P_l) = V^{-l}$. This completes the proof of the corollary. □

3. Two-sided cells

The aim of this section is to show that, if x and y are two elements of W such that $\ell_t(x) = \ell_t(y) = l$, then $x \leqslant_{\mathcal{LR}} y$ if and only if $\sigma_x \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y$ (see Theorem 3.5). Here, $\leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}}$ is the preorder $\leqslant_{\mathcal{LR}}$ defined inside the parabolic subgroup $\mathfrak{S}_{l,n-l}$. For this, we adapt an argument of Geck [7] who was considering the preorder $\leqslant_{\mathcal{L}}$.

We start by defining an order relation $\preccurlyeq$ on W. Let x and y be two elements of W. Then $x \prec y$ if the following conditions are fulfilled:

(1) $\ell_t(x) = \ell_t(y)$,
(2) $x \leqslant y$,
(3) $a_x < a_y$ or $b_x < b_y$,
(4) $\sigma_x \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y$.

We write $x \preccurlyeq y$ if $x \prec y$ or $x = y$. If $y \in W_n$, we set

$$\Gamma_y = T_{a_y} C_{a_{\ell_t(y)}} C_{\sigma_y} T_{b_y^{-1}}.$$

Lemma 3.1. *Let $y \in W$. Then $\Gamma_y \in T_y + \bigoplus_{x \prec y} A_{<0} T_x$.*

Proof. First, Γ_y is a linear combination of elements of the form $T_{a_y} T_z T_{b_y^{-1}}$ with $z \leqslant a_{\ell_t(y)}\sigma_y$, so it is a linear combination of elements of the form T_x with $x \leqslant y$.

Let $l = \ell_t(y)$ and $\sigma = \sigma_y$. We have

$$\Gamma_y = T_{a_y} T_{a_l} T_\sigma T_{b_y^{-1}} + \left(\sum_{\tau < \sigma} p_{\tau,\sigma}^* T_{a_y} T_{a_l} T_\tau T_{b_y^{-1}}\right) + \left(\sum_{a < a_l,\, \tau \leqslant \sigma} p_{a,a_l}^* p_{\tau,\sigma}^* T_{a_y} T_a T_\tau T_{b_y^{-1}}\right).$$

If $\tau < \sigma$, then $T_{a_y} T_{a_l} T_\tau T_{b_y^{-1}} = T_{a_y a_l \tau b_{y^{-1}}}$ by [3, §4.6]. On the other hand, if $a < a_l$, then $\ell_t(a) < l$ so $T_{a_y} T_a T_\tau T_{b_y^{-1}}$ is a linear combination, with coefficients in $\mathbb{Z}[v, v^{-1}]$ of elements T_w with $\ell_t(w) = \ell_t(a) < l$ (because a_y, τ and b_y^{-1} are elements of $\mathfrak{S}_n$). Since $V^{l-\ell_t(a)} p^*_{a,a_l} \in \mathbb{Z}[v, v^{-1}]$ by [3, Theorem 6.3(a)], this proves the lemma. □

Lemma 3.2. *If $y \in W_n$, then*

$$\overline{\Gamma}_y = \Gamma_y + \sum_{x \prec y} \rho_{x,y} \Gamma_x,$$

where the $\rho_{x,y}$'s belong to $\mathbb{Z}[v, v^{-1}]$.

Proof. Let $l = \ell_t(y)$. Then

$$T^{-1}_{a_y^{-1}} = T_{a_y} + \sum_{\substack{a \in Y_{l,n-l} \\ x \in \mathfrak{S}_{l,n-l} \\ ax < a_y}} R_{ax,a_y} T_a T_x.$$

Moreover, if $a \in Y_{l,n-l}$ and $x \in \mathfrak{S}_{l,n-l}$ are such that $ax < a_y$, then $a < a_y$ (see [16, Lemma 9.10(f)]). Thus,

$$\overline{\Gamma}_y = \Gamma_y + \sum_{\substack{a,b \in Y_{l,n-l} \\ x,x' \in \mathfrak{S}_{l,n-l} \\ ax < a_y \text{ or } bx' < b_y}} R_{ax,a_y} R_{bx',b_y} T_a (T_x C_{a_l \sigma_y} T_{x'^{-1}}) T_{b^{-1}}.$$

The result now follows from Lemma 3.1. □

Corollary 3.3. *If $x \preccurlyeq y$, then $\sum_{x \preccurlyeq z \preccurlyeq y} \bar{\rho}_{x,z} \rho_{z,y} = \delta_{x,y}$.*

Proof. This follows immediately from Lemma 3.2 and from the fact that $\mathcal{H} \to \mathcal{H}$, $h \mapsto \bar{h}$ is an involution. □

Corollary 3.4. *If $w \in W$, then*

$$C_w = \Gamma_w + \sum_{y \prec w} \pi^*_{y,w} \Gamma_y,$$

where $\pi^*_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}] \subset A_{<0}$ if $y \prec w$.

