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Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian

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To Ryoshi Hotta on the occasion of his 65th birthday

Abstract

Let LG_n denote the Lagrangian Grassmannian parametrizing maximal isotropic (Lagrangian) subspaces of a fixed symplectic vector space of dimension $2n$. For each strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \leqslant n$ there is a Schubert variety $X(\lambda)$. Let T denote a maximal torus of the symplectic group acting on LG_n. Consider the T-equivariant cohomology of LG_n and the T-equivariant fundamental class $\sigma(\lambda)$ of $X(\lambda)$. The main result of the present paper is an explicit formula for the restriction of the class $\sigma(\lambda)$ to any torus fixed point. The formula is written in terms of factorial analogue of the Schur Q-function, introduced by Ivanov. As a corollary to the restriction formula, we obtain an equivariant version of the Giambelli-type formula for LG_n. As another consequence of the main result, we obtained a presentation of the ring $H_T^*(LG_n)$.

Keywords: Equivariant cohomology; Schubert classes; Lagrangian Grassmannian; Factorial Q-functions

1. Introduction

Let LG_n denote the *Lagrangian Grassmannian* parametrizing n-dimensional isotropic subspaces of a fixed $2n$-dimensional symplectic vector space. The *Schubert classes* give a linear basis for the integral cohomology ring of LG_n. These classes can be parametrized by sequences $\lambda = (\lambda_1, \ldots, \lambda_k)$ of integers such that $n \geqslant \lambda_1 > \cdots > \lambda_k \geqslant 1$. The same set of sequences also parametrizes the T-fixed points in LG_n, where T denotes a maximal torus of the symplectic

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group acting on LG_n. Here we consider the T-equivariant integral cohomology ring $H_T^*(LG_n)$. We are interested in the *T-equivariant Schubert classes* $\sigma(\lambda)$ in $H_T^*(LG_n)$. The classes $\sigma(\lambda)$ form a free basis of $H_T^*(LG_n)$ over the ring $\mathcal{S}$ of T-equivariant cohomology of a point. It is known that $\mathcal{S}$ is naturally identified with the polynomial ring $\mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n]$ (for the definition of ε_i see Section 4.1). Let $e(\mu)$ be the T-fixed point corresponding to an index μ. The inclusion map $i_\mu : \{e(\mu)\} \hookrightarrow LG_n$ induces the *restriction* morphism $i_\mu^* : H_T^*(LG_n) \to H_T^*(\{e(\mu)\}) \cong \mathcal{S}$. The main result of the present paper (Theorem 6.2) is an explicit formula for $i_\mu^*\sigma(\lambda)$, the restriction of the equivariant Schubert class $\sigma(\lambda)$ to a T-fixed point $e(\mu)$, as a polynomial in $\mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n]$.

For the classical Grassmannian $G_{d,n}$ of d-dimensional subspaces of $\mathbb{C}^n$, Knutson and Tao [21] derived a formula for the restriction of the equivariant Schubert classes to a torus fixed point. This formula is written in terms of the *factorial analogue* of the Schur function introduced by Biedenharn and Louck [3] and studied by other authors (see Section 5). Recently, Lakshmibai et al. [29] also proved the restriction formula by a different method. They also derived *equivariant Giambelli formulas*, determinantal formulas in the global ring $H_T^*(G_{d,n})$ of equivariant cohomology, that express the equivariant Schubert classes.

As for the ordinary cohomology of the Lagrangian Grassmannian, Hiller and Boe [16] proved Pieri-type formulas. Using the result in [16], Pragacz [38] proved Giambelli-type formulas that express each Schubert class in a Schur-type Pfaffian form. The derivation of the Pfaffian formula is based on a comparison of the formula in [16] and the Pieri formula for the Schur Q-function. Our formula (Theorem 6.2) is written in terms of the factorial analogue of the Schur Q-function, introduced by Ivanov [18,19]. This leads to an equivariant Giambelli-type formula (6.4) analogous to Pragacz's result.

Here is a brief summary of the paper. In Section 2 we recall some results on the Weyl group of type C_n and fix some standard notation. The set of minimal coset representatives introduced in this section will be used as an index set that parametrizes the main objects of this paper, the Schubert varieties, T-fixed points, etc. We also present several combinatorial descriptions of this set. Section 3 is devoted to basic geometric settings, where we introduce the Schubert varieties. We proceed by studying the T-equivariant cohomology of LG_n in Section 4. A recurrence relation (4.10) arising from the equivariant Pieri–Chevalley formula (4.8) plays the central role. In Section 5 we give a definition of the factorial analogue of the Schur Q-function and present some properties. Finally, in Section 6 we prove our main theorem. The proof is performed by a comparison of Ivanov's Pieri-type formula (5.2) for factorial Q-function and the recurrence relation for the restricted Schubert class. Using the restriction formula, we prove the Giambelli-type formula (6.4) for the equivariant Schubert classes of LG_n. In Section 7, we also give a supplementary discussion on expressing the Schubert classes of *two-row* type diagrams in terms of a polynomial in the *special* Schubert classes $\sigma(i)$ $(1 \leqslant i \leqslant n)$. A formula in Proposition 7.1 seems to be new. It provides an explicit form of any two-row type factorial Q-function. In Section 8, we prove a ring presentation for $H_T^*(LG_n)$ as a quotient of the polynomial ring over $\mathcal{S}$. In Appendix A, we present a brief introduction to Ivanov's functions and prove a vanishing property crucial to the main body of the paper.

It is well-known that the cohomology ring of the Grassmannian of orthogonal type is very similar to that of LG_n. It is natural to expect the equivariant Schubert classes for the orthogonal Grassmannian is described by factorial Schur P-functions (cf. [20,38]). Indeed, after the first version of this paper is written, the author and H. Naruse succeed in deriving such formulas. We will discuss the subject in a separate paper [17] which focuses on some combinatorial aspects of the equivariant Schubert calculus.

It should be mentioned that a recent result due to Ghorpade and Raghavan [10] provides an alternative combinatorial approach to our formula (6.2). They developed the standard monomial theory for the coordinate ring of the tangent cone of $X(\lambda)$ (see [22] for the corresponding result for the ordinary Grassmannian). Note that a preceding result of Conca [8] corresponds to the case with the fixed point is the identity coset. This description immediately leads to a combinatorial formula for $\sigma(\lambda)|_{\mu}$, which is quite similar to a tableau type formula for $Q_{\lambda}(x|a)$ given in [19]. Details of these issues will be discussed in [17]. Recently, the result, namely a combinatorial expression for the restriction of a Schubert class in LG_n to a T-fixed point, was independently proved by Kreiman [25] (see also [24] for type A).

2. Preliminaries

We first recall some basic notions about the Weyl group of type C_n, in order to fix our notation. The purpose of this section is to introduce an index set for the main ingredients of this paper, Schubert varieties, torus fixed points, etc. References for this section are [4,15,16].

2.1. Weyl group of type C_n

Let S_{2n} be the symmetric group of all permutations of $2n$ letters $\{1,\dots,2n\}$. Set $\bar{i}=2n-i+1$. Let W be the subgroup of $w\in S_{2n}$ such that $w(i)=j \Leftrightarrow w(\bar{i})=\bar{j}$. Then $w\in W$ can be determined by $w(1),\dots,w(n)$. A standard set of generators of W is given by $s_i=(i,i+1)(\overline{i+1},\bar{i})$ $(1\leqslant i\leqslant n-1)$ and $s_n=(n,\bar{n})$, where we denote by (i,j) the transposition. The *length* $\ell(w)$ of an element w in W is the smallest number of the generators $s_1,\dots,s_n$ (the simple reflections) whose product is w.

Let W_P denote the *parabolic subgroup* of W consisting of the element w such that $w(\{1,\dots,n\})\subset\{1,\dots,n\}$. Clearly W_P is isomorphic to S_n. Let W^P denote the set of $w\in W$ such that $w(1)<\cdots<w(n)$. Let $u\in W$. The coset uW_P contains a unique element w in W^P. Actually w is the unique element in the coset uW_P of minimal length. The longest element in W is denoted by w_0. If the coset $uW_P\in W/W_P$ is represented by $w\in W^P$, then w_0uW_P is represented by $w^{\vee}=(\overline{w(n)},\dots,\overline{w(1)})\in W^P$.

2.2. Combinatorial description of W^P

By a *symmetric* Young diagram, we mean a sequence $D=(d_1,\dots,d_n)$ with $d_1\geqslant d_2\geqslant\cdots\geqslant d_n\geqslant 0$ such that

$$d_i=\sharp\{j\mid d_j\geqslant i\}\quad(1\leqslant i\leqslant n).$$

By $\mathcal{Y}_n^{\mathrm{sym}}$ we denote the set of symmetric Young diagrams $D=(d_1,\dots,d_n)$ contained in the square $n\times n$, where by the last condition we mean $d_1\leqslant n$.

Let $w\in W^P$. Then the sequence

$$D(w)=\bigl(n+1-w(1),n+2-w(2),\dots,2n-w(n)\bigr), \tag{2.1}$$

is an element of $\mathcal{Y}_n^{\mathrm{sym}}$. Note that here we consider $w(i)$ simply as an element of $\{1,\dots,2n\}$ without "*bar*." For example if $n=5$ and $w=(1,3,4,\bar{5},\bar{2})=(1,3,4,6,9)$, then the corresponding Young diagram is $D(w)=(5,4,4,3,1)$. See Fig. 1.

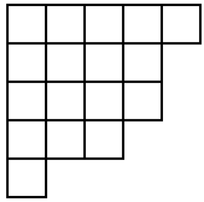

Fig. 1.

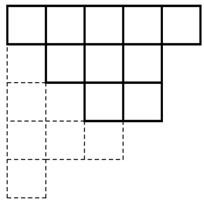

Fig. 2.

