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ON THE PROPERTY OF KELLEY IN THE HYPERSPACE AND WHITNEY CONTINUA

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In this paper, we introduce the notion of property $[K]^*$ which implies property $[K]$, and we show the following: Let X be a continuum and let ω be any Whitney map for $C(X)$. Then the following are equivalent. (1) X has property $[K]^*$. (2) $C(X)$ has property $[K]^*$. (3) The Whitney continuum $\omega^{-1}(t)$ $(0 \leq t < \omega(X))$ has property $[K]^*$.

As a corollary, we obtain that if a continuum X has property $[K]^*$, then $C(X)$ has property $[K]$ and each Whitney continuum in $C(X)$ has property $[K]$. These are partial answers to Nadler's question and Wardle's question ([10, (16.37)] and [11, p. 295]).

Also, we show that if each continuum X_n $(n = 1, 2, 3, \ldots)$ has property $[K]^*$, then the product $\prod X_n$ has property $[K]^*$, hence $C(\prod X_n)$ and each Whitney continuum have property $[K]^*$. It is known that there exists a curve X such that X has property $[K]$, but $X \times X$ does not have property $[K]$ (see [11]).

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property $[K]$ eqi-LC^0

Introduction

By a *continuum* we mean a compact connected metric space. For given continuum X with metric d, by the *hyperspaces* of X we mean

$$2^X = \{A \subset X \mid A \text{ is nonempty closed subset of } X\}$$

and

$$C(X) = \{A \in 2^X \mid A \text{ is connected}\},$$

which have the *Hausdorff metric* d_H defined by $d_H(A, B) = \inf\{\varepsilon > 0 \mid B \subset U(A; \varepsilon)$ and $A \subset U(B; \varepsilon)\}$, where $A, B \in 2^X$ and $U(A; \varepsilon)$ is the ε-neighborhood of A in X. A mapping $\omega : C(X) \to [0, \omega(X)]$ is called a *Whitney map* for $C(X)$ provided that the following conditions are satisfied:

(1) $\omega(\{x\}) = 0$ for each $x \in X$, and

(2) if $A, B \in C(X)$, $A \subset B$ and $A \neq B$, then $\omega(A) < \omega(B)$.

In [12], Whitney showed that there always exists a Whitney map on any continuum. Then $\omega^{-1}(t)$ $(0 \leq t < \omega(X))$ is called a *Whitney continuum*. Let $a \in A \in C(X)$. We shall say that X has *property* $[K]$ with respect to (A, a) if X satisfies the following

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condition: given any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $b \in X$ and $d(a, b) < \delta(\varepsilon)$, then there exists $B \in C(X)$ such that $b \in B$ and $d_H(A, B) < \varepsilon$.

If X has property [K] with respect to each (A, a), we say that X has *property* [K].

The notion of property [K] is important in hyperspaces theory (see the references).

In [10, (16.37), p. 558], Nadler had the following questions:

(a) If a continuum X has property [K], then does 2^X have property [K]?

(b) If a continuum X has property [K], then does $C(X)$ have property [K]?

(c) If 2^X has property [K], then does $C(X)$ have property [K]?

(d) If $C(X)$ has property [K], then does 2^X have property [K]?

In [2], W.J. Charatonik showed that there is a curve X such that X and $C(X)$ have property [K], while 2^X does not have property [K]. Hence (a) and (d) are answered in the negative. Thus only two questions (b) and (c) have remained open in this area. Also, it is important to note that property [K] is not closed concerning the operation of product, i.e., there is a continuum X such that X has property [K], but $X \times X$ does not have property [K] (see [11]). In fact, the curve in [2] is the same as in the curve in [11]. This curve is described in Example 3.6.

For a given concrete continuum Y, it is complicated to determine whether $C(Y)$ has property [K] or not. For example, in [2, p. 458], Charatonik used the following fact without proof: if X is the curve in [2], then the cone of X has property [K]. But it seems that the proof is not easy. In fact, the author does not know whether $Y \times [0, 1]$ and the cone of Y have property [K] or not, when Y has property [K].

In this paper, we introduce the notion of property [K]* and we show the following: Let X be a continuum and let ω be any Whitney map for $C(X)$. Then the following are equivalent.

(1) X has property [K]*.

(2) $C(X)$ has property [K]*.

(3) The Whitney continuum $\omega^{-1}(t)$ $(0 \leq t < \omega(X))$ has property [K]*.

As a corollary, we have that if X has property [K]*, then $C(X)$ and each Whitney continuum $\omega^{-1}(t)$ have property [K]. These are partial answers to Nadler's question [10, (16.37)] and Wardle's question [11, p. 295]. Also, we show the following:

Let X_n be a continuum $(n = 1, 2, \ldots)$. Then the following are equivalent.

(1) Each X_n has property [K]*.

(2) The product $\prod X_n$ has property [K]*.