Proof. By Corollary 3.3, there exists a unique family $(\pi^*_{y,w})_{y \prec w}$ of elements of $v^{-1}\mathbb{Z}[v^{-1}]$ such that $\Gamma_w + \sum_{y \prec w} \pi^*_{y,w} \Gamma_y$ is stable under the involution $h \mapsto \bar{h}$ of $\mathcal{H}_n$ (see [6, p. 214]: this contains a general setting for including the arguments in [13, Proposition 2] or in [7, Proposition 3.3]).

But, by Lemma 3.1, we have

$$\Gamma_w + \sum_{y \prec w} \pi^*_{y,w} \Gamma_y \in T_w + \left(\bigoplus_{y < w} A_{<0} T_y \right).$$

So $C_w = \Gamma_w + \sum_{y \prec w} \pi^*_{y,w} \Gamma_y$. □

We are now ready to prove the main theorem of this section.

Theorem 3.5. *Let x and y be two elements of W such that $\ell_t(x) = \ell_t(y) = l$. Then $x \leqslant_{\mathcal{LR}} y$ if and only if $\sigma_x \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y$.*

Proof. Assume first that $\sigma_x \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y$. Decompose $\sigma_x = (\sigma'_x, \sigma''_x)$ with $\sigma'_x \in \mathfrak{S}_l$ and $\sigma''_x \in \mathfrak{S}_{n-l}$. Then $\sigma'_x \leqslant_{\mathcal{LR}}^{\mathfrak{S}_l} \sigma'_y$ so $\sigma_l \sigma'_y \leqslant_{\mathcal{LR}}^{\mathfrak{S}_l} \sigma_l \sigma'_x$ so $w_l \sigma_l \sigma'_x \leqslant_{\mathcal{LR}}^{W_l} w_l \sigma_l \sigma'_y$. In other words, $a_l \sigma'_x \leqslant_{\mathcal{LR}}^{W_l} a_l \sigma'_y$. Therefore, $a_l \sigma_x \leqslant_{\mathcal{LR}} a_l \sigma_y$. But, by [3, Theorem 7.7], we have $x \sim_{\mathcal{LR}} a_l \sigma_x$ and $y \sim_{\mathcal{LR}} a_l \sigma_y$. So $x \leqslant_{\mathcal{LR}} y$.

To show the converse statement, it is sufficient to show that

$$I = \left(\bigoplus_{\substack{u \in W_n \\ \ell_t(u) = l \text{ and } \sigma_u \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y}} A C_u \right) \oplus \mathcal{C}_{\geqslant l+1}$$

is a two-sided ideal. But, by Corollary 3.4, we have

$$I = \left(\bigoplus_{\substack{u \in W_n \\ \ell_t(u) = l \text{ and } \sigma_u \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y}} A \Gamma_u \right) \oplus \mathcal{C}_{\geqslant l+1}.$$

By symmetry, we only need to prove that I is a left ideal. Let $h \in \mathcal{H}_n$ and let $u \in W_n$ such that $\ell_t(u) = l$ and $\sigma_u \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y$. We want to prove that $h \Gamma_u \in I$. For simplification, let $a = a_u$, $b = b_u$, $\sigma = \sigma_u$. Let

$$X_l = \{ x \in W_n \mid \forall w \in W_{l,n-l},\ \ell(xw) \geqslant \ell(w) \}.$$

Then, by [7, Proposition 3.3] and [3, Lemma 7.3 and Corollary 7.4],

$$T_a C_{a_l \sigma} \in \bigoplus_{\substack{x \in X_l \\ \tau \leqslant_{\mathcal{L}}^{\mathfrak{S}_{l,n-l}} \sigma}} A C_{x a_l \tau}.$$

Let I' be the right-hand side of the previous formula. By [7, Corollary 3.4], I' is a left ideal. Therefore, $hT_a C_{a_l\sigma} \in I'$. On the other hand,

$$I' \subset \Bigg(\bigoplus_{\substack{x \in Y_{l,n-l} \\ \tau \leqslant_{\mathcal{L}}^{\mathfrak{S}_{l,n-l}} \sigma}} AC_{xa_l\tau} \Bigg) \oplus \mathcal{C}_{\geqslant l+1}.$$