For any symmetric Young diagram $D = (d_1, \dots, d_n)$ in $\mathcal{Y}_n^{\mathrm{sym}}$, its *upper shifted diagram* $S(D)$ is obtained from D by discarding the boxes strictly lower than the diagonal, i.e.

$$S(D) = \left\{(i, j) \in \mathbb{Z}^2 \;\middle|\; 1 \leqslant i \leqslant j \leqslant d_i\right\},$$

which we regard as an array of boxes in the plane with matrix-style coordinates. For example if $D = (5, 4, 4, 3, 1)$, its upper shifted diagram $S(D)$ is depicted as Fig. 2.

Let λ_i be the number of boxes in the ith row of $S(D)$. Then the sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ is a *strict partition*. Namely there is k such that $\lambda_1 > \cdots > \lambda_k > 0$ and $\lambda_j = 0$ for $j > k$. Let $\mathcal{SP}_n$ denote the set of strict partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ contained in the *staircase* $\rho(n) = (n, n-1, \dots, 1)$, namely $\lambda_1 \leqslant n$. For D in $\mathcal{Y}_n^{\mathrm{sym}}$, its upper shifted diagram $S(D)$ is thus considered to be a strict partition in $\mathcal{SP}_n$. For example, the diagram of Fig. 2 is considered to be the strict partition $\lambda = (5, 3, 2)$.

We let $\mathcal{M}_n$ denote the set $\{0, 1\}^n$. We use $\delta = (\delta_1, \dots, \delta_n)$ to denote an element in $\mathcal{M}_n$. For $w \in W^P$, we set $\delta_i = 1$ if $i \in \{w(1), \dots, w(n)\}$ and $\delta_i = 0$ if $i \notin \{w(1), \dots, w(n)\}$. Then we associate $\delta = (\delta_1, \dots, \delta_n) \in \mathcal{M}_n$ to $w \in W^P$.

Proposition 2.1. *By the above correspondences, we have bijections between the following sets*:

(i) *The coset representatives* W^P;
(ii) *The set* $\mathcal{Y}_n^{\mathrm{sym}}$ *of symmetric Young diagrams contained in the square* $n \times n$;
(iii) *The set* $\mathcal{M}_n$ *of sequences* $\delta = (\delta_1, \dots, \delta_n)$ *with* $\delta_i \in \{0, 1\}$;
(iv) *The set* $\mathcal{SP}_n$ *of strict partitions* λ *contained in* $\rho(n)$.

Proof. It is clear that each correspondence is one to one. Also it is easy to see that the cardinality of each of the sets $W^P, \mathcal{Y}_n^{\mathrm{sym}}, \mathcal{M}_n, \mathcal{SP}_n$ is 2^n. Hence the claim follows. □

The following result is well known (see e.g. [15]).

Lemma 2.2. *For $w \in W^P$, the length $\ell(w^\vee)$ is equal to $|\lambda| = \sum_{i=1}^k \lambda_i$, where $\lambda \in \mathcal{SP}_n$ corresponds to $w \in W^P$.*

3. Lagrangian Grassmannians and Schubert varieties

This section is devoted to the set up of geometric objects.

3.1. Lagrangian Grassmannians

Let V be a vector space spanned by the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \boldsymbol{e}_{\bar{n}}, \ldots, \boldsymbol{e}_{\bar{1}}$. Introduce a symplectic form by $(\boldsymbol{e}_i, \boldsymbol{e}_j) = (\boldsymbol{e}_{\bar{i}}, \boldsymbol{e}_{\bar{j}}) = 0$ and $(\boldsymbol{e}_i, \boldsymbol{e}_{\bar{j}}) = -(\boldsymbol{e}_{\bar{j}}, \boldsymbol{e}_i) = \delta_{ij}$. Let V_i denote the subspace spanned by the first i vectors in $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \boldsymbol{e}_{\bar{n}}, \ldots, \boldsymbol{e}_{\bar{1}}$. A subspace W in V is *isotropic* if $(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $\boldsymbol{u}, \boldsymbol{v} \in W$. Note that V_i $(1 \leqslant i \leqslant n)$ are isotropic of dimension i and $V_{n+i} = (V_{n-i})^\perp$ $(1 \leqslant i \leqslant n)$. Denote by LG_n the set of n-dimensional isotropic subspaces of V. Then LG_n is a closed subvariety of the Grassmannian of n-dimensional subspaces of V, and is called the *Lagrangian Grassmannian*.

The group $G = Sp(V)$ of linear automorphisms of V preserving $(,)$ acts transitively on LG_n. So LG_n is identified with the quotient of G by the stabilizer of any point. We identify LG_n with the homogeneous space G/P, where P denotes the stabilizer of the point V_n, the span of $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. The elements of G that are diagonal with respect to the basis we took form a maximal torus T of G. The elements of G that are upper triangular matrices form a Borel subgroup B of G.

3.2. Schubert varieties

The T-fixed points of LG_n are parametrized by W^P: for w in W^P, the corresponding T-fixed point, denoted by $e(w)$, is the span of $\boldsymbol{e}_{w(1)}, \ldots, \boldsymbol{e}_{w(n)}$. Let $X(w)^\circ$ denote the B-orbit of $e(w)$. It is known that $X(w)^\circ$ is an affine space of dimension $\ell(w)$, called a *Schubert cell*. The Zariski closures $X(w) = \overline{X(w)^\circ}$ are called the *Schubert varieties*.

We have the following description:

$$X(w) = \bigl\{L \in LG_n \bigm| \dim\bigl(L \cap V_{w(i)}\bigr) \geqslant i \text{ for } 1 \leqslant i \leqslant n\bigr\}.$$

Let $\lambda = (\lambda_1 > \cdots > \lambda_k > 0) \in \mathcal{SP}_n$ be a strict partition corresponding to $w \in W^P$. Then we also denote the variety $X(w)$ by $X(\lambda)$. We have

$$X(\lambda) = \bigl\{L \in LG_n \bigm| \dim(L \cap V_{n+1-\lambda_i}) \geqslant i \text{ for } 1 \leqslant i \leqslant k\bigr\},$$

whose codimension is given as $|\lambda| = \sum_{i=1}^k \lambda_i$ by Lemma 2.2.

3.3. Bruhat–Chevalley order

For $w, v \in W^P$, we say $w \geqslant v$ if the torus fixed point $e(v)$ belongs to $X(w)$. This is a partial order called the *Chevalley–Bruhat order*. The condition $w \geqslant v$ is given by the following two equivalent forms: (1) $w(i) \geqslant v(i)$ $(1 \leqslant i \leqslant n)$, (2) $\lambda \subset \mu$, namely $\lambda_i \leqslant \mu_i$ $(1 \leqslant i \leqslant n)$, where $\lambda, \mu \in \mathcal{SP}_n$ correspond to w, v, respectively. The Schubert variety $X(\lambda)$ admits a canonical partition into Schubert cells $X(\mu)^\circ$ with $\mu \leqslant \lambda$.

4. Equivariant cohomology

We are interested in the T-equivariant integral cohomology ring $H_T^*(LG_n)$. For general facts on the equivariant cohomology, we refer to Brion [5], Goresky et al. [11] and references therein.

4.1. Equivariant Schubert classes

Let $\mathcal{S}$ denote the T-equivariant integral cohomology ring of a point $\{pt\}$ (namely the ordinary integral cohomology ring of the classifying space of T). The natural map $LG_n \to \{pt\}$ induces an $\mathcal{S}$-algebra structure on $H_T^*(LG_n)$. Given $w \in W^P$, denote by $\sigma(w)$ the T-equivariant fundamental class of $X(w)$, called the *equivariant Schubert class*. We also denote $\sigma(w)$ by $\sigma(\lambda)$, where $\lambda \in \mathcal{SP}_n$ corresponds to $w \in W^P$. It is known that $H_T^*(LG_n)$ is a free $\mathcal{S}$-module with the basis $\sigma(w)$ $(w \in W^P)$:

$$H_T^*(LG_n) = \bigoplus_{w \in W^P} \mathcal{S} \cdot \sigma(w).$$

For each T-fixed point $e(v)$, $v \in W^P$, we have an embedding $i_v : \{e(v)\} \hookrightarrow LG_n$. This yields a homomorphism $i_v^* : H_T^*(LG_n) \to H_T^*(\{e(v)\}) \cong \mathcal{S}$. The direct product of these is an injection of rings:

$$\prod_v i_v^* : H_T^*(LG_n) \longrightarrow \prod_v H_T^*\bigl(\{e(v)\}\bigr). \tag{4.1}$$

The injectivity is a consequence of "equivariant formality" (cf. [11]) of the T-variety LG_n. For $w, v \in W^P$, denote by $\sigma(w)|_v$ the image $i_v^*\sigma(w)$. The goal of this paper is to give an explicit formula for $\sigma(w)|_v \in \mathcal{S}$.

Remark. A remarkable characterization of the image of the morphism $\prod_v i_v^*$ has been obtained by [11]. However we shall not use the result in the present paper.

Let $\mathfrak{t} = \mathrm{Lie}(T)$ be the Lie algebra of the torus T. An element of $\mathfrak{t}$ takes the form $h = \mathrm{diag}(h_1, \dots, h_n, -h_n, \dots, -h_1)$. Define linear functionals $\varepsilon_i \in \mathfrak{t}^*$ by $\varepsilon_i(h) = h_i$ for $1 \leqslant i \leqslant n$. Let $\widehat{T}$ be the free abelian group generated by $\varepsilon_1, \dots, \varepsilon_n$. Each element $\sum m_i\varepsilon_i$ in $\widehat{T}$ determines a character of T via $T \ni \exp(h) \mapsto e^{\sum m_i\varepsilon_i(h)} \in \mathbb{C}^\times$. By this correspondence we can identify $\widehat{T}$ with the character group of T. There is a canonical map $\widehat{T} \to \mathcal{S}$ that extends to an isomorphism of the symmetric algebra $\mathrm{Sym}(\widehat{T})$ onto $\mathcal{S}$ (see e.g. [5, Section 1]). Thus we identify $\mathcal{S}$ with the polynomial ring $\mathbb{Z}[\varepsilon_1, \dots, \varepsilon_n]$. We shall also use the variables $x_i = -\varepsilon_i$ for $1 \leqslant i \leqslant n$. They are convenient for positivity reasons (cf. Section 4.7).

