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Projective completions of Jordan pairs Part I. The generalized projective geometry of a Lie algebra

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Abstract

We prove that the projective completion (X^+, X^-) of the Jordan pair $(\mathfrak{g}_1, \mathfrak{g}_{-1})$ corresponding to a 3-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ can be realized inside the space $\mathcal{F}$ of inner 3-filtrations of $\mathfrak{g}$ in such a way that the remoteness relation on $X^+ \times X^-$ corresponds to transversality of flags. This realization is used to give geometric proofs of structure results which will be used in Part II of this work in order to define on X^+ and X^- the structure of a smooth manifold (in arbitrary dimension and over general base fields or -rings).

Keywords: Jordan pair; Generalized projective geometry; 3-graded Lie algebra; Projective elementary group; Symmetric space; Projective completion

Introduction

A basic construction in linear algebra permits to imbed an affine space V into a projective space X as the complement of a "hyperplane at infinity"—let us assume here for simplicity that everything is defined over a commutative field $\mathbb{K}$, so that X may be seen as the projective space $\mathbb{P}(W)$ with $W \cong V \oplus \mathbb{K}$. In the real or complex case, if the

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dimension is finite or if V is, e.g., a Banach space, the projective space X is a smooth manifold with V as a typical chart domain. An atlas of X is obtained by taking all affine parts of X (all complements of hyperplanes of X); as is well known, change of charts is then given by rational and hence differentiable expressions. Similar constructions are known for other manifolds X such as Grassmannians, spaces of Lagrangians or conformal quadrics.

In the present work we will construct such manifolds in a very general context, in arbitrary dimension, and over general base fields or -rings instead of $\mathbb{R}$ or $\mathbb{C}$. The present and first part contains the algebraic theory, and Part II [6] contains the analytic theory. For the case of base fields other than $\mathbb{R}$ or $\mathbb{C}$, we use in Part II suitable concepts of differential calculus and of smooth manifolds developed in [5] which, in the case of locally convex real or complex model spaces—in particular, for Banach and Fréchet spaces—agree with the usual concepts (but work more generally for manifolds modeled on any Hausdorff topological vector space). The present Part I is of independent interest since indeed a good deal of the above mentioned constructions is purely algebraic and admits a beautiful Lie- and Jordan theoretic interpretation. Geometrically, we work in the context of *generalized projective geometries* (introduced in [3]), and algebraically, in the context of 3-*graded Lie algebras* which in turn correspond to *Jordan pairs* (however, the paper is self-contained, and we assume only basic knowledge of Lie-algebras). As in the ordinary projective case, it is a purely algebraic problem to define the chart domains, to give the precise description of the intersection of chart domains and to find explicit formulas for the transition maps between different charts. Once this is established, differential calculus can be applied in order to show in Part II that these structures are differentiable under some suitable and natural assumptions. In this way we not only obtain, e.g., Grassmannian manifolds, Lagrangian manifolds, or conformal quadrics in arbitrary dimension over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \ldots,$ but also a wealth of *symmetric spaces* (*over* $\mathbb{K}$) which generalize the *symmetric Banach manifolds* (see the monograph [19]) but include many completely new examples that had not been accessible before. The symmetric spaces thus constructed are precisely those which are in the image of the *Jordan–Lie functor* (cf. [1,3]).

Let us now describe the contents in some more detail. Our basic objects are, on the one hand, 3-*graded Lie algebras*, i.e., Lie algebras of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ satisfying the relations $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and on the other hand, 3-*filtered Lie algebras*, i.e., Lie algebras $\mathfrak{g}$ with a flag $\mathfrak{f} : 0 = \mathfrak{f}_2 \subset \mathfrak{f}_1 \subset \mathfrak{f}_0 \subset \mathfrak{g}$ of subalgebras such that $[\mathfrak{f}_\alpha, \mathfrak{f}_\beta] \subset \mathfrak{f}_{\alpha+\beta}$. For simplicity we shall also write these flags as pairs $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0)$. If $\mathfrak{g}$ is 3-graded, then $D(X) = iX$ ($X \in \mathfrak{g}_i$) defines a derivation of $\mathfrak{g}$ such that $D^3 = D$, called the *characteristic element*, and if D is inner, $D = \mathrm{ad}(E)$, E will be called an *Euler operator*. The *space of inner* 3-*gradings of* $\mathfrak{g}$ is

$$\mathcal{G} = \left\{\mathrm{ad}(E)\colon\ E \in \mathfrak{g},\ \mathrm{ad}(E)^3 = \mathrm{ad}(E)\right\}.$$

As usual in algebra, graded structures have underlying *filtered structures*. However, for every 3-grading, there are *two* naturally associated filtrations, $\mathfrak{f}^+ := \mathfrak{f}^+(D) : \mathfrak{g}_1 \subset \mathfrak{g}_1 \oplus \mathfrak{g}_0 \subset \mathfrak{g}$ and $\mathfrak{f}^- := \mathfrak{f}^-(D) = \mathfrak{f}^+(-D) : \mathfrak{g}_{-1} \subset \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \subset \mathfrak{g}$. If

$$\mathcal{F} = \left\{\mathfrak{f}^+(D)\colon\ D \in \mathcal{G}\right\}$$

denotes the *space of inner* 3-*filtrations of* $\mathfrak{g}$, then we have an injection

$$\mathcal{G} \hookrightarrow \mathcal{F} \times \mathcal{F}, \qquad D \mapsto \big(\mathfrak{f}^+(D), \mathfrak{f}^-(D)\big).$$

The spaces $\mathcal{G}$ and $\mathcal{F}$ carry many interesting geometric structures; one may say that the pair $(\mathcal{F} \times \mathcal{F}, \mathcal{G})$ is a "universal model of the generalized projective geometry associated to $\mathfrak{g}$." On $\mathcal{F} \times \mathcal{F}$ there is a natural relation of being transversal: two flags $\mathfrak{e} = (\mathfrak{e}_1, \mathfrak{e}_0)$ and $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0)$ are transversal if

$$\mathfrak{g} = \mathfrak{e}_1 \oplus \mathfrak{f}_0 = \mathfrak{f}_1 \oplus \mathfrak{e}_0.$$

Our key result on the structure of 3-graded Lie algebras (Theorem 1.6) affirms that $\mathcal{G} \subset \mathcal{F} \times \mathcal{F}$ is exactly the set of pairs of transversal inner 3-filtrations of $\mathfrak{g}$, and the set $\mathfrak{f}^\top$ of filtrations transversal to a given filtration $\mathfrak{f}$ carries canonically the structure of an affine space over $\mathbb{K}$ with translation group $(\mathfrak{f}_1, +)$. The *elementary projective group* $G = G(D)$ of the 3-graded Lie algebra $(\mathfrak{g}, D)$ is the group of automorphisms of $\mathfrak{g}$ generated by the abelian groups $U^\pm = e^{\mathrm{ad}(\mathfrak{g}_{\pm 1})}$; it acts on $\mathcal{F}$ and on $\mathcal{G}$. We realize the *projective completion* (X^+, X^-) of the pair $(\mathfrak{g}_1, \mathfrak{g}_{-1})$ as the G-orbits in $\mathcal{F}$ of the base points $\mathfrak{f}^-$ and $\mathfrak{f}^+$ such that $V^\pm := U^\pm.\mathfrak{f}^\mp = (\mathfrak{f}^\pm)^\top$ are "affine parts of $X^\pm$" (Theorem 1.12). Summing up, the "generalized projective geometry (X^+, X^-)" is imbedded as a subgeometry in $(\mathcal{F}, \mathcal{F})$.

Using this model, we have a natural definition of the "tangent bundle" $T\mathcal{F}$ of $\mathcal{F}$ and of a "structure bundle" $T'\mathcal{F}$ (taking the rôle of a cotangent bundle), and of sections of these bundles. Thus we can define, in a purely algebraic context, *vector fields* on $\mathcal{F}$ as well as a certain operator between $T'\mathcal{F}$ and $T\mathcal{F}$ called the *canonical kernel* (Section 2). Over the affine parts $V^\pm$, the bundles and their sections can be trivialized, and it is seen that our vector fields are actually quadratic polynomial and that the canonical kernel coincides with the well-known *Bergman operator* from Jordan theory (see below). Thus we get a very natural interpretation of the "Koecher construction" which consists of realizing a 3-graded Lie algebra by quadratic polynomial vector fields (cf. also [1, Chapter VII], where in the finite-dimensional real case another natural interpretation of this construction is given by using the *integrability of almost* (*para*-) *complex structures*). This approach naturally leads to one of the main results to be used in [6], namely the chart description of the action of $\mathrm{Aut}(\mathfrak{g})$ by "fractional quadratic maps" (Theorem 2.8).

In Section 3 we explain the link of the preceding results with Jordan theory: the pair $(V^+, V^-) = (\mathfrak{g}_1, \mathfrak{g}_{-1})$ together with the trilinear maps $T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm$ given by triple Lie brackets is a (*linear*) *Jordan pair*, and one can express in a straightforward way all relevant formulas from the preceding chapter by these maps. Thus we obtain in a calculation-free way the Bergman-operator, the quasi-inverse, and many of their fundamental relations and thus get new and "geometric" proofs of many Jordan theoretic results.

In Section 4 we add a new structure feature, namely an *involution* of the 3-graded Lie algebra. It leads to a bijection $p : X^+ \to X^-$ which is called a *polarity* in case that there exist non-isotropic points x (i.e., $p(x) \top x$). Then the space of all non-isotropic points carries the structure of *symmetric space over* $\mathbb{K}$. We prove that the structure maps of this symmetric space are given by suitable Jordan-theoretic formulas (Theorem 4.4), which will allow to conclude in Part II of this work that these structure maps are differentiable and so we really deal with symmetric spaces in the category of smooth manifolds.

Sections 5 up to 8 contain further material that is not strictly necessary for Part II of this work: in Section 5 we discuss those geometries that correspond to *unital Jordan algebras*: using our realization of $X^{\pm}$ as G-orbits in $\mathcal{F}$, they are characterized by the simple property that $V^+ \cap V^-$ is non-empty; in particular, $X^+ = X^-$. An axiomatic characterization of the "canonical identification of X^+ and X^-" has been given in [4]; thanks to our model, things are considerably easier here than in the axiomatic approach.

In Section 6 some functorial aspects of our constructions are investigated. It is shown that surjective homomorphisms of 3-graded Lie algebras induce equivariant maps of the associated geometries and we also show that inclusions of inner 3-graded subalgebras containing $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ induce isomorphisms of the corresponding geometries.

In Section 7 we discuss central extensions of inner 3-graded Lie algebras. We show that for each central extension $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ of an inner 3-graded Lie algebra $\mathfrak{g}$ the extended Lie algebra $\hat{\mathfrak{g}}$ carries a natural structure of an inner 3-graded Lie algebra for which q is a morphism of 3-graded Lie algebras. We further construct a universal inner 3-graded central extension of $\mathfrak{g}$. We know from Section 6 that quotient maps induce maps on the level of geometries. For central extensions we show that these maps are isomorphisms.

In the final Section 8, we look at an important class of geometries, the *Grassmannian geometries*: let R be an associative algebra over the commutative ring $\mathbb{K}$, V be a right R-module, $\mathcal{P}$ the space of all R-linear projectors $V \to V$, and $\mathcal{C}$ be the space of all R-submodules of V that admit a complement. Then, by elementary linear algebra, the pair $(\mathcal{C} \times \mathcal{C}, \mathcal{P})$ has the main features of a generalized projective geometry (Proposition 8.2, cf. also [2]), and in fact there is a homomorphism into the geometry $(\mathcal{F} \times \mathcal{F}, \mathcal{G})$ with $\mathfrak{g} = \mathfrak{gl}_R(V)$ which induces isomorphisms on subgeometries that are homogeneous under the elementary projective groups (Theorem 8.4). Such geometries, called *Grassmannian geometries*, correspond to *special Jordan pairs*, i.e., to subpairs of *associative pairs*. In particular, if $V = R$, then the Grassmannian geometry can also be called the "geometry of right ideals of the associative algebra R;" it corresponds to R, seen as a Jordan algebra over $\mathbb{K}$.

Finally, we would like to add some comments on related work and on some open problems. The elementary projective group and the projective completion of a general Jordan pair have been introduced by J. Faulkner [10], and results closely related to ours have been obtained by O. Loos [18]. Their results are based on the axiomatic theory of Jordan pairs [16] and hence work even for base rings in which 2 or 3 are not invertible. In contrast, we work in a Lie theoretic context and hence assume throughout that 2 and 3 are invertible in $\mathbb{K}$. However, it is possible to extend our approach also to the case of a general base ring $\mathbb{K}$—see Remark 3.9. Our results are more general in the sense that they apply to general 3-graded Lie algebras (not only to the Tits–Kantor–Koecher algebra of a Jordan pair) and to the general automorphism group $\mathrm{Aut}(\mathfrak{g})$ (and not only to the important special case given by transformations corresponding to quasi-inverses). As a by-product, we get new proofs of many Jordan theoretic results. It is an interesting open problem whether it is possible to derive "all" Jordan theoretic formulas in a similar geometric way—in particular, we would like to have in our model a "geometric" proof of the fundamental identity (PG2) of a generalized projective geometry (cf. [3]) which is very closely related to the *fundamental formula* of Jordan theory.

Closely related results have also been obtained by Kaup [13] and Upmeier [19] in the complex case in presence of a Jordan–Banach structure. In fact, some arguments used to prove our structure Theorem 1.6 have been used by Kaup in the proof of his Riemann Mapping Theorem (see the proof of [13, Proposition (2.14)] and the detailed version of this in [19, Lemma 9.16]). Our proofs are much simpler since we work directly with the 3-graded Lie algebra, whereas Kaup and Upmeier always use its homomorphic image realized by quadratic polynomial vector fields (called *binary Lie algebras* in [19]).

The special case of Grassmannians, especially in the context of Banach manifolds, has attracted much attention since it plays an important rôle in differential geometry and is related to several interesting differential equations—see, e.g., [8,9]; our constructions are similar to, but much more general than the ones described there. For further references to constructions of manifolds in contexts related to Jordan theory see Part II [6]; cf. also [11] for an extensive bibliography.

Notation. Throughout this paper, $\mathbb{K}$ is a commutative ring with unit 1 such that 2 and 3 are invertible in $\mathbb{K}$. In Section 8, R denotes a possibly non-commutative ring which is a $\mathbb{K}$-algebra.

1. Three-graded and three-filtered Lie algebras

1.1. Three-graded Lie algebras

A 3-*graded Lie algebra* (over $\mathbb{K}$) is a Lie algebra over $\mathbb{K}$ of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ such that $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$, i.e., $\mathfrak{g}_{\pm 1}$ are abelian subalgebras which are $\mathfrak{g}_0$-modules, in the following often denoted by $V^{\pm}$ or $\mathfrak{g}_{\pm}$, and $[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}_0$. The map $D : \mathfrak{g} \to \mathfrak{g}$ defined by $DX = iX$ for $X \in \mathfrak{g}_i$ is a derivation of $\mathfrak{g}$, called the *characteristic element of the grading*. It satisfies the relation $(D - \mathrm{id})D(D + \mathrm{id}) = 0$, i.e., $D^3 = D$; we say that it is a *tripotent derivation*. Conversely, any tripotent derivation $D : \mathfrak{g} \to \mathfrak{g}$ is diagonizable with possible eigenvalues $-1, 0, 1$ and corresponding decomposition of $X \in \mathfrak{g}$:

$$X = X_1 + X_0 + X_{-1}, \qquad X_0 = X - D^2X,$$
$$X_1 = \frac{DX + D^2X}{2}, \qquad X_{-1} = \frac{-DX + D^2X}{2}. \tag{1.1}$$

Since D is a derivation, this eigenspace decomposition is a 3-grading. Therefore, we may identify the *space of* 3-*gradings of* $\mathfrak{g}$ with the set

$$\widetilde{\mathcal{G}} := \{ D \in \mathrm{der}(\mathfrak{g}) \colon\ D^3 = D \}$$

of tripotent derivations. If $D = \mathrm{ad}(E)$ is an inner tripotent derivation, then E is called an *Euler operator*, and we denote by

$$\mathcal{G} := \{ \mathrm{ad}(E) \colon\ E \in \mathfrak{g},\ \mathrm{ad}(E)^3 = \mathrm{ad}(E) \} \tag{1.2}$$

the *space of inner* 3-*gradings of* $\mathfrak{g}$. The *odd part* of the 3-graded Lie algebra $(\mathfrak{g}, D)$ is $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, and we say that $(\mathfrak{g}, D)$ is *minimal* if it is generated by its odd part, that is, $\mathfrak{g}_0$ is generated by the brackets $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$.

The following degenerate cases may arise: $D^2 = \mathrm{id}$, then $\mathfrak{g}$ must be abelian, and we have merely a decomposition of a $\mathbb{K}$-module into complementary subspaces; $D^2 = D$, then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the split null extension of $\mathfrak{g}_0$ by a $\mathfrak{g}_0$-module $\mathfrak{g}_1$, in particular, $D = 0$ corresponds to the case $\mathfrak{g}_1 = \{0\}$.

1.2. The projective elementary group

Let $(\mathfrak{g}, D)$ be a 3-graded Lie algebra over $\mathbb{K}$. For $x \in \mathfrak{g}_{\pm 1}$, the operator $e^{\operatorname{ad} x} = \mathbf{1} + \operatorname{ad} x + \frac{1}{2}(\operatorname{ad} x)^2$ is a well-defined automorphism of $\mathfrak{g}$. (In order to see that $e^{\operatorname{ad}(x)}$ is an automorphism, we need that $\mathbb{K}$ has no 3-torsion.) The group generated by these operators,

$$G := G(D) := \mathrm{PE}(\mathfrak{g}, D) := \left\langle e^{\operatorname{ad} x}\colon\ x \in \mathfrak{g}_{\pm 1} \right\rangle \subseteq \mathrm{Aut}(\mathfrak{g}),$$

is called the *projective elementary group of* $(\mathfrak{g}, D)$ (see Section 3.2 for the relation with the projective elementary group defined in Jordan theoretic terms, as in [10,18]). Sometimes it will be useful to have a matrix notation for elements of G: if $g \in \mathrm{Aut}(\mathfrak{g})$, we let, with respect to the fixed 3-grading,

$$g_{ij} := \mathrm{pr}_i \circ g \circ \iota_j : \mathfrak{g}_j \to \mathfrak{g}_i, \quad i, j = 1, 0, -1,$$

where $\iota_j : \mathfrak{g}_j \to \mathfrak{g}$ are the inclusion maps and $\mathrm{pr}_i := \mathrm{pr}_i(D) : \mathfrak{g} \to \mathfrak{g}_i$ the projections, given by

$$\mathrm{pr}_1 = \frac{D + D^2}{2}, \qquad \mathrm{pr}_0 = \mathbf{1} - D^2, \qquad \mathrm{pr}_{-1} = \frac{D^2 - D}{2}, \tag{1.3}$$

and write g in "matrix form"

$$g = \begin{pmatrix} g_{11} & g_{10} & g_{1,-1} \\ g_{01} & g_{00} & g_{0,-1} \\ g_{-1,1} & g_{-1,0} & g_{-1,-1} \end{pmatrix}. \tag{1.4}$$

The subgroups $U^{\pm} := U^{\pm}(D) := e^{\operatorname{ad} \mathfrak{g}_{\pm}}$ of G are abelian and generate G. If the grading derivation is inner, $D = \operatorname{ad}(E)$, then

$$\exp : \mathfrak{g}_{\pm 1} \to U^{\pm}, \quad X \mapsto e^{\operatorname{ad}(X)}$$

is injective since $v \in \mathfrak{g}_{\pm}$ implies $e^{\operatorname{ad} v}.E = E \mp v$. In the general case, we define the *automorphism group of* $(\mathfrak{g}, D)$ to be

$$\mathrm{Aut}(\mathfrak{g}, D) = \left\{ g \in \mathrm{Aut}(\mathfrak{g})\colon\ g \circ D = D \circ g \right\},$$

and we further define subgroups $H := H(D)$ and $P^\pm := P^\pm(D)$ of G via

$$H := G(D) \cap \operatorname{Aut}(\mathfrak{g}, D) \quad \text{and} \quad P^\pm := HU^\pm = U^\pm H. \tag{1.5}$$

(If D is inner, $D = \operatorname{ad}(E)$, then $H = \{h \in G\colon\ h \circ \operatorname{ad} E \circ h^{-1} = \operatorname{ad} E\} = \{h \in G\colon\ h.E - E \in \mathfrak{z}(\mathfrak{g})\}$.) The groups $U^\pm$ are abelian, and since the group H commutes with D, it preserves the grading, hence normalizes $U^\pm$, so that $P^\pm$ are subgroups of G. Using notation from Eq. (1.4), the generators of G are represented by the following matrices (where $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_{-1}$, $h \in H$):

$$e^{\operatorname{ad} x} = \begin{pmatrix} \mathbf{1} & \operatorname{ad} x & \frac{1}{2}\operatorname{ad}(x)^2 \\ 0 & \mathbf{1} & \operatorname{ad} x \\ 0 & 0 & \mathbf{1} \end{pmatrix}, \qquad e^{\operatorname{ad} y} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \operatorname{ad} y & \mathbf{1} & 0 \\ \frac{1}{2}\operatorname{ad}(y)^2 & \operatorname{ad} y & \mathbf{1} \end{pmatrix},$$
$$h = \begin{pmatrix} h_{11} & & \\ & h_{00} & \\ & & h_{-1,-1} \end{pmatrix}.$$

More information on the group $G(D)$ for inner 3-gradings D is given in Theorem 1.12.