Now, by Corollary 3.4, we have

$$I' \subset \Bigg(\bigoplus_{\substack{x \in Y_{l,n-l} \\ \tau \leqslant_{\mathcal{L}}^{\mathfrak{S}_{l,n-l}} \sigma}} AT_x C_{a_l\tau} \Bigg) \oplus \mathcal{C}_{\geqslant l+1}.$$

Therefore, $h\Gamma_u \in I' T_{b^{-1}} \subset I$, as desired. $\square$

Corollary 3.6. *Let x and y be two elements of W_n. Then $x \sim_{\mathcal{LR}} y$ if and only if $\ell_t(x) = \ell_t(y)$ $(= l)$ and $\sigma_x \sim_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_y$.*

Remark 3.7. We associate to each element $w \in W_n$ a pair $(P(w), Q(w))$ of standard bi-tableaux as in [3, §3]. Let $l = \ell_t(w)$. Write $Q(w) = (Q^+(w), Q^-(w))$ and denote by $\lambda^?(w)$ the shape of $Q^?(w)$ for $? \in \{+, -\}$. The map $w \mapsto (P(w), Q(w))$ is a generalization of the Robinson–Schensted correspondence (see [18, Theorem 3.3] or [3, Theorem 3.3]). Then $\lambda^+(w)$ is a partition of $n - l$ and $\lambda^-(w)$ is a partition of l, so that $\lambda(w) = (\lambda^+(w), \lambda^-(w))$ is a bipartition of n. If we write $\sigma_w = \sigma_w^- \times \sigma_w^+$ with $\sigma_w^- \in \mathfrak{S}_l$ and $\sigma_w^+ \in \mathfrak{S}_{n-l}$, note that $\lambda^+(w)$ is the shape of the standard tableau associated to σ_w^+ by the classical Robinson–Schensted correspondence while $\lambda^-(w)^*$ (the partition conjugate to $\lambda^-(w)$) is the shape of the standard tableau associated to σ_w^-. Let $\trianglelefteqslant$ denote the *dominance order* on partitions: if $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots)$ and $\beta = (\beta_1 \geqslant \beta_2 \geqslant \cdots)$ are two partitions of the same natural number, we write $\alpha \trianglelefteqslant \beta$ if

$$\sum_{j=1}^{i} \alpha_j \leqslant \sum_{j=1}^{i} \beta_j$$

for every $i \geqslant 1$. Now, let x and y be two elements of W_n. If $\ell_t(x) = \ell_t(y)$, then Theorem 3.5 is equivalent to:

$$x \leqslant_{\mathcal{LR}} y \quad \text{if and only if} \quad \lambda^+(x) \trianglelefteqslant \lambda^+(y) \text{ and } \lambda^-(y) \trianglelefteqslant \lambda^-(x). \tag{3.8}$$

This follows from [17, 3.2] and [5, 2.13.1] (see also [10, Exercise 5.6]). Then, for general x and y, Corollary 3.6 is equivalent to:

$$x \sim_{\mathcal{LR}} y \quad \text{if and only if} \quad \lambda(x) = \lambda(y). \tag{3.9}$$

4. Around Lusztig's conjectures

In this section, we prove some results which are related to Lusztig's conjectures. If $\sigma \in \mathfrak{S}_n$, we denote by $\mathbf{a}_{\mathfrak{S}}(\sigma)$ the function $\mathbf{a}$ evaluated on σ but computed in $\mathfrak{S}_n$. It is given by the following formula. Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ be the shape of the left cell of σ. Then

$$\mathbf{a}_{\mathfrak{S}}(\sigma) = \sum_{i \geqslant 1} (i-1)\lambda_i.$$

We denote by $\mathbf{a}_\lambda$ the right-hand side of the previous formula. If $z \in W$, we set

$$\boldsymbol{\alpha}(z) = \big(\ell_t(z), 2\mathbf{a}_{\mathfrak{S}}(\sigma_z) - \mathbf{a}_{\mathfrak{S}}(\sigma_{\ell_t(z)}\sigma_z)\big) \in \mathbb{N}^2.$$

In terms of partitions (using the notation introduced in Remark 3.7), we have

$$\boldsymbol{\alpha}(z) = \big(\big|\lambda^-(z)\big|, \mathbf{a}_{\lambda^+(z)} + 2\mathbf{a}_{\lambda^-(z)^*} - \mathbf{a}_{\lambda^-(z)}\big).$$

We now study some properties of the function $\boldsymbol{\alpha}$.