4.2. T-stable affine neighborhood of $e(v)$

Let $U(v)$ denote the set of points of LG_n whose matrix representatives $\xi = (\xi_{i,j})_{2n\times n}$ satisfy $\xi_{v(i),j} = \delta_{i,j}$. This is a T-stable affine space isomorphic to $\mathbb{A}^{n(n+1)/2}$ containing the point $e(v)$ as the origin. The coordinate function on $U(v)$ determined by the matrix entry $\xi_{i,j}$ with $i \notin \{v(1), \dots, v(n)\}$ is an eigenvector for T of the weight $-(\varepsilon_i - \varepsilon_{v(j)})$, here we understand $\varepsilon_{\bar{i}} = -\varepsilon_i$. Let $\varXi$ be the square matrix $(\xi_{v^\vee(i),v(j)})_{1\leqslant i,j\leqslant n}$. Not all the entries of $\varXi$ are independent, since the column vectors of ξ should span an isotropic subspace. As a set of free parameters on the affine space $U(v)$, we can take the set of entries of weakly upper triangular part of Ξ with respect to the anti-diagonal. Thus the coordinate ring of $U(v)$ is $R(v)=\mathbb{C}[\xi_{v^\vee(i),v(j)} \mid v^\vee(i) \geqslant v(j)]$.

It is convenient to consider $v^\vee$ and v as the (ordered) index sets corresponding to the rows and columns of Ξ respectively. In this notation we write $\Xi=(\xi_{r,c})_{r\in v^\vee, c\in v}$. Note that the weight of the coordinate function $\xi_{r,c}$ is given by $-(\varepsilon_r-\varepsilon_c)$.

4.3. Gröbner degeneration

Following [29, Section 6], we briefly explain the idea of Gröbner degeneration. Let U denote the affine space $\mathbb{A}^N$ equipped with an action of algebraic torus T. Suppose the origin $o \in U$ is the only T-fixed point. Choose a T-diagonalizable coordinate system $\xi_1,\dots,\xi_N$ such that ξ_i has weight $-\chi_i \in \widehat{T}$. Let $1 \leqslant i_1 < \cdots < i_k \leqslant n$ be a sequence. The prime ideal $(\xi_{i_1},\dots,\xi_{i_k})$ of the coordinate ring $\mathbb{C}[U]=\mathbb{C}[\xi_1,\dots,\xi_N]$ defines a simplest possible T-stable subvariety $\mathcal{V}$ in U, a *coordinate subspace*. In the T-equivariant cohomology ring of $H_T^*(U)$ we have the class $[\mathcal{V}] \in H_T^{2k}(U)$ of $\mathcal{V}$. The image of the restriction map to the origin o is given by

$$[\mathcal{V}]|_o=\prod_{j=1}^{k}\chi_{i_j}. \tag{4.2}$$

Let $\mathcal{V}_1,\dots,\mathcal{V}_m$ be distinct coordinate subspaces of the same codimension k. Consider the T-stable subvariety $Y=\bigcup_{j=1}^{k}\mathcal{V}_j$. Then we have

$$[Y]|_o=\sum_{j=1}^{m}[\mathcal{V}_j]|_o. \tag{4.3}$$

Let X be a smooth projective variety on which T acts with finitely many fixed points. Let Z be a T-stable subvariety and p be a T-fixed point on Z. Suppose we have a T-stable affine neighborhood U of p such that $U \cong \mathbb{A}^N$ with the origin $o \in \mathbb{A}^N$ corresponding to p. Since the restriction to the point p factors through the restriction to the open set U, we have $[Z]|_p = [Z\cap U]|_o$. The idea of Gröbner degeneration is to hope that the affine variety $Z \cap U$ can be deformed into $\bigcup_{j=1}^{k}\mathcal{V}_j$ such that $\mathcal{V}_j$ $(1 \leqslant j \leqslant m)$ are distinct coordinate subspaces of the same codimension, so that we can calculate $[Z]|_p$ by the above formula (4.3). If the coordinate ring $\mathbb{C}[Z\cap U]$ degenerates to a Stanley–Reisner ring by taking the initial ideal of the defining ideal of $Z\cap U$ with respect to some monomial order, then we have a desired deformation.

4.4. A product formula

We shall prove a formula that expresses $\sigma(w)|_w$ as a product of negative roots. This is a special case of the main result (Theorem 6.2). In the subsequent of the paper, we need only the fact that $\sigma(w)|_w$ is a non-zero polynomial (see the proof of Lemma 4.9).

Lemma 4.1. *We have the following formula*:

$$\sigma(w)|_w=\prod_{(i,j)\in\lambda}(x_{w(i)}-x_{\overline{w(j)}}), \tag{4.4}$$

where $\lambda \in \mathcal{SP}_n$ is the upper shifted diagram corresponding to $w \in W^P$.

	$\bar{1}$	$\bar{3}$	5	4	2
1	$2x_1$	x_1+x_3	x_1-x_5	x_1-x_4	x_1-x_2
3		$2x_3$	x_3-x_5	x_3-x_4	
$\bar{5}$					
$\bar{4}$					
$\bar{2}$					

Fig. 3.

Proof. The variety $U(w) \cap X(w)$ is just a "coordinate subspace" in $U(w)$ defined by $\xi_{r,c} = 0$ for $r \in w^\vee$, $c \in w$, and $r > c$. We denote by $\mathcal{I}(w)$ the set of such pairs. For each $(r, c) \in \mathcal{I}(w)$, we associate (i, j) by $w(i) = c$ and $\overline{w(j)} = r$. Then (i, j) is a box in the upper shifted diagram λ corresponding to w. This establishes a bijection from $\mathcal{I}(w)$ to the set of boxes of λ. Recall that $\xi_{r,c}$ has the weight $-(\varepsilon_r - \varepsilon_c)$. Then by using (4.2) we obtain the formula. □

For example, let $w = (1, 3, \bar{5}, \bar{4}, \bar{2})$ (see Fig. 3). The corresponding strict partition is $\lambda = (5, 3)$. Then $\sigma(w)|_w = 2x_1(x_1 + x_3)(x_1 - x_5)(x_1 - x_4)(x_1 - x_2) \times 2x_3(x_3 - x_5)(x_3 - x_4)$.

4.5. The divisor class

Let $\mathrm{div} = (n, \overline{n-1}, \ldots, \bar{2}, \bar{1})$. The corresponding $\sigma(\mathrm{div})$ is the unique Schubert class of codimension one. So we call it the *divisor class*. We know the following explicit form of this class restricted to any T-fixed point $e(v)$.

Lemma 4.2. *The restriction of the divisor class $\sigma(\mathrm{div})$ to a T-fixed point $e(v)$ is given by*

$$\sigma(\mathrm{div})|_v = 2\sum_{i=1}^{n} \delta_i x_i \quad \bigl(v \in W^P\bigr), \tag{4.5}$$

where $\delta = (\delta_1, \ldots, \delta_n) \in \mathcal{M}_n$ corresponds to $v \in W^P$.

Proof. Consider the closed subvariety $U(v) \cap X(\mathrm{div})$ of $U(v) \cong \mathbb{A}^{n(n+1)/2}$. Let $\xi = [\boldsymbol{\xi}_{v(1)}, \ldots, \boldsymbol{\xi}_{v(n)}]$ be a matrix representative of a point L in $U(v)$. The condition for L to be in $X(\mathrm{div})$ is equivalent to $\dim(V_n + L) \leqslant 2n - 1$. If we define the $n \times n$ matrix X by

$$[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \boldsymbol{\xi}_{v(1)}, \ldots, \boldsymbol{\xi}_{v(n)}] = \begin{bmatrix} 1_n & * \\ 0 & X \end{bmatrix},$$

then the last condition says that $\det X$ should vanish. Let k be such a number that $v(k) \leqslant n$ and $v(k+1) > n$. Then by elementary manipulations of a determinant, we have $\det X = \pm \det Y$ where we denote by $Y = (\xi_{r,c})$ the $k \times k$ submatrix of X with $c \in \{v(1), \ldots, v(k)\}$, and $r \in \{\overline{v(k)}, \ldots, \overline{v(1)}\}$. Choose a monomial order on $R(v)$ such that the initial term of $\det Y$ is the product of anti-diagonal entries $\pm\prod_{i=1}^{k}\xi_{\overline{v(i)},v(i)}$. Now applying (4.3) we have $\sigma(\mathrm{div})|_v = -\sum_{i=1}^{k} 2\varepsilon_{v(i)}$. Hence the claim follows. $\square$

Remark. To the flag variety of the Kac–Moody groups, Kostant and Kumar derived the corresponding formula ([23, Proposition 4.24(c)], see also [27, Section 11]).

4.6. Chevalley's multiplicities

Let us recall the Chevalley multiplicities [7]. Let $w, w' \in W^P$, such that $X(w')$ is a *Schubert divisor* of $X(w)$, i.e., $X(w')$ is a codimension one subvariety in $X(w)$. Then there is a positive root β such that $w' = ws_\beta$ and $\ell(w') = \ell(w) - 1$, where s_β is the reflection corresponding to β. Let $(,)$ be the inner product on $\widehat{T}\otimes_{\mathbb{Z}}\mathbb{R} = \bigoplus_{i=1}^{n}\mathbb{R}\varepsilon_i$ such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, and $\beta^\vee$ be $2\beta/(\beta,\beta)$. Then the Chevalley multiplicity $c(w, w')$ is defined

$$c(w, w') = \big(\varpi_n, \beta^\vee\big), \tag{4.6}$$

where $\varpi_n = \sum_{i=1}^{n}\varepsilon_i$, the nth fundamental weight. We can describe $c(w, w')$ in a combinatorial way.