Sometimes it will be useful to replace G by a slightly bigger group: if $D \in \widetilde{\mathcal{G}}$ and $r \in \mathbb{K}^\times$, then, using the matrix notation (1.4),

$$h^{(D,r)} := \begin{pmatrix} r & & \\ & 1 & \\ & & r^{-1} \end{pmatrix} = r\,\mathrm{pr}_1 + \mathrm{pr}_0 + r^{-1}\,\mathrm{pr}_{-1}, \tag{1.6}$$

with the pr_i as in Eq. (1.3), defines an automorphism of $(\mathfrak{g}, D)$ normalizing $U^\pm$ and commuting with all elements of the group $\operatorname{Aut}(\mathfrak{g}, D)$. The group G^{ext} generated by G and the group $\{h^{(D,r)}\colon\ r \in \mathbb{K}^\times\}$ will be called the *extended projective elementary group*.

1.3. Three-filtered Lie algebras

A 3-*filtration* of a Lie algebra $\mathfrak{g}$ is a flag of subspaces

$$0 = \mathfrak{f}_2 \subset \mathfrak{f}_1 \subset \mathfrak{f}_0 \subset \mathfrak{f}_{-1} = \mathfrak{g}$$

such that

$$[\mathfrak{f}_k, \mathfrak{f}_l] \subset \mathfrak{f}_{k+l}. \tag{1.7}$$

Suppressing the trivial parts $\mathfrak{f}_2$ and $\mathfrak{f}_{-1}$ in the notation, we will denote such a flag by $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0)$ or $\mathfrak{f} : (\mathfrak{f}_1 \subset \mathfrak{f}_0)$. Let $\widetilde{\mathcal{F}}$ be the set of such flags $\mathfrak{f}$, called the *space of* 3-*filtrations of* $\mathfrak{g}$. Conditions (1.7) are equivalent to the following requirements:

- $\mathfrak{f}_0$ is a subalgebra, and $\mathfrak{f}_1$ is an abelian subalgebra of $\mathfrak{g}$,
- $\mathfrak{f}_1$ is an ideal in $\mathfrak{f}_0$, and $[\mathfrak{g}, \mathfrak{f}_1] \subset \mathfrak{f}_0$.

It follows that the operators $\mathrm{ad}(X)$ with $X \in \mathfrak{f}_1$ are 3-step nilpotent and hence the automorphism $e^{\mathrm{ad}(X)}$ of $\mathfrak{g}$ is well-defined. We denote by

$$U(\mathfrak{f}) := e^{\mathrm{ad}(\mathfrak{f}_1)} = \left\{e^{\mathrm{ad}(X)}\colon X \in \mathfrak{f}_1\right\} \subset \mathrm{Aut}(\mathfrak{g}) \tag{1.8}$$

the corresponding abelian group. From (1.7) it follows that $U(\mathfrak{f})$ preserves the filtration $\mathfrak{f}$. The filtration $\mathfrak{f}$ is also stable under the action of the subalgebra $\mathfrak{f}_0$.

1.4. *Relation between* 3*-gradings and* 3*-filtrations*

To any 3-grading $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ of $\mathfrak{g}$ with characteristic derivation $D \in \widetilde{\mathcal{G}}$ we may associate *two* 3-filtrations of $\mathfrak{g}$, called the *associated plus- and minus-filtration*, given by the two flags

$$\mathfrak{f}^+(D) := (\mathfrak{g}_1, \mathfrak{g}_0 \oplus \mathfrak{g}_1), \qquad \mathfrak{f}^-(D) := (\mathfrak{g}_{-1}, \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}). \tag{1.9}$$

Clearly, $\mathfrak{f}^\pm(D) = \mathfrak{f}^\mp(-D)$. We will say that a 3-filtration is *inner* if it is of the form $\mathfrak{f} = \mathfrak{f}^+(\mathrm{ad}(E)) = \mathfrak{f}^-(\mathrm{ad}(-E))$ for some Euler operator E, and the *space of inner* 3*-filtrations* will be denoted by

$$\mathcal{F} := \left\{\mathfrak{f}^+(D)\colon D \in \mathcal{G}\right\}. \tag{1.10}$$

By these definitions, the maps $\mathcal{G} \to \mathcal{F}$, $D \mapsto \mathfrak{f}^\pm(D)$ are surjective, and the map

$$\mathcal{G} \to \mathcal{F} \times \mathcal{F}, \quad D \mapsto \left(\mathfrak{f}^+(D), \mathfrak{f}^-(D)\right) \tag{1.11}$$

is injective (since $\mathfrak{g}_{\pm 1}$ are recovered by the filtration and $\mathfrak{g}_0 = (\mathfrak{g}_0 \oplus \mathfrak{g}_1) \cap (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$).

1.5. *Transversality*

Two flags $\mathfrak{e} = (\mathfrak{e}_1, \mathfrak{e}_0)$ and $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0)$ as above are called *transversal* if

$$\mathfrak{g} = \mathfrak{e}_1 \oplus \mathfrak{f}_0 = \mathfrak{f}_1 \oplus \mathfrak{e}_0.$$

It is clear by construction that the two filtrations $\mathfrak{f}^+(D)$ and $\mathfrak{f}^-(D)$ associated to a 3-grading D of $\mathfrak{g}$ are transversal. We will prove that, conversely, any pair of transversal inner 3-filtrations arises in this way. If $\mathfrak{f} \in \mathcal{F}$, we will use the notation

$$\mathfrak{f}^\top := \{\mathfrak{e} \in \mathcal{F}\colon \mathfrak{e} \top \mathfrak{f}\} \tag{1.12}$$

for the set of inner 3-filtrations that are transversal to $\mathfrak{f}$, and

$$(\mathcal{F} \times \mathcal{F})^\top = \left\{(\mathfrak{e}, \mathfrak{f}) \in \mathcal{F} \times \mathcal{F}\colon \mathfrak{e} \top \mathfrak{f}\right\} \tag{1.13}$$

for the set of transversal pairs.

Theorem 1.6 (Structure theorem for the space of 3-filtrations). *With the notation introduced above, the following holds for any Lie algebra* $\mathfrak{g}$ *over* $\mathbb{K}$:

(1) *The space of inner* 3*-gradings can be canonically identified with the space of transversal pairs of inner* 3*-filtrations*:

$$\mathcal{G} = (\mathcal{F} \times \mathcal{F})^{\top}.$$

In other words, two inner 3*-filtrations* $\mathfrak{e}$ *and* $\mathfrak{f}$ *are transversal if and only if there exists an Euler operator* E *such that* $\mathfrak{f} = \mathfrak{f}^{+}(\mathrm{ad}(E))$ *and* $\mathfrak{e} = \mathfrak{f}^{-}(\mathrm{ad}(E))$.

(2) *For any inner* 3*-filtration* $\mathfrak{f}$, *the space* $\mathfrak{f}^{\top}$ *carries a natural structure of an affine space over* $\mathbb{K}$ *with translation group* $(\mathfrak{f}_1, +)$. *The group* $\mathfrak{f}_1$ *acts simply transitively on* $\mathfrak{f}^{\top}$ *by* $x.\mathfrak{e} := e^{\mathrm{ad}\,x}.\mathfrak{e}$.

Proof. (1) We have already remarked that $\mathcal{G} \subset (\mathcal{F} \times \mathcal{F})^{\top}$. In order to prove the other inclusion, let us assume that $(\mathfrak{e}, \mathfrak{f})$ is transversal. We have to show that $(\mathfrak{e}, \mathfrak{f}) \in \mathcal{G}$.

Since $\mathfrak{f}$ is inner, there exists an Euler operator $E' \in \mathfrak{g}$ such that $\mathfrak{f} = \mathfrak{f}^{+}(\mathrm{ad}(E'))$. By the first transversality condition $\mathfrak{g} = \mathfrak{f}_1 \oplus \mathfrak{e}_0$, there exists an element $Z \in \mathfrak{f}_1$ with $E' - Z \in \mathfrak{e}_0$. Now let

$$E := e^{\mathrm{ad}\,Z}.E' = E' + [Z, E'] = E' - Z.$$

Then, since $Z \in \mathfrak{f}_1$,

$$\mathfrak{f}^{+}(\mathrm{ad}\,E) = e^{\mathrm{ad}\,Z}.\mathfrak{f}^{+}(\mathrm{ad}\,E') = e^{\mathrm{ad}\,Z}.\mathfrak{f} = \mathfrak{f}.$$

It remains to show that $\mathfrak{e}$ is the flag

$$\mathfrak{f}^{-}(\mathrm{ad}\,E) = e^{\mathrm{ad}\,Z}.\mathfrak{f}^{-}(\mathrm{ad}\,E').$$

In order to prove this, note first that, by our choice of Z, E belongs $\mathfrak{e}_0$, and hence the flag $\mathfrak{e}$ is stable under $\mathrm{ad}(E)$. By transversality of $\mathfrak{e}$ and $\mathfrak{f}$, we can write $\mathfrak{g} = \mathfrak{e}_1 \oplus \mathfrak{f}_0$, and since the flag $\mathfrak{f} = \mathfrak{f}^{+}(\mathrm{ad}\,E)$ is also stable under $\mathrm{ad}(E)$, this decomposition is $\mathrm{ad}(E)$-stable. But the only $\mathrm{ad}(E)$-stable complement of $\mathfrak{f}_0$ is the -1-eigenspace of $\mathrm{ad}(E)$, and hence $\mathfrak{e}_1 = \{X \in \mathfrak{g}\colon [E, X] = -X\}$. Next, we use again the first transversality condition $\mathfrak{g} = \mathfrak{e}_0 \oplus \mathfrak{f}_1$ in order to conclude that the $\mathrm{ad}(E)$-invariant complement $\mathfrak{e}_0$ of $\mathfrak{f}_1 = \{X \in \mathfrak{g}\colon [E, X] = X\}$ must be equal to the complement given by the sum of the 0- and the -1-eigenspace of $\mathrm{ad}(E)$. Thus $\mathfrak{e}_0 = (\mathfrak{f}^{-}(\mathrm{ad}\,E))_0$, and hence $\mathfrak{e} = \mathfrak{f}^{-}(\mathrm{ad}\,E)$.

(2) Using the same notation as above, we have just proved that an arbitrary element $\mathfrak{e} \in \mathfrak{f}^{\top}$ is of the form $\mathfrak{e} = e^{\mathrm{ad}(Z)}\mathfrak{f}^{-}(\mathrm{ad}\,E')$ with $Z \in \mathfrak{f}_1$, where $\mathfrak{f}^{-}(\mathrm{ad}\,E')$ is some fixed base point in $\mathfrak{f}^{\top}$. Thus $\mathfrak{f}_1$ acts transitively on $\mathfrak{f}^{\top}$. This action is simply transitive: if E' and $E = e^{\mathrm{ad}\,Z}.E' = E' - Z$ with $Z \in \mathfrak{f}_1$ are such that $\mathfrak{f}^{+}(\mathrm{ad}\,E) = f^{+}(\mathrm{ad}\,E')$ and $\mathfrak{f}^{-}(\mathrm{ad}\,E) = \mathfrak{f}^{-}(\mathrm{ad}\,E')$, then $\mathrm{ad}\,E' = \mathrm{ad}\,E$, hence $\mathrm{ad}\,Z = 0$, $Z = [E', Z] = 0$ and $E = E'$. $\square$

Corollary 1.7. *Let* $D_1 = \mathrm{ad}(E_1)$, $D_2 = \mathrm{ad}(E_2) \in \mathcal{G}$, *and* $\mathfrak{g}_1 := \{X \in \mathfrak{g}\colon [E_1, X] = X\}$. *Then the following are equivalent*:

(1) *E_1 and E_2 have the same associated $+$-filtration*: $\mathfrak{f}^+(\operatorname{ad}(E_1)) = \mathfrak{f}^+(\operatorname{ad}(E_2))$.
(2) *There is $v \in \mathfrak{g}_1$ such that $D_2 = e^{\operatorname{ad} v} D_1 e^{-\operatorname{ad}(v)}$.*
(3) $D_1 - D_2 \in \operatorname{ad}(\mathfrak{g}_1)$.
(4) $[D_1, D_2] = D_2 - D_1$.

Proof. (2) implies (1) since $U(\mathfrak{f}^+(D_1))$ preserves $\mathfrak{f}^+(D_1)$. Conversely, if (1) holds, then $\mathfrak{f}^-(D_2)$ is transversal to $\mathfrak{f}^+(D_2) = \mathfrak{f}^+(D_1)$, and now (2) follows from part (2) of Theorem 1.6.

(2) $\Leftrightarrow$ (3). If $v \in \mathfrak{g}_1$, then

$$e^{\operatorname{ad} v} D_1 e^{-\operatorname{ad} v} = \operatorname{ad}\left(e^{\operatorname{ad} v}.E_1\right) = \operatorname{ad}(E_1 - v) = D_1 - \operatorname{ad} v$$

shows that (2) and (3) are equivalent.

(3) $\Leftrightarrow$ (4). Let $x := E_1 - E_2$. Then $D_1 - D_2 = \operatorname{ad} x$ and $\operatorname{ad}[E_1, x] = \operatorname{ad}[E_2, E_1] = [D_2, D_1]$. So (4) is equivalent to $[E_1, x] - x \in \mathfrak{z}(\mathfrak{g})$, and (3) is equivalent to $x \in \mathfrak{g}_1 + \mathfrak{z}(\mathfrak{g})$. Writing $x = x_1 + x_0 + x_{-1}$ with $[E_1, x_i] = i x_i$, we have $x - [E_1, x] = x_0 + 2x_{-1}$, so that (4) is equivalent to $x_{-1} = 0$ and $x_0 \in \mathfrak{z}(\mathfrak{g})$. In view of $\mathfrak{z}(\mathfrak{g}) \subseteq \ker \operatorname{ad} E_1$, this in turn is equivalent to (3). $\square$

Next we state a "matrix version" of part (1) of Theorem 1.6, using the matrix notation introduced in Eq. (1.4).

Corollary 1.8. *With respect to a fixed inner* 3*-grading given by the Euler operator E, with corresponding pair of* 3*-filtrations* $(\mathfrak{f}^-, \mathfrak{f}^+) = (\mathfrak{f}^-(D), \mathfrak{f}^+(D)) = ((\mathfrak{g}_{-1}, \mathfrak{g}_0 + \mathfrak{g}_{-1}), (\mathfrak{g}_1, \mathfrak{g}_0 + \mathfrak{g}_1))$*, the following statements are equivalent*:

(1) $(g.\mathfrak{f}^-, \mathfrak{f}^+) \in \mathcal{G}$.
(2) *$\mathfrak{f}^+$ and $g.\mathfrak{f}^-$ are transversal.*
(3) *$g_{-1,-1}$ and $(g^{-1})_{11}$ are invertible in* $\operatorname{End}(\mathfrak{g}_{-1})$*, respectively in* $\operatorname{End}(\mathfrak{g}_1)$.

Proof. The equivalence of (1) and (2) is given by Theorem 1.6(1). Now, (2) is equivalent to (4) and to (5):

(4) $g(\mathfrak{g}_{-1})$ is a complement of $\mathfrak{g}_1 \oplus \mathfrak{g}_0$ and $g(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ is a complement of $\mathfrak{g}_1$,
(5) $g(\mathfrak{g}_{-1})$ is a complement of $\mathfrak{g}_1 \oplus \mathfrak{g}_0$ and $g^{-1}(\mathfrak{g}_1)$ is a complement of $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$,

and clearly (5) is equivalent to (3). $\square$

Definition 1.9. For $x \in \mathfrak{g}_1$ and $g \in \operatorname{Aut}(\mathfrak{g})$, we define

$$d_g(x) := \left(e^{-\operatorname{ad}(x)} g^{-1}\right)_{11}, \qquad c_g(x) := \left(g e^{\operatorname{ad}(x)}\right)_{-1,-1}.$$

Then

$$d_g^+ := d_g : \mathfrak{g}_1 \to \operatorname{End}(\mathfrak{g}_1), \qquad c_g^+ := c_g : \mathfrak{g}_1 \to \operatorname{End}(\mathfrak{g}_{-1})$$

are quadratic polynomial maps, called the *denominator* and *co-denominator* of g (w.r.t. the fixed inner grading defined by $\mathrm{ad}(E)$). In a similar way, d_g^- and c_g^- are defined.

Writing g and $e^{\mathrm{ad}(x)}$ in matrix form (1.4), the denominator for g^{-1} is given by

$$d_{g^{-1}}(x) = g_{11} - \mathrm{ad}(x) \circ g_{01} + \frac{1}{2}\,\mathrm{ad}(x)^2 \circ g_{-1,1},$$

and similarly for the co-denominator. For the generators of G we get the following (co-) denominators (where $v \in \mathfrak{g}_1$, $w \in \mathfrak{g}_{-1}$):

$$\begin{aligned}
g = e^{\mathrm{ad}(v)}:&\quad d_g(x) = \mathrm{id}_{\mathfrak{g}_1}, \qquad c_g(x) = \mathrm{id}_{\mathfrak{g}_{-1}},\\
g = e^{\mathrm{ad}(w)}:&\quad d_g(x) = \mathrm{id}_{\mathfrak{g}_1} + \mathrm{ad}(x)\,\mathrm{ad}(w) + \frac{1}{4}\,\mathrm{ad}(x)^2\,\mathrm{ad}(w)^2,\\
&\quad c_g(x) = \mathrm{id}_{\mathfrak{g}_{-1}} + \mathrm{ad}(w)\,\mathrm{ad}(x) + \frac{1}{4}\,\mathrm{ad}(w)^2\,\mathrm{ad}(x)^2,\\
g = h \in H:&\quad d_h(x) = (h_{11})^{-1} = \left(h^{-1}\right)_{11}, \qquad c_h(x) = h_{-1,-1}.
\end{aligned} \tag{1.14}$$

For $g = e^{\mathrm{ad}(w)}$ as in the second equation, we introduce the notation

$$B_+(x, w) := d_g(x), \qquad B_-(w, x) := c_g(x). \tag{1.15}$$

These linear maps define the *Bergman operator*, see Section 3.3.

Corollary 1.10. *With respect to a fixed inner* 3*-grading given by the Euler operator* E*, we identify* $V^+ := \mathfrak{g}_1$ *with the set* $(\mathfrak{f}^+)^\top = e^{\mathrm{ad}(V^+)}\mathfrak{f}^-$*. Then for* $x \in V^+$ *the following statements are equivalent*:

(1) $(g.x, \mathfrak{f}^+) \in \mathcal{G}$.
(2) $\mathfrak{f}^+$ *and* $g.x$ *are transversal, i.e.*, $g.x \in V^+$.
(3) $c_g(x)$ *and* $d_g(x)$ *are invertible in* $\mathrm{End}(\mathfrak{g}_{-1})$, *respectively in* $\mathrm{End}(\mathfrak{g}_1)$.

Proof. This follows by applying Corollary 1.8 to the element $ge^{\mathrm{ad}(x)} \in \mathrm{Aut}(\mathfrak{g})$. □

In particular, for $g = e^{\mathrm{ad}(w)}$ with $w \in \mathfrak{g}_{-1}$, it follows that $g.x \in V^+$ if and only if $B_+(x, w)$ and $B_-(w, x)$ are invertible.