Remark 4.1. Geck and Iancu [9] have proved, using the result of this section (and especially Proposition 4.2), that $\mathbf{a} = \boldsymbol{\alpha}$. They have deduced, using the notion of *orthogonal representations*, that Lusztig's conjectures P_i hold for $i = \{1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14\}$. After that, Geck [8] proved P_9 and P_{10}.

The first proposition shows that $\boldsymbol{\alpha}$ is decreasing with respect to $\leqslant_{\mathcal{LR}}$ (compare with Lusztig's conjecture P_4).

Proposition 4.2. *Let z and z' be two elements of W. Then*:

(a) *If $z \leqslant_{\mathcal{LR}} z'$, then $\boldsymbol{\alpha}(z') \leqslant \boldsymbol{\alpha}(z)$.*
(b) *If $z \leqslant_{\mathcal{LR}} z'$ and $\boldsymbol{\alpha}(z) = \boldsymbol{\alpha}(z')$ then $z \sim_{\mathcal{LR}} z'$.*

Proof. Since $z \leqslant_{\mathcal{LR}} z'$, we have $\ell_t(z) \geqslant \ell_z(z')$ by [3, Corollary 6.7]. Therefore, if $\ell_t(z) > \ell_t(z')$, then $\boldsymbol{\alpha}(z) > \boldsymbol{\alpha}(z')$ and $z \nsim_{\mathcal{LR}} z'$. This proves (a) and (b) in this case.

So, assume that $\ell_t(z) = \ell_t(z') = l$. Then, by Theorem 3.5, we have $\sigma_z \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \sigma_{z'}$. Write $\sigma_z = (\sigma, \tau)$ and $\sigma_{z'} = (\sigma', \tau')$ where $\sigma, \sigma' \in \mathfrak{S}_l$ and $\tau, \tau' \in \mathfrak{S}_{n-l}$. Then

$$\boldsymbol{\alpha}(z) = \big(l, 2\mathbf{a}_{\mathfrak{S}}(\sigma) - \mathbf{a}_{\mathfrak{S}}(\sigma_l\sigma) + \mathbf{a}_{\mathfrak{S}}(\tau)\big)$$

and

$$\boldsymbol{\alpha}(z') = \big(l, 2\mathbf{a}_{\mathfrak{S}}(\sigma') - \mathbf{a}_{\mathfrak{S}}(\sigma_l\sigma') + \mathbf{a}_{\mathfrak{S}}(\tau')\big).$$

But $\sigma \leqslant_{\mathcal{LR}}^{\mathfrak{S}_l} \sigma'$ and $\tau \leqslant_{\mathcal{LR}}^{\mathfrak{S}_{l,n-l}} \tau'$. Moreover, $\sigma_l\sigma' \leqslant_{\mathcal{LR}}^{\mathfrak{S}_l} \sigma_l\tau'$. Therefore, since Lusztig's conjecture P_4 holds in the symmetric groups, we obtain (a).

If moreover $\boldsymbol{\alpha}(z) = \boldsymbol{\alpha}(z')$, then $\mathbf{a}_{\mathfrak{S}_l}(\sigma) = \mathbf{a}_{\mathfrak{S}_l}(\sigma')$ so $\sigma \sim_{\mathcal{LR}} \sigma'$ by property P_{11} for the symmetric group. Similarly, $\tau \sim_{\mathcal{LR}} \tau'$ so $\sigma_z \sim_{\mathcal{LR}} \sigma_{z'}$. So, by Corollary 3.6, $z \sim_{\mathcal{LR}} z'$. $\square$

The next proposition relates the functions $\boldsymbol{\alpha}$ and Δ.

Proposition 4.3. *Let $z \in W$. Then $\boldsymbol{\alpha}(z) \leqslant \Delta(z)$. Moreover, $\boldsymbol{\alpha}(z) = \Delta(z)$ if and only if $z^2 = 1$.*

Proof. Let us start with two results concerning the degree of $\tau(\Gamma_z)$ for $z \in W_n$:

Lemma 4.4. *Let $z \in W_n$. Then*

$$\tau(\Gamma_z) = \begin{cases} 0, & \text{if } a_z \neq b_z, \\ V^{-\ell_t(z)}\tau(T_{\sigma_{\ell_t(z)}}^{-1} C_{\sigma_z}), & \text{if } a_z = b_z. \end{cases}$$