Lemma 4.3. *(See [16].) Let $X(w')$ be a Schubert divisor in $X(w)$. Let $D(w), D(w')$ be the corresponding symmetric diagrams. Exactly one of the following holds.*

(1) *$D(w')$ is obtained from $D(w)$ adding two boxes at the positions (i, j) and (j, i) $(i \neq j)$. Then the corresponding positive root β is $\varepsilon_i + \varepsilon_j$, and we have $c(w, w') = 2$.*
(2) *$D(w')$ is obtained from $D(w)$ adding a box at the diagonal position (i, i), then the corresponding positive root β is $2\varepsilon_i$, and we have $c(w, w') = 1$.*

In Fig. 4, the numbers indicate the Chevalley multiplicities, where $n = 3$.

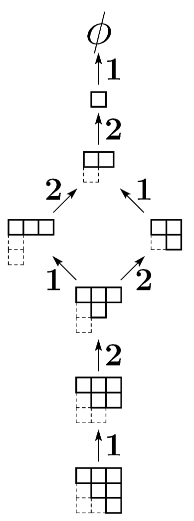

Fig. 4.

One can easily verify the following rule.

Lemma 4.4. *We assume $w' \to w$ $(w, w' \in W^P)$. Let $\lambda, \lambda' \in \mathcal{SP}_n$ correspond to w, w', respectively. Let k, k' be the numbers of non-zero parts of λ, λ', respectively. Then $c(w, w') = 1$ if $k' = k + 1$, and $c(w, w') = 2$ if $k' = k$.*

4.7. Equivariant Pieri–Chevalley formula

Since $\{\sigma(w)\}_{w \in W^P}$ forms a basis of $H_T^*(LG_n)$ over the ring $\mathcal{S}$, we can define the structure constants $c_{w,v}^u \in \mathcal{S}$ for all $w, v, u \in W^P$ by the formula

$$\sigma(w) \cdot \sigma(v) = \sum_u c_{w,v}^u \sigma(u). \tag{4.7}$$

The structure constant $c_{w,v}^u$ has degree $\ell(w) + \ell(v) - \ell(u)$ and vanishes unless $u \geqslant w, v$ and $\ell(u) \leqslant \ell(w) + \ell(v)$. It should be remarked that $c_{w,v}^u$ has a remarkable positivity property conjectured by D. Peterson and proved by Graham [14]. Namely each $c_{w,v}^u$ can be written as a linear combination of monomials in the negative roots with non-negative integer coefficients.

Lemma 4.5 *(The equivariant Pieri–Chevalley formula). Let $w \in W^P$. Then the following formula holds*:

$$\sigma(\mathrm{div}) \cdot \sigma(w) = c_{\mathrm{div},w}^w \sigma(w) + \sum_{w' : w' \to w} c(w, w') \sigma(w'), \tag{4.8}$$

where $w' \to w$ means that $X(w')$ is a Schubert divisor of $X(w)$.

Proof. By the same argument of [21, Proposition 2], the claim follows from the Pieri–Chevalley-type formulas for the ordinary integral cohomology, which is a special case Hiller and Boe's Pieri formula from [16] (see also Fulton and Woodward [9, Lemma 8.1]). □

For the flag variety of an arbitrary Kac-Moody group, the corresponding formula of Lemma 4.5 has been appeared in the context of the nil-Hecke algebra by Kostant and Kumar, see [23]. Later Arabia [2] established the fact that the equivariant cohomology is isomorphic to the dual of the nil-Hecke algebra. The parabolic analogue is also studied in [27, Section 11]. See also Robinson [39], Andersen et al. [1, Appendix D].

4.8. Recurrence relation

In this section, we prove a key lemma (Lemma 4.7) to the proof of our main result. First we need a simple lemma on structure constants.

Lemma 4.6. *The structure constant $c_{\mathrm{div},w}^w$ is given by*

$$c_{\mathrm{div},w}^w = \sigma(\mathrm{div})|_w. \tag{4.9}$$

Proof. If we restrict (4.8) to $e(w)$, we have

$$\sigma(\mathrm{div})|_w \cdot \sigma(w)|_w = c^w_{\mathrm{div},w}\sigma(w)|_w + \sum_{w' : w' \to w} c(w, w')\sigma(w')|_w.$$

For w' such that $w' \to w$, we have $\sigma(w')|_w = 0$ since $w \not\leqslant w'$. Hence the sum in the right-hand side vanishes. The claim follows since $\sigma(w)|_w$ is non-zero as we see from Lemma 4.4. □

Now the equivariant Pieri–Chevalley formula (4.8) gives directly the following recurrence relation on the family of restricted classes $\sigma(w)|_v$ ($w \in W^P$) for any fixed $v \in W^P$.

Lemma 4.7. *Let $e(v)$ be any T-fixed point. The polynomials $\sigma(w)|_v$ ($w \in W^P$) satisfy the following recurrence relation*:

$$d(w, v) \cdot \sigma(w)|_v = \sum_{w' : w' \to w} c(w, w')\, \sigma(w')|_v, \tag{4.10}$$

where $d(w, v) = \sigma(\mathrm{div})|_v - \sigma(\mathrm{div})|_w$.

Since $d(w, v)$ is non-zero if $w \geqslant v$ and $w \neq v$, the recurrence relation (4.10) and the initial condition $\sigma(\phi)|_v = 1$ determine the polynomials $\sigma(w)|_v$ ($w \in W^P$) uniquely. An analogous recurrence relation was used by Rosenthal and Zelevinsky [40] to prove a determinantal formula of the multiplicity of a T-fixed point in a Schubert variety in the Grassmannian.

Remark. The ordinary-cohomology version of Lemma 4.7 has been obtained by Lakshmibai and Weyman [28], and Hiller [15].

5. The factorial Schur Q-functions

Let $x = (x_1, \ldots, x_n)$ be a finite sequence of variables and let $a = (a_i)_{i \geqslant 1}$ be any sequence such that $a_1 = 0$. Let

$$(x|a)^k = \prod_{i=1}^{k} (x - a_i)$$

for each $k \geqslant 1$ and $(x|a)^0 = 1$. The *factorial Schur Q-function* for a strict partition $\lambda = (\lambda_1 > \cdots > \lambda_k > 0)$ of *length* $k \leqslant n$ is defined as follows [19].

Definition 5.1. Let $A(x)$ denote the skew-symmetric $n \times n$ matrix $((x_i - x_j)/(x_i + x_j))_{1 \leqslant i,j \leqslant n}$ and let $B_\lambda(x|a)$ denote the $n \times k$ matrix $((x_i|a)^{\lambda_{k-j+1}})$. Let

$$A_\lambda(x|a) = \begin{bmatrix} A(x) & B_\lambda(x|a) \\ -{}^t B_\lambda(x|a) & 0 \end{bmatrix}$$

which is a skew-symmetric $(n + k) \times (n + k)$ matrix. Put

$$\mathrm{Pf}_\lambda(x|a) = \begin{cases} \mathrm{Pf}(A_\lambda(x_1, \ldots, x_n|a)) & \text{if } n + k \text{ is even;} \\ \mathrm{Pf}(A_\lambda(x_1, \ldots, x_n, 0|a)) & \text{if } n + k \text{ is odd.} \end{cases}$$

Then put

$$P_\lambda(x|a) = \frac{\mathrm{Pf}_\lambda(x|a)}{D_n(x)}, \qquad Q_\lambda(x|a) = 2^k P_\lambda(x|a), \tag{5.1}$$

where $D_n(x) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)/(x_i + x_j)$.

Remark. Note that the ordinary Schur Q-functions are obtained from Ivanov's Q-functions by setting $a_i = 0$. In fact the above definition is a factorial analogue of Nimmo's formula [35] (see also [31, Chapter III, 8, Example 13]) for the Schur Q-functions. The reader can find other expressions for $Q_\lambda(x|a)$ in [19].

The functions $Q_\lambda(x|a)$ were introduced by Ivanov[1] [18,19]. He established some fundamental properties of the functions (combinatorial presentations, Schur-type Pfaffian formulas, vanishing and characterization properties, etc.). In particular, a Pieri-type formula is available, which is crucial to our consideration. Note that $P_{(1)}(x|a)$ does not depend on the parameter $a = (a_i)$ and actually we have $P_{(1)}(x|a) = \sum_{i=1}^n x_i$. So we simply denote $P_{(1)}(x|a)$ by $P_{(1)}(x)$. Let λ and λ' be strict partitions of length $\leqslant n$. We will write $\lambda' \to \lambda$ if $\lambda \subset \lambda'$ and $|\lambda'| = |\lambda| + 1$.

Proposition 5.2 *(A Pieri-type formula). (See [19].) For any strict partition $\lambda = (\lambda_1 > \cdots > \lambda_k > 0)$ of length $k \leqslant n$, we have*

$$\left(P_{(1)}(x) - \sum_{j=1}^k a_{\lambda_j+1}\right) \cdot P_\lambda(x|a) = \sum_{\lambda' : \lambda' \to \lambda} P_{\lambda'}(x|a), \tag{5.2}$$

where λ' runs for all strict partitions of length less than or equal to n such that $\lambda' \to \lambda$.

Proof. The above formula corresponds to [19, Theorem 6.2]. The only difference is that we use n-variables $x = (x_1, \ldots, x_n)$ here. Then we can consistently set $P_\mu(x|a)$ to be zero for any strict partition μ of length strictly greater than n (see [19, Definition 2.10]). □

Factorial analogues of the Schur S-functions were introduced by Biedenharn and Louck [3] and further studied by Chen and Louck [6], Goulden and Greene [12], Goulden and Hamel [13], Macdonald [32], and Molev and Sagan [34] (see also Macdonald [31, Chapter I, 3, Examples 20, 21]). In these works it was shown that several important facts about the Schur S-functions (combinatorial presentations, Jacobi–Trudi identities, Pieri-type formulas, Littlewood–Richardson rules, etc.) can be transferred to the factorial Schur S-functions. The factorial Schur S-functions also play a central role in the study of the center of the universal enveloping algebra of $\mathfrak{gl}_n$ (see Okounkov and Olshanski [37], Okounkov [36] and references therein).