This implies that property [K] is not equal to property [K]*. By definitions, we can easily see that property [K]* implies property [K] (see (1.1)).

1. Property [*K*]*

Let X be a continuum and let $a \in A \in C(X)$. A finite sequence $p_1, p_2, \ldots, p_m$ of points of X is called an *ε-chain* $(\varepsilon > 0)$ if $d(p_i, p_{i+1}) < \varepsilon$ for each i. Let $\mathscr{A}$ be a finite open covering of A. A finite sequence $U_1, U_2, \ldots, U_m$ of $\mathscr{A}$ is called a *chain* if

$U_i \cap U_{i+1} \neq \emptyset$ for each $i = 1, 2, \ldots, m-1$. For each $x \in A$, we consider the set $\mathrm{Chain}_{ax}(\mathscr{A})$ of all finite chains $\langle U_1, U_2, \ldots, U_m \rangle$ of $\mathscr{A}$ with $a \in U_1$ and $x \in U_m$.

Let $\varepsilon > 0$. Consider the following conditions $(A, a, \varepsilon)^*$: there exists $\delta(\varepsilon) > 0$ such that if $b \in U(a, \delta(\varepsilon))$, then for each $x \in A$ and $\zeta > 0$ there is a finite open covering $\mathscr{A}$ of A with mesh $\mathscr{A} < \varepsilon$ such that if $\langle U_1, U_2, \ldots, U_m \rangle \in \mathrm{Chain}_{ax}(\mathscr{A})$, then there is a ζ-chain $b = b_1^1, b_1^2, \ldots, b_1^{i(1)}, b_2^1, b_2^2, \ldots, b_2^{i(2)}, \ldots, b_m^1, b_m^2, \ldots, b_m^{i(m)}$ of points of X such that $d_H(b_j^k, \mathrm{Cl}\, U_j) < \varepsilon$ for each $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, i(j)$. We say X has *property* $[K]^*$ with respect to (A, a) if X satisfies the condition $(A, a, \varepsilon)^*$ for each $\varepsilon > 0$. Also, we say X has *property* $[K]^*$ if X has property $[K]^*$ with respect to each (A, a) $(a \in A \in C(X))$.

Proposition 1.1. *If a continuum X has property $[K]^*$ with respect to (A, a), then X has property $[K]$ with respect to (A, a), where $a \in A \in C(X)$. Hence, if X has property $[K]^*$, then X has property $[K]$.*

Remark 1.2. In Proposition 1.1, the converse assertion is not true. The curve X in [2] or [11] has property $[K]$, but not property $[K]^*$ (see Example 3.6).

A continuum X is *equi-homogeneous* with respect to mappings if for any $\varepsilon > 0$ there is $\delta > 0$ such that if $a, b \in X$ and $d(a, b) < \delta$, then there is a mapping $f: X \to X$ such that $f(a) = b$ and $d(f(x), x) < \varepsilon$ for each $x \in X$. Clearly, if X is equi-homogeneous with respect to mappings, X has property $[K]^*$. By the theorem of Effros [13, (2.1)], we can easily see that each homogeneous continuum has property $[K]^*$. Also, each Peano continuum has property $[K]^*$. Note that the $\sin(1/x)$-curve has property $[K]^*$.

We refer readers to [5] and [10] for hyperspaces theory.

2. Property $[K]^*$ and equi-LC^0 mappings

A mapping $f: X \to Y$ between metric spaces is *equi*-LC^0 provided that f is surjective and for given $x \in X$ and a neighborhood V of x in X there is a neighborhood U of x in V such that if $a, b \in f^{-1}(y) \cap U$ for some $y \in Y$, then there is a path $\alpha(a, b)$ from a to b in $f^{-1}(y) \cap V$.

First, we show the following:

Theorem 2.1. *Let $f: X \to Y$ be an open mapping between continua. Suppose that f is* equi-LC^0. *Let $a \in A \in C(X)$. If Y has property $[K]^*$ with respect to $(f(A), f(a))$, then X has property $[K]^*$ with respect to (A, a).*

Proof. Let $\varepsilon > 0$. Since f is equi-LC^0, there is $\varepsilon' > 0$ $(7\varepsilon' < \varepsilon)$ such that if $x, x' \in f^{-1}(y)$ for some $y \in Y$ and $d(x, x') < 7\varepsilon'$, then there is a path $\alpha(x, x')$ from x to x' in $f^{-1}(y)$ such that $\mathrm{diam}\, \alpha(x, x') < \frac{1}{2}\varepsilon$.

For each $x \in A$, define the following: set Z_x by $Z_x = U(x, \varepsilon') \cap A$. Since A is compact, there are points $x_1, x_2, \ldots, x_n$ of A such that $\bigcup_{i=1}^n U(x_i, \varepsilon') \supset A$. Since f is an open mapping, there is $\gamma > 0$ such that if y, $y' \in Y$ and $d(y, y') < \gamma$, then $d_H(f^{-1}(y), f^{-1}(y')) < \varepsilon'$.