Proof of Lemma 4.4. Write $l = \ell_t(z)$. Then, $\tau(\Gamma_z) = \tau(\Pi_0^T(\Gamma_z))$. So, by Proposition 2.1(a) and Corollary 2.6, we have $\tau(\Gamma_z) = V^{-l}\tau(T_{a_z}T_{\sigma_l}^{-1}C_{\sigma_z}T_{b_z^{-1}})$. Therefore, $V^l\tau(\Gamma_z)$ is equal to the coefficient of T_{b_z} in $T_{a_z}T_{\sigma_l}^{-1}C_{\sigma_z}$. Write $T_{\sigma_l}^{-1}C_{\sigma_z} = \sum_{x \in \mathfrak{S}_{l,n-l}} \beta_x T_x$. Then $T_{a_z}T_{\sigma_l}^{-1}C_{\sigma_z} = \sum_{x \in \mathfrak{S}_{l,n-l}} \beta_x T_{a_z x}$. Thus, if $a_z \neq b_z$, then $b_z \notin a_z\mathfrak{S}_{l,n-l}$ so $\tau(\Gamma_z) = 0$. If $a_z = b_z$, then $\tau(\Gamma_z) = V^{-l}\beta_1 = V^{-l}\tau(T_{\sigma_l}^{-1}C_{\sigma_z})$. □

Corollary 4.5. *Let $z \in W_n$. Then*:

(a) $\deg\tau(\Gamma_z) \leqslant -\boldsymbol{\alpha}(z)$.
(b) $\deg\tau(\Gamma_z) = -\boldsymbol{\alpha}(z)$ *if and only if z is an involution.*

Proof of Corollary 4.5. This follows from Lemma 4.4 and Corollary 1.6 (recall that Lusztig's conjectures $(P_i)_{1 \leqslant i \leqslant 15}$ hold in the symmetric group). □

Let us now come back to the computation of $\Delta(z)$. By Corollary 3.4, we have

$$\tau(C_z) = \tau(\Gamma_z) + \sum_{y \prec z} \pi_{y,z}^* \tau(\Gamma_y).$$

But, if $y \prec z$, then $\boldsymbol{\alpha}(z) \leqslant \boldsymbol{\alpha}(y)$ (see Proposition 4.2(a)). Therefore, by Corollary 4.5(a), we have $\deg\pi_{y,z}^*\tau(\Gamma_y) < -\boldsymbol{\alpha}(z)$. So $\deg\tau(C_z) \leqslant -\boldsymbol{\alpha}(z)$ and $\deg\tau(C_z) = -\boldsymbol{\alpha}(z)$ if and only if $\deg\tau(\Gamma_z) = -\boldsymbol{\alpha}(z)$ that is, if and only if z is an involution (see Corollary 4.5(b)). □

5. Specialization

We fix now a totally ordered abelian group Γ° and a weight function $L^\circ : W_n \to \Gamma^\circ$ such that $L^\circ(s) > 0$ for every $s \in S_n$. Let $A^\circ = \mathbb{Z}[\Gamma^\circ]$ be denoted exponentially and let $\mathcal{H}_n^\circ = \mathcal{H}(W_n, S_n, L^\circ)$. Let $(T_w^\circ)_{w \in W_n}$ denote the usual A°-basis of $\mathcal{H}_n^\circ$ and let $(C_w^\circ)_{w \in W_n}$ denote the Kazhdan–Lusztig basis of $\mathcal{H}_n^\circ$.

Let $b = L^\circ(t)$ and $a = L^\circ(s_1) = \cdots = L^\circ(s_{n-1})$. Let $\theta_\Gamma : \Gamma \to \Gamma^\circ$, $(r,s) \mapsto ar + bs$. It is a morphism of groups which induces a morphism of $\mathbb{Z}$-algebras $\theta_A : A \to A^\circ$ such that $\theta_A(V) = v^b$ and $\theta_A(v) = v^a$. If $\mathcal{H}_n^\circ$ is viewed as an A-algebra through θ_A, then there is a unique morphism of A-algebras $\theta_{\mathcal{H}} : \mathcal{H}_n \to \mathcal{H}_n^\circ$ such that $\theta_{\mathcal{H}}(T_w) = T_w^\circ$ for every $w \in W_n$. The main result of this section is the following:

Proposition 5.1. *If $b > (n-1)a$, then $\theta_{\mathcal{H}}(C_w) = C_w^\circ$ for every $w \in W_n$.*