In a geometric context, the factorial Schur functions appeared in [21,29]. They present the restriction to torus fixed points of the Schubert classes in the equivariant cohomology of the Grassmannian. Recently Mihalcea [33] obtained a presentation by generators and relations for the equivariant *quantum* cohomology ring of the Grassmannian. In this work the factorial Schur

[1] According to Ivanov [18,19], A. Okounkov defined them for the special parameter a with $a_i = i - 1$.

S-functions[2] appeared as the polynomial representatives of the equivariant quantum Schubert classes. A similar presentation for the quantum cohomology ring of the Lagrangian Grassmannian was given by Kresch and Tamvakis [26]. It will be an interesting problem to extend their result to the quantum equivariant cohomology ring.

6. Restriction and Giambelli-type formulas

6.1. Restriction formula

Let us take the following particular parameters:

$$a_1 = 0, \qquad a_i = x_{n-i+2} \quad (2 \leqslant i \leqslant n+1), \qquad a_i = 0 \quad (i > n+1). \tag{6.1}$$

We denote by $x_{\langle n\rangle}$ the specialization of $a = (a_i)_{i=1}^{\infty}$ given by (6.1). Let $\mu \in W^P$ and $\delta = (\delta_1, \ldots, \delta_n) \in \mathcal{M}_n$ correspond to μ. Then we set $x_\mu = (\delta_1 x_1, \ldots, \delta_n x_n)$.

Definition 6.1. A specialization $Q_\lambda(x_\mu|x_{\langle n\rangle})$ of $Q_\lambda(x|a)$ is given as follows. First we substitute x_μ for $x = (x_1, \ldots, x_n)$ to obtain $Q_\lambda(x_\mu|a)$, then we specialize a_i's as in (6.1) to get $Q_\lambda(x_\mu|x_{\langle n\rangle})$.

Theorem 6.2. *For strict partitions $\lambda, \mu \in \mathcal{SP}_n$, we have*

$$\sigma(\lambda)|_\mu = Q_\lambda(x_\mu|x_{\langle n\rangle}). \tag{6.2}$$

Proof. It suffices to show that the right-hand side of (6.2) satisfies the recurrence relation (4.10) and the initial condition $\sigma(\phi)|_\mu = 1$. As for the initial condition, we have $Q_\phi(x|a) = 1$ by definition. Hence the proof is completed by a comparison of (4.10) and (5.2). We first specialize $P_\lambda(x|a)$ to $P_\lambda(x_\mu|a)$ and then to $P_\lambda(x_\mu|x_{\langle n\rangle})$. By applying this specialization to (5.2), we have

$$\left(P_{(1)}(x_\mu) - \sum_{j=1}^{k} x_{n-\lambda_j+1}\right) \cdot P_\lambda(x_\mu|x_{\langle n\rangle}) = \sum_{\lambda'} P_{\lambda'}(x_\mu|x_{\langle n\rangle}), \tag{6.3}$$

where the sum is taken over those $\lambda' \in \mathcal{SP}_n$ such that $\lambda' \to \lambda$ because $P_{\lambda'}(x_\mu|x_{\langle n\rangle})$ vanishes unless $\lambda' \in \mathcal{SP}_n$ (Proposition A.1). Now we multiply the both hand sides of (6.3) by 2^{k+1}, where k is the number of non-zero parts of λ. By Lemma 4.2, we have $\sigma(\mathrm{div})|_w = 2\sum_{j=1}^{k} x_{n-\lambda_j+1}$ and $\sigma(\mathrm{div})|_v = 2P_{(1)}(x_\mu)$. Therefore we have

$$d(w, v) = 2P_{(1)}(x_\mu) - 2\sum_{i=1}^{k} x_{n-\lambda_i+1}.$$

Now let $\lambda' \in \mathcal{SP}_n$ be such that $\lambda' \to \lambda$ and k' be the number of non-zero parts of λ'. From Lemmas 4.3 and 4.4, we can see that $2^{k+1} P_{\lambda'}(x|a) = c(w, w') Q_{\lambda'}(x|a)$. Note also $2^k P_\lambda(x|a) = Q_\lambda(x|a)$. Thus we proved that $Q_\lambda(x_\mu|x_{\langle n\rangle})$ ($\lambda \in \mathcal{SP}_n$) satisfy (4.10). □

[2] It has come to my knowledge via [33, §5, Remark 2] that the factorial Schur S-functions coincide with the double Schubert polynomials by Lascoux and Schützenberger [30] when indexed by a Grassmann permutation. However, the details of this connection seem to be missing from the literature.

6.2. Giambelli-type formula

We can prove an equivariant analogue of Pragacz' Giambelli-type formula. Let $\lambda \in \mathcal{SP}_n$. We write $\lambda = (\lambda_1, \dots, \lambda_{2r})$ with $\lambda_1 > \cdots > \lambda_{2r} \geqslant 0$.

Theorem 6.3. *The equivariant Schubert class $\sigma(\lambda)$ is expressed as a Pfaffian of the following form*:

$$\sigma(\lambda) = \mathrm{Pf}\big(\sigma(\lambda_i, \lambda_j)\big)_{1 \leqslant i,j \leqslant 2r}. \tag{6.4}$$

Proof. Because of the injection (4.1), it is enough to show that, for arbitrary μ in $\mathcal{SP}_n$, the restrictions to $e(\mu)$ of the both hand sides of (6.4) coincide. We have

$$\sigma(\lambda)|_\mu = Q_\lambda(x_\mu | x_{\langle n \rangle}) = \mathrm{Pf}\big(Q_{\lambda_i,\lambda_j}\big(x_\mu | x_{\langle n \rangle}\big)\big)_{1 \leqslant i,j \leqslant 2r} = \mathrm{Pf}\big(\sigma(\lambda_i, \lambda_j)|_\mu\big)_{1 \leqslant i,j \leqslant 2r}.$$

In the second equality, we use the Pfaffian formula for factorial Q-functions (A.1). Since the restriction i_μ^* is a ring homomorphism and we are done. □

The above formula has a striking character in contrast to the ordinary Grassmannian case [29], where the equivariant Giambelli formula is given in a Jacobi–Trudi type determinant, with matrix entries of *linear combinations* of (equivariant) special Schubert classes. In our formula, each matrix entry of the Pfaffian is itself an equivariant Schubert class. In spite of this simplicity, if we wish to express the equivariant Schubert class as a polynomial of the *special Schubert classes* $\sigma(k)$ $(1 \leqslant k \leqslant n)$, we need some work to be done. We will treat the problem in the next subsection.

7. On the two-row type classes

The formula (6.4) looks the same as the classical one shown by Pragacz ([38, Proposition 6.6], see also Józefiak [20]), where the Q-functions $Q_\lambda(x)$ $(\lambda \in \mathcal{SP}_n)$ represent the Schubert classes in the ordinary cohomology ring of LG_n. Recall that we have the following formula for $r > s \geqslant 0$:

$$Q_{r,s}(x) = Q_r(x)Q_s(x) + 2\sum_{i=1}^{s}(-1)^i Q_{r+i}(x)Q_{s-i}(x). \tag{7.1}$$

Therefore the Pragacz' formula gives an expression for each Schubert class as a polynomial in the special Schubert classes.

Now in our setting of equivariant cohomology ring, Eq. (6.4) actually provides an expression for each $\sigma(\lambda)$ as a polynomial in $\sigma(\lambda_i, \lambda_j)$. If $\lambda_j = 0$ then $\sigma(\lambda_i, \lambda_j) = \sigma(\lambda_i)$ is a *special* class. For the *two-row* type classes, i.e. $\sigma(\lambda_i, \lambda_j)$ with $\lambda_j > 0$, we want to express them as a polynomial in the special classes $\sigma(k)$ $(1 \leqslant k \leqslant n)$. In fact, we have the following expression for two-row type classes $\sigma(k, 1)$ in $H_T^*(LG_n)$:

$$\sigma(k, 1) = \sigma(k)\sigma(1) - 2\sigma(k+1) - 2x_{n-k+1}\sigma(k) \quad (2 \leqslant k \leqslant n), \tag{7.2}$$

where $\sigma(j) = 0$ for $j > n$. The above expression is a consequence of the following formula for Ivanov's functions:

$$Q_{k,1}(x|a) = Q_k(x|a)Q_1(x|a) - 2Q_{k+1}(x|a) - 2a_{k+1}Q_k(x|a). \tag{7.3}$$

As illustrated by this example, we need a correction term to classical formula (7.1).

To generalize (7.2), we prove the next proposition, which is also interesting from a purely combinatorial point of view. In this section, $x = (x_1, x_2, \ldots)$ and $a = (a_2, a_3, \ldots)$ are two sequences of infinite variables. We can define $Q_\lambda(x|a)$ for any strict partition λ. They are in the ring $\mathbb{Z}[a_2, a_3, a_4, \ldots] \otimes_{\mathbb{Z}} \Gamma$, where Γ denotes a distinguished subring spanned by the Schur's Q-functions in "the ring of symmetric functions Λ" [31]. For the details of definition for $Q_\lambda(x|a)$, see [19]. Note that, if we substitute $x_j = 0$ $(j > n)$ for $Q_\lambda(x|a)$ $(\lambda \in \mathcal{SP}_n)$ we can recover the polynomial introduced by Definition 5.1. Let h_r (respectively e_r) denote the rth complete (respectively elementary) symmetric function.