1.11. The projective geometry of a 3-graded Lie algebra

Recall from Section 1.2 the definition of the projective elementary group $G := G(D)$. Using Theorem 1.6(1), we may identify an inner grading $D = \mathrm{ad}(E)$ with the corresponding pair $(\mathfrak{f}, \mathfrak{e}) = (\mathfrak{f}^+(D), \mathfrak{f}^-(D))$ of inner filtrations; hence we may also write $G(\mathfrak{f}, \mathfrak{e})$ for the elementary group $G(D)$, and similarly for $H(D)$ and $P^\pm(D)$. If $\mathfrak{f}, \mathfrak{e}, \mathfrak{e}'$ are inner 3-filtrations such that $\mathfrak{e} \top \mathfrak{f}$ and $\mathfrak{e}' \top \mathfrak{f}$, then Theorem 1.6(2) implies that $\mathfrak{e}$ and $\mathfrak{e}'$ are conjugate under $G(\mathfrak{f}, \mathfrak{e})$, and hence we have $G(\mathfrak{f}, \mathfrak{e}) = G(\mathfrak{f}, \mathfrak{e}')$. Therefore, we may define the *projective elementary group of the inner* 3*-filtration* $\mathfrak{f}$ to be $G(\mathfrak{f}) := G(\mathfrak{f}, \mathfrak{e})$, where $\mathfrak{e} \in \mathcal{F}$ is any filtration that is transversal to $\mathfrak{f}$. Note that

$$U^+(\mathfrak{f}, \mathfrak{e}) = U^+(\mathfrak{f})$$

is the abelian group defined by Eq. (1.8) and hence is independent of $\mathfrak{e}$, whereas the groups $U^- = U^-(\mathfrak{f}, \mathfrak{e})$, $H = H(\mathfrak{f}, \mathfrak{e})$, and $P^- = P^-(\mathfrak{f}, \mathfrak{e})$ depend on the choice of $\mathfrak{e}$. (We will see below that P^+ does not depend on $\mathfrak{e}$.) We define the following homogeneous spaces:

$$X^\pm := G/P^\mp, \qquad M := G/H. \tag{1.16}$$

For reasons that will be explained below, the data (X^+, X^-, M) are called the (*generalized*) *projective geometry associated to the graded Lie algebra* $(\mathfrak{g}, D)$. The base point (P^-, P^+) in $X^+ \times X^-$ will often be denoted by (o^+, o^-).

Theorem 1.12 (Structure theorem for the projective geometry of a 3-graded Lie algebra). *With the notation introduced above, the following holds*:

(1) *The orbits of $D := \mathrm{ad}(E) \in \mathcal{G}$, respectively of $\mathfrak{f}^\pm \in \mathcal{F}$, under the action of G are isomorphic to M, respectively to $X^\pm$. In other words,*

$$H = \{g \in G(D)\colon\ g.(\mathfrak{f}^-, \mathfrak{f}^+) = (\mathfrak{f}^-, \mathfrak{f}^+)\} \quad \textit{and} \quad P^\pm = \{g \in G(D)\colon\ g.\mathfrak{f}^\pm = \mathfrak{f}^\pm\}.$$

Moreover, $P^+ \cap P^- = H$, $P^\pm \cap U^\mp = \{\mathbf{1}\}$, and

$$P^\pm = \{g \in G\colon\ gDg^{-1} - D \in \mathrm{ad}(\mathfrak{g}_{\pm 1})\} = \{g \in G\colon\ g.E - E \in \mathfrak{z}(\mathfrak{g}) + \mathfrak{g}_{\pm 1}\}.$$

(2) *If we identify $X^\pm$ with the corresponding orbits in $\mathcal{F}$, then*

$$\mathcal{G} \cap (X^+ \times X^-) = M.$$

(3) *For every element $\mathfrak{e} \in X^-$, the set $\mathfrak{e}^\top$ is contained in X^+ and carries a well-defined structure of an affine space over $\mathbb{K}$ with translation group $\mathfrak{e}_1 = \mathfrak{g}_1$. In particular, $(o^-)^\top$ is canonically identified with $V^+ = e^{\mathrm{ad}(\mathfrak{g}_1)}.o^+$.*

(4) *Consider the set Ω^+ of elements of G sending the base point $o^+ \in X^+$ to a point of the affine part $V^+ \subset X^+$,*

$$\Omega^+ := \{g \in G\colon\ g.o^+ \in V^+\}.$$

Then the map

$$\mathfrak{g}_1 \times H \times \mathfrak{g}_{-1} \to \Omega^+, \quad (v, h, w) \mapsto e^{\mathrm{ad}(v)} h e^{\mathrm{ad}(w)}$$

is a bijection, and, moreover,

$$\Omega^+ = \{g \in G\colon\ d_g(o^+) \in \mathrm{GL}(\mathfrak{g}_1),\ c_g(o^+) \in \mathrm{GL}(\mathfrak{g}_{-1})\}.$$

(5) *The spaces $X^{\pm} \subset \mathcal{F}$ and $M \subset \mathcal{G}$ are stable under the action of the automorphism group* $\mathrm{Aut}(\mathfrak{g}, D)$ *and of the extended projective elementary group G^{ext}.*

Proof. (1) An element $g \in G$ stabilizes $(\mathfrak{f}^+, \mathfrak{f}^-)$ if and only if it commutes with $D = \mathrm{ad}(E)$ which means that it belongs to H.

It is clear that P^+ stabilizes $\mathfrak{f}^+$. Conversely, assume that $g \in G$ satisfies $g.\mathfrak{f}^+ = \mathfrak{f}^+$. Then $g.\mathfrak{f}^+ = \mathfrak{f}^+$ is transversal to $g.\mathfrak{f}^-$, and hence by Theorem 1.6(2) there exists $v \in \mathfrak{g}_1$ such that $g.\mathfrak{f}^- = e^{\mathrm{ad}(v)}\mathfrak{f}^-$. Then $h := e^{-\mathrm{ad}(v)}g$ preserves $(\mathfrak{f}^+, \mathfrak{f}^-)$ and thus belongs to H. Therefore $g = e^{\mathrm{ad}(v)}h$ belongs to P^+. Hence P^+ is the stabilizer of $\mathfrak{f}^+$. Similarly for P^-.

It follows that $P^+ \cap P^-$ is the stabilizer of $(\mathfrak{f}^+, \mathfrak{f}^-)$ which is H. Next, assume $g \in P^+ \cap U^-$. Write $g = e^{\mathrm{ad}(v)}$ with $v \in \mathfrak{g}_{-1}$. Since $v \mapsto e^{\mathrm{ad}(v)}\mathfrak{f}^+$ is injective (Theorem 1.6(2)), it follows from $g\mathfrak{f}^+ = \mathfrak{f}^+$ that $v = 0$ and hence $g = \mathbf{1}$.

Finally, g stabilizes $\mathfrak{f}^+$ if and only if D and gDg^{-1} have the same associated $+$-filtration, if and only if $gDg^{-1} - D$ belongs to $\mathrm{ad}(\mathfrak{g}_1)$ (Corollary 1.7), whence the last claim of part (1) for P^+, and similarly for P^-.

(2) It is clear that the G-orbit $G.(\mathfrak{f}^+, \mathfrak{f}^-)$ belongs both to $X^+ \times X^-$ and to $\mathcal{G}$. In order to prove the converse, let $(\mathfrak{f}, \mathfrak{e}) \in (X^+ \times X^-) \cap \mathcal{G}$. We may write $\mathfrak{f} = g.\mathfrak{f}^+$ for some $g \in G$. Then $g^{-1}(\mathfrak{f}, \mathfrak{e}) = (\mathfrak{f}^+, g^{-1}\mathfrak{e})$ again belongs to $(X^+ \times X^-) \cap \mathcal{G}$. According to Theorem 1.6, there exists $v \in \mathfrak{g}_1$ such that $g^{-1}\mathfrak{e} = e^{\mathrm{ad}(v)}\mathfrak{f}^-$. It follows that $(\mathfrak{f}, \mathfrak{e}) = ge^{\mathrm{ad}(v)}(\mathfrak{f}^+, \mathfrak{f}^-)$ belongs to the G-orbit $G.(\mathfrak{f}^+, \mathfrak{f}^-)$.

(3) As in the proof of (2), we translate by an element $g \in G$ such that $g\mathfrak{e} = \mathfrak{f}^-$, and then the claim is precisely the one of part (2) of Theorem 1.6.

(4) Assume $g \in \Omega^+$ and let $v := g.o^+ \in V^+$. Then $e^{-\mathrm{ad}(v)}g.o^+ = o^+$, and according to part (1), it follows that then $p := e^{-\mathrm{ad}(v)}g \in P^-$, whence the decomposition $g = e^{\mathrm{ad}(v)}p = e^{\mathrm{ad}(v)}he^{\mathrm{ad}(w)}$. Uniqueness follows from the fact that $P^+ \cap P^- = H$. Also, it is clear that any element $g \in U^+P^-$ belongs to Ω^+.

The second claim is a reformulation of Corollary 1.10.

(5) Assume $h \in \mathrm{Aut}(\mathfrak{g}, D)$. From the relation $he^{\mathrm{ad}(x)}h^{-1} = e^{\mathrm{ad}(hx)}$ $(x \in \mathfrak{g}_{\pm})$ it follows that h normalizes G. Since h stabilizes $\mathfrak{f}^{\pm}$, it follows that, for all $g \in G$, $hg.\mathfrak{f}^- = hgh^{-1}.\mathfrak{f}^- \in G.\mathfrak{f}^- = X^+$. It follows that X^+, X^-, and M are stable under h. Since G^{ext} is generated by G and all $h^{(D,r)}$ (cf. Eq. (1.6)), stability under G^{ext} also follows. $\square$

1.13. The space of flags seen as a generalized projective geometry

Theorem 1.6 may be reformulated by saying that the data $(\mathcal{F}, \mathcal{F}, \top)$ define an *affine pair geometry over* $\mathbb{K}$ in the sense of [3, Section 1.4], where the term *remote* is used instead of “transversal:” for any $\mathfrak{f} \in \mathcal{F}$ the set of elements remote to $\mathfrak{f}$ is non-empty and carries a canonical structure of an affine space over $\mathbb{K}$, and $\mathcal{F}$ is covered by these “affine parts”. The inclusion $(X^+, X^-) \subset (\mathcal{F}, \mathcal{F})$ is compatible with this structure: since $P^+ \cap P^- = H$,

$$M \to X^+ \times X^-, \quad gH \mapsto \big(gP^-, gP^+\big) \tag{1.17}$$

is a well-defined imbedding, and the following diagram commutes:

$$\begin{tikzcd} G/H \arrow[r, hook] \arrow[d] & G/P^- \times G/P^+ \arrow[d] \\ \mathcal{G} \arrow[r, hook] & \mathcal{F} \times \mathcal{F}. \end{tikzcd} \tag{1.18}$$

Thus we may say that the data (X^+, X^-, M) defines a subgeometry of $(\mathcal{F}, \mathcal{F}, \mathcal{G})$ on which the elementary projective group G acts transitively. On every affine pair geometry there is a natural relation of *connectedness* (two elements are connected if there is a sequence of affine parts, each one having non-empty intersection with the preceding one, joining these two points, cf. [3, Section 5.6]), and in this sense (X^+, X^-) is simply the connected component of $(\mathcal{F}, \mathcal{F})$ containing the base point (o^+, o^-).

Generalized projective geometries are distinguished among more general affine pair geometries by additional algebraic properties. Namely, assume $(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)$ is a "generic triple" of inner 3-filtrations; by this we mean that it belongs to the space

$$(\mathcal{F} \times \mathcal{F} \times \mathcal{F})^\top := \{(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3) \in \mathcal{F} \times \mathcal{F} \times \mathcal{F}\colon\ \mathfrak{f}_1 \top \mathfrak{f}_2,\ \mathfrak{f}_3 \top \mathfrak{f}_2\}. \tag{1.19}$$

Since $\mathfrak{f}_2^\top$ carries a natural structure of an affine space over $\mathbb{K}$, we may take $\mathfrak{f}_1$ as origin in $\mathfrak{f}_2^\top$, i.e., we turn $\mathfrak{f}_2^\top$ into a $\mathbb{K}$-module with zero vector $\mathfrak{f}_1$. Let $r \in \mathbb{K}$ and $r\mathfrak{f}_3$ be the ordinary multiple of $\mathfrak{f}_3$ in this $\mathbb{K}$-module. Since it depends on $\mathfrak{f}_1$ and on $\mathfrak{f}_2$, we write

$$\mu_r(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3) := r_{\mathfrak{f}_1,\mathfrak{f}_2}(\mathfrak{f}_3) := r\mathfrak{f}_3 = (1-r)\mathfrak{f}_1 + r\mathfrak{f}_3,$$

where the latter expression only refers to the affine structure. The map

$$\mu_r : (\mathcal{F} \times \mathcal{F} \times \mathcal{F})^\top \to \mathcal{F}, \quad (\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3) \mapsto \mu_r(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3) \tag{1.20}$$

is called the *structure map of the affine pair geometry* $(\mathcal{F}, \mathcal{F}, \mathcal{G})$. If $r \in \mathbb{K}^\times$, then we have

$$\mu_r(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3) = h^{(D,r)} \cdot \mathfrak{f}_3, \tag{1.21}$$

where $h^{(D,r)}$ is the automorphism defined by Eq. (1.6) and D corresponds to the 3-grading defined by the transversal pair $(\mathfrak{f}_1, \mathfrak{f}_2)$. The structure map (1.20) can be restricted to the subgeometry (X^+, X^-, M) and then gives rise to *two* maps

$$\mu_r^\pm : (X^\pm \times X^\mp \times X^\pm)^\top \to X^\pm$$

(which are well-defined because for $\mathfrak{f}_2 \in X^\mp$ we have $\mathfrak{f}_2^\top \subseteq X^\pm$ by Theorem 1.12(3)). In [3, Theorem 10.1] it is shown that these maps satisfy two remarkable identities (PG1) and (PG2) which axiomatically define the category of generalized projective geometries. The case $r = -1$ is of particular interest since it leads to associated *symmetric spaces*, cf. [3, Theorem 4.2] for the general case and Section 4 for the flag model.

2. Tangent bundle, structure bundle, and the canonical kernel

2.1. Tangent bundle and structure bundle

We continue to use the notation $\mathcal{G}$, respectively $\mathcal{F}$, for the space of inner 3-gradings (respectively 3-filtrations) of a Lie algebra $\mathfrak{g}$. For a 3-filtration $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_0)$, we define $\mathbb{K}$-modules by

$$T_{\mathfrak{f}}\mathcal{F} := \mathfrak{g}/\mathfrak{f}_0, \qquad T'_{\mathfrak{f}}\mathcal{F} := \mathfrak{f}_1, \tag{2.1}$$

called the *tangent space of $\mathcal{F}$ at $\mathfrak{f}$*, respectively the *structural space of $\mathcal{F}$ at $\mathfrak{f}$*. If $\mathfrak{f} = \mathfrak{f}^-(\mathrm{ad}(E))$ is the minus-filtration w.r.t. an Euler operator E, then $\mathfrak{f}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, and hence

$$T_{\mathfrak{f}}\mathcal{F} \cong \mathfrak{g}_1, \qquad T'_{\mathfrak{f}}\mathcal{F} = \mathfrak{g}_{-1}.$$

It is not misleading to think of $T'_{\mathfrak{f}}\mathcal{F}$ as a sort of "cotangent space" of $\mathcal{F}$ at $\mathfrak{f}$. We let

$$T\mathcal{F} := \bigcup_{\mathfrak{f}\in\mathcal{F}} T_{\mathfrak{f}}\mathcal{F}, \qquad T'\mathcal{F} := \bigcup_{\mathfrak{f}\in\mathcal{F}} T'_{\mathfrak{f}}\mathcal{F} \tag{2.2}$$

(disjoint union), called the *tangent bundle* of $\mathcal{F}$, respectively the *structure bundle* of $\mathcal{F}$. The group $\mathrm{Aut}(\mathfrak{g})$ acts on $\mathcal{G}$ and on $\mathcal{F}$, and for any $g \in \mathrm{Aut}(G)$, the following maps are well-defined and linear:

$$\begin{aligned} &T_{\mathfrak{f}}g : T_{\mathfrak{f}}\mathcal{F} \to T_{g.\mathfrak{f}}\mathcal{F}, \quad Y \bmod \mathfrak{f}_0 \mapsto gY \bmod g\mathfrak{f}_0, \\ &T'_{\mathfrak{f}}g : T'_{\mathfrak{f}}\mathcal{F} \to T'_{g.\mathfrak{f}}\mathcal{F}, \quad Y \mapsto gY, \end{aligned} \tag{2.3}$$

and if we define now $Tg : T\mathcal{F} \to T\mathcal{F}$, $T'g : T'\mathcal{F} \to T'\mathcal{F}$ in the obvious way, then clearly the functorial properties $T(g \circ h) = T(g) \circ T(h)$, and $T'(g \circ h) = T'(g) \circ T'(h)$ hold. Finally, if a base point $D \in \mathcal{G}$ is fixed and $X^\pm \subset \mathcal{F}$ are as in Corollary 1.10, then the tangent spaces $T_{\mathfrak{f}}X^\pm$, $T'_{\mathfrak{f}}X^\pm$ and the corresponding bundles $TX^\pm$, $T'X^\pm$ are defined. The natural group acting on these spaces is the normalizer of $G(D)$ in $\mathrm{Aut}(\mathfrak{g})$.

2.2. Vector fields and the canonical kernel

If $Y \in \mathfrak{g}$ and $\mathfrak{f} \in \mathcal{F}$ is as above, we say that

$$Y_{\mathfrak{f}} := Y \bmod \mathfrak{f}_0 \in T_{\mathfrak{f}}\mathcal{F} \tag{2.4}$$

is *the value of Y at $\mathfrak{f}$*, and the assignment $\widetilde{Y} : \mathcal{F} \to T\mathcal{F}$, $\mathfrak{f} \mapsto Y_{\mathfrak{f}}$ defines a *vector field* on $\mathcal{F}$. The space of vector fields on $\mathcal{F}$ is denoted by $\mathfrak{X}(\mathcal{F})$; it is a $\mathbb{K}$-module in the obvious way such that the surjection

$$\mathfrak{g} \to \mathfrak{X}(\mathcal{F}), \quad Y \mapsto \widetilde{Y}$$

becomes a $\mathbb{K}$-linear map which is equivariant w.r.t. the natural actions of $\mathrm{Aut}(\mathfrak{g})$ on both spaces. In particular, the structural spaces $T'_{\mathfrak{f}}\mathcal{F}$ are subspaces of $\mathfrak{g}$ and hence give rise to vector fields. Composing with evaluation at another point, we are lead to define, for $(\mathfrak{f},\mathfrak{e}) \in \mathcal{F} \times \mathcal{F}$, a $\mathbb{K}$-linear map by

$$K_{\mathfrak{f},\mathfrak{e}} : T'_{\mathfrak{e}}\mathcal{F} = \mathfrak{e}_1 \to T_{\mathfrak{f}}\mathcal{F} = \mathfrak{g}/\mathfrak{f}_0, \quad Y \mapsto Y_{\mathfrak{f}} = Y \bmod \mathfrak{f}_0. \tag{2.5}$$

The collection of maps $(K_{\mathfrak{e},\mathfrak{f}}, K_{\mathfrak{f},\mathfrak{e}})$, $(\mathfrak{f},\mathfrak{e}) \in \mathcal{F} \times \mathcal{F}$, is called the *canonical kernel*. Note that $K_{\mathfrak{f},\mathfrak{e}}$ is bijective if and only if $\mathfrak{e}_1$ is a $\mathbb{K}$-module complement of $\mathfrak{f}_0$ in $\mathfrak{g}$. In particular, if $\mathfrak{f} = \mathfrak{f}^-(\mathrm{ad}(E))$, $\mathfrak{e} = \mathfrak{f}^+(\mathrm{ad}(E))$, then $K_{\mathfrak{f},\mathfrak{e}}$ is identified with a linear map $\mathfrak{g}_1 \to \mathfrak{g}_1$ which is simply the identity.

Theorem 2.3. *For $\mathfrak{e},\mathfrak{f} \in \mathcal{F}$ the following statements are equivalent*:

(1) $(\mathfrak{e},\mathfrak{f}) \in \mathcal{G}$,
(2) $K_{\mathfrak{f},\mathfrak{e}} : T'_{\mathfrak{e}}\mathcal{F} \to T_{\mathfrak{f}}\mathcal{F}$ *and* $K_{\mathfrak{e},\mathfrak{f}} : T'_{\mathfrak{f}}\mathcal{F} \to T_{\mathfrak{e}}\mathcal{F}$ *are bijective.*

Proof. The second condition clearly is equivalent to saying that $\mathfrak{e}$ and $\mathfrak{f}$ are transversal, and therefore Theorem 2.3 is a restatement of part (1) of Theorem 1.6. □

2.4. Trivialization over affine parts, and quadratic polynomial vector fields

In the following we will often fix an Euler operator E, the associated 3-grading of $\mathfrak{g}$, and the associated pair $(\mathfrak{f}^-,\mathfrak{f}^+) = (\mathfrak{f}^-(\mathrm{ad}(E)), \mathfrak{f}^+(\mathrm{ad}(E)))$ of filtrations. The pair $(\mathfrak{f}^-,\mathfrak{f}^+)$ then serves as a base point in $\mathcal{G}$ and in the homogeneous space $G.(\mathfrak{f}^-,\mathfrak{f}^+) \cong G/H \subset X^+ \times X^-$ (cf. Theorem 1.12) and will also often be denoted by (o^+,o^-). The spaces $V^\pm := \mathfrak{g}_{\pm 1}$ are imbedded into $X^\pm = G.\mathfrak{f}^\mp \cong G/P^\mp$ via $X \mapsto e^{\mathrm{ad}(X)}\mathfrak{f}^\mp$; this imbedding will be considered as an inclusion, so that, for $x \in X^+$, the condition $x \in V^+$ means that $(x,o^-) \in \mathcal{G}$.

