The Numerical Range of 3 \times 3 Matrices

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ABSTRACT

Let $A$ be an $n \times n$ complex matrix. Then the numerical range of $A$, $W(A)$, is defined to be $\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$. In this article a series of tests is given, allowing one to determine the shape of $W(A)$ for $3 \times 3$ matrices. Reconstruction of $A$, up to unitary similarity, from $W(A)$ is also examined. © Elsevier Science Inc., 1997

1. INTRODUCTION

Let $A$ be an $n \times n$ matrix with complex entries. The numerical range of $A$, $W(A)$, is defined as $\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$. It is well known (see [1, 2]) that $W(A)$ is a convex compact subset of $\mathbb{C}$, which contains all the

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eigenvalues of $A$ and therefore its convex hull $\text{conv}(\sigma(A))$. For $A$ normal (that is, commuting with $A^*$), $W(A) = \text{conv}(\sigma(A))$; the converse statement holds when $n \leq 4$.

For $2 \times 2$ matrices $A$ a complete description of the numerical range $W(A)$ is well known. Namely, $W(A)$ is an ellipse with foci at the eigenvalues $\lambda_1, \lambda_2$ of $A$ and a minor axis of the length

$$s = \left( \text{trace}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2 \right)^{1/2}. \quad (1.1)$$

Of course, $s = 0$ for normal $A$, and the ellipse in this case degenerates into a line segment connecting $\lambda_1$ with $\lambda_2$. On the other hand, for $2 \times 2$ matrices $A$ with coinciding eigenvalues the ellipse $W(A)$ degenerates into a disk.

For general $n$, the following procedure is useful: Write $A = H + iK$ with $H, K$ Hermitian, and let

$$L_A(u, v, w) = \det(uH + vK + \omega I).$$

The equation $L_A(u, v, w) = 0$, with $u, v, w$ viewed as homogeneous line coordinates, defines an algebraic curve of class $n$. The real part of this curve we denote by $C(A)$ and call the associated curve (the randerzeugende curve, in German terminology of [3]) of $A$. The $n$ real foci of $C(A)$ correspond to the eigenvalues of $A$ [3, 4] and $W(A) = \text{conv} C(A)$ [3]; see [5, Sect. 3] for a detailed discussion of the connections between the polynomial $L_A$ and the numerical range of $A$. Note that the usual point equation $f(x, y) = 0$ of the curve $C(A)$ also can be written down (see [6]), but for $n > 2$ it is much more complicated than the line equation.

For $n = 3$, the following classification, based on factorability of $L_A$, was given by Kippenhahn in [3]:

**Case 1.** $L_A$ factors into three linear factors. Then $C(A)$ consists of three (not necessarily distinct) points, $A$ is normal (and therefore reducible$^1$), and $W(A)$ is the convex hull of its eigenvalues.

**Case 2.** $L_A$ factors into a linear factor and a quadratic factor. Then $C(A)$ consists of a point $\lambda_0$ (the eigenvalue of $A$ corresponding to the linear factor)

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$^1$We say that a matrix $A$ is reducible if there exists a unitary matrix $U$ such that $U^*AU = \text{diag}[A_1, A_2]$, where both diagonal blocks are of nonzero size. For reducible $A$, $W(A) = \text{conv}(W(A_1), W(A_2))$. 
and an ellipse \( E \). The numerical range \( W(A) \) is either an ellipse (if \( \lambda_0 \) lies inside \( E \)) or a "cone-like" figure otherwise; in the latter case \( A \) is reducible (but not normal).

In the next two cases the polynomial \( L_A \) (and therefore the matrix \( A \)) is irreducible.

Case 3. The degree of \( C(A) \) (that is, the degree of its point equation) equals 4. Then \( C(A) \) has a "double tangent," and the boundary of \( W(A) \) contains one flat portion but no angular points.

Case 4. The degree of \( C(A) \) equals 6. Then \( C(A) \) consists of two parts, one inside another; an outer part (and therefore \( W(A) \)) has an "ovular" shape.

This classification is complete: the same article [3] contains examples of matrices \( A \) falling in each of the above-mentioned cases.