Proposition 7.1. *Let $k > \ell > 0$. We have*

$$Q_{k,\ell}(x|a) = Q_k(x|a)Q_\ell(x|a) + 2\sum_{i=1}^{\ell}(-1)^i Q_{k+i}(x|a)Q_{\ell-i}(x|a) + G_{k,\ell}(x|a), \tag{7.4}$$

where

$$G_{k,\ell}(x|a) = \sum_{r=k}^{k+\ell-1}\sum_{s=0}^{k+\ell-1-r} f_{k,\ell}^{r,s}(a)Q_r(x|a)Q_s(x|a), \tag{7.5}$$

and the coefficient $f_{k,\ell}^{r,s}(a)$ is given by

$$f_{k,\ell}^{r,s}(a) = (-1)^{\ell-s}\sum_{j=0}^{k+\ell-r-s} 2h_{k+\ell-r-s-j}(a_{k+1}, a_{k+2}, \ldots, a_{r+1})e_j(a_{s+2}, \ldots, a_{\ell-1}, a_\ell). \tag{7.6}$$

Proof. We use Eq. (8.2) of [19] that reads

$$Q_{k+1,\ell} + Q_{k,\ell+1} + (a_{k+1} + a_{\ell+1})Q_{k,\ell} = Q_kQ_{\ell+1} - Q_{k+1}Q_\ell + (a_{\ell+1} - a_{k+1})Q_kQ_\ell, \tag{7.7}$$

for $k > \ell > 0$, where we denote $Q_{r,s}(x|a)$ simply by $Q_{r,s}$. By this equation, it is easy to see that each function $Q_{k,\ell}$ is a linear combination of the functions Q_rQ_s $(r > s \geqslant 0)$. Note that the functions Q_rQ_s $(r > s \geqslant 0)$ are linearly independent over the ring $\mathbb{Z}[a_2, a_3, \ldots]$ (this fact can be seen from [19, Proposition 2.11] and [31, III, (8.9)]).

We shall prove the proposition by induction on ℓ. The case $\ell = 1$ is true by (7.3). Let $\ell > 1$. Suppose the proposition holds for ℓ. We have an expansion

$$Q_{k,\ell+1} = \sum_{r>s\geqslant 0} g_{k,\ell+1}^{r,s}(a)Q_rQ_s, \tag{7.8}$$

with coefficients $g^{r,s}_{k,\ell+1}(a) \in \mathbb{Z}[a_2, a_3, \ldots]$. Our task is to show $g^{r,s}_{k,\ell+1} = f^{r,s}_{k,\ell+1}$. By extracting the coefficient of $Q_k Q_\ell$ in both hand sides of (7.7), we have

$$g^{k,\ell}_{k,\ell+1} + (a_{\ell+1} + a_{k+1}) = a_{\ell+1} - a_{k+1}.$$

Hence we have $g^{k,\ell}_{k,\ell+1} = -2a_{k+1} = f^{k,\ell}_{k,\ell+1}$. Let $(r,s) \neq (k,\ell)$ with $r+s<k+\ell$. By comparing the coefficients of $Q_r Q_s$ in both hand sides of (7.7), we have

$$f^{r,s}_{k+1,\ell} + g^{r,s}_{k,\ell+1} + (a_{k+1} + a_{\ell+1}) f^{r,s}_{k,\ell} = 0. \tag{7.9}$$

We shall prove $g^{r,s}_{k,\ell+1} = f^{r,s}_{k,\ell+1}$ by showing

$$f^{r,s}_{k+1,\ell} + f^{r,s}_{k,\ell+1} + (a_{k+1} + a_{\ell+1}) f^{r,s}_{k,\ell} = 0. \tag{7.10}$$

This follows from the equality

$$\frac{\prod_{\alpha=s+2}^{\ell}(1+a_\alpha z)}{\prod_{\beta=k+2}^{r+1}(1-a_\beta z)} + z(a_{k+1}+a_{\ell+1})\frac{\prod_{\alpha=s+2}^{\ell}(1+a_\alpha z)}{\prod_{\beta=k+1}^{r+1}(1-a_\beta z)} = \frac{\prod_{\alpha=s+2}^{\ell+1}(1+a_\alpha z)}{\prod_{\beta=k+1}^{r+1}(1-a_\beta z)}. \qquad \square$$

For example, we have

$$\begin{aligned}
G_{k,1} &= -2a_{k+1}Q_k,\\
G_{k,2} &= 2(a_{k+1}+a_{k+2}+a_2)Q_{k+1} - 2a_{k+1}Q_kQ_1 + 2\big(a_{k+1}^2 + a_2a_{k+1}\big)Q_k,\\
G_{k,3} &= -2(a_{k+1}+a_{k+2}+a_{k+3}+a_2+a_3)Q_{k+2} + 2(a_{k+1}+a_{k+2}+a_3)Q_{k+1}Q_1\\
&\quad - 2\big(a_{k+1}^2 + a_{k+1}a_{k+2} + a_{k+2}^2 + (a_{k+1}+a_{k+2})(a_2+a_3) + a_2a_3\big)Q_{k+1}\\
&\quad - 2a_{k+1}Q_kQ_2 + 2\big(a_{k+1}^2 + a_{k+1}a_3\big)Q_kQ_1 - 2\big(a_{k+1}^3 + a_{k+1}^2(a_2+a_3) + a_{k+1}a_2a_3\big)Q_k.
\end{aligned}$$

Proposition 7.1 combined with (6.2) gives rise to a polynomial expression for $\sigma(r,s)$ with $n \geqslant r > s > 0$ in terms of the special classes $\sigma(k)$ $(1 \leqslant k \leqslant n)$. For example, we have

$$\begin{aligned}
\sigma(k,2) &= \sigma(k)\sigma(2) - 2\sigma(k+1)\sigma(1) + 2\sigma(k+2) - 2x_{n-k+1}\sigma(k)\sigma(1)\\
&\quad + 2(x_{n-k+1} + x_{n-k} + x_n)\sigma(k+1) + 2\big(x_{n-k+1}^2 + x_{n-k+1}x_n\big)\sigma(k)
\end{aligned}$$

for $2 < k \leqslant n$, with $\sigma(j) = 0$ for $j > n$ (cf. Proposition A.1).

The next proposition will be used in Section 8.

Proposition 7.2. *We have*

$$Q_k(x|a)^2 + 2\sum_{i=1}^{k}(-1)^i Q_{k+i}(x|a)Q_{k-i}(x|a) + \sum_{r=k}^{2k-1}\sum_{s=0}^{2k-1-r} f^{r,s}_{k,k}(a)Q_r(x|a)Q_s(x|a) = 0. \tag{7.11}$$

Proof. The proof of Lemma 7.1 is valid also for $k=\ell$ with $Q_{k,k}(x|a) = 0$ for $k \geqslant 1$. $\square$

8. Presentation of the ring $H_T^*(LG_n)$

As an application of Theorems 6.2 and 6.3, we obtain a presentation of the ring $H_T^*(LG_n)$ in terms of generators and relations. Consider the ring $\mathbb{Z}[a]=\mathbb{Z}[a_2,a_3,\ldots,a_{n+1}]$. Throughout the section, we identify $\mathcal{S}=\mathbb{Z}[x_1,\ldots,x_n]$ and $\mathbb{Z}[a]$ by the isomorphism $\iota_n:\mathbb{Z}[a]\to\mathcal{S}$ of rings given by

$$\iota_n(a_j)=x_{n-j+2}\quad (2\leqslant i\leqslant n). \tag{8.1}$$

8.1. Statement of the result

Let $X_1,\ldots,X_n$ denote a set of indeterminates. Set $X_0=1$ and $X_j=0$ for $j>n$ (cf. Proposition A.1). Let k,ℓ be $n\geqslant k\geqslant\ell\geqslant 0$. Consider the following elements of the polynomial ring $\mathcal{S}[X_1,\ldots,X_n]$:

$$X_{k,\ell}=X_kX_\ell+2\sum_{i=1}^{\min(n-k,\ell)}(-1)^iX_{k+i}X_{\ell-i}+\sum_{r=k}^{\min(n,k+\ell-1)}\sum_{s=0}^{k+\ell-1-r}f_{k,\ell}^{r,s}(a)X_rX_s, \tag{8.2}$$

where $f_{k,\ell}^{r,s}(a)$ is given by the right-hand side of (7.6). Since we restrict $r\leqslant n$, we can consider $f_{k,\ell}^{r,s}(a)$ to be in $\mathcal{S}$ via the isomorphism ι_n. Note also that we also consider the case of $\ell=k$. Define an ideal $\mathcal{I}_n=\langle X_{1,1},\ldots,X_{n,n}\rangle$ and consider the quotient ring

$$\mathcal{R}_n=\mathcal{S}[X_1,\ldots,X_n]/\mathcal{I}_n.$$

We shall define a morphism of $\mathcal{S}$-algebras $\phi:\mathcal{R}_n\to H_T^*(LG_n)$ by setting X_i to $\sigma(i)$ $(1\leqslant i\leqslant n)$.

Lemma 8.1. *The map ϕ is well defined.*

Proof. Define a morphism of $\mathcal{S}$-algebras $\tilde\phi:\mathcal{S}[X_1,\ldots,X_n]\to H_T^*(LG_n)$ by $\tilde\phi(X_i)=\sigma(i)$. For k with $1\leqslant k\leqslant n$, and $\mu\in W^P$, we have

$$\begin{aligned}\tilde\phi(X_{k,k})|_\mu&=\sigma(k)^2|_\mu+2\sum_{i=1}^{\min(n-k,k)}(-1)^i\sigma(k+i)|_\mu\sigma(k-i)|_\mu\\&\quad+\sum_{r=k}^{\min(n,2k-1)}\sum_{s=0}^{2k-1-r}\iota_n\bigl(f_{k,k}^{r,s}(a)\bigr)\sigma(r)|_\mu\sigma(s)|_\mu\\&=Q_k(x_\mu|x_{\langle n\rangle})^2+2\sum_{i=1}^{k}(-1)^iQ_{k+i}(x_\mu|x_{\langle n\rangle})Q_{k-i}(x_\mu|x_{\langle n\rangle})\\&\quad+\sum_{r=k}^{2k-1}\sum_{s=0}^{2k-1-r}f_{k,k}^{r,s}(x_{\langle n\rangle})\,Q_r(x_\mu|x_{\langle n\rangle})Q_s(x_\mu|x_{\langle n\rangle}),\end{aligned}$$

where in the second equality, we used Theorem 6 and a vanishing property (Proposition A.1). We can see the last expression is zero by specializing (7.11) (see Definition 6.1). Thus we have

$\tilde{\phi}(X_{k,k})|_\mu = 0$ for all $\mu \in W^P$, and hence $\tilde{\phi}(X_{k,k}) = 0$. So $\tilde{\phi}$ induces $\phi : \mathcal{R}_n \to H_T^*(LG_n)$ such that $\phi(X_i) = \sigma(i)$ $(1 \leqslant i \leqslant n)$. $\square$

Definition 8.2. Let $\lambda = (\lambda_1 > \cdots > \lambda_{2r} \geqslant 0)$ be in $\mathcal{SP}_n$. We introduce the following Schur-type Pfaffian:

$$X_\lambda = \mathrm{Pf}(X_{\lambda_i,\lambda_j})_{1 \leqslant i,j \leqslant 2r}.$$

Theorem 8.3. *There exists an isomorphism of $\mathcal{S}$-algebras*:

$$\phi : \mathcal{R}_n \longrightarrow H_T^*(LG_n)$$

sending X_i to $\sigma(i)$ $(1 \leqslant i \leqslant n)$ and the Pfaffian X_λ to the equivariant Schubert class $\sigma(\lambda)$.