The reader may think of $X^\pm$ as a kind of "manifolds" modeled on the $\mathbb{K}$-modules $V^\pm$: we will say that

$$\mathcal{A} := \bigl\{\bigl(g\bigl(V^+\bigr), g\bigr):\ g \in G\bigr\}, \qquad \varphi_g : g\bigl(V^+\bigr) \to V^+, \quad g.x \mapsto x \tag{2.6}$$

is the *natural atlas of* X^+. Having this in mind, a natural question is to describe the structures introduced so far by a "trivialized picture" in the charts of the atlas $\mathcal{A}$. Since the spaces $X^\pm$ are homogeneous under G, one can describe $TX^\pm$ and $T'X^\pm$ as *associated bundles*: if $\pi : P^\pm \to \mathrm{GL}(W)$ is a homomorphism of $P^\pm$ into the linear group of a $\mathbb{K}$-module W, let

$$G \times_{P^\pm} W = G \times W/\sim$$

with $(g,w) \sim (gp, \pi(p)^{-1}w)$ for $p \in P^\pm$. If π is the natural representation of P^- on $W := \mathfrak{g}/(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \cong \mathfrak{g}_1$ given by

$$\pi(p) := p_{11} : \mathfrak{g}_1 \to \mathfrak{g}_1, \quad X \mapsto \mathrm{pr}_1(pX) \tag{2.7}$$

(this is the action of P^- on $T_{\mathfrak{f}^-}X^+$), then

$$G \times_{P^-} \mathfrak{g}_1 \to TX^+, \quad [g, X] \mapsto (T_{o^+}g)(X) \tag{2.8}$$

is a G-equivariant bijection. Similarly, if π is the natural representation of P^- on $W := \mathfrak{g}_{-1}$ given simply by $\pi(p)X = pX = p_{-1,-1}X$, then

$$G \times_{P^-} \mathfrak{g}_{-1} \to T'X^+, \quad [g, X] \mapsto gX \tag{2.9}$$

is a G-equivariant bijection. For TX^- and $T'X^-$ we have similar formulas. If $f : G \to W$ is a function such that $f(gp) = \pi(p)^{-1}.f(g)$ for all $g \in G$ and $p \in P^-$, then via

$$s_f\big(gP^-\big) = \big[g, f(g)\big]$$

we get a well-defined section of the natural projection $G \times_{P^-} W \to G/P^-$, and every section arises in this way. For instance, for $Y \in \mathfrak{g}$, the corresponding vector field $\widetilde{Y}$ on X^+ is given by the function

$$\widetilde{Y}_G : G \to \mathfrak{g}_1, \quad g \mapsto g^{-1}Y \bmod (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) = \mathrm{pr}_1\big(g^{-1}Y\big), \tag{2.10}$$

where for the last equality we identified $\mathfrak{g}/(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$ and $\mathfrak{g}_1$. In fact, considering (2.8) as an identification, we have

$$\begin{aligned}\widetilde{Y}_{g.o^+} &= Y \bmod \big(g(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})\big) = g\big(g^{-1}Y \bmod (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})\big)\\ &= \big[g, g^{-1}Y \bmod (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})\big] = \big[g, \widetilde{Y}_G(g)\big].\end{aligned}$$

We consider the special case $g = e^{\mathrm{ad}(v)}$ with $v \in \mathfrak{g}_1$. We identify the restriction of $\widetilde{Y}$ to $V^+ \subset X^+$ with the map

$$\widetilde{Y}^+ : V^+ \to V^+, \quad v \mapsto \mathrm{pr}_1\big(e^{-\mathrm{ad}\,v}.Y\big) = \mathrm{pr}_1\bigg(Y - [v, Y] + \frac{1}{2}\big[v, [v, Y]\big]\bigg). \tag{2.11}$$

Note that the map $\widetilde{Y}^+$ is a *quadratic* map from V^+ to V^+. In particular, it immediately follows from this formula that for $Y \in \mathfrak{g}_1$ this map is constant, for $Y \in \mathfrak{g}_0$ it is linear, and for $Y \in \mathfrak{g}_{-1}$ it is homogeneous quadratic:

$$\widetilde{Y}^+(v) = \begin{cases} Y & \text{for } Y \in \mathfrak{g}_1,\\ [Y, v] & \text{for } Y \in \mathfrak{g}_0,\\ \frac{1}{2}\big[v, [v, Y]\big] & \text{for } Y \in \mathfrak{g}_{-1}.\end{cases} \tag{2.12}$$

Similarly, $Y \in \mathfrak{g}$ gives rise to a quadratic map $\widetilde{Y}^- : V^- \to V^-$. Summing up, associating to $Y \in \mathfrak{g}$ the quadratic polynomial map $\widetilde{Y}^+ \times \widetilde{Y}^- : V^+ \times V^- \to V^+ \times V^-$ gives rise to a *trivialization map*

$$\mathfrak{g} \to \mathrm{Pol}_2\big(V^+, V^+\big) \times \mathrm{Pol}_2\big(V^-, V^-\big),$$

where $\mathrm{Pol}_2(W, W)$ is space of polynomial selfmappings of degree at most two of a $\mathbb{K}$-module w. Elements of $\mathfrak{g}_0$ are mapped onto linear polynomials; in particular, the Euler operator E is mapped onto $(\mathrm{id}_{V^+}, -\mathrm{id}_{V^-})$. The following result will not be used in the sequel, but is recorded here for the sake of completeness.

Proposition 2.5. *The trivialization map is a homomorphism of Lie algebras if we define the bracket of two quadratic polynomial maps $p, q : W \to W$ on a $\mathbb{K}$-module W by*

$$[p, q](x) = dp(x)\, q(x) - dq(x)\, p(x),$$

where the (algebraic) differentials $dp(x)$, $dq(x)$ of a (quadratic) polynomial mapping are defined in the usual way.

Proof. The commutator relations are directly checked by choosing p, q in the homogeneous parts $\mathfrak{g}_1, \mathfrak{g}_0, \mathfrak{g}_{-1}$ of $\mathfrak{g}$. $\square$

For the corresponding result on the group level, recall from Definition 1.9 the nominator and co-denominator of an element $g \in G$.

Proposition 2.6. *If $g \in \mathrm{Aut}(\mathfrak{g})$ and $x \in V^+ \subset X^+$ are such that $d_g(x)$ and $c_g(x)$ are invertible (equivalently, if $g.x \in V^+$), then for all $Y \in \mathfrak{g}$,*

$$\widetilde{\left(g^{-1}Y\right)}^+(x) = d_g(x)\widetilde{Y}^+(g.x).$$

In particular, for $Y = v \in \mathfrak{g}_1$ we have

$$\widetilde{\left(g^{-1}v\right)}^+(x) = d_g(x)v.$$

If x, $g_1.x$, and $g_1g_2.x$ belong to V^+, then the cocycle relation

$$d_{g_1g_2}(x) = d_{g_2}(x) \circ d_{g_1}(g_2.x)$$

holds.

Proof. The assumption that $g.x \in V^+$ means that $g \circ e^{\mathrm{ad}(x)}$ belongs to the set $\Omega^+ \subset G$ defined in Theorem 1.12, part (4). Therefore, according to this theorem, there exists a unique element $p(g, x) \in P^-$ such that $ge^{\mathrm{ad}(x)} = e^{\mathrm{ad}(g.x)}p(g, x)$ and hence $p(g, x) = e^{-\mathrm{ad}(g.x)}ge^{\mathrm{ad}(x)}$. From this we get

$$\begin{aligned}\left(p(g, x)^{-1}\right)_{11} &= \left(e^{-\mathrm{ad}(x)}g^{-1}e^{\mathrm{ad}(g.x)}\right)_{11} = \mathrm{pr}_1 \circ e^{-\mathrm{ad}(x)}g^{-1}e^{\mathrm{ad}(g.x)} \circ \iota_1 \\ &= \mathrm{pr}_1 \circ e^{-\mathrm{ad}(x)}g^{-1} \circ \iota_1 = \left(e^{-\mathrm{ad}(x)}g^{-1}\right)_{11} = d_g(x).\end{aligned}$$

This will be used in the last line of the following calculation (cf. also [1, VIII.B.2] for the general framework):

$$\begin{aligned}\widetilde{\left(g^{-1}Y\right)}^{+}(x) &= \widetilde{Y}_G\big(ge^{\operatorname{ad}(x)}\big) = \widetilde{Y}_G\big(e^{\operatorname{ad}(g.x)}p(g,x)\big) = \pi\big(p(g,x)\big)^{-1}\widetilde{Y}_G\big(e^{\operatorname{ad}(g.x)}\big)\\ &= \big(p(g,x)^{-1}\big)_{11}\widetilde{Y}_G\big(e^{\operatorname{ad}(g.x)}\big) = d_g(x)\widetilde{Y}^{+}(g.x).\end{aligned}$$

The second assertion follows since $\tilde{v}^{+}$ is a constant vector field on V^{+}, see Eq. (2.12). The cocycle relation now follows:

$$d_{g_1g_2}(x)v = \widetilde{\big(g_2^{-1}g_1^{-1}v\big)}^{+}(x) = d_{g_2}(x)\widetilde{\big(g_1^{-1}v\big)}^{+}(g_2.x) = d_{g_2}(x)\circ d_{g_1}(g_2.x)v. \qquad \square$$

Proposition 2.6 implies in particular that the action of g on the tangent bundle TX^{+} is given in the canonical trivialization on V^{+} by the expression $T_xg\cdot v = d_g(x)^{-1}v$; in Part II of this work we will show that, in presence of a differentiable structure, this really corresponds to the differential $dg(x)$ of g at x, applied to v. Similarly as in the proof of Proposition 2.6, it is seen that the action of g on $T'X^{+}$ is, in the trivialization $T'(V^{+})\cong V^{+}\times V^{-}$ over the affine part $V^{+}\subset X^{+}$, given by

$$T'_x g\cdot w = c_g(x)w,$$

and that the co-denominators also satisfy a cocycle relation $c_{g_1g_2}(x) = c_{g_1}(g_2.x)\circ c_{g_2}(x)$.

2.7. *Nominators*

We apply the preceding proposition in the case where Y is an Euler operator E inducing the fixed 3-grading of $\mathfrak{g}$: for $g\in\operatorname{Aut}(\mathfrak{g})$ consider the vector field $\widetilde{g^{-1}E}$ on X^{+} and define the *nominator of* g to be the quadratic polynomial map

$$n_g : V^{+}\to V^{+},\quad x\mapsto \widetilde{g^{-1}.E}^{+}(x) = \operatorname{pr}_1\big(e^{-\operatorname{ad}(x)}g^{-1}E\big) = \big(e^{-\operatorname{ad}(x)}g^{-1}\big)_{10}.E. \tag{2.13}$$

Using the matrix notation (1.4), we can also write

$$n_{g^{-1}}(x) = (g_{10} - \operatorname{ad}(x)\circ g_{00} + \frac{1}{2}\operatorname{ad}(x)^2\circ g_{-1,0})(E).$$

For the generators of G, we get the following nominators: if $v\in\mathfrak{g}_1$, $w\in\mathfrak{g}_{-1}$, $h\in H$,

$$n_g(x) = \begin{cases} x+v & \text{for } g = e^{\operatorname{ad}(v)},\\ x - \frac{1}{2}\operatorname{ad}(x)^2 w & \text{for } g = e^{\operatorname{ad}(w)},\\ x & \text{for } g = h.\end{cases} \tag{2.14}$$

Note that the nominators will not depend on the Euler operator E such that $\operatorname{ad}(E) = D$ as long as g acts trivially on the center of $\mathfrak{g}$; this is the case for all elements $g\in G$. For general $g\in\operatorname{Aut}(\mathfrak{g})$ such that $g.x\in V^{+}$, we can apply the preceding proposition and get, using that $\widetilde{E}^{+}(z) = z$ for all $z\in V^{+}$,

$$n_g(x) = d_g(x)\widetilde{E}^{+}(g.x) = d_g(x)(g.x).$$

Since $d_g(x)$ is invertible, it follows that $g.x = d_g(x)^{-1} n_g(x)$.

Theorem 2.8. *Let $g \in \mathrm{Aut}(\mathfrak{g})$ and $x \in V^+$. Then $g.x \in V^+$ if and only if $d_g(x)$ and $c_g(x)$ are invertible, and then the value $g.x \in V^+$ is given by*

$$g.x = d_g(x)^{-1} n_g(x).$$

Using matrix notation (1.4) *and replacing g by g^{-1}, this can explicitly be written as an action of* $\mathrm{Aut}(\mathfrak{g})$ *on V^+ by "fractional quadratic maps:" if $g^{-1}.x \in V^+$, then*

$$\begin{aligned} g^{-1}.x = {} & \left(g_{11} - \mathrm{ad}(x) \circ g_{01} + \frac{1}{2} \mathrm{ad}(x)^2 \circ g_{-1,1}\right)^{-1} \\ & \times \left(g_{10} - \mathrm{ad}(x) \circ g_{00} + \frac{1}{2} \mathrm{ad}(x)^2 \circ g_{-1,0}\right)(E). \end{aligned}$$

Proof. For the first claim, see Corollary 1.10, and the second claim is proved by the calculation preceding the statement of the theorem. □

Using the formulas (1.14) for the denominators and (2.14) for the nominators, we can now explicitly describe the fractional quadratic action of the generators of G:

$$\begin{aligned} g = e^{\mathrm{ad}(v)}: \quad & g(x) = x + v, \\ g = e^{\mathrm{ad}(w)}: \quad & g(x) = \left(\mathrm{id}_{V^+} + \mathrm{ad}(x)\,\mathrm{ad}(w) + \frac{1}{4} \mathrm{ad}(x)^2\,\mathrm{ad}(w)^2\right)^{-1} \left(x - \frac{1}{2} \mathrm{ad}(x)^2 w\right), \\ g = h: \quad & g(x) = h_{11} x. \end{aligned}$$

2.9. The automorphism group

The group $\mathrm{Aut}(\mathfrak{g}, D)$ acts on $V^+ \times V^-$ by

$$\begin{aligned} & \mathrm{Aut}(\mathfrak{g}, D) \to \mathrm{GL}(V^+) \times \mathrm{GL}(V^-), \\ & \quad h \mapsto (h_{11}, h_{-1,-1}) = \left(d_{h^{-1}}(o^+), c_h(o^-)\right) = \left(d_h(o^+)^{-1}, c_h(o^-)\right). \end{aligned}$$

We denote by $\mathrm{Aut}_{\mathfrak{g}}(V^+, V^-) \subset \mathrm{GL}(V^+) \times \mathrm{GL}(V^-)$ the image of this homomorphism (this is the automorphism group of the associated Jordan pair; see Section 3.1 for Jordan pairs), and by $\mathrm{Str}(V^+) := \mathrm{pr}_1 \circ \mathrm{Aut}_{\mathfrak{g}}(V^+, V^-) \circ \iota_1$ its projection to the first factor (sometimes called the *structure group of* V^+).

Theorem 2.10. *If $x \in V^+$ and $g \in \mathrm{Aut}(\mathfrak{g})$ satisfy $g.x \in V^+$, then $d_g(x) \in \mathrm{Str}(V^+)$; more precisely,*

$$\left(d_g(x)^{-1}, c_g(x)\right) \in \mathrm{Aut}_{\mathfrak{g}}\left(V^+, V^-\right).$$

Proof. If $g.x \in V^+$, then $g' := ge^{\mathrm{ad}(x)}$ belongs to the set $\Omega^+ \subset G$ defined in Theorem 1.12. According to part (4) of this theorem, we decompose

$$g' = e^{\mathrm{ad}(v)} h e^{\mathrm{ad}(w)} \tag{2.15}$$

with a unique $h = h(g,x) \in H$ depending on g and x. From the definition of the (co-) denominators it follows then that

$$d_g(x) = d_{g'}(0) = h_{11}^{-1}, \qquad c_g(x) = c_{g'}(0) = h_{-1,-1},$$

and hence $(d_g(x)^{-1}, c_g(x)) = (h_{11}, h_{-1,-1}) \in \mathrm{Aut}_{\mathfrak{g}}(V^+, V^-)$. $\square$

As remarked after Proposition 2.6, the linear map $d_g(x)^{-1}$ can be interpreted as the tangent map of g at x, and so Theorem 2.10 means that $\mathrm{Aut}(\mathfrak{g})$ acts on X^+ by mappings that are *conformal with respect to the linear group* $\mathrm{Str}(V^+)$ (in the sense defined in [1, Section VIII.1.2]). In some cases this already characterizes the group $\mathrm{Aut}(\mathfrak{g})$ as "the conformal group of X^+;" this is the content of the *Liouville theorem*, see [1, Chapter IX].

3. The Jordan theoretic formulation

3.1. Jordan pairs

If $(\mathfrak{g}, D)$ is a 3-graded Lie algebra and $V^\pm = \mathfrak{g}_{\pm 1}$, the following $\mathbb{K}$-trilinear maps are well-defined:

$$\begin{aligned} &T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm, \\ &\quad (X,Y,Z) \mapsto T^\pm(X,Y,Z) := -\big[[X,Y],Z\big] = \mathrm{ad}(Z)\,\mathrm{ad}(X)Y = -\,\mathrm{ad}(X)\,\mathrm{ad}(Y)Z, \end{aligned} \tag{3.1}$$

and they satisfy the following identities, where we use the notation $T^\pm(X,Y)Z := T^\pm(X,Y,Z)$:

$$\begin{aligned} T^\pm(X,Y,Z) &= T^\pm(Z,Y,X), \\ T^\pm(X,Y)T^\pm(U,V,W) &= T^\pm\big(T^\pm(X,Y,U),V,W\big) - T^\pm\big(U,T^\mp(Y,X,V),W\big) \\ &\quad + T^\pm\big(U,V,T^\pm(X,Y,W)\big), \end{aligned} \tag{3.2}$$

which mean that $((V^+,V^-),(T^+,T^-))$ is a *linear Jordan pair over* $\mathbb{K}$ (if 2 and 3 are invertible in $\mathbb{K}$, these two identities imply all other identities valid in Jordan pairs, cf. [16, Proposition 2.2(b)]). In the following we shall omit the adjective *linear*, when dealing with Jordan pairs. Conversely, if $(V^\pm, T^\pm)$ is a Jordan pair over $\mathbb{K}$, then for $v \in V^\pm$ and $w \in V^\mp$ we define the operator $(v,w) \in \mathrm{End}(V^\pm)$ by $T^\pm(v,w).x := T^\pm(v,w,x)$ and let $\mathrm{ider}(V^+,V^-) \subseteq \mathfrak{gl}(V^+) \times \mathfrak{gl}(V^-)$ be the Lie subalgebra generated by the operators

$(-T^+(v,w), T^-(w,v))$, $v \in V^+$, $w \in V^-$. The elements of this Lie algebra are called *inner derivations*. The algebra of *derivations of* (V^+, V^-) is defined by

$$\begin{aligned}\operatorname{der}\bigl(V^+, V^-\bigr) = \bigl\{\bigl(A^+, A^-\bigr) \in \operatorname{End}_{\mathbb{K}}\bigl(V^+\bigr) \times \operatorname{End}_{\mathbb{K}}\bigl(V^+\bigr)\colon\ (\forall u, v, w)\\ A^\pm T^\pm(u,v,w) = T^\pm\bigl(A^\pm u, v, w\bigr) + T^\pm\bigl(u, A^\mp v, w\bigr)\\ + T^\pm\bigl(u, v, A^\pm w\bigr)\bigr\},\end{aligned} \tag{3.3}$$

and it follows from (3.2) that it contains $\operatorname{ider}(V^+, V^-)$. Clearly, it contains also the element

$$E := (\mathrm{id}_{V^+}, -\mathrm{id}_{V^-}), \tag{3.4}$$

called the *Euler operator of the Jordan pair* $V^\pm$.

If we are given a Jordan pair (V^+, V^-), and $\mathfrak{g}_0 \subseteq \operatorname{der}(V^+, V^-)$ is a Lie subalgebra containing all inner derivations, then there is a unique structure of a 3-graded Lie algebra on $V^+ \oplus \mathfrak{g}_0 \oplus V^-$ whose associated Jordan pair is (V^-, V^+), and where the bracket satisfies

$$[v,w] = \bigl(-T^+(v,w), T^-(w,v)\bigr), \quad v \in V^+,\ w \in V^-, \tag{3.5}$$

and the grading element is the Euler operator E given by (3.4). The subalgebra

$$\operatorname{TKK}\bigl(V^+, V^-\bigr) := V^+ \oplus \bigl(\operatorname{ider}\bigl(V^+, V^-\bigr) + \mathbb{K}E\bigr) \oplus V^-$$

is called the *Tits–Kantor–Koecher algebra of the Jordan pair* (V^+, V^-). This choice for the 3-graded Lie algebra associated to (V^+, V^-) has the advantage that $\mathfrak{z}(\mathfrak{g}) = 0$.

