Some Problems on Dedekind-Type Groups

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1. INTRODUCTION

Knowing the properties of subgroups of a group, what can we say about the group? This problem has been of consistent interest in the theory of groups from the beginning of the century. Miller and Moreno [16] studied groups all of whose proper subgroups are abelian. Schmidt [29] and Golffond [10] determined all finite groups with all proper subgroups nilpotent.

The above authors were interested in groups all of whose proper subgroups have a given property, no matter how the subgroups are embedded in the group. The characterisation of groups all of whose subgroups are embedded in a given manner also has a long history. The results of Dedekind [6] and Baer [2] characterize Dedekind groups (groups all of whose subgroups are normal). Roseblade [28] studied groups all of whose subgroups are subnormal.

The study of the structure of groups which have subgroups of mixed type (i.e., a subgroup of the group either has a prescribed property or is embedded in a given manner) has drawn the interest of several authors. Nagrebeckii [17–19], Romalis and Seskin [25–27], Černikov [5], Garasčuk [9], and Mahanev [15] were interested in meta-Hamiltonian groups (groups all of whose subgroups are normal or abelian). Phillips and Wilson [21] have determined groups (with some finiteness condition) in which every subgroup is either serial or abelian. Bruno and Phillips [4] deal with non-locally nilpotent groups (with some finiteness condition) in which every subgroup is either normal or locally nilpotent.

The transversals of a subgroup of a group determine the embedding of the subgroup in the group. The object of this paper is to introduce new types of subgroups by specifying the properties of transversals and initiate
a programme of research dealing with the problems of the above type in relation to these types of subgroups.

Let \( H \) be a subgroup of a group \( G \) and \( S \) a right transversal of \( H \) in \( G \) which contains the identity. Then \( S \) is a right quasigroup with identity with respect to the binary operation \( \circ \) defined by \( \{x \circ y\} = S \cap Hxy; \ x, y \in S \). Conversely, we have

**Theorem [22]**. Let \( S \) be a right quasigroup with identity. Then there is a group \( G^S \) containing \( S \) as a right transversal such that given any other group \( G \) containing \( S \) as a right transversal, there exists a unique homomorphism from \( G \) to \( G^S \) which is the identity map on \( S \).

If a subgroup \( H \) is normal in \( G \), then all the transversals of \( H \) in \( G \) are isomorphic as right quasigroups (they are isomorphic to the quotient group \( G/H \)). Let us call a subgroup \( H \) a perfectly stable subgroup if all the right transversals of \( H \) in \( G \) are isomorphic as right quasigroups. For finite groups we have

**Theorem [23]**. Every perfectly stable subgroup of a finite group is normal.

For infinite groups we do not know the answer to the preceding result.

Consider the subgroup \( \langle S \rangle \) of a group \( G \) generated by a right transversal \( S \) of a subgroup \( H \) of a group \( G \). Let \( H_S = \langle S \rangle \cap H \). Then \( H_S = \langle \{xy(\circ y)^{-1}:x, y \in S\} \rangle \), where \( \circ \) is the induced binary operation on \( S \) (as discussed earlier) and \( H_S S = \langle S \rangle \). Identifying \( S \) with the set \( G/H \) of right cosets of \( H \) in \( G \), we obtain the permutation representation \( \phi: G \to \text{Sym}(S) \) \((\phi(g)(x)) = S \cap H_{Sg}; \ g \in G, x \in S \). Let \( G_S = \phi(H_S) \).

Then \( G_S \) is the group torsion [22] (or associator) of the induced right quasigroup structure on \( S \) (depends only on the right quasigroup structure on \( S \)). Clearly, a subgroup \( H \) of a group \( G \) is normal if and only if \( G_S \) is trivial for every right transversal \( S \) of \( H \) in \( G \).

**Definition 1.1**. A subgroup \( H \) of a group \( G \) will be called a stable subgroup of \( G \) if for any right transversals \( S_1 \) and \( S_2 \) of \( H \) in \( G \), \( G_{S_1} \) is isomorphic to \( G_{S_2} \).