Since Y has property $[K]^*$ with respect to $(f(A), f(a))$, there is $\delta(\gamma) > 0$ satisfying the condition $(f(A), f(a), \gamma)^*$. Take $\delta > 0$ $(\delta < \varepsilon')$ such that $f(U(a, \delta)) \subset U(f(a), \delta(\gamma))$. Suppose that $b \in U(a, \delta)$. Let $x \in A$ and let $\zeta > 0$. Choose $\zeta' > 0$ such that if y, $y' \in Y$ and $d(y, y') < \zeta'$, then $d_H(f^{-1}(y), f^{-1}(y')) < \zeta$. Choose a finite open covering $\mathscr{A}$ of $f(A)$ with mesh $\mathscr{A} < \gamma$ such that if $\langle U_1, U_2, \ldots, U_m \rangle \in \mathrm{Chain}_{f(a)f(x)}(\mathscr{A})$, then there is a ζ'-chain $f(b) = y_1^1, y_1^2, \ldots, y_1^{i(1)}, \ldots, y_m^1, \ldots, y_m^{i(m)}$ of points of Y satisfying that $d_H(y_j^k, \mathrm{Cl}\, U_j) < \gamma$ for each j, k. Consider the set $\mathscr{B} = \{D \mid D = f^{-1}(U) \cap Z_{x_i} \neq \emptyset$ for $i = 1, 2, \ldots, n$ and $U \in \mathscr{A}\}$. Then $\mathscr{B}$ is a finite open covering of A with mesh $\mathscr{B} < \varepsilon$.

Suppose that $\langle D_1, D_2, \ldots, D_m \rangle \in \mathrm{Chain}_{ax}(\mathscr{B})$. Set $D_i = f^{-1}(U_i) \cap Z_{x_{n(i)}}$ $(i = 1, 2, \ldots, m)$. Then $U_i \cap U_{i+1} \neq \emptyset$ $(i = 1, 2, \ldots, m-1)$, hence $\langle U_1, U_2, \ldots, U_m \rangle \in \mathrm{Chain}_{f(a)f(x)}(\mathscr{A})$. Thus there is a ζ'-chain $f(b) = y_1^1$, $y_1^2, \ldots, y_1^{i(1)}, \ldots, y_m^1$, $y_m^2, \ldots, y_m^{i(m)}$ of points of Y such that $d_H(y_j^k, \mathrm{Cl}\, U_j) < \gamma$ for each $j = 1, 2, \ldots, m$. By the choice of γ, we have a finite sequence $b = p_1^1, p_1^2, \ldots, p_1^{i(1)}, \ldots, p_m^1, p_m^2, \ldots, p_m^{i(m)}$ of points of X such that each p_j^k belongs to $f^{-1}(y_j^k)$ and $d(x_{n(j)}, p_j^k) < 2\varepsilon'$. Let C_j^k be the component of $\mathrm{Cl}\, U(p_j^k; \frac{1}{2}\varepsilon) \cap f^{-1}(y_j^k)$ which contains p_j^k. Note that $b \in C_1^1$, $C_j^k \subset U(x_{n(j)}; \varepsilon)$, $\mathrm{Cl}\, U(p_j^k; 7\varepsilon') \cap f^{-1}(y_j^k) \subset C_j^k$ and $d(p_j^k, p_j^{k+1}) < 4\varepsilon'$ and