The preceding construction may also be interpreted in the context of *Lie triple systems* (cf., e.g., [1, Section III.3]): it is essentially the *standard imbedding* of the (polarized) Lie triple system $\mathfrak{q} := V^+ \oplus V^-$ into the corresponding Lie algebra $\mathfrak{g} = \mathfrak{q} \oplus [\mathfrak{q}, \mathfrak{q}]$. The standard imbedding yields a bijection between Lie triple systems and Lie algebras with involution, generated by the -1-eigenspace of the involution. See Section 6 concerning functorial properties of these constructions.

For any $\mathfrak{g}_0$ as above, the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ will be faithful, so that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_0 = \{0\}$. It may happen for central extensions $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ that the corresponding subalgebra $\hat{\mathfrak{g}}_0$ does not act faithfully on $\hat{\mathfrak{g}}_{-1} \oplus \hat{\mathfrak{g}}_1 \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ (see Section 7).

3.2. *Projective elementary group and projective completion*

For the rest of Section 3, we fix a Jordan pair (V^+, V^-) and let $\mathfrak{g} := \operatorname{TKK}(V^+, V^-)$. The projective elementary group $\operatorname{PE}(V^+, V^-) := G(\operatorname{ad}(E))$ is defined as in Section 1.2. Using the notation, with $x, y \in V^\pm$, $v \in V^\mp$,

$$\begin{aligned}Q^\pm(x)v &:= \frac{1}{2}\operatorname{ad}(x)^2 v = \frac{1}{2}\bigl[x, [x,v]\bigr] = \frac{1}{2}T(x,v,x),\\ Q^\pm(x,y) &:= Q^\pm(x+y) - Q^\pm(x) - Q^\pm(y) = T(x,\cdot,y)\\ &= \operatorname{ad}(x)\operatorname{ad}(y) : V^\mp \to V^\pm,\end{aligned} \tag{3.6}$$

the operators $e^{\operatorname{ad} x} = \mathbf{1} + \operatorname{ad} x + \frac{1}{2}(\operatorname{ad} x)^2$ ($x \in V^{\pm}$) are given in matrix notation by Eq. (1.5), with $\frac{1}{2}\operatorname{ad}(x)^2$ replaced by $Q^+(x)$ and $\frac{1}{2}\operatorname{ad}(y)^2$ replaced by $Q^-(y)$. Our definition of $\mathrm{PE}(V^+, V^-)$ follows the one by O. Loos from [18]. The projective linear group of a Jordan pair has been introduced by Faulkner in [10] in a slightly different context (without Euler operator). The groups $P^{\pm}$ and the spaces $X^{\pm} = G/P^{\mp}$ are defined as in Section 1.11; the embedding $V^+ \times V^- \to X^+ \times X^-$ is called the *projective completion of the Jordan pair* (V^+, V^-).

3.3. The Bergman operator

Recall from Section 2.2 the canonical kernel: for $(x, y) \in X^+ \times X^-$,

$$K_{x,y} : T_y' X^- \to T_x X^+, \quad Y \mapsto Y_x. \tag{3.7}$$

Of course, there is a similarly defined map $K_{y,x}$; we will also use the notation $(K^+_{x,y}, K^-_{y,x})$ for $(K_{x,y}, K_{y,x})$. Using the description via associated bundles, the kernel is given by

$$K_{g_1 P^-, g_2 P^+} : T'_{g_2 P^+} X^- \to T_{g_1 P^-} X^+, \quad [g_2, v] \mapsto \big[g_1, \mathrm{pr}_1\big(g_1^{-1} g_2.v\big)\big], \tag{3.8}$$

and hence the trivialized picture is

$$K^+_{x,y} = \big(e^{-\operatorname{ad} x} e^{\operatorname{ad} y}\big)_{11} = d_{\exp -y}(x) : V^+ \to V^+. \tag{3.9}$$

In matrix form,

$$e^{-\operatorname{ad} x} e^{\operatorname{ad} y} = \begin{pmatrix} \mathbf{1} & -\operatorname{ad}(x) & Q^+(x) \\ 0 & \mathbf{1} & -\operatorname{ad}(x) \\ 0 & 0 & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \operatorname{ad}(y) & \mathbf{1} & 0 \\ Q^-(y) & \operatorname{ad}(y) & \mathbf{1} \end{pmatrix},$$

so that we get for the coefficient with index 11, using that on V^+ we have for $x \in V^+$ and $y \in V^-$ the relation $\operatorname{ad} x \operatorname{ad} y = \operatorname{ad}[x, y] = -T^+(x, y)$:

$$K^+_{x,-y} = B_+(x, y) = \mathrm{id}_{V^+} - T^+(x, y) + Q^+(x) Q^-(y). \tag{3.10}$$

We likewise get

$$K^-_{x,-y} = B_-(y, x) = \mathrm{id}_{V^-} - T^-(y, x) + Q^-(y) Q^+(x)$$

(cf. the definition in (1.15)). This expression is known as the *Bergman operator of the Jordan pair* (V^+, V^-). Theorem 2.3 now implies that *the pair* (v, w) *is transversal if and only if* $(B_+(v, -w), B_-(-w, v))$ *is invertible in* $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$. It is known in Jordan theory that $B_+(v, -w)$ is invertible if and only if so is $B_-(-w, v)$ (the symmetry principle, cf. [16, Proposition I.3.3]), and hence (v, w) is transversal if and only if $B_+(v, -w)$ is invertible. So far we do not know a "Lie theoretic" proof of this fact.

3.4. The quasi-inverse

Let $y \in \mathfrak{g}_{-1}$. Then for $g = e^{\mathrm{ad}(y)}$ and $x \in V^+$, formulas (1.14) and (2.14) show that denominator, codenominator, and nominator of g are given by

$$d_g(x) = B_+(x, y), \qquad c_g(x) = B_-(y, x), \qquad n_g(x) = x - Q^+(x)y, \tag{3.11}$$

and hence, according to Theorem 2.8, $g(x) \in V^+$ if and only if $(B_+(x, y), B_-(y, x))$ is invertible, and then

$$g.x = B_+(x, y)^{-1}\bigl(x - Q^+(x)y\bigr). \tag{3.12}$$

Following [17], we will use also the notation $t_v(x) = x + v$ for translations on V^+ and

$$\tilde{t}_w(x) := e^{\mathrm{ad}(w)}.x = B_+(x, w)^{-1}\bigl(x - Q^+(x)w\bigr) \tag{3.13}$$

for "dual translations" or "quasi-inverses." In Jordan theory the notation $x^y := e^{\mathrm{ad}\, y}.x$ is also widely used (cf. [16]), and one says that (x, y) is *quasi-invertible* if $(B_+(x, y), B_-(y, x))$ is invertible, i.e., if $(x, -y)$ is a transversal pair. Our definitions of the Bergman operator via the canonical kernel and of the quasi-inverse are natural in the sense that they have natural transformation properties with respect to elements g of the group $\mathrm{Aut}(\mathfrak{g})$; taking for g typical generators of G, we get Jordan theoretic results such as the "shifting principle" (see [1, Section VIII.A] for the precise form of the argument).

3.5. Automorphism and structure group

The group $\mathrm{Aut}_{\mathfrak{g}}(V^+, V^-)$ defined in Section 2.9 coincides for $\mathfrak{g} = \mathrm{TKK}(V^+, V^-)$ with the automorphism group $\mathrm{Aut}(V^+, V^-)$ of (V^+, V^-) in the Jordan theoretic sense. It follows from Theorem 2.10 that if $(x, -y)$ is transversal, then $\beta(x, y) := (B_+(x, y), B_-(y, x)^{-1})$ belongs to $\mathrm{Aut}(V^+, V^-)$. The subgroup generated by these elements is called the *inner automorphism group*. Projecting to the first factor, one gets the *structure group*, respectively the *inner structure group of* V^+.

3.6. Jordan fractional quadratic transformations

An $\mathrm{End}(V^+)$*-valued Jordan matrix coefficient* (*of type* $(1, 1)$, *respectively of type* $(1, 0)$) is a map of the type

$$q : V^\sigma \times V^\nu \to \mathrm{End}\bigl(V^+\bigr), \quad (x, y) \mapsto \bigl(e^{\mathrm{ad}(x)} g e^{\mathrm{ad}(y)} h\bigr)_{11},$$

where $\sigma, \nu \in \{\pm\}$ and g, h belong to the extended elementary projective group G^{ext} (cf. Section 1.2), respectively

$$p : V^\sigma \times V^\nu \to V^+, \quad (x, y) \mapsto \bigl(e^{\mathrm{ad}(x)} g e^{\mathrm{ad}(y)} h\bigr)_{10} E.$$

These maps are quadratic polynomial in x and in y. Nominators and denominators of elements of G are partial maps obtained of maps of the type of p or q by fixing one of the arguments to be zero. A *Jordan fractional quadratic map* is a map of the form

$$f : V^\sigma \times V^\nu \supset U \to V^+, \quad (x, y) \mapsto q(x, y)^{-1} p(x, y),$$

where q, p are Jordan matrix coefficients of type $(1, 1)$, respectively $(1, 0)$, and $U = \{(x, y) \in V^\sigma \times V^\nu\colon\ q(x, y) \in \mathrm{GL}(V^+)\}$. In the following, we also use the notation $\exp(x) := e^{\mathrm{ad}(x)}$ for $x \in V^\pm$.

Theorem 3.7. *The actions*

$$V^+ \times X^+ \to X^+ \quad \textit{and} \quad V^- \times X^+ \to X^+$$

are given, with respect to all charts from the atlas $\mathcal{A}$ (cf. Eq. (2.6))*, by Jordan fractional quadratic maps. In other words, for all $g, h \in G$, the maps*

$$(v, y) \mapsto \big(h \circ \exp(v) \circ g\big).y, \quad (w, y) \mapsto \big(h \circ \exp(w) \circ g\big).y$$

are Jordan fractional quadratic.

Proof. As to the first action, we write

$$\big(h \circ \exp(v) \circ g\big).y = \big(d_{h\circ\exp(v)\circ g}(y)\big)^{-1} n_{h\circ\exp(v)\circ g}(y) = q(v, y)^{-1} p(v, y)$$

with

$$q(v, y) = d_{h\circ\exp(v)\circ g}(y) = \big(e^{-\mathrm{ad}(y)} g^{-1} e^{-\mathrm{ad}(v)} h^{-1}\big)_{11} \quad \text{and}$$
$$p(v, y) = n_{h\circ\exp(v)\circ g}(y) = \big(e^{-\mathrm{ad}(y)} g^{-1} e^{-\mathrm{ad}(v)} h^{-1}\big)_{10} E,$$

and hence the action is Jordan fractional quadratic. For the action of $e^{\mathrm{ad}(V^-)}$, we use the same arguments. □

We may say that $H_\infty := X^+ \setminus V^+$ is the "hyperplane at infinity;" then H_∞ is stable under the action of V^+. In case $(X^+, X^-) = (\mathbb{KP}^n, (\mathbb{KP}^n)^*)$ is an ordinary projective geometry, the action of the translation group on the hyperplane at infinity is the trivial action. However, already in the case of more general Grassmannian geometries this is no longer true, as can be seen from the explicit formulas for this case given in [2].

Corollary 3.8. *With respect to the charts from the atlas $\mathcal{A}$, the structure maps μ_r for $r \in \mathbb{K}^\times$ defined in Section* 1.14 *are given by a composition of Jordan fractional quadratic maps and diagonal maps $\delta(x) = (x, x)$.*

Proof. According to [3, Corollary 5.8], the multiplication maps can be written as a composition of maps of the type described in the preceding theorem, diagonal maps and one dilation $h^{(D,r)}$ (defined in Section 1.2). But this dilation comes from an element of G^{ext} and hence the composition with such a dilation is again Jordan fractional quadratic. □

3.9. Case of a general base ring

Even if $\mathbb{K}$ is a general base ring (i.e., possibly with 2 or 3 not invertible), there still is a 3-graded Lie algebra $\mathrm{TKK}(V^+, V^-)$ and a group $\mathrm{PE}(V^+, V^-)$ associated to a general (quadratic) Jordan pair, cf. [18]. The main difference is that in the matrix expression of $e^{\mathrm{ad}(x)}$ ($x \in \mathfrak{g}_1$) the term $\frac{1}{2}\,\mathrm{ad}(x)^2$ has to be replaced by $Q^+(x)$. Once one has checked that the abelian groups $U^\pm$ obtained in this way are well-defined groups of automorphisms, one can essentially proceed as we did in Section 1, replacing the space $\mathcal{G}$ by the $\mathrm{PE}(V^+, V^-)$-orbit of $\mathrm{ad}(E)$ in $\mathrm{der}(\mathfrak{g})$ and the space $\mathcal{F}$ by the space of inner filtrations belonging to gradings from $\mathcal{G}$.

4. Involutions, symmetric spaces, and Jordan triple systems

4.1. Symmetric spaces attached to a Lie algebra

An (*abstract*) *reflection space* is a set S together with a map $\mu : S \times S \to S$ such that, if we let $\sigma_x(y) := \mu(x, y)$,

(S1) $\mu(x, x) = x$,
(S2) $\sigma_x^2 = \mathrm{id}_S$,
(S3) σ_x is an automorphism of μ, i.e. $\sigma_x(\mu(y, z)) = \mu(\sigma_x(y), \sigma_x(z))$.

(Differentiable reflection spaces, i.e., manifolds with a smooth reflection space structure μ, have been introduced by O. Loos in [14].) In Part II [6] of this work we define a *symmetric space* (*over* $\mathbb{K}$) to be a reflection space (S, μ) such that S is a smooth manifold over $\mathbb{K}$ (in the sense of [5]) and μ is smooth and satisfies

(S4) the tangent map $T_x\sigma_x$ of σ_x at x is given by $-\mathrm{id}_{T_x S}$.

(See [6] for the basic theory of symmetric spaces and for a comparison with the approach by O. Loos [15].) To any Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ we may associate a reflection space as follows. Let $S = \widetilde{\mathcal{G}} = \{D \in \mathrm{der}(\mathfrak{g}){:}\ D^3 = D\}$ be the space of 3-gradings of $\mathfrak{g}$ and recall from Section 1.2 the definition of the extended projective elementary group G^{ext} which is generated by its normal subgroup G and the subgroup $\{h^{(D,r)}{:}\ r \in \mathbb{K}^\times\}$. Taking $r = -1$, we get the *reflection elements*

$$\sigma^{(D)} := h^{(D,-1)} = \mathbf{1} - 2D^2 \in \mathrm{Aut}(\mathfrak{g}, D). \tag{4.1}$$

We define the map μ by

$$\mu : S \times S \to S, \quad \mu(D, D') := \sigma^{(D)} D' \sigma^{(D)} = (\mathbf{1} - 2D)D'(\mathbf{1} - 2D). \tag{4.2}$$

Then (S1) follows from the fact that D and $\sigma^{(D)}$ commute, (S2) holds because $\sigma^{(D)}$ is an involution, and (S3) follows from the fact that $\mathrm{Aut}(\mathfrak{g})$ clearly acts as a group of automorphisms of μ, and all reflection elements $\sigma^{(D)}$ belong to $\mathrm{Aut}(\mathfrak{g})$. It is clear that the subset $\mathcal{G} \subset \widetilde{\mathcal{G}}$ is stable under μ. Also, $M \subset \mathcal{G}$ is stable under μ because M is stable under the action of G^{ext} (Theorem 1.12(5)), and G^{ext} contains the reflection element $\sigma^{(D)}$ corresponding to the base point and hence contains also all reflection elements corresponding to points of M. Property (S4) is also satisfied in a purely algebraic sense: since $\sigma^{(D)}$ acts by -1 on the complement $\mathfrak{g}_\mp$ of $\mathfrak{g}_\pm \oplus \mathfrak{g}_0$, it follows readily from the definition of the tangent map in Section 2.1 that

$$T_{\mathfrak{f}^+(D)}\sigma^{(D)} = -\mathrm{id}_{T_{\mathfrak{f}^+(D)}}\mathcal{F}, \qquad T_{\mathfrak{f}^-(D)}\sigma^{(D)} = -\mathrm{id}_{T_{\mathfrak{f}^-(D)}}\mathcal{F},$$

and hence the tangent map $T_D(\sigma^{(D)})$ will be minus one if we define tangent map and tangent space at D to be the direct product of the ones defined with respect to $\mathfrak{f}^+(D)$ and $\mathfrak{f}^-(D)$.

The restriction of μ to $\mathcal{G} \times \mathcal{G}$ is related to the ternary map μ_{-1} from Section 1.13 as follows: assume D_1 corresponds to the transversal pair $(\mathfrak{f}_1, \mathfrak{f}_2)$ and D_2 to the transversal pair $(\mathfrak{f}_3, \mathfrak{f}_4)$. Then

$$\mu\big((\mathfrak{f}_1, \mathfrak{f}_2), (\mathfrak{f}_3, \mathfrak{f}_4)\big) = \big(\mu_{-1}(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3), \mu_{-1}(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_4)\big), = \big(h^{(D,-1)}.\mathfrak{f}_3, h^{(D,-1)}.\mathfrak{f}_4\big), \tag{4.3}$$

which is the same as the product map on M considered in [3, Corollary 4.4].

4.2. *Involutions and symmetric subspaces*

An *involution* of a 3-graded Lie algebra is a Lie algebra automorphism θ of order 2 reversing the grading, i.e., such that $\theta(\mathfrak{g}_{\pm 1}) = \mathfrak{g}_{\mp 1}$ and $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$. An involution θ induces by conjugation an automorphism of the elementary projective group G, again denoted by θ, such that $\theta(P^-) = P^+$. Therefore, it induces a bijection

$$p : X^+ \to X^-, \quad gP^- \mapsto \theta(g)P^+, \tag{4.4}$$

compatible with the map $\mathcal{F} \to \mathcal{F}$, $\mathfrak{f} \mapsto \theta(\mathfrak{f})$, and such that $p(o^+) = o^-$. We say that $\mathfrak{f} \in \mathcal{F}$ is *non-isotropic* (*with respect to* θ) if $\theta(\mathfrak{f}) \top \mathfrak{f}$. In particular, the base point $o^+ = \mathfrak{f}^-$ is non-isotropic; thus there exist non-isotropic points, and p is a *polarity* in the sense of [3]. Since θ is an automorphism normalizing G, the spaces $\mathcal{G}$ and $M \subset \mathcal{G}$ are stable under θ, and the naturality of the product μ implies that θ is an automorphism of μ. Therefore, the θ-fixed subspace M^θ is a symmetric subspace of M, which as a set is in bijection with the set of non-isotropic points of X^+, i.e.,

$$M^{(p)} := \big\{\mathfrak{f} \in X^+ \colon \mathfrak{f} \text{ non-isotropic w.r.t. } \theta\big\} \to M^\theta, \quad \mathfrak{f} \mapsto \big(\mathfrak{f}, \theta(\mathfrak{f})\big)$$

is a bijection. By forward transport of structure, the symmetric space structure of M^θ corresponds to the structure on $M^{(p)}$ given by

$$\mu(x, y) = \mu_{-1}\big(x, p(x), y\big) \tag{4.5}$$

(this is the formula used in [3] to define the symmetric space structure). The symmetry w.r.t. the point x is now induced by the element $\sigma^{(D)}$, where $D \in \mathcal{G}$ corresponds to the point $(x, \theta(x)) \in \mathcal{G}$; as noticed above, the algebraically defined tangent map $T_x(\sigma^{(D)})$ equals minus the identity, and hence (S4) is again satisfied in an algebraic sense.

Theorem 4.3. *For a fixed polarity $p : X^+ \to X^-$, we identify X^+ and X^- via p. Then the multiplication map μ on $M^{(p)}$ is a composition of Jordan fractional quadratic maps and diagonal maps $\delta(x) = (x, x)$.*

Proof. By Corollary 3.8, the map μ_{-1} is of the form mentioned in the claim. According to formula (4.5), μ is related to μ_{-1} via

$$\mu(x, y) = \mu_{-1}(x, x, y), \quad \text{i.e.,} \quad \mu = \mu_{-1} \circ (\delta \times \mathrm{id}),$$

which proves the claim. □

In [6] it will be shown that Theorem 4.3 implies, in very general situations, smoothness of μ.

4.4. Involutions and Jordan triple systems

If θ is an involution of the 3-graded Lie algebra $\mathfrak{g}$, the trilinear map on V^+ defined by

$$T(X, Y, Z) := -\big[[X, \theta(Y)], Z\big] \tag{4.6}$$

is a *Jordan triple product*, i.e., it satisfies the identities (3.1) with the superscripts $\pm$ omitted. Conversely, given a *Jordan triple system over* $\mathbb{K}$ (abbreviated JTS) (i.e., a $\mathbb{K}$-module with a $\mathbb{K}$-trilinear map satisfying the above mentioned identities), we can define an involution on the Lie algebra $V^+ \oplus \mathrm{der}(V^+, V^-) \oplus V^-$ by

$$\theta\big(v, (A, B), w\big) = \big(w, (B, A), v\big), \tag{4.7}$$

and the associated JTS is the one we started with. In this way we get a *bijection between Jordan triple systems over* $\mathbb{K}$ *and minimal* 3*-graded Lie algebras with involution* (see Section 1.1).