Thus all normal subgroups are stable. Every subgroup of order two of a non-abelian simple group is stable. In fact if \( G \) is a non-abelian simple group of order \( n \) and \( H \) is a subgroup of order \( k \) such that \( k! < 2n \), then \( H \) is a stable subgroup of \( G \). In particular, every finite group can be embedded as a stable subgroup into a finite simple group. Tarski groups (two generator infinite simple groups all of whose proper subgroups are cyclic of same prime (sufficiently large) order) [20] are infinite simple groups all of whose subgroups are stable. Here we initiate a program of characterizing finite groups all of whose subgroups are stable. In this paper, we establish the following two results:
**Theorem 2.4.** A finite solvable group has all its subgroups stable if and only if it is a Dedekind group.

**Theorem 3.9.** A finite simple group has all its subgroups stable if and only if it is non-factorisable.

The complete list of non-factorisable simple groups can be found in [14].

### 2. Finite Solvable Groups with All Subgroups Stable

The following two theorems respectively due to Baer [2] and Hall [11] will be used in this section.

**Theorem A.** All subgroups of a non-abelian group $G$ are normal if and only if $G$ is a direct product of the quaternion group of order 8, an elementary abelian 2-group and an abelian group with all elements of odd order.

**Theorem B.** A finite group $G$ is solvable if and only if for every set $\pi$ of prime divisors of $|G|$, $G$ has a Hall $\pi$-subgroup (a $\pi$-subgroup whose index is a $\pi'$-number).

**Proposition 2.1.** Let $H$ be a normal subgroup of $G$. Then a subgroup $K$ of $G$ containing $H$ is a stable subgroup of $G$ if and only if $K/H$ is a stable subgroup of $G/H$.

**Proof.** It can be easily checked that $S \rightarrow \nu(S) = \{Hx|x \in S\}$, where $\nu$ is the quotient map from $G$ to $G/H$, is a surjective map from the set of all right transversals of $K$ in $G$ to the set of all right transversals of $K/H$ in $G/H$ such that the corresponding right quasigroups are isomorphic. The result now follows.

The proof of Theorem 2.4 depends on two lemmas.

**Lemma 2.2.** All subgroups of a finite $p$-group, where $p$ is an odd prime, are stable if and only if it is an abelian $p$-group.

**Proof.** The proof is by induction on $|G|$. If $|G| = p$, then there is nothing to do. Assume that the result is true for all those $p$-groups whose order is less than $|G|$. Suppose there is a subgroup $K$ of $G$ of order $p$ which is not contained in some maximal subgroup $M$ of $G$. Then $K \cap M = \{e\}$, $KM = G$, and so $M$ can be chosen as a right transversal of $K$ in $G$ whose group torsion $G_M = \{e\}$. Since $K$ is a stable subgroup of $G$, the group torsion $G_S = \{e\}$ for every right transversal $S$ of $K$ in $G$ and so $K$ is normal. Thus $G = K \times M$ (direct product) and $M$ is isomorphic to $G/K$. 


By Proposition 2.1, all subgroups of $M$ are stable. By the induction assumption $M$, and hence $G$, is abelian.

Suppose, if possible, that $G$ is non-abelian and all maximal subgroups of $G$ contain all elements of order $p$. Let $a$ be an element of the center $Z(G)$ of $G$ of order $p$. Then $G/\langle a \rangle$ is a $p$-group all of whose subgroups are stable (Proposition 2.1). By the induction assumption $G/\langle a \rangle$ is abelian. Since $G$ is assumed to be non-abelian, the commutator subgroup $G^2 = \langle a \rangle$.