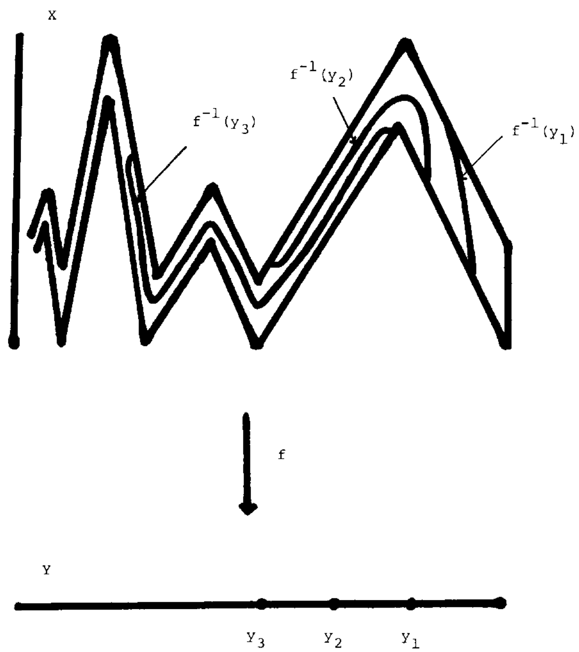

Fig. 1

$d(p_j^{i(j)}, p_{j+1}^1) < 6\varepsilon'$. Since $d_H(f^{-1}(y_j^k), f^{-1}(y_j^{k+1})) < \zeta$ and $d_H(f^{-1}(y_j^{i(j)}), f^{-1}(y_{j+1}^1)) < \zeta$, we can choose a point q_j^k of C_j^k such that $d(p_j^k, q_j^{k+1}) < \zeta$ and $d(p_j^{i(j)}, q_{j+1}^1) < \zeta$. Since each C_j^k is connected, we can easily choose a ζ-chain $b = b_1^1, b_1^2, \ldots, b_1^{i'(1)}, \ldots, b_m^1, \ldots, b_m^{i'(m)}$ of points of X such that $d_H(b_j^k, \mathrm{Cl}\, D_j) < \varepsilon$ for each $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, i'(j)$. This implies that X satisfies the condition $(A, a, \varepsilon)^*$. Hence X has property $[K]^*$ with respect to (A, a). □

Example 2.2. In the statement of Theorem 2.1, we cannot omit the condition that f is equi-LC^0. Let X be the two-dimensional continuum as below (see Fig. 1) and $Y = [0, 1]$. Then there is an open mapping $f: X \to Y$ such that $f^{-1}(y)$ is an arc for each $y \in Y$. Since Y is a Peano continuum, Y has property $[K]^*$, but X does not have property $[K]$.

Theorem 2.3. *Let $f: X' \to X$ be an open monotone mapping between continua. If X' has property $[K]^*$, then X has property $[K]^*$.*

Proof. Let $a \in A \in C(X)$ and let $\varepsilon > 0$. We shall show that X satisfies the condition $(A, a, \varepsilon)^*$. Set $A' = f^{-1}(A)$ and choose a point $a' \in f^{-1}(a)$. Take a positive number ε' such that if $x_1, x_2 \in X'$ and $d(x_1, x_2) < \varepsilon'$, then $d(f(x_1), f(x_2)) < \varepsilon$. Since X' has property $[K]^*$, we can choose a neighborhood U' of a' in X' satisfying the condition $(A', a', \varepsilon')^*$. Set $U = f(U')$. Then U is a neighborhood of a in X. Suppose that $b \in U$. Let $x \in A$ and let $\zeta > 0$. Choose $\zeta' > 0$ such that if $x', x'' \in X'$ and $d(x', x'') < \zeta'$, then $d(f(x'), f(x'')) < \zeta$. Choose a point $b' \in f^{-1}(b) \cap U'$. Let $x' \in f^{-1}(x)$. Then there is a finite open covering $\mathscr{A}'$ of A' with mesh $\mathscr{A}' < \varepsilon'$ such that if $\langle U_1', U_2', \ldots, U_m' \rangle \in \mathrm{Chain}_{a'x'}(\mathscr{A}')$, there is a ζ'-chain $b_1^1, b_1^2, \ldots, b_1^{i(1)}, \ldots, b_m^1, \ldots, b_m^{i(m)}$ of points of X' such that $d_H(b_j^k, \mathrm{Cl}\, U_j') < \varepsilon'$.

For each $y \in A$, choose an open neighborhood U_y of y in A such that $f^{-1}(U_y) \subset \bigcup\{U' \mid U' \cap f^{-1}(y) \neq \emptyset, U' \in \mathscr{A}'\}$ and $U_y \subset \bigcap\{f(U') \mid U' \cap f^{-1}(y) \neq \emptyset, U' \in \mathscr{A}'\}$. Since A is compact, there are points $y_1, y_2, \ldots, y_n$ of A such that $\bigcup_{i=1}^n U_{y_i} = A$. Set $\mathscr{A} = \{U \mid U = U_{y_i}\ (i = 1, 2, \ldots, n)\}$. Suppose that $\langle U_1, U_2, \ldots, U_m \rangle \in \mathrm{Chain}_{ax}(\mathscr{A})$. By the choice of U_{y_i}, we can see that there is a chain $\langle U_1', U_2', \ldots, U_m' \rangle$ of $\mathscr{A}'$ such that $a' \in U_1'$ and $f(U_j') \supset U_j$. Since $f^{-1}(x)$ is a continuum, there is a chain $\langle U_m', \ldots, U_r' \rangle$ of $\mathscr{A}'$ such that $U_k' \cap f^{-1}(x) \neq \emptyset$ $(k \geq m)$ and $x' \in U_r'$. Then $\langle U_1', U_2', \ldots, U_m', \ldots, U_r' \rangle \in \mathrm{Chain}_{a'x'}(\mathscr{A}')$. Hence there is a ζ'-chain $b' = b_1^1, b_1^2, \ldots, b_1^{i(1)}, \ldots, b_r^1, \ldots, b_r^{i(r)}$ of points of X' such that $d_H(b_j^k, \mathrm{Cl}\, U_j') < \varepsilon'$. Then $b = f(b_1^1), f(b_1^2), \ldots, f(b_1^{i(1)}), \ldots, f(b_m^1), \ldots, f(b_m^{i(m)})$ is a ζ-chain of points of X such that $d_H(f(b_j^k), \mathrm{Cl}\, U_j) < \varepsilon$. This implies that X satisfies the condition $(A, a, \varepsilon)^*$. Hence X has property $[K]^*$. □