5. Self-dual geometries and Jordan algebras

5.1. Self-dual geometries

We fix a 3-graded Lie algebra $\mathfrak{g}$ with grading induced by the Euler operator E. Recall our realization of X^+ and X^- as G-orbits in the space $\mathcal{F}$ of 3-filtrations of $\mathfrak{g}$. Two cases can arise: either $X^+ \cap X^-$ is empty, or $X^+ = X^-$. In the latter case we let $X := X^+ = X^-$, and again two cases are possible: either

(a) $V^+ \cap V^-$ is empty, or
(b) $V^+ \cap V^-$ is not empty; then we say that the geometry given by $(\mathfrak{g}, E)$ is *self-dual*, and we let $V^\times := V^+ \cap V^-$.

An equivalent characterization of self-dual geometries is: there are three points $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3 \in X^+$ such that $\mathfrak{f}_1 \top \mathfrak{f}_2, \mathfrak{f}_2 \top \mathfrak{f}_3, \mathfrak{f}_3 \top \mathfrak{f}_1$ (namely, take $(\mathfrak{f}_1, \mathfrak{f}_2) = (\mathfrak{f}^-, \mathfrak{f}^+)$ to be the base point and $\mathfrak{f}_3$ some element of $V^+ \cap V^-$). In this situation, the identity map $\mathrm{id} : X^+ \to X^-$ is called the *absolute identification* or the *central null-system* (cf. [4, 1.4]).

5.2. The Jordan inverse

Assume that $(\mathfrak{g}, E)$ is self-dual and fix some point $\mathfrak{f} \in V^+ \cap V^-$. We claim that there exists an involution j of $\mathfrak{g}$ (cf. Section 4.2) such that $j(V^+) \cap V^+ \neq \emptyset$. In fact, let $W := \mathfrak{f}^\top$; then $W \subset X$ carries a natural structure of an affine space over $\mathbb{K}$ (Theorem 1.12(3)), and by assumption, o^+ and o^- belong to W. Let $\mathfrak{e} \in W$ be the midpoint of o^+ and o^- in the affine space W. Since $\mathfrak{e} \in W$, the pair $(\mathfrak{e}, \mathfrak{f})$ is transversal and hence corresponds to a 3-grading $\mathfrak{g} = \mathfrak{g}'_1 \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_{-1}$, i.e., to an element $D' \in \mathcal{G}$. Let $j := h^{(D',-1)} \in G^{\mathrm{ext}}$ be the automorphism that is minus one on $\mathfrak{g}'_1 \oplus \mathfrak{g}'_{-1}$ and one on $\mathfrak{g}'_0$. Then j fixes $(\mathfrak{e}, \mathfrak{f})$ and acts by the scalar minus one on the $\mathbb{K}$-module W with zero vector $\mathfrak{e}$. Since $\mathfrak{e}$ is the midpoint of o^- and o^+, it follows that $j(o^-) = o^+$, and since obviously j is of order two, it is an involution. The condition $j(\mathfrak{f}^-) = \mathfrak{f}^+$ implies that $j(V^+) = j((\mathfrak{f}^+)^\top) = (\mathfrak{f}^-)^\top = V^-$. In particular, $V^+ \cap j(V^+) = V^+ \cap V^- = V^\times$ is non-empty by assumption. It contains the point $\mathfrak{e} = j(\mathfrak{e})$.

Now we apply Theorem 2.8 in order to derive an explicit formula for j in the chart V^+: for $v \in \mathfrak{g}_1$, let $\mathbf{v} = jv \in \mathfrak{g}_{-1}$; by Eq. (2.12), $\mathbf{v}$ gives rise to the homogeneous quadratic vector field $\mathbf{v}^+(x) = Q^+(x)\mathbf{v}$ on V^+. From Proposition 2.6 we now get

$$d_j(x)v = \big(j^{-1}v\big)(x) = \mathbf{v}^+(x) = Q^+(x)\mathbf{v} = Q^+(x)jv.$$

In a similar way we see that $c_j(x)w = Q^-(\mathbf{x})w = Q^-(jx)w$. (In fact, since j is an involution, $c_j(x) = jd_j(-x)j = jd_j(x)j$, and this is invertible if and only if so is $d_j(x)$.) Corollary 1.10 now shows that $j(x) \in V^+$ if and only if the operator $Q^+(x)$ is invertible. On the other hand, $j(x) \in V^+$ iff $x \in V^\times = V^+ \cap V^-$, and hence

$$V^\times = \big\{x \in V^+ \colon\ Q^+(x) \text{ invertible}\big\} \tag{5.1}$$

is the *set of invertible elements in* V^+ in the Jordan theoretic sense [16, I.1.10]. The nominator of j is $n_j(x) = -x$ since j reverses the grading (i.e., $jE + E \in \mathfrak{z}(\mathfrak{g})$). Now Theorem 2.8 shows that, for $x \in V^\times$, $j(x) = -Q^+(x)^{-1}x$. Therefore, the (non-linear) description of j in the chart V^+ is

$$V^+ \supset V^\times \to V^\times \subset V^+, \quad x \mapsto j(x) := -Q^+(x)^{-1}x,$$

which is called the (Jordan algebraic) *Jordan inverse*.

5.3. *Jordan algebras*

If (V^+, V^-) is a Jordan pair and $y \in V^-$ any element, then V^+ with squaring and product given by

$$x^2 = Q^+(x)y, \qquad x \cdot z = \frac{1}{2}\big((x+z)^2 - x^2 - z^2\big) = \frac{1}{2}Q^+(x,z)y \tag{5.2}$$

is a Jordan algebra in the usual sense (in the linear case, this result is known as “Meyberg’s theorem;” cf. [16, I.1.9] for the general, quadratic case). This algebra is unital if and only if y is invertible, and then the unit element is $Q^-(y)^{-1}y$ [16, Proposition I.1.11]. Moreover, every unital Jordan algebra arises in this way. Comparing with the situation from Section 5.2, we choose $y = \mathbf{e} \in V^-$ to be the element of V^- that corresponds to the point $\mathfrak{e}$ which in turn corresponds to the point $e \in V^+$. Then e is the unit element in the corresponding Jordan algebra structure on V^+, and j is the inversion map associated to this Jordan algebra. Note that for all these constructions it is not necessary to identify V^+ and V^- which would be somewhat dangerous because there are several ways to do so (cf. [4]).

5.4. *The self dual geometry associated to a unital Jordan algebra*

Now assume that $(\mathfrak{g}, E)$ is 3-graded and there exists $e^- \in \mathfrak{g}_{-1}$ such that $Q^-(e^-): V^+ \to V^-$ is a bijection. Let $g := e^{\mathrm{ad}(e^-)}$. We claim that the flag $\mathfrak{f}^+ : \mathfrak{g}_1 \subset \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is transversal to the flag $g(\mathfrak{f}^+)$: first of all, for $v \in \mathfrak{g}_1$,

$$\mathrm{pr}_{-1}\big(g(v)\big) = \mathrm{pr}_{-1}\big(v + [e^-, v] + Q^-(e^-)v\big) = Q^-(e^-)v,$$

hence $\mathrm{pr}_{-1} \circ g \circ \iota_1$ is bijective and thus $g(\mathfrak{g}_1)$ is a complement of $\mathfrak{g}_1 \oplus \mathfrak{g}_0$. Next, $g(\mathfrak{g}_1 \oplus \mathfrak{g}_0)$ is a complement of $\mathfrak{g}_1$: equivalently, $e^{-\mathrm{ad}(e^-)}\mathfrak{g}_1$ is a complement of $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, which is true by the same argument. Hence, $g\mathfrak{f}^+ \top \mathfrak{f}^+$. With $(o^+, o^-) = (\mathfrak{f}^-, \mathfrak{f}^+)$, this means that $g.o^- \in V^+ \subset X^+$; but since $g \in G$, this means that $X^- = X^+$. Moreover, $o^- \in V^-$, and $e^{\mathrm{ad}(e^-)}$ acts as a translation on V^-; therefore $g.o^- \in V^- \cap V^+$, and it follows that the geometry is self-dual. Summing up:

Theorem 5.5. *For a Lie algebra* $\mathfrak{g}$ *with Euler operator* E, *the following are equivalent*:

(1) *The geometry given by* $(\mathfrak{g}, E)$ *is self-dual.*

(2) *There is an involution j of $(\mathfrak{g}, E)$ such that $j(V^+) \cap V^+ \neq \emptyset$.*
(3) *The Jordan pair (V^+, V^-) contains invertible elements.*
(4) *The Jordan pair (V^+, V^-) comes from a unital Jordan algebra (V, E).*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) has been shown in Section 5.2, and (3) $\Rightarrow$ (1) has been shown in Section 5.4. The equivalence of (3) and (4) is well-known (cf. [16, I.1.10]; see Section 5.3). $\square$

We do not know whether the condition $X^+ = X^-$ alone already implies that we have $V^+ \cap V^- \neq \emptyset$—in the finite-dimensional case over a field this certainly is true since then the "hyperplane at infinity" $X^+ \setminus V^+$ is an algebraic hypersurface, and hence V^+ and V^- must intersect if they are both included in X^+. However, in infinite dimension the "hyperplane at infinity" may become rather "big" and may very well contain some affine parts—this problem is discussed in [4, Section 1.9].

6. Functorial properties

6.1. Functoriality problems

So far we have considered the following categories: Jordan pairs (V^+, V^-) over $\mathbb{K}$; 3-graded Lie algebras $(\mathfrak{g}, D)$ over $\mathbb{K}$; generalized projective geometries (X^+, X^-) (these may be defined here simply as the geometries (X^+, X^-) associated to a 3-graded Lie algebra); associated reflection spaces (M, μ); elementary projective groups $G = G(\mathfrak{g}, D)$ associated to 3-graded Lie algebras. What are the functorial relations between these categories? It is obvious that homomorphisms of 3-graded Lie algebras induce, by restriction to the pair $(\mathfrak{g}_1, \mathfrak{g}_{-1})$, homomorphisms of Jordan pairs. Other functoriality problems are less trivial:

(FP1) When does a homomorphism of Jordan pairs induce a homomorphism of the associated Tits–Kantor–Koecher algebras?
(FP2) When does a homomorphism of Jordan pairs induce a homomorphism of the associated generalized projective geometries, respectively of the associated reflection spaces?
(FP3) When does a homomorphism of Tits–Kantor–Koecher algebras induce a homomorphism of the associated elementary projective groups?
(FP4) When does a homomorphism of general 3-graded Lie algebras induce a homomorphism of the associated elementary projective groups?

6.2. Functoriality of the Tits–Kantor–Koecher algebra

In general, a homomorphism of Jordan pairs does not induce a homomorphism of the associated Tits–Kantor–Koecher algebra. In fact, as remarked in Section 3.1, the Tits–Kantor–Koecher algebra $\mathrm{TKK}(V^+, V^-)$ may be seen as the standard imbedding of the polarized Lie triple system $V^+ \oplus V^-$; but the standard imbedding of a Lie triple system does in general not depend functorially on the Lie triple system. However, for *surjective* homomorphisms this is the case (cf. [18, Proposition 1.6]), and it is also true for finite-dimensional semisimple Lie triple systems over fields (cf. [1, Theorem V.1.9]).

6.3. Functoriality of the projective geometry and of the reflection spaces

Any Jordan pair homomorphism $\varphi^{\pm} : V^{\pm} \to (V')^{\pm}$ induces, in a functorial way, a well-defined map of geometries

$$\tilde{\varphi}^{\pm} : X^{\pm} \to (X')^{\pm},$$
$$e^{\operatorname{ad}(v_1)} e^{\operatorname{ad}(w_1)} \cdots e^{\operatorname{ad}(v_k)} e^{\operatorname{ad}(w_k)}.o^{+} \mapsto e^{\operatorname{ad}(\varphi^{+}(v_1))} e^{\operatorname{ad}(\varphi^{-}(w_1))} \cdots e^{\operatorname{ad}(\varphi^{+}(v_k))} e^{\operatorname{ad}(\varphi^{-}(w_k))}.(o')^{+},$$

where $v_i \in V^{+}$, $w_i \in V^{-}$, $i = 1, \dots, k$, $k \in \mathbb{N}$ [3, Theorem 10.1]; the main point here is that the geometry (X^{+}, X^{-}) can be described by generators (namely (V^{+}, V^{-})) and relations (with respect to the product maps μ_r from Section 1.13), and Jordan pair homomorphisms are compatible with the relations. (If the geometry is *stable* in the sense of [18], then these relations are given by *projective equivalence*, cf. [17,18].) A homomorphism of geometries in the sense of [3] induces a homomorphism of the corresponding reflection spaces (because the reflection space structure is defined via the maps μ_r); therefore, Jordan pair homomorphisms always induce homomorphisms of associated reflection spaces.

In particular, an isomorphism of Jordan pairs induces a bijection of geometries. Therefore, if two 3-graded Lie algebras have the same Jordan pair $(\mathfrak{g}_1, \mathfrak{g}_{-1})$, then there is a canonical bijection between the associated geometries (cf. Theorem 6.6 below for another, elementary proof). In particular, as long as we are only interested in the associated geometry (X^{+}, X^{-}) (e.g., in Part II of this work) we may without loss of generality assume that $\mathfrak{g}$ is a Tits–Kantor–Koecher algebra.

6.4. Functoriality problem for the projective elementary group

Let $\varphi : \mathfrak{g} \to \mathfrak{g}'$ be a morphism of 3-graded Lie algebras. One would like to define a homomorphism $\tilde{\varphi} : G \to G'$ of the associated elementary projective groups by requiring that $\tilde{\varphi}(e^{\operatorname{ad}(v^{\pm})}) = e^{\operatorname{ad}(\varphi v^{\pm})}$, but in general this will not be well-defined. Therefore, we introduce the group

$$G(\varphi) := \bigl\{ g = (g_1, g_2) \in G \times G' \colon\ (\forall X \in \mathfrak{g})\ g_2 \varphi(X) = \varphi(g_1 X) \bigr\}.$$

Then the projection $\mathrm{pr}_1 : G(\varphi) \to G$ onto the first factor is surjective: in fact, the image of pr_1 contains the generators of G because all $g_1 := e^{\operatorname{ad}(x)}$, $x \in \mathfrak{g}_{\pm}$, preserve the ideal $\ker(\varphi)$, and so with $g_2 := e^{\operatorname{ad}(\varphi(x))}$ the pair (g_1, g_2) belongs to $G(\varphi)$. Since G is generated by $e^{\operatorname{ad}(\mathfrak{g}_{\pm})}$, it follows that the projection pr_1 is surjective. The kernel of the projection pr_1 is given by all elements of the form $(\mathbf{1}, g_2)$ where g_2 acts trivially on the subalgebra $\varphi(\mathfrak{g}) \subset \mathfrak{g}'$. Therefore, if φ is surjective, then pr_1 is a bijection, and $\mathrm{pr}_2 \circ (\mathrm{pr}_1)^{-1} : G \to G'$ is the desired homomorphism (see [1, Section I.3] for similar considerations on the level of symmetric spaces). Combining with Section 6.2, we see that surjective Jordan pair homomorphisms induce (surjective) homomorphisms of associated elementary projective groups (this result is also contained in [18, Proposition 1.6]).

The functoriality problem is now reduced to the case of injective homomorphisms. In good cases, one may then hope to recognize $\mathrm{pr}_1 : G(\varphi) \to G$ as a sort of covering of G, and thus to view pr_2 as a sort of lift of the desired homomorphism to a covering group.

6.5. *Problem (FP4) for isomorphisms of Jordan pairs*

Let $\mathfrak{g}$ be a 3-graded Lie algebra $\mathfrak{g}$ with grading element E and $\underline{\mathfrak{g}} \subseteq \mathfrak{g}$ an inner 3-graded subalgebra containing $\mathfrak{g}_\pm$. We denote by G, respectively by $\underline{G}$ the associated elementary projective groups. In the present section we will see that the injective homomorphism $\underline{\mathfrak{g}} \to \mathfrak{g}$ (which induces an isomorphism of associated Jordan pairs) induces a surjective homomorphism "in the opposite sense:" $G \to \underline{G}$. In particular, we shall give another and more elementary proof of the fact that the associated homogeneous spaces are the same (cf. Section 6.3). As $\underline{\mathfrak{g}}$ contains $\mathfrak{g}_\pm$, it is invariant under the group G generated by $e^{\mathrm{ad}\,\mathfrak{g}_\pm}$. Moreover, G acts trivially on the quotient space $\mathfrak{g}/\underline{\mathfrak{g}}$, because its generators have this property, i.e., $g.x - x \in \underline{\mathfrak{g}}$ for each $x \in \mathfrak{g}$ and $g \in G$.

Theorem 6.6. *There is a surjective restriction homomorphism*

$$R : G \to \underline{G}, \quad g \mapsto g|_{\underline{\mathfrak{g}}} \quad \textit{with} \quad R^{-1}(\underline{H}) = H \quad \textit{and} \quad R^{-1}\big(\underline{P}^\pm\big) = P^\pm.$$

For the corresponding homogeneous spaces, we have

$$\underline{G}/\underline{P}^\pm \cong G/P^\pm \quad \textit{and} \quad \underline{G}/\underline{H} \cong G/H$$

as homogeneous spaces of G.

Proof. First we observe that $R(U^\pm) = \underline{U}^\pm$ implies that R is surjective.

Let $\mathrm{ad}_{\underline{\mathfrak{g}}} : \mathfrak{g} \to \mathrm{der}(\underline{\mathfrak{g}})$ be given by $\mathrm{ad}_{\underline{\mathfrak{g}}}(x) := \mathrm{ad}\,x|_{\underline{\mathfrak{g}}}$ and let E be an Euler operator defining the grading of $\mathfrak{g}$, respectively an Euler operator $\underline{E} \in \underline{\mathfrak{g}}$ defining the grading on $\underline{\mathfrak{g}}$. Then the ideal $\ker \mathrm{ad}_{\underline{\mathfrak{g}}}$ of $\mathfrak{g}$ is invariant under $\mathrm{ad}\,E$, hence adapted to the grading. For $x \in \mathfrak{g}_\pm$ we have $\mathrm{ad}_{\underline{\mathfrak{g}}}(x)(E') = [x, E'] = \mp x$, so that

$$\ker \mathrm{ad}_{\underline{\mathfrak{g}}} \subseteq \mathfrak{g}_0,$$

and in particular $\mathrm{ad}_{\underline{\mathfrak{g}}}$ is injective on $\mathfrak{g}_+ + \mathfrak{g}_-$. For $x = x_+ + x_0 + x_-$ with $x_\pm \in \mathfrak{g}_\pm$ and $x_0 \in \mathfrak{g}_0$ we have

$$[\mathrm{ad}_{\underline{\mathfrak{g}}} E, \mathrm{ad}_{\underline{\mathfrak{g}}} x] = \mathrm{ad}_{\underline{\mathfrak{g}}}[E, x] = \mathrm{ad}_{\underline{\mathfrak{g}}}(x_+ - x_-).$$

If this bracket vanishes, then $x_+ - x_- \in \ker \mathrm{ad}_{\underline{\mathfrak{g}}} \subseteq \mathfrak{g}_0$ implies $x = x_0 \in \mathfrak{g}_0$, i.e., we obtain the refined information

$$\ker \mathrm{ad}_{\mathfrak{g}} \subseteq \mathrm{ad}_{\mathfrak{g}}^{-1}\big(\mathfrak{z}_{\mathrm{ad}_{\underline{\mathfrak{g}}}}(\mathrm{ad}_{\underline{\mathfrak{g}}} E)\big) = \mathfrak{g}_0.$$

Now let $g \in G$ with $R(g) \in \underline{H}$. For $x \in \mathfrak{g}_0$ we then have

$$\mathrm{ad}_{\underline{\mathfrak{g}}}(g.x) = R(g) \circ \mathrm{ad}_{\underline{\mathfrak{g}}}(x) \circ R(g)^{-1},$$

and all three factors on the right-hand side commute with the grading derivation $\mathrm{ad}_{\underline{\mathfrak{g}}} E$ of $\underline{\mathfrak{g}}$. Hence $\mathrm{ad}_{\underline{\mathfrak{g}}}(g.x)$ commutes with $\mathrm{ad}_{\underline{\mathfrak{g}}} E$, and the argument from above implies that $g.x \in \mathfrak{g}_0$. On the other hand, $R(g)$ preserves the grading of $\underline{\mathfrak{g}}$, and hence in particular the subspaces $\mathfrak{g}_\pm$. This means that g preserves all eigenspaces of $\mathrm{ad}\, E$ on $\mathfrak{g}$, and therefore that g commutes with $\mathrm{ad}\, E$, so that $g \in H$. We conclude that $R^{-1}(\underline{H}) \subseteq H$, and the converse inclusion follows from the fact that the action of H on $\underline{\mathfrak{g}}$ preserves the grading $\underline{\mathfrak{g}} = \mathfrak{g}_+ \oplus (\underline{\mathfrak{g}} \cap \mathfrak{g}_0) \oplus \mathfrak{g}_-$ of $\underline{\mathfrak{g}}$.