This also shows that there is a unique subgroup of order $p$ in $Z(G)$. Hence $Z(G)$ is cyclic. Suppose $|Z(G)| = p^i$. Since $G^2 \subseteq Z(G)$ and $|G^2| = p$, we get $[x, g^p] = [x, g]^p = e$ for every $x, g \in G$. This shows that $g^p \in Z(G)$ for every $g \in G$ and $G/Z(G)$ is an elementary abelian $p$-group. Let $x$ be an element of $G$ of order $p$. Then $x \in Z(G)$. For, if not, $xZ(G)$ will be an element of $G/Z(G)$ of order $p$. Since $G/Z(G)$ is an elementary abelian $p$-group there will be a maximal subgroup $M$ of $G$ such that $xZ(G)$ does not belong to $M/Z(G)$. But then there is a maximal subgroup $M$ of $G$ such that $x$ does not belong to $M$, a contradiction to the supposition that every maximal subgroup of $G$ contains every element of order $p$. Thus, every element of $G$ of order $p$ is contained in $Z(G)$. Hence $Z(G)$ is cyclic. Suppose $|Z(G)| = p^i$. Since $G^2 \subseteq Z(G)$ and $|G^2| = p$, we get $[x, g^p] = [x, g]^p = e$ for every $x, g \in G$. This shows that $g^p \in Z(G)$ for every $g \in G$ and $G/Z(G)$ is an elementary abelian $p$-group. Let $x$ be an element of $G$ of order $p$. Then $x \in Z(G)$. For, if not, $xZ(G)$ will be an element of $G/Z(G)$ of order $p$. Since $G/Z(G)$ is an elementary abelian $p$-group there will be a maximal subgroup $M$ of $G$ such that $xZ(G)$ does not belong to $M/Z(G)$. But then there is a maximal subgroup $M$ of $G$ such that $x$ does not belong to $M$, a contradiction to the supposition that every maximal subgroup of $G$ contains every element of order $p$.

In particular, $H/\langle a \rangle$ is normal in $G/\langle a \rangle$. This shows that every subgroup of $G$ is normal in $G$. From Theorem A, it follows that $G$ is abelian.

**Lemma 2.3.** Let $G$ be a finite 2-group all of whose subgroups are stable. Then $G$ is a Dedekind group.

**Proof.** The proof is again by induction on $|G|$. If $|G| = 2$, then there is nothing to do. Assume that the result is true for all those 2-groups whose order are less than $|G|$. Then all proper quotient groups of $G$ are Dedekind (Proposition 2.1). If all proper quotient groups of $G$ are abelian, then as in the Lemma 2.2 we can show that all subgroups of $G$ are normal. Suppose, now, that not all proper quotient groups of $G$ are abelian. We may also suppose that all maximal subgroups of $G$ contain all elements of order 2, for otherwise as in the proof of Lemma 2.2, $G$ will be the direct product of a Dedekind group and a cyclic group of order 2 which again will be a Dedekind group (Theorem A). Since every non-trivial normal subgroup of $G$ intersects the center $Z(G)$ nontrivially, there is a subgroup $L = \langle a \rangle \subseteq Z(G)$ of order 2 such that $G/L$ is a non-abelian Dedekind group. By Theorem A, $G/L = A \times B_1 \times \cdots \times B_n$, where $A$ is the quaternion group of order 8 and $B_i$ are groups of order 2. Let $\nu: G \to G/L$ be the quotient map. Put $\nu^{-1}(A) = K$, $\nu^{-1}(B_i) = H_i$. Then $K$ and $H_i$ are
normal subgroups of $G$ containing $L$. We also have the following:

(i) $G = KH_1H_2 \cdots H_r$

(ii) $L = K \cap H_1H_2 \cdots H_r$

(iii) $K$ is a subgroup of order 16 such that $K/L$ is the quaternion group of order 8.

(iv) $G/K$ is an elementary abelian 2-group.

(v) All elements of $G$ of order 2 are contained in $K$: If there is an element $x$ of $G$ of order 2 such that $x$ does not belong to $K$, then there is a maximal subgroup $M/K$ of $G/K$ such that $xK$ does not belong to $M/K$. This gives us a maximal subgroup $M$ of $G$ missing $x$, a contradiction to the supposition that all maximal subgroups of $G$ contain all elements of $G$ of order 2.