By definition of ϕ and Giambelli formula (6.4), we have $\phi(X_\lambda) = \sigma(\lambda)$. Moreover, since $\sigma(\lambda)$ $(\lambda \in \mathcal{SP}_n)$ generates $H_T^*(LG_n)$ as an $\mathcal{S}$-module, ϕ is surjective. The rest of this section is devoted to the proof of injectivity of ϕ.

8.2. A monomial ordering

Here we give a preliminary discussion to prove Theorem 8.3. The argument below is quite similar to the one in Macdonald [31, III, 8], however a different ordering on the partitions will be used, which proves to be useful in our situation.

For any partition $\lambda = (1^{e_1} 2^{e_2} \cdots n^{e_n})$,

$$X^\lambda = X_1^{e_1} \cdots X_n^{e_n}.$$

By $\deg(\lambda)$ we denote the degree $\sum_{i=1}^n e_i$ of the monomial X^λ. Let $\mu = (1^{e'_1} \cdots n^{e'_n})$ be another partition. We write $\lambda \succ \mu$ if

$$\deg(\lambda) > \deg(\mu), \quad \text{or}$$
$$\deg(\lambda) = \deg(\mu) \text{ and there is } k \text{ such that } e_1 = e'_1, \ldots, e_k = e'_k \text{ and } e_{k+1} < e'_{k+1}.$$

Then we also write $X^\lambda \succ X^\mu$. This is a monomial ordering called the *grevlex order* with $X_1 \prec X_2 \prec \cdots \prec X_n$. In particular, if we have $\lambda \succ \mu$, then $\lambda + \nu \succ \mu + \nu$ for any partition ν.

Lemma 8.4. *Let $\lambda = (1^{e_1} 2^{e_2} \cdots n^{e_n})$ be a partition. If λ is not strict, then X^λ is an $\mathcal{S}$-linear combination of the X^μ with $\mu \in \mathcal{SP}_n$, and $\mu \prec \lambda$. In particular, the monomials X^λ $(\lambda \in \mathcal{SP}_n)$ generate $\mathcal{R}_n$ as an $\mathcal{S}$-module.*

Proof. First note that if λ is strict then we have $\lambda \in \mathcal{SP}_n$. We prove the first statement by induction, assuming the claim for all partition μ such that $\mu \prec \lambda$. If λ is not strict then for some k we have $e_k \geqslant 2$. We have the following relation:

$$X_k^2 = -2 \sum_{i=1}^{\min(n-k,k)} (-1)^i X_{k+i} X_{k-i} - \sum_{r=k}^{\min(n,2k-1)} \sum_{s=0}^{2k-1-r} f_{k,k}^{r,s}(a) X_r X_s. \tag{8.3}$$

We can see that the monomials appearing in the right-hand side of the above equation are strictly lower than X_k^2 in the grevlex order $\prec$. Replacing the factor X_k^2 in X^λ by the right-hand side of (8.3), we can express X^λ as an $\mathcal{S}$-linear combination of the X^μ's where each μ is a partition such that $\mu \prec \lambda$. By the inductive hypothesis the claim is true for each X^μ, and the proof completes. $\square$

Lemma 8.5. *Let $\lambda \in \mathcal{SP}_n$. The Pfaffian X_λ is written in the form*

$$X_\lambda = X^\lambda + \sum_\mu b_{\lambda\mu}(a) X^\mu$$

with coefficients $b_{\lambda\mu}(a) \in \mathcal{S}$, where the sum is over $\mu \in \mathcal{SP}_n$ such that $\mu \prec \lambda$.

Proof. Let $\lambda = (\lambda_1 > \cdots > \lambda_{2r} \geqslant 0)$ be a strict partition in $\mathcal{SP}_n$. We proceed by induction on r. Let $r = 1$. If $\lambda = (i)$ with $1 \leqslant i \leqslant n$ the lemma is clear. For two-row type the lemma is true by (8.2). Let $r \geqslant 2$ and assume the lemma holds for all $\mu = (\mu_1 > \cdots > \mu_{2s} \geqslant 0) \in \mathcal{SP}_n$ with $s < r$. From the definition of the Pfaffian it follows that

$$X_\lambda = \sum_{j=2}^{2r} (-1)^j X_{\lambda_1,\lambda_j} X_{\lambda_2,\ldots,\widehat{\lambda_j},\ldots,\lambda_{2r}}.$$

By the inductive hypothesis, we have

$$X_{\lambda_2,\ldots,\widehat{\lambda_j},\ldots,\lambda_{2r}} = X_{\lambda_2} \cdots \widehat{X_{\lambda_j}} \cdots X_{\lambda_{2r}} + F_j,$$

where F_j is a $\mathcal{S}$-linear combination of X^μ's with $\mu \in \mathcal{SP}_n$ such that $\mu \prec (\lambda_2, \ldots, \widehat{\lambda_j}, \ldots, \lambda_{2r})$. Then it is easy to see that the lemma holds for λ. $\square$

From Lemmas 8.4 and 8.5, we have the following.

Lemma 8.6. *The Pfaffians X_λ $(\lambda \in \mathcal{SP}_n)$ generate $\mathcal{R}_n$ as an $\mathcal{S}$-module.*

8.3. Completion of the proof of Theorem 8.3

It remains to prove the injectivity of ϕ. Let F be in $\mathrm{Ker}(\phi)$. By Lemma 8.6 we have

$$F = \sum_{\lambda \in \mathcal{SP}_n} c_\lambda(a) X_\lambda$$

with coefficients $c_\lambda(a) \in \mathcal{S}$. We know $\phi(X_\lambda) = \sigma(\lambda)$. So we have $0 = \sum_\lambda \iota_n(c_\lambda(a))\sigma(\lambda)$. Since $\sigma(\lambda)$ are linearly independent over $\mathcal{S}$, $\iota_n(c_\lambda(a)) = 0$ for all $\lambda \in \mathcal{SP}_n$. Hence we have $c_\lambda(a) = 0$ $(\lambda \in \mathcal{SP}_n)$ and $F = 0$.

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Appendix A

For the reader's convenience, we provide a summary of some properties of $Q_\lambda(x|a)$. We also prove a vanishing property (Proposition A.1) essentially used in the main body of the paper.

We use standard notation for symmetric functions as in Macdonald's book [31]. Let Λ denote the *ring of symmetric functions* in infinitely many indeterminates $x=(x_1,x_2,\dots)$. The ring Λ is graded as $\Lambda=\bigoplus_{k=0}^{\infty}\Lambda^k$ and each graded part Λ^k has a $\mathbb{Z}$-basis consisting of the *monomial symmetric functions* $m_\lambda=m_\lambda(x)$ (for all partitions λ of k).

Recall an expression for the $Q_k(x)$ the Schur's Q-functions for the one-row partition

$$Q_k(x)=\sum_{\lambda}2^{\ell(\lambda)}m_\lambda(x),$$

where the sum runs over the all partitions λ of k and $\ell(\lambda)$ is the length of λ, the number of non-zero parts of λ. Let Γ be the subring of Λ generated by Q_k:

$$\Gamma=\mathbb{Z}[Q_1,Q_2,Q_3,\dots].$$

We have a gradation $\Gamma=\bigoplus_{k=0}^{\infty}\Gamma^k$ where $\Gamma^k=\Gamma\cap\Lambda^k$. The Schur's Q-functions $Q_\lambda(x)$, with λ strict partition of k, form a distinguished $\mathbb{Z}$-basis of Γ^k.

Let $a_2,a_3,a_4,\dots$ be an infinite sequence of independent variables. We set $a_1=0$. Ivanov introduced a factorial analogue of Q-functions $Q_\lambda(x|a)$ defined for any strict partition λ. Each $Q_\lambda(x|a)$ is an element of the ring $\mathbb{Z}[a_2,a_3,\dots]\otimes_{\mathbb{Z}}\Gamma$. In particular, we have, by Ivanov [19, Theorem 8.2],

$$Q_k(x|a)=\sum_{j=0}^{k-1}(-1)^j e_j(a_2,a_3,\dots,a_k)Q_{k-j}(x).$$

For $k>\ell>0$, we can *define* $Q_{k,\ell}(x|a)$ by Proposition 7.1. Moreover, for arbitrary strict partition λ, we have

$$Q_\lambda(x|a)=\mathrm{Pf}\big(Q_{\lambda_i,\lambda_j}(x|a)\big)_{1\leqslant i<j\leqslant 2r},\tag{A.1}$$

where we write $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2r})$ with $\lambda_1>\dots>\lambda_{2r}\geqslant 0$.

The following result is very important. See Section 6.1 for the meaning of the specialization $Q_\lambda(x_\mu|x_{\langle n\rangle})$.