From $\underline{P} = \underline{H}\underline{U}^\pm$ and $R(U^\pm) = \underline{U}^\pm$, we obtain

$$R^{-1}\big(P^\pm\big) = R^{-1}(\underline{H})U^\pm \subseteq HU^\pm = P^\pm.$$

Since $R(P^\pm) = R(H)R(U^\pm) \subseteq \underline{H}\underline{U}^\pm = \underline{P}^\pm$, the first assertion follows.

For the homogeneous spaces, we now get

$$\underline{G}/\underline{P}^\pm \cong G/R^{-1}\big(\underline{P}^\pm\big) = G/P^\pm \quad \text{and} \quad \underline{G}/\underline{H} \cong G/R^{-1}(\underline{H}) = G/H. \qquad \square$$

7. Central extensions of three-graded Lie algebras

In this section $\mathbb{K}$ denotes a field with $2, 3 \in \mathbb{K}^\times$.

7.1. Let $\mathfrak{g}$ be a 3-graded Lie algebra with grading element E. In this section we assume that $\mathfrak{g}$ is generated by E and $\mathfrak{g}_\pm$, i.e., that

$$\mathfrak{g}_0 = \mathbb{K}E + [\mathfrak{g}_+, \mathfrak{g}_-]. \tag{7.1}$$

We shall show that the homogeneous spaces associated of the elementary projective group of $\mathfrak{g}$ do not change for central extensions. Combining these results with those of the preceding section, it follows that they only depend on the Jordan pair $(\mathfrak{g}_+, \mathfrak{g}_-)$.

Lemma 7.2. *Let $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ be a central extension of $\mathfrak{g}$, i.e., q is surjective and* $\ker q$ *is a central subspace of $\hat{\mathfrak{g}}$. We pick an element $\widehat{E} \in \hat{\mathfrak{g}}$ with $q(\widehat{E}) = E$. Then* $\mathrm{ad}\,\widehat{E}$ *is diagonalizable with the eigenvalues* $\{\pm 1, 0\}$ *and defines a* 3*-grading*

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0 \oplus \mathfrak{g}_-$$

such that q is a morphism of 3*-graded Lie algebras.*

Proof. First we observe that $q \circ \mathrm{ad}\,\widehat{E} = \mathrm{ad}\, E \circ q$. From the relation $(\mathrm{ad}\, E)^3 = \mathrm{ad}\, E$ we derive that

$$0 = \big((\mathrm{ad}\, E)^3 - \mathrm{ad}\, E\big) \circ q = q \circ \big((\mathrm{ad}\,\widehat{E})^3 - \mathrm{ad}\,\widehat{E}\big),$$

and hence that

$$\big((\operatorname{ad}\widehat{E})^3 - \operatorname{ad}\widehat{E}\big)(\hat{\mathfrak{g}}) \subseteq \ker q \subseteq \mathfrak{z}(\hat{\mathfrak{g}}).$$

Applying $\operatorname{ad}\widehat{E}$, we see that

$$(\operatorname{ad}\widehat{E})^4 = (\operatorname{ad}\widehat{E})^2,$$

i.e.,

$$(\operatorname{ad}\widehat{E})^2(\operatorname{ad}\widehat{E} - \mathbf{1})(\operatorname{ad}\widehat{E} + \mathbf{1}) = 0.$$

Let

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_{-1}$$

be the generalized eigenspace decomposition for $\operatorname{ad}\widehat{E}$. Then

$$\operatorname{ad}\widehat{E}|_{\hat{\mathfrak{g}}_\pm} = \pm\operatorname{id}_{\hat{\mathfrak{g}}_\pm} \quad \text{and} \quad (\operatorname{ad}\widehat{E})^2.\hat{\mathfrak{g}}_0 = \{0\}.$$

From $\ker q \subseteq \mathfrak{z}(\hat{\mathfrak{g}}) \subseteq \hat{\mathfrak{g}}_0$, we derive that $q|_{\hat{\mathfrak{g}}_\pm}$ is injective and maps $\hat{\mathfrak{g}}_\pm$ bijectively onto $\mathfrak{g}_\pm$. Therefore $\mathfrak{g}_0 = \mathbb{K}E + [\mathfrak{g}_+, \mathfrak{g}_-]$ leads to

$$\hat{\mathfrak{g}}_0 = q^{-1}(\mathfrak{g}_0) = \ker q + \mathbb{K}\widehat{E} + [\hat{\mathfrak{g}}_+, \hat{\mathfrak{g}}_-].$$

As $[\hat{\mathfrak{g}}_+, \hat{\mathfrak{g}}_-] \subseteq \ker\operatorname{ad}\widehat{E}$, we conclude that

$$\hat{\mathfrak{g}}_0 \subseteq \ker\operatorname{ad}\widehat{E},$$

and hence that $\widehat{E}$ is a grading element for the 3-grading $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_-$. □

7.3. If $\mathfrak{g}$ is 3-graded with grading element E and $\mathfrak{z} \subseteq \mathfrak{g}$ is a central subspace, then $\mathfrak{z} \subseteq \ker\operatorname{ad}E = \mathfrak{g}_0$, and the quotient map $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$ is a central extension which is a morphism of 3-graded Lie algebras.

This implies that for a central extension $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ for which $\hat{\mathfrak{g}}$ is 3-graded with grading element $\widehat{E}$, the Lie algebra $\mathfrak{g}$ is 3-graded with grading element $E := q(\widehat{E})$, and Lemma 7.2 provides the converse information, that if $\mathfrak{g}$ is 3-graded with grading element E and generated by E and $\mathfrak{g}_\pm$, then the Lie algebra $\hat{\mathfrak{g}}$ has a natural 3-grading defined by an element $\widehat{E}$ with $q(\widehat{E}) = E$ and q is a morphism of 3-graded Lie algebra. Passing to the subalgebra generated by $\widehat{E}$ and $\hat{\mathfrak{g}}_\pm$, we even obtain a 3-grading satisfying the same condition as $\mathfrak{g}$. In fact, $\mathfrak{h} := \hat{\mathfrak{g}}_+ + \hat{\mathfrak{g}}_- + [\hat{\mathfrak{g}}_+, \hat{\mathfrak{g}}_-] + \mathbb{K}\widehat{E} \subseteq \hat{\mathfrak{g}}$ is a 3-graded subalgebra with $q(\mathfrak{h}) = \mathfrak{g}$, so that $\mathfrak{g} \subseteq \mathfrak{h} + \ker q \subseteq \mathfrak{h} + \mathfrak{z}(\hat{\mathfrak{g}})$. In particular, $\mathfrak{h}$ is an ideal of $\hat{\mathfrak{g}}$.

These consideration show that to understand central extensions of 3-graded Lie algebras, a natural context is given by those central extensions $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ which are morphisms of 3-graded Lie algebras with grading element satisfying (7.1).

Lemma 7.4. *Let* $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ *be a central extension of* 3*-graded Lie algebras with grading elements* $\widehat{E}$ *and* $E = q(\widehat{E})$ *satisfying* (7.1)*. Then*

$$q^{-1}\big(\mathfrak{z}(\mathfrak{g})\big) = \mathfrak{z}(\hat{\mathfrak{g}})$$

and therefore $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \hat{\mathfrak{g}}/\mathfrak{z}(\hat{\mathfrak{g}})$.

Proof. Since q is surjective, we have $\mathfrak{z}(\hat{\mathfrak{g}}) \subseteq q^{-1}(\mathfrak{z}(\mathfrak{g}))$. If, conversely, $q(x) \in \mathfrak{z}(\mathfrak{g})$, then $[x, \hat{\mathfrak{g}}] \subseteq \ker q \subseteq \mathfrak{z}(\hat{\mathfrak{g}}) \subseteq \hat{\mathfrak{g}}_0$. In particular, we obtain $[x, \widehat{E}] \in \hat{\mathfrak{g}}_0$ and therefore $x \in \hat{\mathfrak{g}}_0$. This in turn implies $[x, \hat{\mathfrak{g}}_\pm] \subseteq \hat{\mathfrak{g}}_\pm$. As $q|_{\hat{\mathfrak{g}}_\pm}$ is injective, $[x, \hat{\mathfrak{g}}_\pm] \subseteq \ker q \cap \hat{\mathfrak{g}}_\pm = \{0\}$. Therefore x commutes with $\hat{\mathfrak{g}}_\pm$ and $\widehat{E}$, hence is central because $\hat{\mathfrak{g}}$ is generated by $\widehat{E}$ and $\hat{\mathfrak{g}}_\pm$. □

Corollary 7.5. *If* $\mathfrak{g}$ *satisfies* (7.1)*, then* $\mathfrak{z}(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) = \{0\}$.

Proof. The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{ad}\,\mathfrak{g} \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is a central extension satisfying the assumptions of Lemma 6.3. Therefore, $\ker \mathrm{ad} = \mathfrak{z}(\mathfrak{g}) = \mathrm{ad}^{-1}(\mathfrak{z}(\mathrm{ad}\,\mathfrak{g}))$ implies $\mathfrak{z}(\mathrm{ad}\,\mathfrak{g}) = \{0\}$. □

Remark 7.6.

(a) Let $\mathfrak{g}$ a 3-graded Lie algebra with grading element E and $\underline{\mathfrak{g}} \trianglelefteq \mathfrak{g}$ the ideal $\underline{\mathfrak{g}}$ generated by E and $\mathfrak{g}_\pm$ (see Section 6.5). We consider the Lie algebra homomorphism

$$\underline{\mathrm{ad}} : \mathfrak{g} \to \mathrm{der}(\underline{\mathfrak{g}}), \quad x \mapsto \mathrm{ad}\,x|_{\underline{\mathfrak{g}}}.$$

In view of Corollary 7.5, $(\underline{\mathrm{ad}})(\underline{\mathfrak{g}}) \cong \mathrm{ad}\,\underline{\mathfrak{g}} \cong \underline{\mathfrak{g}}/\mathfrak{z}(\underline{\mathfrak{g}})$ is a center-free 3-graded Lie algebra satisfying (7.1).

(b) If $\mathfrak{g}$ is a center-free 3-graded Lie algebra satisfying (7.1) and $(V^+, V^-) = (\mathfrak{g}_+, \mathfrak{g}_-)$ is the corresponding Jordan pair, then the representation

$$\mathrm{ad}_{V^\pm} : \mathfrak{g}_0 \to \mathrm{der}\big(V^+, V^-\big), \quad x \mapsto (\mathrm{ad}\,x|_{V^+}, \mathrm{ad}\,x|_{V^-})$$

is injective, and

$$\mathfrak{g} \to V^+ \oplus \mathrm{der}\big(V^+, V^-\big) \oplus V^-, \quad x_+ + x_0 + x_- \mapsto (x_+, \mathrm{ad}_{V^\pm} x_0, x_-)$$

is an embedding of Lie algebras, where the right hand side carries the bracket defined in Section 3.1.

On the other hand, the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\pm$ is isomorphic to the corresponding subalgebra of $V^+ \oplus \mathrm{der}(V^+, V^-) \oplus V^-$, which is $\mathrm{TKK}(V^+, V^-)$.

Definition 7.7. Let $\mathfrak{g}$ be a Lie algebra. We write $\langle \mathfrak{g}, \mathfrak{g} \rangle$ for the quotient of $\Lambda^2(\mathfrak{g})$ by the subspace generated by the elements of the form

$$[x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y,$$

and write $\langle x, y\rangle$ for the image of $x \wedge y$ in $\langle \mathfrak{g}, \mathfrak{g}\rangle$. Then $\langle \mathfrak{g}, \mathfrak{g}\rangle$ carries a natural Lie algebra structure satisfying

$$\bigl[\langle x, y\rangle, \langle x', y'\rangle\bigr] = \bigl\langle [x, y], [x', y']\bigr\rangle,$$

and the map

$$b_{\mathfrak{g}} : \langle \mathfrak{g}, \mathfrak{g}\rangle \to \mathfrak{g}, \quad \langle x, y\rangle \mapsto [x, y]$$

is a homomorphism of Lie algebras.

Theorem 7.8. *Suppose that $\mathfrak{g}$ is 3-graded with grading element E satisfying* (7.1). *If $\mathfrak{g}$ is perfect, then we put $\tilde{\mathfrak{g}} := \langle \mathfrak{g}, \mathfrak{g}\rangle$, and if $\mathfrak{g}$ is not perfect, then we define*

$$\tilde{\mathfrak{g}} := \langle \mathfrak{g}, \mathfrak{g}\rangle \rtimes \mathbb{K}\widetilde{E},$$

where $\operatorname{ad}\widetilde{E}$ *satisfies*

$$\bigl[\widetilde{E}, \langle x, y\rangle\bigr] := \bigl\langle [E, x], y\bigr\rangle + \bigl\langle x, [E, y]\bigr\rangle = \bigl\langle E, [x, y]\bigr\rangle. \tag{7.2}$$

Then there is a unique Lie algebra homomorphism

$$q_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g} \quad \text{with} \quad q_{\mathfrak{g}}\bigl(\langle x, y\rangle\bigr) = [x, y] \quad \text{and} \quad q_{\mathfrak{g}}(\widetilde{E}) = E.$$

This homomorphism is surjective with central kernel, hence a central extension of $\mathfrak{g}$. Moreover, it is weakly universal in the sense that for any central extension $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ with a 3-graded Lie algebra $\hat{\mathfrak{g}}$ with grading element $\widehat{E} \in \hat{\mathfrak{g}}$ there exists a unique Lie algebra homomorphism $\alpha : \tilde{\mathfrak{g}} \to \hat{\mathfrak{g}}$ with $q \circ \alpha = q_{\mathfrak{g}}$ and, if $\mathfrak{g}$ is not perfect, with $\alpha(\widetilde{E}) = \widehat{E}$.

Proof. First we observe that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathbb{K}E$. If $\mathfrak{g}$ is perfect, then

$$b_{\mathfrak{g}} : \tilde{\mathfrak{g}} := \langle \mathfrak{g}, \mathfrak{g}\rangle \to \mathfrak{g}$$

is the universal central extension of $\mathfrak{g}$. If $\mathfrak{g}$ is not perfect, then $E \notin [\mathfrak{g}, \mathfrak{g}] = \operatorname{im}(b_{\mathfrak{g}})$. Therefore $\mathfrak{g} \cong [\mathfrak{g}, \mathfrak{g}] \rtimes \mathbb{K}E$.

The Lie algebra $\operatorname{der}(\mathfrak{g})$ acts in a natural way by derivations on $\langle \mathfrak{g}, \mathfrak{g}\rangle$ via

$$d.\langle x, y\rangle = \langle d.x, y\rangle + \langle x, d.y\rangle.$$

We may therefore form the Lie algebra $\tilde{\mathfrak{g}} := \langle \mathfrak{g}, \mathfrak{g}\rangle \rtimes \mathbb{K}\widetilde{E}$, where $\operatorname{ad}\widetilde{E}$ satisfies (7.2).

In both cases we obtain quotient homomorphisms $q_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ with $\ker q_{\mathfrak{g}} = \ker b_{\mathfrak{g}} \subseteq \mathfrak{z}(\langle \mathfrak{g}, \mathfrak{g}\rangle)$. From (7.2) we derive that the action of E on $\langle \mathfrak{g}, \mathfrak{g}\rangle$ annihilates $\ker b_{\mathfrak{g}}$, so that $\ker q_{\mathfrak{g}}$ is central in both cases. This means that $q_{\mathfrak{g}}$ is a central extension, and Lemma 7.2 implies that $\tilde{\mathfrak{g}}$ is 3-graded with grading element $\widetilde{E}$. Moreover,

$$\underline{\tilde{\mathfrak{g}}} := \tilde{\mathfrak{g}}_+ + \tilde{\mathfrak{g}}_- + [\tilde{\mathfrak{g}}_+, \tilde{\mathfrak{g}}_-] + \mathbb{K}\widetilde{E}$$

is an ideal of $\tilde{\mathfrak{g}}$ with $\underline{\tilde{\mathfrak{g}}} + \ker q_{\mathfrak{g}} = \tilde{\mathfrak{g}}$. In view of $\langle E, E\rangle = 0$, we have

$$[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \big\langle [\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]\big\rangle + \big\langle E, [\mathfrak{g}, \mathfrak{g}]\big\rangle = \big\langle \mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]\big\rangle + \big\langle [\mathfrak{g}, \mathfrak{g}], E\big\rangle = \langle \mathfrak{g}, \mathfrak{g}\rangle.$$

Therefore $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subseteq \underline{\tilde{\mathfrak{g}}}$ implies $\tilde{\mathfrak{g}} = \underline{\tilde{\mathfrak{g}}}$ and hence that $\tilde{\mathfrak{g}}$ satisfies (7.1).

We claim that $q_{\mathfrak{g}}$ is weakly universal as a central extension of 3-graded Lie algebras satisfying (7.1). So let $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ be a central extension. Then the bracket map $\hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ factors through an alternating bilinear map

$$b : \mathfrak{g} \times \mathfrak{g} \to \hat{\mathfrak{g}} \quad \text{with} \quad b\big(q(x), q(y)\big) = [x, y],\ x, y \in \hat{\mathfrak{g}}.$$

Then the Jacobi identity in $\hat{\mathfrak{g}}$ implies that b satisfies the cocycle condition

$$b\big([x, y], z\big) + b\big([y, z], x\big) + b\big([z, x], y\big) = 0.$$

Hence there exists a unique linear map

$$\varphi : \langle \mathfrak{g}, \mathfrak{g}\rangle \to \hat{\mathfrak{g}} \quad \text{with } \varphi\big(\langle x, y\rangle\big) = b(x, y),$$

and it is easy to see that φ is a homomorphism of Lie algebras. Moreover, φ is a morphism of 3-graded Lie algebras, because the grading on $\hat{\mathfrak{g}}$ is induced by the map $x \mapsto b(E, q(x))$. If $\mathfrak{g}$ is not perfect, then $\hat{\mathfrak{g}}$ is not perfect, and no grading element $\widehat{E} \in \hat{\mathfrak{g}}$ is contained in $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$. We may therefore extend φ to a Lie algebra homomorphism

$$\varphi : \tilde{\mathfrak{g}} \to \hat{\mathfrak{g}} \quad \text{with } \varphi(\widetilde{E}) = \widehat{E}.$$

This proves the weak universality of $\tilde{\mathfrak{g}}$ as a 3-graded Lie algebra with grading element $\widetilde{E}$. The map $\varphi : \tilde{\mathfrak{g}} \to \hat{\mathfrak{g}}$ is not uniquely determined by the requirement that $q \circ \varphi = q_{\mathfrak{g}}$ because we may add any Lie algebra homomorphism $\psi : \tilde{\mathfrak{g}} \to \ker q$, which corresponds to the ambiguity in the choice of the grading element $\widehat{E} \in \hat{\mathfrak{g}}$. Note that the commutator algebra of $\tilde{\mathfrak{g}}$ is a hyperplane, so that ψ is determined by $\psi(\widetilde{E})$. □

7.9. Central extensions have isomorphic geometries

Next we compare the groups

$$G \subseteq \big\langle e^{\operatorname{ad} \mathfrak{g}_\pm}\big\rangle \subseteq \operatorname{Aut}(\mathfrak{g}) \quad \text{and} \quad \widehat{G} \subseteq \big\langle e^{\operatorname{ad} \hat{\mathfrak{g}}_\pm}\big\rangle \subseteq \operatorname{Aut}(\hat{\mathfrak{g}}),$$

where $q : \hat{\mathfrak{g}} \to \mathfrak{g}$ is a central extension of 3-graded Lie algebras satisfying (7.1). Since each element of $\widehat{G}$ fixes the kernel $\mathfrak{z} := \ker q$ pointwise, it induces an automorphism of $\mathfrak{g}$, and we thus obtain a group homomorphism

$$q_G : \widehat{G} \to G \quad \text{with } q_G(g) \circ q = q \circ g,\ g \in \widehat{G},$$

because $e^{\operatorname{ad} \hat{\mathfrak{g}}_\pm}$ is mapped onto $e^{\operatorname{ad} \mathfrak{g}_\pm}$. The following theorem provides a short direct argument for the isomorphy of the geometries associated to central extensions. Since the corresponding Jordan pairs are the same, this could also be deduced from the general result mentioned in Section 6.2.

Theorem 7.10. *We have* $q_G^{-1}(H) = \widehat{H}$ *and* $q_G^{-1}(P^\pm) = \widehat{P}^\pm$. *For the corresponding homogeneous spaces, we have*

$$\widehat{G}/\widehat{P}^\pm \cong G/P^\pm \quad \textit{and} \quad \widehat{G}/\widehat{H} \cong G/H$$

as homogeneous spaces of $\widehat{G}$.

Proof. Since q_G maps a generating subset of $\widehat{G}$ onto a generating subset of G, it is surjective.