Suppose, now, that there is a maximal subgroup $M$ of $K$ which does not contain $L$. Then $K = ML$ and $M \cap L = \{e\}$, where $M$ is the quaternion group of order 8. Now $G = KH_1H_2 \cdots H_r = MLH_1H_2 \cdots H_r$. Further, $M \cap H_1H_2 \cdots H_r \subseteq K \cap H_1H_2 \cdots H_r = L$ and since $M \cap L = \{e\}$, $M \cap H_1H_2 \cdots H_r = \{e\}$. Thus $H_1H_2 \cdots H_r$, which is a normal subgroup of $G$ (being a product of normal subgroups) is a right transversal of $M$ in $G$ whose group torsion is trivial. Since every subgroup of $G$ is stable and in particular $M$ is stable, group torsion of each right transversal of $M$ in $G$ is trivial. This shows that $M$ is normal and so $G = M \times H_1H_2 \cdots H_r$. By Proposition 2.1, all subgroups of $H_1H_2 \cdots H_r$ are stable and therefore, by the induction assumption $H_1H_2 \cdots H_r$ is Dedekind. This implies that all subgroups of $G$ of order 2 are normal in $G$. Let $H$ be a non-trivial subgroup of $G$. Then it contains a subgroup $N$ of order 2 which is normal in $G$. By the induction assumption $G/N$ will have all its subgroups normal and in particular, $H/N$ is normal. Hence $H$ is normal in $G$ and so $G$ will be Dedekind.

We may assume, therefore, that $L$ is contained in every maximal subgroup of $K$. Let $b$ be an element of $K$ of order 2 and $M$ a maximal subgroup of $K$. Since $MH_1H_2 \cdots H_r$ being the inverse image of the maximal subgroup $M/L$ of $K/L$ under the surjective homomorphism $kh_1h_2 \cdots h_r \mapsto kL$ from $G$ to $K/L$, is a maximal subgroup of $G$, $b \in MH_1H_2 \cdots H_r$. This shows that $m^{-1}b \in K \cap H_1H_2 \cdots H_r = L \subseteq M$ for some $m \in M$ and so $b \in M$. Thus every maximal subgroup $M$ of $K$ contains every involution of $K$ and it follows from this that $M$ contains every involution of $G$. The subgroup $K$ cannot be cyclic for $K/L$ is the quaternion group of order 8. Suppose $K$ contains a cyclic subgroup of order 8. Then, it being a maximal subgroup of $K$, contains all elements of $K$ which are of order 2. Since a cyclic 2-group contains a unique subgroup of order 2, $K$ contains a unique subgroup of order 2. By [24, 5.3.6, p. 138],

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$K$ is cyclic or generalised quaternion. Since $K/L$ is the quaternion group of order 8, $K$ is neither of these. Thus $K$ cannot contain any element of order 8 and therefore $K$ has a presentation of the type

$$\langle a, b, c; a^2 = b^4 = c^4 = e, [a, b] = [a, c] = e, \quad b^2 = c^2 e, cbc^{-1} = b^{-1} \eta \rangle,$$

(1)

where $e = e$ or $a$ and $\eta = e$ or $a$ (observe that $b^2 \neq a$ and $c^2 \neq a$). Now let us look at the elements of $K$ of order 2. An element of $K$ can be written as $a^i b^j c^k$, where $i, j, k = 0, 1, 2, 3$. Consider

$$b^i c^j b^k = b^i (c^j b c^{-1})^j c^{2j}$$

$$= \begin{cases} b^{2j} & \text{if } j \text{ is even} \\ c^{2j} \eta & \text{if } j \text{ is odd} \end{cases}.$$