Proof. Assume that $b > (n-1)a$. Since $\overline{\theta_{\mathcal{H}}(C_w)} = \theta_{\mathcal{H}}(C_w)$, it is sufficient to show that $\theta_{\mathcal{H}}(C_w) \in T_w^\circ + (\bigoplus_{y<w} A_{<0}^\circ T_y^\circ)$. Since $\theta_A(\pi_{y,w}^*) \in A_{<0}^\circ$ for every $y < w$, it is sufficient to show that $\theta_{\mathcal{H}}(\Gamma_w) \in T_w^\circ + (\bigoplus_{y<w} A_{<0}^\circ T_y^\circ)$. For simplification, we set $l = \ell_t(w)$, $a = a_w$, $b = b_w$ and $\sigma = \sigma_w$. We set $\Gamma'_w = T_a C_{a_l} T_\sigma T_{b^{-1}}$. Then $\Gamma_w = \sum_{\tau \leqslant \sigma} p_{\tau,\sigma}^* \Gamma'_{a a_l \tau b^{-1}}$, with $p_{\tau,\sigma}^* \in v^{-1}\mathbb{Z}[v^{-1}]$ if $\tau < \sigma$ and $p_{\sigma,\sigma}^* = 1$. So it is sufficient to show that $\theta_{\mathcal{H}}(\Gamma'_w) \in T_w^\circ + (\bigoplus_{y<w} A_{<0}^\circ T_y^\circ)$. By Proposition 2.5, we have

$$\begin{aligned}
\Gamma'_w &= T_a P_l T_{\sigma_l}^{-1} T_\sigma T_{b^{-1}} \\
&= \sum_{k=0}^{l} V^{k-l} \Biggl(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l} T_a T_{t_{i_1} t_{i_2} \ldots t_{i_k}} T_{\sigma_l}^{-1} T_{\sigma b^{-1}} \Biggr) \\
&= \sum_{k=0}^{l} V^{k-l} \Biggl(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l} T_{a\alpha(i_1,\ldots,i_k)} T_{a_k} T_{\beta(i_1,\ldots,i_k)} T_{\sigma_l}^{-1} T_{\sigma b^{-1}} \Biggr),
\end{aligned}$$

where $t_{i_1} \ldots t_{i_k} = \alpha(i_1,\ldots,i_k) a_k \beta(i_1,\ldots,i_k)$ with $\alpha(i_1,\ldots,i_k) \in Y_{k,n-k} \cap \mathfrak{S}_l$ and $\beta(i_1,\ldots, i_k) \in \mathfrak{S}_l$. Note that $\alpha(i_1,\ldots,i_k) a_k = r_{i_1} \ldots r_{i_k}$ (recall that r_i is defined as in [3, §4.1]) so that $\ell(\beta(i_1,\ldots,i_k)) = (i_1 - 1) + \cdots + (i_k - 1)$. Now, let $\gamma(i_1,\ldots,i_k) = \sigma_l \beta(i_1,\ldots,i_k)^{-1}$. Then

$$\Gamma'_w = T_w + \sum_{k=0}^{l-1} V^{k-l} \Biggl(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l} T_{a\alpha(i_1,\ldots,i_k)} T_{a_k} T_{\gamma(i_1,\ldots,i_k)}^{-1} T_{\sigma b^{-1}} \Biggr).$$

If $0 \leqslant k \leqslant l-1 \leqslant n-1$, we define

$$Y_{k,l-k,n-l} = \bigl\{ \sigma \in \mathfrak{S}_n \mid \forall i \in \{1,2,\ldots,n-1\} \setminus \{k,l\},\ \sigma s_i > \sigma \bigr\}.$$

Then $Y_{k,l-k,n-l} = Y_{l,n-l}(Y_{k,n-k} \cap \mathfrak{S}_l)$. Therefore, $a\alpha(i_1,\ldots,i_k) \in Y_{k,l-k,n-l}$. But, $Y_{k,l-k,n-l} = Y_{k,n-k}(Y_{l,n-l} \cap \mathfrak{S}_{k,n-k})$. So we can write $a\alpha(i_1,\ldots,i_k) = \alpha_{i_1,\ldots,i_k} \alpha'(i_1,\ldots,i_k)$ with $\alpha_{i_1,\ldots,i_k} \in Y_{k,n-k}$ and $\alpha'(i_1,\ldots,i_k) \in Y_{l,n-l} \cap \mathfrak{S}_{k,n-k}$. Then $\ell(\alpha'(i_1,\ldots,i_k)) \leqslant (l-k)(n-l)$ (indeed, $Y_{l,n-l} \cap \mathfrak{S}_{k,n-k}$ may be identified with the set of minimal length coset representatives of $\mathfrak{S}_{n-k}/\mathfrak{S}_{l-k,n-l}$). Note also that a_k and $\alpha'(i_1,\ldots,i_k)$ commute. So