Now, we need the following:

2.4 (cf., [6, (2.3)]). *Let X be a continuum and let ω be any Whitney map for $C(X)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $A, B \in C(X)$, $|\omega(A) - \omega(B)| < \delta$ and $B \subset U(A, \delta)$, then $d_H(A, B) < \varepsilon$.*

2.5 [8, Lemma 1]. *Let X be a continuum and let* ω *be any Whitney map for* $C(X)$. *Let* $0<t<\omega(X)$. *If* A, $B\in\omega^{-1}(t)$ *such that* $A\cap B\neq\emptyset$, *then there is a path* α *in* $[\omega^{-1}(t)\cap C(A\cup B)]$ *such that end points of* α *are A and B.*

Theorem 2.6. *Let X be a continuum and let* ω *be any Whitney map for* $C(X)$. *Then the following are equivalent.*

(1) *X has property* $[K]^*$.

(2) $C(X)$ *has property* $[K]^*$.

(3) *Each Whitney continuum* $\omega^{-1}(t)$ $(0\leq t<\omega(X))$ *has property* $[K]^*$.

Proof. First, we shall show that (1) implies (2). Consider the following subset of $X\times C(X)$:

$$Z=\{(x,A)\,|\,x\in X, A\in C(X) \text{ and } x\in A\}.$$

Then Z is a continuum.

Let $p:Z\to X$ be the projection, i.e., $p((x,A))=x$. By Wardle's result [11, (2.2)], p is an open mapping. Also, By 2.4 and 2.5, we can see that p is equi-LC^0. By Theorem 2.1, Z has property $[K]^*$. Let $q:Z\to C(X)$ be the projection, i.e., $q((x,A))=A$. Since $q^{-1}(A)=\{(x,A)\,|\,x\in A\}$ for each $A\in C(X)$, we can easily see that q is an open monotone mapping. By Theorem 2.3, $C(X)$ has property $[K]^*$.

Next, we shall show that (1) implies (3). Consider the following set in $X\times\omega^{-1}(t)$:

$$Z'=\{(x,A)\,|\,x\in X, A\in\omega^{-1}(t) \text{ and } x\in A\}.$$

Then Z' is a continuum. By the similar arguments as above, we see that $\omega^{-1}(t)$ has property $[K]^*$.

Clearly, (3) implies (1).

The remainder of the proof follows from the next lemma.

Lemma 2.7. *Let X be a continuum. If* $C(X)$ *has property* $[K]^*$, *then X has property* $[K]^*$.

Proof. Let $a\in A\in C(X)$ and let $\varepsilon>0$. Since $C(X)$ has property $[K]^*$, there is a neighborhood $\mathscr{U}$ of $\{a\}$ in $C(X)$ satisfying the condition $(A,\{a\},\varepsilon)^*$ $(A\subset X\subset C(X))$. Set $U=\mathscr{U}\cap X$. Suppose that $b\in U$. Let $\zeta>0$ and let $x\in A$. Then there is a finite open covering $\mathscr{A}$ of A with mesh $\mathscr{A}<\varepsilon$ such that if $\langle U_1,U_2,\ldots,U_m\rangle\in \text{Chain}_{ax}(\mathscr{A})$, then there is a ζ-chain $\{b\}=B_1^1, B_1^2,\ldots,B_1^{i(1)},\ldots,B_m^1,\ldots,B_m^{i(m)}$ of points of $C(X)$ such that $d_H(B_j^k,\text{Cl}\,U_j)<\varepsilon$. Then there is a ζ-chain $b=b_1^1,\ldots,b_1^{i(1)},\ldots,b_m^1,\ldots,b_m^{i(m)}$ of points of X such that $b_j^k\in B_j^k$. Note that $d_H(b_j^k,\text{Cl}\,U_j)<\varepsilon$. Hence X satisfies the condition $(A,a,\varepsilon)^*$. □

In relation to (2.6), we have the following proposition.

Proposition 2.8 (cf., [11, (2.8)]). *Let X be a continuum and let* ω *be any Whitney map for* $C(X)$. *If* $C(X)$ *has property* $[K]$, *then* $\omega^{-1}(t)$ *has property* $[K]$ *for each* $0\leq t\leq\omega(X)$.