Since $b^{2j} = e \iff i$ is even and $c^{2j} \eta$ can never be $e$ for $j$ odd, the elements of $K$ of order 2 are precisely $a, b^2$ and $ab^2$. By (v), these are the only elements of $G$ of order 2. It is clear from the presentation (1) that these all lie in the center of $K$. We show that they all lie in the center of $G$ also. The element $a$ is already in the center of $G$. Let $h_i \in H_i$. Clearly, $bh_i b^{-1} h_i^{-1} \in K \cap H_i = L = \langle a \rangle$. Hence $bh_i = \mu h_i b$ where $\mu = e$ or $a$. Now, $b^2 h_i = b \mu h_i b = \mu b h_i b = \mu^2 h_i b^2 = h_i b^2$. This shows that $b^2$ and so also $ab^2$ commutes with each element of $G$. Thus every element of $G$ of order 2 lies in the center. Hence, every non-trivial subgroup of $G$ contains a non-trivial normal subgroup. Using the induction as before, we find that all subgroups of $G$ are normal.

**Theorem 2.4.** A finite solvable group all of whose subgroups are stable is a Dedekind group.

**Proof.** Let $G$ be a finite solvable group all of whose subgroups are stable. Let $p$ be a prime dividing the order of $G$. By Theorem B, $G$ has a Hall $p'$-subgroup $H$. Let $P$ be a Sylow $p$-subgroup of $G$. Then $G = PH$ and $P \cap H = \{e\}$. Thus $H$ is a right transversal of $P$ in $G$ whose group torsion is trivial. Since $P$ is stable, the group torsion of each right transversal of $P$ in $G$ is trivial and so $P$ is normal. Thus all Sylow subgroups of $G$ are normal and $G$ is the direct product of its Sylow subgroups. The result now follows from Prop. 2.1, Lemma 2.2, Lemma 2.3, and Theorem A.

**Corollary 2.5.** A finitely generated solvable group all of whose subgroups are stable is a nilpotent group.

**Proof.** From a result of Robinson and Wehrfritz (for the proof, see [24, 15.5.3, p. 459]), a non-nilpotent finitely generated solvable group has a
finite image which is not nilpotent. Since an image of a group all of whose subgroups are stable, has all its subgroups stable (Proposition 2.1) and there is no finite non-nilpotent solvable group all of whose subgroups are stable (Theorem 2.4), the result follows.

**Corollary 2.6.** Let G be a non-Dedekind finite group all of whose subgroups are stable and all of whose proper quotient groups are Dedekind. Then G is simple.

**Proof.** Suppose that G is not simple. Let H be a nontrivial proper normal subgroup of G. Then \( G/H \) will be Dedekind and so G will have a normal subgroup L of index p for some prime p. Further there exists a cyclic subgroup \( K = \langle a \rangle \) of G of order a power of p such that \( G = K.L \). Choose a right transversal S of K in G out of elements of L. Then the group torsion \( G_s \) of S is not isomorphic to K. By Theorem 2.4, G is not solvable and therefore, there is a prime \( q \neq p, q > 2 \) such that \( q \) divides \( |G| \). Let x be an element of G of order \( q \). Choose a right transversal T of K in containing \( \langle e, x, ax^2 \rangle \). Then \( \langle T \rangle = G \) and so \( K_T = \langle T \rangle \cap K = K \). Let \( \phi: G = K_T T \rightarrow G_T T \) be the surjective homomorphism given by \( \phi(kr) = \rho(k)r \), where \( \rho \) is the permutation representation of K on T and \( G_T \), the group torsion of T. Clearly, \( \text{Ker } \phi = \text{Ker } \rho|K \) is a cyclic normal subgroup of G contained in K. Since K is stable, \( G_s \cong G_s \neq K \). This shows that \( \text{Ker } \phi \) is non-trivial. By the supposition, \( G/\text{Ker } \phi \) is Dedekind and so G is solvable. This is a contradiction to the Theorem 2.4.

### 3. SIMPLE GROUPS

In this section, we classify finite simple groups with all subgroups stable.

**Lemma 3.1.** Let G be a simple group and H a subgroup of G. If S is a right transversal of H in G which generates G, then the group torsion \( G_S \) of S is isomorphic to H.