Table 1
$[X_w]|_v$ for LG_3

$v\backslash w$	$\bar{3}\bar{2}\bar{1}$	$3\bar{2}\bar{1}$	$2\bar{3}\bar{1}$	$1\bar{3}\bar{2}$	$23\bar{1}$	$13\bar{2}$	$12\bar{3}$	123
$\bar{3}\bar{2}\bar{1}$	1	0	0	0	0	0	0	0
$3\bar{2}\bar{1}$	1	$2x_3$	0	0	0	0	0	0
$2\bar{3}\bar{1}$	1	$2x_2$	$2x_2x_{2\bar{3}}$	0	0	0	0	0
$1\bar{3}\bar{2}$	1	$2x_1$	$2x_1x_{1\bar{3}}$	$2x_1x_{1\bar{2}}x_{1\bar{3}}$	0	0	0	0
$23\bar{1}$	1	$2x_{23}$	$2x_2x_{23}$	0	$4x_2x_3x_{23}$	0	0	0
$13\bar{2}$	1	$2x_{13}$	$2x_1x_{13}$	$2x_1x_{1\bar{2}}x_{13}$	$4x_1x_3x_{13}$	$4x_1x_3x_{1\bar{2}}x_{13}$	0	0
$12\bar{3}$	1	$2x_{12}$	$2x_{12}x_{12\bar{3}}$	$2x_1x_{12}x_{1\bar{3}}$	$4x_1x_2x_{12}$	$4x_1x_2x_{12}x_{1\bar{3}}$	$4x_1x_2x_{12}x_{1\bar{3}}x_{2\bar{3}}$	0
123	1	$2x_{123}$	$2x_{12}x_{123}$	$2x_1x_{12}x_{13}$	4Δ	$4x_1\Delta$	$4x_1x_2\Delta$	$8x_1x_2x_3\Delta$

Proposition A.1. *$Q_\lambda(x_\mu|x_{\langle n\rangle})$ vanishes identically unless $\lambda \in \mathcal{SP}_n$.*

Proof. We prove the proposition for $\lambda = (k)$. In [19], Ivanov derived the following equation [19, Theorem 8.2]:

$$\sum_{k=0}^{\infty} \frac{Q_k(x|a)z^k}{\prod_{j=1}^{k}(1-a_{j+1}z)} = \prod_{i=1}^{\infty} \frac{1+x_i z}{1-x_i z}. \tag{A.2}$$

If we specialize the variables as in the statement of the proposition, we have

$$\sum_{k=0}^{n} \frac{Q_k(x_\mu|x_{\langle n\rangle})z^k}{\prod_{j=1}^{k}(1-x_{n+1-j}z)} + \frac{\sum_{k>n} Q_k(x_\mu|x_{\langle n\rangle})z^k}{\prod_{j=1}^{n}(1-x_j z)} = \prod_{1\leqslant i\leqslant n,\delta_i=1} \frac{1+x_i z}{1-x_i z}.$$

Multiplying by $\prod_{1\leqslant i\leqslant n}(1-x_i z)$ the both hand sides, we have

$$\sum_{k>n} Q_k(x_\mu|x_{\langle n\rangle})z^k = -\sum_{k=0}^{n} Q_k(x_\mu|x_{\langle n\rangle})z^k \prod_{j=1}^{n-k}(1-x_j z) + \prod_{i=1}^{n}\bigl(1+(-1)^{\delta_i+1}x_i z\bigr).$$

The right-hand side of the equation is a polynomial in z of degree lower than n and we are done. For general λ, the proposition follows from Proposition 7.1 and the Pfaffian formula (A.1) for $Q_\lambda(x|a)$. $\square$

Here we record Table 1 for $[X_w]|_v$ in the case of LG_3. We set $x_{ij} = x_i + x_j$, $x_{ijk} = x_i + x_j + x_k$ and $\Delta = x_{12}x_{13}x_{23}$. For example $x_{12} = x_1 + x_2$, $x_{12\bar{3}} = x_1 + x_2 - x_3$, etc.

References

[1] H.H. Andersen, J.C. Jantzen, W. Soergel, Representations of quantum groups at a pth root of unity and semisimple groups in characteristic p, Astérisque 220 (1994) 1–321.
[2] A. Arabia, Cohomologie T-équivariante de G/B pour an groupe G de Kac–Moody, C. R. Acad. Sci. Paris Sér. I Math. 302 (1986) 631–634.
[3] L. Biedenharn, J.D. Louck, A new class of symmetric polynomials defined in terms of tableaux, Adv. in Appl. Math. 10 (1989) 396–438.
[4] N. Bourbaki, Groupes et algèbres de Lie, Ch. IV, V, VI, Hermann, Paris, 1968.

[5] M. Brion, Equivariant cohomology and equivariant intersection theory, in: Representation Theories and Algebraic Geometry, Montreal, PQ, 1997, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, pp. 1–37.
[6] W.Y.C. Chen, J.D. Louck, The factorial Schur functions, J. Math. Phys. 34 (1993) 4144–4160.
[7] C. Chevalley, Sur les décompositions cellulaires des espaces G/B, in: W. Haboush, B. Parhall (Eds.), Algebraic Groups and Their Generalizations: Classical Methods, Part 1, in: Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 1–23.
[8] A. Conca, Gröbner bases of ideals of minors of a symmetric matrix, J. Algebra 166 (1994) 406–421.
[9] W. Fulton, C. Woodward, On the quantum product of Schubert classes, J. Algebraic Geom. 13 (4) (2004) 641–661.
[10] S.R. Ghorpade, K.N. Raghavan, Hilbert functions of points on Schubert varieties in the symplectic Grassmannian, Trans. Amer. Math. Soc. 358 (12) (2006) 5401–5423.
[11] M. Goresky, R. Kottwitz, R. MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, Invent. Math. 131 (1998) 25–83.
[12] I. Goulden, C. Greene, A new tableau representation for supersymmetric Schur functions, J. Algebra 170 (1994) 687–703.
[13] I.P. Goulden, A.M. Hamel, Shift operators and factorial symmetric functions, J. Combin. Theory Ser. A 69 (1995) 51–60.
[14] W. Graham, Positivity in equivariant Schubert calculus, Duke Math. J. 109 (3) (2001) 599–614.
[15] H. Hiller, Combinatorics and intersections of Schubert varieties, Comment. Math. Helv. 57 (1982) 41–59.
[16] H. Hiller, B. Boe, Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n, Adv. Math. 62 (1986) 49–67.
[17] T. Ikeda, H. Naruse, Excited Young diagrams and equivariant Schubert calculus, in preparation.
[18] V.N. Ivanov, Dimensions of skew shifted Young diagrams and projective characters of the infinite symmetric group, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 240 (1997) 116–136 (in Russian); English translation in: J. Math. Sci. 96 (1999) 3517–3530.
[19] V.N. Ivanov, Interpolation analogue of Schur Q-functions, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 307 (2004) 99–119, available at arXiv: math.CO/0305419.
[20] T. Józefiak, Schur Q-functions and cohomology of isotropic Grassmannians, Math. Proc. Cambridge Philos. Soc. 109 (1991) 471–478.
[21] A. Knutson, T. Tao, Puzzles and (equivariant) cohomology of Grassmannians, Duke Math. J. 119 (2) (2003) 221–260.
[22] V. Kodiyalam, K.N. Raghavan, Hilbert functions of points on Schubert varieties in Grassmannians, J. Algebra 270 (2003) 28–54.
[23] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac–Moody group G, Adv. Math. 62 (1986) 187–237.
[24] V. Kreiman, Schubert classes in the equivariant K-theory and equivariant cohomology of the Grassmannian, math.AG/0512204.
[25] V. Kreiman, Schubert classes in the equivariant K-theory and equivariant cohomology of the Lagrangian Grassmannian, math.AG/0602245.
[26] A. Kresch, H. Tamvakis, Quantum cohomology of the Lagrangian Grassmannians, J. Algebraic Geom. 12 (2003) 777–810.
[27] S. Kumar, Kac–Moody Groups, Their Flag Varieties and Representation Theory, Progr. Math., vol. 204, Birkhäuser Boston, Boston, MA, 2002.
[28] V. Lakshmibai, J. Weyman, Multiplicities of points on a Schubert variety in a minuscule G/P, Adv. Math. 84 (1990) 179–208.
[29] V. Lakshmibai, K.N. Raghavan, P. Sankaran, Equivariant Giambelli and determinantal restriction formulas for the Grassmannian, in: Special issue: In Honor of Robert Macpherson, Part 1 of 3, Pure Appl. Math. Quart. 2 (3) (2006) 699–717.
[30] A. Lascoux, M.-P. Schützenberger, Interpolation de Newton à plusieurs variables, in: Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin, 36ème anné, Paris, 1983–1984, in: Lecture Notes in Math., vol. 1146, Springer, Berlin, 1985, pp. 161–175.
[31] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford Univ. Press, Oxford, 1995.
[32] I.G. Macdonald, Schur functions: Theme and variations, in: Actes 28-e Séminaire Lotharingien, in: Publ. Inst. Rech. Math. Av., Univ. Louis Pasteur, Strasbourg, 1992, pp. 5–39.
[33] L.C. Mihalcea, Giambelli formulae for the equivariant quantum cohomology of the Grassmannian, arXiv: math.CO/0506335, Trans. Amer. Math. Soc., in press.
[34] A.I. Molev, B.E. Sagan, A Littlewood–Richardson rule for factorial Schur functions, Trans. Amer. Math. Soc. 351 (1990) 4429–4443.

[35] J.J.C. Nimmo, Hall–Littlewood symmetric functions and the BKP equation, J. Phys. A 23 (1990) 751–760.
[36] A.Yu. Okounkov, Quantum immanants and higher Capelli identities, Transform. Groups 1 (1996) 99–126.
[37] A. Okounkov, G. Olshanski, Shifted Schur functions, St. Petersburg Math. J. 9 (1998) 239–300.
[38] P. Pragacz, Algebro-geometric applications of Schur S- and Q-polynomials, in: Séminaire d'Algèbre Dubreil–Malliavin 1989–1990, in: Springer Lecture Notes in Math., vol. 1478, Springer, Berlin, 1991, pp. 130–191.
[39] S. Robinson, A Pieri-type formula for $H_T^*(SL_n(\mathcal{C})/B)$, J. Algebra 249 (2002) 38–58.
[40] J. Rosenthal, A. Zelevinsky, Multiplicities of points on Schubert varieties in Grassmannians, J. Algebraic Combin. 13 (2001) 213–218.