Proof. Let $A \in \mathcal{A} \in C(\omega^{-1}(t))$ and let $\varepsilon > 0$. Since $C(X)$ has property $[K]$, there is $\delta > 0$ such that if $B \in C(X)$ and $d_H(A, B) < \delta$, then there is $\mathcal{B} \in C(C(X))$ such that $B \in \mathcal{B}$ and $d_H(\mathcal{A}, \mathcal{B}) < \varepsilon$. Let $B \in \omega^{-1}(t)$ such that $d_H(A, B) < \delta$. Then there is a continuum $\mathcal{B}' \in C(C(X))$ such that $B \in \mathcal{B}'$, $d_H(\mathcal{A}, \mathcal{B}') < \varepsilon$. By [11, (2.8)], X has property $[K]$. By [5], there is a retraction $r: C(X) \to \omega^{-1}([t, \omega(X)]$ such that $D \subset r(D)$ for each $D \in C(X)$. By 2.4, we may assume that $d_H(\mathcal{A}, r^*(\mathcal{B}')) < \varepsilon$, where $r^*: C(C(X)) \to C(C(X))$ is the mapping induced by r. Then $r^*(\mathcal{B}') \subset \omega^{-1}([t, \omega(X)])$. Consider the following subset Z of $r^*(\mathcal{B}') \times \omega^{-1}(t)$:

$$Z = \{(D, D'): D \in \gamma^*(\mathcal{B}'), D' \in \omega^{-1}(t) \cap C(D)\}.$$

Then Z is a compactum. Let $p: Z \to r^*(\mathcal{B}')$ be the projection. Since p is a monotone mapping, Z is a continuum. Let $q: Z \to \omega^{-1}(t)$ be the projection. Then $\mathcal{B} = q(Z)$ is a continuum contained in $\omega^{-1}(t)$ such that $B \in \mathcal{B}$. By 2.4, we may assume that $d_H(\mathcal{A}, \mathcal{B}) < \varepsilon$. This completes the proof. □

As a corollary of Theorem 2.6, we have partial answers to Nadler's question and Wardle's question ([10, (16.37)] and [11, p. 295]).

Corollary 2.9. *If a continuum X has property $[K]^*$, then $C(X)$ and $\omega^{-1}(t)$ have property $[K]$, where ω is any Whitney map for $C(X)$ and $0 < t < \omega(X)$.*

3. Property [K]* and product

In this section, we show that if each continuum X_n has property $[K]^*$ $(n = 1, 2, \ldots)$, then the product $\prod X_n$ has property $[K]^*$. Hence $C(\prod X_n)$ and the Whitney continuum have property $[K]^*$ (see Theorem 2.6).

Lemma 3.1. *If X_1 and X_2 are continua which have property $[K]^*$, then $X_1 \times X_2$ has property $[K]^*$.*

Proof. Let d be the metric on $X_1 \times X_2$ defined by $d((x, y), (x', y')) = d_1(x, x') + d_2(y, y')$, where d_i denotes a metric on X_i $(i = 1, 2)$.

Let $a \in A \in C(X_1 \times X_2)$ and let $\varepsilon > 0$. We shall show that $X_1 \times X_2$ satisfies the condition $(A, a, \varepsilon)^*$. Let $p_i: X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection and let $a_i = p_i(a)$ and $A_i = p_i(A)$. Then $a_i \in A_i \in C(X_i)$ $(i = 1, 2)$. Since each X_i has property $[K]^*$, there is $\delta_i = \delta_i(\frac{1}{2}\varepsilon) > 0$ satisfying the condition $(A_i, a_i, \frac{1}{2}\varepsilon)^*$. Set $\delta = \min\{\delta_1, \delta_2\}$. Consider the neighborhood $U(a, \delta)$ of a in $X_1 \times X_2$. Suppose $b = (b_1, b_2) \in U(a, \delta)$. Let ζ be any positive number and let $x = (x_1, x_2) \in A$. Since $b_i \in U(a_i, \delta_i)$, there is a finite open covering $\mathcal{A}_i$ of A_i with mesh $\mathcal{A}_i < \frac{1}{2}\varepsilon$ satisfying the condition $(A_i, a_i, \frac{1}{2}\varepsilon)^*$. Consider the following set $\mathcal{A} = \{U \mid U = (V \times W) \cap A \neq \emptyset,\ V \in \mathcal{A}_1,\ W \in \mathcal{A}_2\}$. Then mesh $\mathcal{A} < \varepsilon$. Let $\langle U_1, U_2, \ldots, U_m \rangle \in \mathrm{Chain}_{ax}(\mathcal{A})$. Set $U_i = (V_i \times W_i) \cap A$. Then $\langle V_1, V_2, \ldots, V_m \rangle \in \mathrm{Chain}_{a_1x_1}(\mathcal{A}_1)$ and $\langle W_1, W_2, \ldots, W_m \rangle \in \mathrm{Chain}_{a_2x_2}(\mathcal{A}_2)$. We may assume that there are $\zeta/2$-chain $a_1 = p_1^1, p_1^2, \ldots, p_1^{i(1)}, \ldots, p_m^1, p_m^2, \ldots, p_m^{i(m)}$ of points of X_1 and $b_2 = q_1^1, q_1^2, \ldots, q_1^{i'(1)}, \ldots, q_m^1, \ldots, q_m^{i'(m)}$ of points of X_2 such that $d_H(p_j^k, \mathrm{Cl}\, V_j) < \frac{1}{2}\varepsilon$ and $d_H(q_j^{k'}, \mathrm{Cl}\, W_j) < \frac{1}{2}\varepsilon$ for each $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, i(m)$ and $k' = 1, 2, \ldots, i'(m)$. We may assume that $i(j) = i'(j)$ for each $j = 1, 2, \ldots, m$. Set $b_j^k = (p_j^k, q_j^k)$. Then $b = b_1^1, b_1^2, \ldots, b_1^{i(1)}, \ldots, b_m^1, \ldots, b_m^{i(m)}$ is a ζ-chain of points of $X_1 \times X_2$ such that $d_H(b_j^k, \mathrm{Cl}\, U_j) < \varepsilon$. This implies that $X_1 \times X_2$ satisfies the condition $(A, a, \varepsilon)^*$. Hence $X_1 \times X_2$ has property $[K]^*$. □