**Proof.** Suppose that \( \langle S \rangle = G \). Then \( H_S = H \cap \langle S \rangle = H \). Since G is simple, the homomorphism \( \phi: G = H_S \rightarrow G_S \) given by \( \phi(hS) = \rho(h)x \), where \( \rho \) is the permutation representation of H on S, is an injective homomorphism. Hence \( G_S \cong H \).

**Lemma 3.2.** Let G be a finite simple group and H a subgroup of G such that \( G \neq H.K \) for any proper subgroup K of G. Then H is a stable subgroup of G.

**Proof.** Under the hypothesis of the Lemma, all right transversals of H in G will generate G. The result follows from Lemma 3.1.
**Theorem 3.3.** Let $G$ be a finite non-abelian simple group generated by two elements. Then all subgroups of $G$ are stable if and only if $G$ admits no factorisation as product of two proper subgroups.

**Proof.** Let $G$ be a finite non-abelian simple group generated by two elements $a$ and $b$. Suppose that $G = A \cdot B$, where $A$ and $B$ are proper subgroups of $G$. We can find a right transversal $S$ of $A$ in $G$ from the elements of $B$ whose group torsion $G_S$ is not isomorphic to $A$.

Suppose that $a \in A$. Then $b \not\in A$. Since $G$ is non-abelian simple group, there is a prime $p \geq 5$ such that $p$ divides $|G|$. Again since $G$ is simple, there is an element $x$ of $G$ of order $p$ such that $x \not\in A$. If $b$ does not lie in the union $Ax \cup Ax^2 \cup \cdots \cup Ax^{p-1}$, then we can find a right transversal $T$ of $A$ in $G$ which contains $\{e, x, ax^2, \ldots, x^{p-1}, b\}$. Clearly the subgroup generated by $T$ in $G$ is $A$ and so the group torsion $G_T$ of $T$ is isomorphic to $A$. This means that $A$ is not stable. Suppose now that $b = cy$, where $c \in A$ and $y = x^i$, $1 \leq i \leq p - 1$. Choose a right transversal $T'$ of $A$ in $G$ which contains $\{e, y, cy^2, ay^3, y^4, \ldots, y^{p-1}\}$. Then, again $G_T$ is isomorphic to $A$. Thus in this case $A$ is nonstable.

Next, suppose that $a \not\in A$. Let $H$ be the subgroup of $G$ generated by a right transversal $U$ of $A$ in $G$ which contains the element $a$. If $H = G$, then the group torsion $G_U$ of $U$ is isomorphic to $A$ and so $A$ will not be stable. If $H \neq G$, then $G = AH = HA$, where $H$ and $A$ are proper subgroups of $G$ such that $a \in H$. As in the previous case, $H$ will not be a stable subgroup. The converse part of this theorem follows from Lemma 3.2.

**Corollary 3.4.** If $G$ is a simple group of alternating type or a simple group of Lie type, then all its subgroups are stable if and only if it cannot be expressed as a product of two proper subgroups.

**Proof.** Since all simple groups of alternating type and also of Lie type are 2-generator groups [30], the result follows from Theorem 3.3.

**Corollary 3.5.** There is no non-abelian finite simple group which has a subgroup of prime power index and all of whose subgroups are stable.

**Proof.** This follows from [1, Theorem 5.8] and Theorem 3.3 if we note that all groups in [1, Theorem 5.8] are factorisable and generated by two elements.

**Corollary 3.6.** Let $G$ be a finite group in which the centralizer of any involution is an elementary abelian 2-group. Then $G$ has all its subgroups stable if and only if $G$ is an elementary abelian 2-group.
Proof. From a result of Brauer et al. [3], under the hypothesis of the corollary,

(i) \(|G| = 2m\), where \(m\) is an odd number or
(ii) the Sylow 2-subgroup of \(G\) is normal in \(G\) or
(iii) \(G = PSL(2, q)\), where \(q\) is a power of 2.