$$\Gamma'_w = T_w + \sum_{k=0}^{l-1} V^{k-l} \Biggl(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l} T_{\alpha_{i_1,\ldots,i_k} a_k} T_{\alpha'(i_1,\ldots,i_k)} T_{\gamma(i_1,\ldots,i_k)}^{-1} T_{\sigma b^{-1}} \Biggr).$$

If we write $T_u T_v^{-1} T_{\sigma b^{-1}} = \sum_{\tau \in \mathfrak{S}_n} \eta_{u,v,\tau} T_\tau$ with $\eta_{u,v,\tau} \in \mathbb{Z}[v, v^{-1}]$, then, by [16, Lemma 10.4(c)], we have $\deg \eta_{u,v,\tau} \leqslant \ell(u) + \ell(v)$. Moreover,

$$\Gamma'_w = T_w + \sum_{k=0}^{l-1} V^{k-l} \Biggl(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l} \Biggl(\sum_{\tau \in \mathfrak{S}_n} \eta_{\alpha'(i_1,\ldots,i_k),\gamma(i_1,\ldots,i_k),\tau} T_{\alpha_{i_1,\ldots,i_k} a_k \tau} \Biggr) \Biggr).$$

So it is sufficient to show that, for every $k \in \{0, 1, \dots, l-1\}$ and every sequence $1 \leqslant i_1 < \dots < i_k \leqslant l$, we have

$$(k-l)b + \big(\ell\big(\alpha'(i_1, \dots, i_k)\big) + \ell\big(\gamma(i_1, \dots, i_k)\big)\big)a < 0. \tag{$*$}$$

But, $\ell(\alpha'(i_1, \dots, i_k)) \leqslant (l-k)(n-l)$ and

$$\begin{aligned}\ell\big(\gamma(i_1, \dots, i_k)\big) &= \ell(\sigma_l) - \ell\big(\beta(i_1, \dots, i_k)\big) \\ &= \frac{l(l-1)}{2} - (i_1 - 1) - \dots - (i_k - 1) \\ &\leqslant \frac{l(l-1)}{2} - \frac{k(k-1)}{2} \\ &= \frac{1}{2}(l-k)(l+k-1).\end{aligned}$$

So, in order to prove $(*)$, it is sufficient to prove that

$$2(k-l)b + (l-k)\big(2(n-l) + (l+k-1)\big)a < 0. \tag{$**$}$$

But,

$$2(k-l)b + a(l-k)\big(2(n-l) + (l+k-1)\big)a = 2(k-l)\big(b - (n-1)a\big) + (l-k)(k+1-l)a.$$

Since $k - l < 0$, $b - (n-1)a > 0$ and $k + 1 - l \leqslant 0$, we get $(**)$. $\square$

If x and y are two elements of W_n, we write

$$C_x^\circ C_y^\circ = \sum_{z \in W_n} h_{x,y,z}^\circ C_z^\circ,$$

where $h_{x,y,z}^\circ \in A^\circ$. We denote by $\leqslant_{\mathcal{L}}^\circ$, $\leqslant_{\mathcal{R}}^\circ$, $\leqslant_{\mathcal{LR}}^\circ$ the preorders $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$ defined in $\mathcal{H}_n^\circ$. Similarly, we define $\sim_{\mathcal{L}}^\circ$, $\sim_{\mathcal{R}}^\circ$ and $\sim_{\mathcal{LR}}^\circ$.

Corollary 5.2. *Assume that $b > (n-1)a$. Let x, y and z be elements of W_n and let $? \in \{\mathcal{L}, \mathcal{R}, \mathcal{LR}\}$. Then*:

(a) $h_{x,y,z}^\circ = \theta_A(h_{x,y,z})$.
(b) *If $x \leqslant_?^\circ y$, then $x \leqslant_? y$.*
(c) *$x \sim_?^\circ y$ if and only if $x \sim_? y$.*

Proof. (a) follows from Proposition 5.1. (b) follows from (a). (c) follows from (b) and from the counting argument in the proof of [3, Theorem 7.7]. $\square$