Theorem 3.2. *Let X_n be a continuum $(n = 1, 2, \ldots)$. Then the product $\prod X_n$ has property $[K]^*$ if and only if each X_n has property $[K]^*$.*

Proof. Suppose that each X_n has property $[K]^*$. Let d_n be the metric on X_n such that $\mathrm{diam}\, X_n < 1$ and let d be the metric on $\prod X_n$ defined by $d((x_n), (y_n)) = \sum_{n=1}^{\infty} d(x_n, y_n)/2^n$. Let $a \in A \in C(\prod X_n)$ and let $\varepsilon > 0$. Choose a natural number k such that $\mathrm{diam}((x_1, x_2, \ldots, x_k) \times \prod_{j=k+1}^{\infty} X_j) < \frac{1}{2}\varepsilon$ for each $(x_1, x_2, \ldots, x_k) \in \prod_{j=1}^{k} X_j$. Let $p : \prod X_n \to \prod_{j=1}^{k} X_j$ be the projection and let $a' = p(a)$ and $A' = p(A)$. Since $\prod_{j=1}^{k} X_j$ has property $[K]^*$ (see Lemma 3.1), there is $\delta > 0$ satisfying the condition $(A', a', \frac{1}{2}\varepsilon)^*$. Suppose $b \in U(a, \delta) \subset \prod X_n$. Let ζ be any positive number and let $x \in A$.

Then there is a finite open covering $\mathscr{A}'$ of A' such that mesh $\mathscr{A}' < \frac{1}{2}\varepsilon$ and if $\langle U_1', U_2', \ldots, U_m' \rangle \in \mathrm{Chain}_{a'p(x)}(\mathscr{A}')$, then there is a ζ-chain $p(b) = p_1^1, p_1^2, \ldots, p_1^{i(1)}, \ldots, p_m^1, \ldots, p_m^{i(m)}$ of points of $\prod_{j=1}^{k} X_j$ such that $d_H(p_j^r, \mathrm{Cl}\, U_j') < \frac{1}{2}\varepsilon$. Set $U = (U' \times \prod_{j=1}^{k} X_j) \cap A$ for each $U' \in \mathscr{A}'$ and set $\mathscr{A} = \{U \mid U' \in \mathscr{A}'\}$. Then $\mathrm{diam}\, \mathscr{A} < \varepsilon$. If $\langle U_1, U_2, \ldots, U_m \rangle \in \mathrm{Chain}_{ax}(\mathscr{A})$, then $\langle U_1', U_2', \ldots, U_m' \rangle \in \mathrm{Chain}_{a'p(x)}(\mathscr{A}')$. Hence there is a ζ-chain $p(b) = p_1^1, p_1^2, \ldots, p_1^{i(1)}, \ldots, p_m^1, \ldots, p_m^{i(m)}$ of points of $\prod_{j=1}^{k} X_j$ such that $d_H(p_j^r, \mathrm{Cl}\, U_j') < \frac{1}{2}\varepsilon$.

Set $b_j^r = p_j^r \times (b_{k+1}, b_{k+2}, \ldots) \in \prod X_n$. Clearly $d_H(b_j^r, \mathrm{Cl}\, U_j) < \varepsilon$ and $b = b_1^1, b_1^2, \ldots, b_1^{i(1)}, \ldots, b_m^{i(m)}$ is a ζ-chain of points of $\prod X_n$. This implies that $\prod X_n$ has property $[K]^*$ with respect to (A, a), hence $\prod X_n$ has property $[K]^*$.

The converse assertion follows from Theorem 2.3. □

Corollary 3.3. *If each continuum X_n $(n = 1, 2, \ldots)$ has property $[K]^*$, then $C(\prod X_n)$ has property $[K]^*$. Also, if ω is any Whitney map for $C(\prod X_n)$, then each Whitney continuum $\omega^{-1}(t)$ has property $[K]^*$.*

Example 3.4. Consider the following set in the plane E^2. $X = \bigcup_{n=0}^{\infty} \langle p_0, q_n \rangle$, where $p_0 = (0, 0) \in E^2$, $q_n = (1, 1/n) \in E^2$ and $\langle p_0, q_n \rangle$ denotes the segment from p_0 to q_n in E^2 (see Fig. 2). Then X has property $[K]^*$.