In the first two cases \(G\) is solvable and so Dedekind (Theorem 2.4). Since \(PSL(2, q)\) is non-factorisable [13] only in the cases \(q \equiv 1(\text{mod } 4)\) and \(q \not\equiv (5, 9, 29)\), case (iii) cannot occur (Corollary 3.4). The result follows if we observe that a Dedekind group satisfies the hypothesis of the corollary if and only if it is an elementary abelian 2-group.

For doubly transitive permutation groups, we have the following result:

**Theorem 3.7.** Let \(G\) be a non-Dedekind doubly transitive permutation group in which the identity is the only element fixing three elements. Then \(G\) has all its subgroups stable if and only if it is one of the following:

(i) \(PSL(2, q)\), \(q \equiv 1(\text{mod } 4)\), \(q \not\equiv (5, 9, 29)\).
(ii) Suzuki group \(Sz(q)\), \(q = 2^{2n+1}, n \geq 1\).

Proof. Since the Frobenius complements of a Frobenius group are non-stable it follows from Theorem 2.4 and Feit [7] that \(G\) has no regular normal subgroup. Thus \(G\) is a Zassenhaus group. From the classification theorem for Zassenhaus groups (see [12]), \(G\) is one of the following:

(i) \(PSL(2, p')\), \(p' > 4\),
(ii) Suzuki groups \(Sz(2^{2n+1})\), \(n \geq 1\),
(iii) \(PGL(2, p')\), \(p' > 3\), \(p \geq 3\),
(iv) \(M(p')\), \(p' > 3\), \(p \geq 3\),

where \(M(p')\) is the group of all mappings on \(P(1, p^{2r}) = GF(p^{2r}) \cup \{\infty\}\) of the form

\[
x \rightarrow \frac{ax^{p'} + b}{cx^{p'} + d}, \quad \text{if } ad - bc \not\in (GF(p^{2r}))^2
\]

and

\[
x \rightarrow \frac{ax + b}{cx + d}, \quad \text{if } ad - bc \in (GF(p^{2r}))^2 - \{0\}.
\]

Now, from Corollary 3.4 and Ito [13], it follows that \(PSL(2, p')\) has all its subgroups stable if and only if \(p' \equiv 1(\text{mod } 4)\) and \(p' \not\equiv (5, 9, 29)\). Further, from Suzuki [31, Theorem 9, p. 137] and Corollary 3.4, it follows that all
subgroups of $Sz(2^{2n+1}), n \geq 1$ are stable. Finally, looking at the structure of the groups $PGL(2, p^n)$ and $M(p^n)$ (see [12]), we find that these groups have nonstable 2-subgroups.

**Theorem 3.8.** Let $G$ be a finite non-abelian simple group of order $p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, $\alpha_i \geq 1$. Let $p_0 = \max(p_1, p_2, \ldots, p_r)$ and suppose that $\alpha_2 + \alpha_3 + \cdots + \alpha_r \leq p_0 - 1$. Then all subgroups of $G$ are stable if and only if it cannot be written as a product of two proper subgroups.

**Proof.** We may assume (Corollary 3.5) that $G$ has no subgroup of prime power index. Suppose that $G = H.K$, where $H$ and $K$ are proper subgroups of $G$. Suppose further, that $|H| = p_1^{\beta_1} \cdots p_r^{\beta_r}$, $\alpha_i \geq \beta_i \geq 0$. Then there exists $j \geq 2$ such that $\beta_j < \alpha_j$ and so $\beta_2 + \beta_3 + \cdots + \beta_j < p_0 - 1$. Since a subgroup of order $p^n$ has a set of generators containing $n$ elements, we can find a set $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ of generators of a Sylow $p_j$-subgroup of $H$. Since $G$ is simple, there is an element $x \in G - H$ such that the order of $x$ is $p_0$ and so $e, x, x^2, \ldots, x^{p_0 - 1}$ will lie in different right cosets of $H$ in $G$. Choose a right transversal $T$ of $H$ in $G$ which contains

$$\{e, x, a_{12}x^2, a_{13}x^{b_2+1}, a_{13}x^{b_2+2}, \ldots, a_{\alpha_1}x^{b_2+\beta_3+\cdots+\beta_j+1}, \ldots,$$

$$a_1x^{b_2+\cdots+\beta_{j-1}+2}, \ldots, a_{\alpha_1}x^{b_2+\cdots+\beta_{j-1}+\beta_j+1}\}.$$