Let $\tau^\circ : \mathcal{H}_n^\circ \to A^\circ$ denote the canonical symmetrizing form. If $z \in W_n$, we set

$$\mathbf{a}^\circ(z) = \max_{x,y \in W_n} \deg h^\circ_{x,y,z},$$
$$\Delta^\circ(z) = -\deg \tau^\circ\big(C^\circ_z\big)$$

and

$$\boldsymbol{\alpha}^\circ(z) = \theta_\Gamma\big(\boldsymbol{\alpha}(z)\big).$$

By Corollary 5.2(b) and by the same argument as in the proof of Proposition 4.2, we have, for every $z, z' \in W_n$ such that $\ell_t(z) = \ell_t(z')$ and $z \leqslant^\circ_{\mathcal{LR}} z'$,

$$\boldsymbol{\alpha}^\circ(z') \leqslant \boldsymbol{\alpha}^\circ(z'). \tag{5.3}$$

Remark. Using the result of this section, Geck and Iancu [9] proved that $\mathbf{a}^\circ = \boldsymbol{\alpha}^\circ$ whenever $b > (n-1)a$.

Proposition 5.4. *Assume that $b > (n-1)a$. Let $z \in W_n$. Then*:

(a) $\Delta^\circ(z) = \theta_\Gamma(\Delta(z)) \geqslant \boldsymbol{\alpha}^\circ(z)$.
(b) $\Delta^\circ(z) = \boldsymbol{\alpha}^\circ(z)$ *if and only if* $z^2 = 1$.

Proof. First, note that $\tau^\circ \circ \theta_{\mathcal{H}} = \theta_{\mathcal{H}} \circ \tau$. Moreover, by Proposition 5.1, we have $\theta_{\mathcal{H}}(C_z) = C^\circ_z$. Since $V^{\ell_t(z)}\tau(C_z) \in \mathbb{Z}[v, v^{-1}]$, we get that $\Delta^\circ(z) = \theta_\Gamma(\Delta(z))$. The other assertions follow easily. □

We conclude this section by showing that the bound given by Proposition 5.1 is optimal.

Proposition 5.5. *If $b \leqslant (n-1)a$, there exists $w \in W_n$ such that $\theta_{\mathcal{H}}(C_w) \neq C^\circ_w$.*

Proof. Assume that $b \leqslant (n-1)a$. To prove the proposition, it is sufficient to show that there exists $w \in W_n$ such that $\theta_{\mathcal{H}}(C_w) \notin T^\circ_w + \bigoplus_{y<w} A^\circ_{<0} T^\circ_y$. Using Corollary 3.4, we see that it is sufficient to show that there exists $w \in W_n$ such that $\theta_{\mathcal{H}}(\Gamma_w) \notin T^\circ_w + \bigoplus_{y<w} A^\circ_{<0} T^\circ_y$. This follows from the next lemma:

Lemma 5.6. *Let $w = s_{n-1} \cdots s_2 s_1 t \sigma_n$. Then $\theta_{\mathcal{H}}(\Gamma_w) \notin T^\circ_w + \bigoplus_{y<w} A^\circ_{<0} T^\circ_y$.*

Proof. We have, by Proposition 2.3,

$$\Gamma_w = T_{s_{n-1} \cdots s_2 s_1 t} C_{\sigma_n} + V^{-1} T_{s_{n-1} \cdots s_2 s_1} C_{\sigma_n}.$$

But, $T_{s_{n-1} \cdots s_2 s_1} C_{\sigma_n} = v^{n-1} C_{\sigma_n}$ (see [16, Theorem 6.6(b)]). Therefore, since $\theta_{\mathcal{H}}(C_\sigma) = C^\circ_\sigma$ for every $\sigma \in \mathfrak{S}_n$, we have

$$\theta_{\mathcal{H}}(\Gamma_w) = \left(\sum_{\tau \in \mathfrak{S}_n} v^{(\ell(\tau) - \ell(\sigma_n))a} T^\circ_{s_{n-1} \cdots s_2 s_1 t \tau} \right) + v^{-b+(n-1)a} C^\circ_{\sigma_n}.$$

(Recall that $C_{\sigma_n} = \sum_{\tau \in \mathfrak{S}_n} v^{(\ell(\tau)-\ell(\sigma_n))} T_\tau$ by [16, Corollary 12.2].) So the coefficient of $\theta_{\mathcal{H}}(\Gamma_w)$ on $T^\circ_{\sigma_n}$ is equal to $v^{-b+(n-1)a}$, which does not belong to $A^\circ_{<0}$. $\square$

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