First we observe that for any $h \in \widehat{G}$ we have

$$q_G(h).E - E = q(h.\widehat{E} - \widehat{E}). \tag{7.3}$$

If $h \in \widehat{H}$, then $h.\widehat{E} - \widehat{E} \in \mathfrak{z}(\hat{\mathfrak{g}})$, and (7.3) leads to

$$q_G(h).E - E = q(h.\widehat{E} - \widehat{E}) \in q\big(\mathfrak{z}(\hat{\mathfrak{g}})\big) = \mathfrak{z}(\mathfrak{g}),$$

and hence $q_G(h) \in H$. Suppose, conversely, that $q_G(h) \in H$. Then (7.3) implies

$$h.\widehat{E} - \widehat{E} \in q^{-1}\big(\mathfrak{z}(\mathfrak{g})\big) = \mathfrak{z}(\hat{\mathfrak{g}}),$$

so that $h \in \widehat{H}$.

Since $\widehat{P} = \widehat{H}\widehat{U}^\pm$ and $q_G(\widehat{U}^\pm) = U^\pm$, we have

$$q_G^{-1}\big(P^\pm\big) = q_G^{-1}(H)\widehat{U}^\pm \subseteq \widehat{H}\widehat{U}^\pm = \widehat{P}^\pm.$$

Further $q_G(\widehat{P}^\pm) = q_G(\widehat{H})q_G(\widehat{U}^\pm) \subseteq HU^\pm = P^\pm$, and we obtain $q_G^{-1}(P^\pm) = \widehat{P}^\pm$.

For the homogeneous spaces, we now get

$$G/P^\pm \cong \widehat{G}/q_G^{-1}\big(\widehat{P}^\pm\big) = \widehat{G}/\widehat{P}^\pm \quad \text{and} \quad G/H \cong \widehat{G}/q_G^{-1}(\widehat{H}) = \widehat{G}/\widehat{H}. \qquad \square$$

Remark 7.11. We take a closer look at the kernel of q_G. Let $g \in \ker q_G \subseteq \widehat{H}$. Then g preserves the grading of $\hat{\mathfrak{g}}$. Since $q|_{\hat{\mathfrak{g}}_\pm}$ is injective, we conclude that $g|_{\hat{\mathfrak{g}}_\pm} = \mathrm{id}_{\hat{\mathfrak{g}}_\pm}$, and hence that $g - \mathrm{id}_{\hat{\mathfrak{g}}}$ vanishes on the subalgebra generated by $\hat{\mathfrak{g}}_\pm$. Moreover, $\mathrm{im}(g - \mathrm{id}_{\hat{\mathfrak{g}}}) \subseteq \ker q = \mathfrak{z}$, so that

$$g = \mathbf{1} + D,$$

where $D : \hat{\mathfrak{g}} \to \mathfrak{z}$ is a linear map. As g is an automorphism, it follows that $D \in \mathrm{der}(\hat{\mathfrak{g}})$, and hence that $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] \subseteq \ker D$. If $\hat{\mathfrak{g}}$ is perfect, then D vanishes, but if $\widehat{E} \notin [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$, then

$$\mathrm{Hom}_{\mathrm{Lie}}(\hat{\mathfrak{g}}, \mathfrak{z}) = \mathrm{Hom}(\mathbb{K}\widehat{E}, \mathfrak{z}) \cong \mathfrak{z}$$

describes the possibilities for D, which is determined by $D(\widehat{E}) \in \mathfrak{z}$.

Since, for $h \in \widehat{H}$ and $v \in \mathfrak{g}_\pm$ we have $he^{\operatorname{ad} v}h^{-1} = e^{\operatorname{ad} h.v}$, the condition $g|_{\hat{\mathfrak{g}}_\pm} = \mathrm{id}_{\hat{\mathfrak{g}}_\pm}$ implies that h commutes with the generating subset $e^{\operatorname{ad} \mathfrak{g}_\pm}$, and hence that

$$\ker q_G \subseteq Z(\widehat{G}).$$

This means that $q_G : \widehat{G} \to G$ is a central extension of groups.

Example 7.12. We consider the case of a trivial Jordan pair (V^+, V^-), i.e., all the maps $T^\pm$ vanish. Then the corresponding 3-graded Lie algebra is the semidirect sum

$$\mathfrak{g} = \left(V^+ \oplus V^-\right) \rtimes \mathbb{K}E,$$

where

$$\left[E, (v_+, v_-)\right] = (v_+, -v_-) \quad \text{and} \quad \left[V^+, V^-\right] = \{0\}.$$

Let $\beta : V^+ \times V^- \to \mathfrak{z}$ be any bilinear map. Then

$$\omega\big((v_+, v_-, \lambda E), (w_+, w_-, \mu E)\big) := \beta(v_+, w_-) - \beta(v_-, w_+)$$

is a Lie algebra cocycle which defines a central extension

$$\hat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$$

with the bracket

$$\left[(x, z), (x', z')\right] := \big([x, x'], \omega(x, x')\big), \quad x, x' \in \mathfrak{g}, \ z, z' \in \mathfrak{z}.$$

The subalgebra of $\hat{\mathfrak{g}}$ generated by $V^\pm$ is 2-step nilpotent and $\hat{\mathfrak{g}}$ is solvable. In $\hat{\mathfrak{g}}$ we have

$$\left[\operatorname{ad} V^+, \operatorname{ad} V^-\right] = \operatorname{ad}\left[V^+, V^-\right] \subseteq \operatorname{ad} \mathfrak{z} = \{0\},$$

so that the groups $\widehat{G}$ and G are both abelian. Considering the orbit of the grading element, it is easy to see that

$$\widehat{G} \cong V^+ \times V^- \cong G.$$

Remark 7.13. Let $\mathfrak{g}$ be a 3-graded Lie algebra with grading element E. We have seen in Section 5 that the homogeneous spaces G/H and $G/P^\pm$ are isomorphic to those associated to the subalgebra $\underline{\mathfrak{g}}$ generated by E and $\mathfrak{g}_\pm$. Furthermore, the results in this section imply that the same holds for the homogeneous spaces associated to the center-free Lie algebra $\underline{\mathfrak{g}}/\mathfrak{z}(\underline{\mathfrak{g}})$. The latter Lie algebra is isomorphic to the Tits–Kantor–Koecher Lie algebra

$$\mathrm{TKK}(\mathfrak{g}_+, \mathfrak{g}_-) = \mathfrak{g}_+ \oplus \big(\mathrm{ider}(\mathfrak{g}_+, \mathfrak{g}_-) + \mathbb{K}E\big) \oplus \mathfrak{g}_-$$

of the Jordan pair $(\mathfrak{g}_+, \mathfrak{g}_-)$. For that we only have to observe that the triviality of the center implies that $\underline{\mathfrak{g}}_0/\mathfrak{z}(\mathfrak{g})$ embeds into $\operatorname{der}(\mathfrak{g}_+, \mathfrak{g}_-)$. We therefore obtain a natural identification of the homogeneous space G/H and $G/P^{\pm}$ with a space of 3-gradings of $\mathrm{TKK}(\mathfrak{g}_+, \mathfrak{g}_-)$, respectively a space of filtrations of this Lie algebra.

8. Grassmannian geometries and associative structures

8.1. Grassmannian geometries

Let R be an associative algebra with unit 1 over the commutative unital ring $\mathbb{K}$ and let V be a right R-module. The *complemented Grassmannian* (*of V over R*) is the space

$$\mathcal{C} := \bigl\{E \subset V\colon\ \exists F\colon\ V = E \oplus F\ (E, F\colon \text{ submodules of } V)\bigr\} \tag{8.1}$$

of R-submodules of V that have a complement. For $V = R$ this is the *space of complemented right ideals* of R (cf. Section 8.6 below). For $E, F \in \mathcal{C}$ we write $E \top F$ if $V = E \oplus F$; we let $E^{\top} = \{F \in \mathcal{C}\colon\ F \top E\}$ be the set of complementary submodules of E and

$$(\mathcal{C} \times \mathcal{C})^{\top} = \bigl\{(E, F) \in \mathcal{C} \times \mathcal{C}\colon\ V = E \oplus F\bigr\}. \tag{8.2}$$

Let

$$\mathcal{P} := \bigl\{p \in \operatorname{End}_R(V)\colon\ p^2 = p\bigr\} = \operatorname{Idem}\bigl(\operatorname{End}_R(V)\bigr) \tag{8.3}$$

be the *space of projectors, respectively idempotents in V*. Taking $I := 2p - \mathrm{id}_V$ instead of p, we may also work with the condition $I^2 = \mathrm{id}_V$ instead of $p^2 = p$ and view $\mathcal{P}$ as the *space of polarizations of V*. In this framework, the following analog of Theorem 1.6 is an easy exercise in Linear Algebra.

Proposition 8.2.

(i) *The map* $\mathcal{P} \to (\mathcal{C} \times \mathcal{C})^{\top}$, $p \mapsto (\operatorname{im}(p), \ker(p)) = (\operatorname{im}(p), \operatorname{im}(\mathbf{1} - p))$ *is a bijection.*
(ii) *For all $E \in \mathcal{C}$, $E^{\top}$ carries canonically the structure of an affine space over $\mathbb{K}$* (*not over R in general*), *modeled on the $\mathbb{K}$-module* $\operatorname{Hom}_R(V/E, E)$.

Moreover, $\mathcal{P}$ clearly is stable under the binary map μ defined by $\mu(p, q) = (2p - \mathbf{1}) \times q(2p - \mathbf{1})$ which defines on $\mathcal{P}$ the structure of a reflection space (cf. 4.1). Using scalar extension by dual numbers over $\mathbb{K}$, one may also define tangent bundles of $\mathcal{P}$ and $\mathcal{C}$, and then property (S4) will also hold for μ; but we will not pursue this construction here.

8.3. Flags and elementary group

We are going to describe the relation between this simple linear algebra model and the model from Theorem 1.6. Let $\mathfrak{g} := \operatorname{End}_R(V)$ with the usual commutator as Lie bracket.

Note that the commutator is not R-bilinear in general, but it is bilinear over the center of R; hence $\mathfrak{g}$ is a $\mathbb{K}$-Lie algebra. An element $p \in \mathcal{P}$ defines a derivation $\mathrm{ad}(p)$ of $\mathfrak{g}$ which is tripotent; with respect to the decomposition $V = E \oplus F := \mathrm{im}(p) \oplus \ker(p)$, i.e., (in the obvious matrix notation) $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and the grading of $\mathfrak{g}$ is described by

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \colon \alpha \in \mathrm{Hom}_R(E, F) \right\}, \qquad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \colon \beta \in \mathrm{Hom}_R(F, E) \right\},$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \colon A \in \mathrm{End}_R(E),\ B \in \mathrm{End}_R(F) \right\}. \tag{8.4}$$

Thus we have a well-defined map from $\mathcal{P}$ to the space $\mathcal{G}$ of inner 3-gradings of $\mathfrak{g}$:

$$\varphi_{\mathcal{P}} : \mathcal{P} \to \mathcal{G}, \quad p \mapsto \mathrm{ad}(p). \tag{8.5}$$

On the other hand, if $E \in \mathcal{C}$, then to the "short flag" $0 \subset E \subset V$ we may associate a "long flag" $\mathfrak{f}_E\colon 0 \subset \mathfrak{f}_1 \subset \mathfrak{f}_0 \subset \mathfrak{g}$ by letting

$$\mathfrak{f}_1 := \{X \in \mathfrak{g}\colon X(V) \subset E,\ X(E) = 0\} \subset \mathfrak{f}_0 := \{X \in \mathfrak{g}\colon X(E) \subset E\} \subset \mathfrak{g}; \tag{8.6}$$

in matrix form:

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \begin{pmatrix} * & * \\ * & * \end{pmatrix}.$$

It is clear that this is a 3-filtration of $\mathfrak{g}$ (even in an associative sense). Thus we have a well-defined map

$$\varphi_{\mathcal{C}} : \mathcal{C} \to \mathcal{F}, \quad E \mapsto \mathfrak{f}_E, \tag{8.7}$$

and it follows from the definitions that the diagram

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \supset & \mathcal{P} \\ \downarrow & & \downarrow \\ \mathcal{F} \times \mathcal{F} & \supset & \mathcal{G} \end{array} \tag{8.8}$$

commutes. All maps in this diagram are obviously equivariant with respect to the natural action of the group $\mathrm{GL}_R(V)$ on all spaces that are involved.

If $E \in \mathcal{C}$ is fixed, then the elements $X \in \mathfrak{f}^1$ (with $\mathfrak{f}^1$ as in (8.6)) are 2-step nilpotent; thus $e^X = \mathbf{1} + X$. Let

$$U_E := e^{\mathfrak{f}_1} = \mathbf{1} + \mathfrak{f}_1 = \left\{ \begin{pmatrix} \mathbf{1} & \beta \\ 0 & \mathbf{1} \end{pmatrix} \colon \beta \in \mathrm{Hom}_R(F, E) \right\}, \tag{8.9}$$

where the latter matrix representation is with respect to a fixed complement F of E. The group U_E acts simply transitively on the set $E^\top$ of complements of E. Therefore, if for such a fixed decomposition $V = E \oplus F$, we let

$$G(E,F) := \langle U_E, U_F \rangle \subset \mathrm{GL}_R(V) \tag{8.10}$$

be the group generated by U_E and U_F, called the *elementary group of* (V, E, F), then $G(E,F) = G(E,F')$ for any two complements F, F' of E, and we may write also G_E for $G(E,F)$. We let

$$P_E := \{g \in G_E\colon\ g(E) = E\}, \tag{8.11}$$

and, for a fixed complement F of E,

$$H(E,F) := \{g \in G(E,F)\colon\ g(E) = E,\ g(F) = F\} = P_E \cap P_F. \tag{8.12}$$

Theorem 8.4. *The equivariant maps $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{C}}$ have the following properties*:

(1) *For all $E \in \mathcal{C}$, $\varphi_{\mathcal{C}}(E^\top) = \varphi_{\mathcal{C}}(E)^\top$.*
(2) *For all $p \in \mathcal{P}$, the restriction of the map $\varphi_{\mathcal{P}}$ to the orbit* $\mathrm{GL}_R(V).p$,

$$\mathcal{P} \supset \mathrm{GL}_R(V).p \to \mathcal{G}, \quad g.p \mapsto \mathrm{ad}(g.p),$$

is injective.
(3) *For all $E \in \mathcal{C}$, the restriction of the map $\varphi_{\mathcal{C}}$ to the orbit* $\mathrm{GL}_R(V).E$,

$$\mathcal{C} \supset \mathrm{GL}_R(V).E \to \mathcal{F}, \quad g.E \mapsto \mathfrak{f}_{g.E},$$

is injective.
(4) *Let $p \in \mathcal{P}$ with associated decomposition $V = E \oplus F = \mathrm{im}(p) \oplus \ker(p)$. The map $\varphi_{\mathcal{P}}$ induces a bijection*

$$G(E,F)/H(E,F) \cong G(E,F).p \to G\big(\mathrm{ad}(p)\big).\mathrm{ad}(p) \cong G\big(\mathrm{ad}(p)\big)/H\big(\mathrm{ad}(p)\big),$$

and the map $\varphi_{\mathcal{C}}$ induces a bijection

$$G(E,F)/P_F \cong G(E,F).E \to G\big(\mathrm{ad}(p)\big).\mathfrak{f}_F = G\big(\mathrm{ad}(p)\big)/P^-.$$

Proof. (1) The action of U_E on $\mathfrak{g}$ is precisely the action of $e^{\mathrm{ad}(\mathfrak{f}_1)}$ on $\mathfrak{g}$. Since U_E acts simply transitively on the set of complements of E, the claim follows from the corresponding fact about $\mathfrak{g}$ (Theorem 1.6(2)).

(2) Let $e \in \mathcal{P}$ and $f := geg^{-1} \in \mathcal{P}$ with $g \in \mathrm{GL}_R(V)$ such that $\mathrm{ad}(e) = \mathrm{ad}(f)$. Then $z := f - e \in Z(A)$ where A is the associative $\mathbb{K}$-algebra $\mathrm{End}_R(V)$. In particular, $ef = fe$ and therefore

$$(e-f)ef = e^2 f - ef^2 = ef - ef = 0.$$

We have

$$(f-e)^2 = f^2 - 2ef + e^2 = f + e - 2ef \quad \text{and}$$
$$(f-e)^3 = (f-e)(f+e-2ef) = f^2 - e^2 - 2(f-e)ef = f - e,$$

i.e., $z^3 = z$. Write $z = z_1 - z_2$ with

$$z_1 = \frac{1}{2}z(z+\mathbf{1}) \quad \text{and} \quad z_2 = \frac{1}{2}z(z-\mathbf{1}).$$

Then z_1 and z_2 are again central, and $z_1^2 = z_1$ and $z_2^2 = z_2$. This implies

$$z_1 = \frac{1}{2}(f-e)(f-e+\mathbf{1}) = \frac{1}{2}(f+e-2ef+f-e) = \frac{1}{2}(2f-2ef) = f - ef = zf$$

and

$$z_2 = \frac{1}{2}(f-e)(f-e-\mathbf{1}) = \frac{1}{2}(f+e-2ef-f+e) = \frac{1}{2}(2e-2ef) = e - ef = -ze.$$

We further obtain

$$\begin{aligned} z_2 &= -z_2 z = z_2\bigl(e - geg^{-1}\bigr) = z_2 e - g z_2 e g^{-1} = -ze^2 + gze^2g^{-1} = -ze + gzeg^{-1} \\ &= z_2 - g z_2 g^{-1} = 0 \end{aligned}$$

because z_2 is central, and likewise

$$\begin{aligned} z_1 &= z_1 z = z_1\bigl(f - g^{-1} f g\bigr) = z_1 f - g^{-1} z_1 f g = zf^2 - g^{-1} z f^2 g = zf - g^{-1} zfg \\ &= z_1 - g^{-1} z_1 g = 0. \end{aligned}$$

Eventually we obtain $z = z_1 - z_2 = 0$ and hence $e = f$, as had to be shown.

(3) This follows by combining (2) and (1) (observing that the fibers of the map $\mathcal{P} \to \mathcal{C}$, $p \mapsto \mathrm{im}(p)$ are of the form $F^\perp$, $F \in \mathcal{C}$, and similarly for $\mathcal{G} \to \mathcal{F}$).

(4) This follows from (2) and (3), observing that the action of $H(E,F)$ on $\mathfrak{g}$ coincides with the action of $H(\mathrm{ad}(p))$. For P_E we argue similarly. □

8.5. *Special Jordan pairs*

If $p \in \mathcal{P}$ and $\mathfrak{g} = \mathfrak{gl}_R(V)$, then the associated Jordan pair is

$$\bigl(\mathrm{Hom}_R(F,E), \mathrm{Hom}_R(E,F)\bigr), \qquad T^{\pm}(X,Y,Z) = XYZ + ZYX.$$

A Jordan pair that is a sub-pair of such a pair is called *special*. The Bergman operator is in this case given by

$$B(X,Y)Z = (\mathbf{1} - XY)Z(\mathbf{1} - YX).$$

The special case where $V = E \oplus E$ gives rise to a self-dual geometry and is related to the $\mathbb{K}$-Jordan algebra $\mathrm{End}_R(E)$.

8.6. Geometry of right ideals

Now let us consider the case of the right R-module $V = R$. In this case (complemented) submodules are the same as (complemented) right ideals, and the Grassmannian geometry should be called the *geometry of right ideals of R*. Via the bijection $R \to \mathrm{Hom}_R(R, R)$, $r \mapsto l_r$ (left multiplication by r), the set $\mathcal{P}$ of projectors is identified with the *set of idempotents of R*,

$$\mathrm{Idem}(R) := \{e \in R\colon\ e^2 = e\}.$$

The pair (R, e) with an idempotent e is also called a *Morita context* (cf. [18, Section 2.1]). In this case, our Theorem 8.4 corresponds essentially to results of Loos [18, Theorem 2.8]. The symmetric space structure on $\mathrm{Idem}(R)$ is described in the same way as after Proposition 8.2: it is given by $\mu(e, f) = (2e - \mathbf{1}) f (2e - \mathbf{1})$.

8.7. Geometry of the projective line

Another interesting case is $V = R \oplus R$, taking this decomposition as base point $p \in \mathcal{P}$. This gives rise to a self-dual geometry (belonging to R seen as a Jordan algebra over $\mathbb{K}$) which is the *projective line over the ring R*, see [12] and the recent work [7]. The corresponding 3-graded Lie algebra is $\mathfrak{g} = \mathfrak{gl}_2(R)$, respectively its subalgebra $\mathfrak{e}_2(R)$ generated by the strict upper and lower triangular matrices.

Finally, let us remark that there exist rings R such that $R \oplus R \cong R$ as right R-modules (e.g., take $R = \mathrm{End}_{\mathbb{K}}(V)$, where V is an infinite dimensional vector space over a field $\mathbb{K}$; then $V \cong V \oplus V$ as a vector space, and hence $R = \mathrm{Hom}_{\mathbb{K}}(V, V) \cong \mathrm{Hom}_{\mathbb{K}}(V, V \oplus V)$ as a right R-module), so that the cases 8.6 and 8.7 have non-empty intersection.

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