If $T$ generates $G$, then the group torsion $G_T$ of $T$ is isomorphic to $H$ (Lemma 3.1). Further, we can find a right transversal $T'$ of $H$ in $G$ out of elements of $K$ whose group torsion will not be isomorphic to $H$. Thus in this case $H$ is non-stable. Next, suppose that $T$ generates a proper subgroup $L$ of $G$. Then $G = H.L$ and $p_1^{\beta_1} \cdots p_r^{\beta_r}$ divides $|H \cap L|$. Hence $p_1^{\beta_1} \cdots p_r^{\beta_r}$ divides the order of $L$ and so the index $[G : L] = p_1^{\delta_1}$ for some $\delta_1 > 0$. From Corollary 3.5, $G$ will have non-stable subgroups.

All sporadic simple groups except the McLaughlin group $Mc$ of order $2^4 3^5 5^2 7.11$ and the Fisher group $Fi_{22}$ of order $2^{17} 3^9 5^2 7.11.13 [14]$ satisfy the hypothesis of the Theorem 3.8. If we look at the maximal subgroups of $Mc$ [8], then it becomes clear that $Mc$ is not factorisable.

Consider $Fi_{22}$. Suppose that $Fi_{22} = H.K$, where $H$ and $K$ are proper subgroups of $Fi_{22}$. We can find a subset $\{a_1, a_2, \ldots, a_r\}$ of $Fi_{22}$, such that the subgroup generated by $a_1, a_2, \ldots, a_r$ contains a Sylow 3-subgroup and a Sylow 5-subgroup of $H$. Let $x$ be an element of $Fi_{22} - H$ of order 13. Let $T$ be a right transversal of $H$ in $Fi_{22}$ which contains $\{e, x, a_1x^2, \ldots, a_rx^{12}\}$. The order of the subgroup generated by $T$ is divisible by $3^9 5^2 13$. Since $Fi_{22}$ contains no maximal subgroup whose order
is divisible by $3^9 \cdot 5^7 \cdot 13$, the subgroup generated by $T$ is $F_{22}$. As before, $H$ is nonstable. This completes the proof of the following theorem:

**Theorem 3.9.** All subgroups of a finite simple group are stable if and only if it is non-factorisable.

**4. SOME PROBLEMS**

Let $S$ be a right transversal of a subgroup $H$ of a group $G$. Let $H^S$ be the smallest normal subgroup of $\langle S \rangle = H^S$ which contains $H$. Let $M(S)$ denote the quotient group $H^S/H^S \cong G^S/\phi(H^S)$, where $\phi$ is as in the proof of the Lemma 3.1. Clearly, $M(S)$ depends only on the right quasigroup structure on $S$ and has a presentation $\langle S; R \rangle$ where $R = \langle xy(x \circ y)^{-1} | x, y \in S \rangle$ is the set of relators of the presentation, $\circ$ being the binary operation induced on $S$ (see the introduction). There is an obvious homomorphism $\alpha: S \to M(S)$ such that given any group $K$ and a homomorphism $\beta: S \to K$, there is a unique homomorphism $\eta: M(S) \to K$ such that $\beta = \eta \circ \alpha$. If $H$ is a normal subgroup of $G$, then $M(S) \cong G/H$ for every right transversal $S$ of $H$ in $G$. Let us call a subgroup $H$ of a group $G$ a **prenormal** subgroup of $G$ if for any two right transversals $S_1$ and $S_2$ of $H$ in $G$, $M(S_1) \cong M(S_2)$. The conclusions of Theorem 2.4 and Corollary 2.5 are valid if we replace stable subgroups by prenormal subgroups. It is an interesting problem to examine the results of the paper after replacing stable by prenormal.

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