Example 3.5. Consider the following set in the plane E^2. $X' = \{(x, \sin 2\pi/x) \mid 0 < x \leq 1$ or $-1 \leq x < 0\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$. Let A be an arc in E^2 from $(-1, 0)$ to $(1, 0)$ such that $A \cap X' = \{(-1, 0), (1, 0)\}$. Let $X = X' \cup A$. Then X has property $[K]^*$ (see Fig. 3). Hence the product $\prod X_n$ $(X_n = X)$ has property $[K]^*$ and $C(\prod X_n)$ and each Whitney continuum has property $[K]^*$.

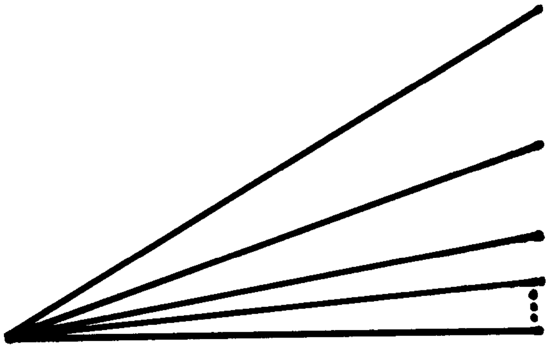

Fig. 2

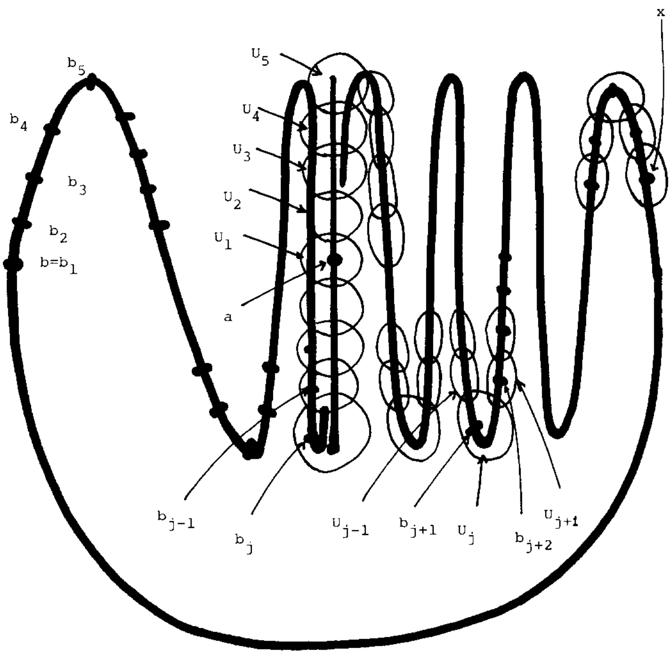

Fig. 3

Example 3.6. Let $X = S^1 \cup \Sigma \cup \Sigma'$, where S^1 is the unit circle in the plane E^2, $\Sigma = \{[1+1/t]\,e^{it}: t \geq 1\}$ and $\Sigma' = \{[1-1/t]\,e^{-it}: t \geq 1\}$ (see Fig. 4). In [11], Wardle showed that X has property $[K]$, but $X \times X$ does not have property $[K]$. By Lemma 3.1, X does not have property $[K]^*$. In fact, X does not have property $[K]^*$ with respect to $(S^1 \cup \Sigma, a)$, where $a \in S^1$.

Finally, we give the following questions.

Question 1. Let $f: X \to Y$ be an open mapping between continua and let f be equi-LC^0. If Y has property $[K]$, is it true that X has property $[K]$? Moreover, if

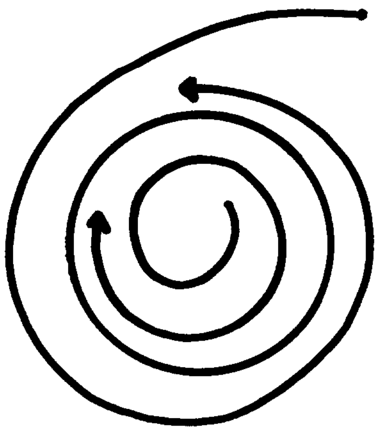

Fig. 4

X is a Peano continuum and a continuum Y has property $[K]$, is it true that $X \times Y$ has property $[K]$?

Question 2. If X is a hereditarily indecomposable continuum, does X have property $[K]^*$? Is it true that $C(X)$ has property $[K]$? It is known that X has property $[K]$ (see [11, (3.1)]), and each Whitney continuum has property $[K]$ (see [5, (8.5)] and [11, (4.3)]).

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