However, it does not provide a constructive procedure that would allow one to determine, for an arbitrary given nonnormal \( 3 \times 3 \) matrix \( A \), the shape of its numerical range. The main purpose of our article is to offer a series of tests, in terms of a matrix \( A \) itself or its canonical unitarily equivalent forms, to determine when \( W(A) \) is an ellipse, a set with a flat portion on its boundary, or an ovular set. This is done in Sections 2 and 3. The results obtained simplify dramatically for matrices with one-point spectrum; this is discussed in Section 4. Section 5 is devoted to the question of when a \( 3 \times 3 \) matrix \( A \) can be restored (of course, up to unitary equivalence) from its numerical range.

2. \( W(A) \) IS AN ELLIPSE

We begin with the following general result concerning matrices of arbitrary size whose associated curve consists of ellipses and points.

Theorem 2.1. Let \( A \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and suppose that its associated curve \( C(A) \) consists of \( k \) ellipses, with minor
axes of lengths $s_1, s_2, \ldots, s_k$, and $n - 2k$ points. Then

$$
\sum_{i=1}^{k} s_i^2 = \text{trace}(A^*A) - \sum_{i=1}^{n} |\lambda_i|^2.
$$

For $n = 2$ the condition imposed in Theorem 2.1 is of no restriction since $C(A)$ always is an ellipse or a pair of points; formula (2.1) in this case takes the form (1.1). In the original version of this article we considered the case $n = 3$, in which, according to Kippenhahn’s classification, the conditions of Theorem 2.1 are satisfied for $C(A)$ being a union of three points (Case 1) or an ellipse and a point (Case 2). Still there is not more than one ellipse, and (2.1) in this case takes the form

$$
s = \left(\text{trace}(A^*A) - \left|\lambda_1\right|^2 - \left|\lambda_2\right|^2 - \left|\lambda_3\right|^2\right)^{1/2}.
$$

The generalization to the case $n > 3$ and its proof were suggested to us by the referee.

Proof. Relabel the eigenvalues of $A$ in such a way that $\lambda_{2i-1}, \lambda_{2i}$ become the foci of the $i$th ellipse ($i = 1, \ldots, k$), and $\lambda_{2k+1}, \ldots, \lambda_n$—the remaining points of $C(A)$.

Along with $A$, consider the matrix

$$
B = \begin{bmatrix} \lambda_1 & s_1 \\ 0 & \lambda_2 \end{bmatrix} \oplus \begin{bmatrix} \lambda_3 & s_2 \\ 0 & \lambda_4 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \lambda_{2k-1} & s_k \\ 0 & \lambda_{2k} \end{bmatrix} \oplus \text{diag}[\lambda_{2k+1}, \ldots, \lambda_n].
$$

Since $C(A) = C(B)$, the polynomials $L_A$ and $L_B$ have to be the same.

Compute now the coefficients of $w^{n-2}$ of these polynomials. When doing that, due to unitarily invariance of $L_A$, we may without loss of generality suppose that $A = [a_{ij}]_{k \times k}$ is in upper-triangular form. The coefficient of $w^{n-2}$ in $L_A$ equals the sum of all $2 \times 2$ principal minors of $uH + vK$, that is,

$$
\sum_{1 \leq i < j \leq n} \left[(uR \lambda_{ii} + vI \lambda_{ii})(uR \lambda_{jj} + vI \lambda_{jj}) - \frac{1}{4}(u^2 + v^2)|a_{ij}|^2\right]
$$

$$
= \sum_{1 \leq i < j \leq n} \left[(uR \lambda_i + vI \lambda_j)(uR \lambda_j + vI \lambda_j) - \frac{1}{4}(u^2 + v^2)|a_{ij}|^2\right].
$$

Here and in what follows, we denote by $Rz$ and $Iz$ the real and imaginary part, respectively, of a complex number $z$. 

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Applying this formula to $B$ (which already is in upper-triangular form) we obtain

$$
\sum_{1 \leq i < j \leq n} \left[ (u \Re \lambda_i + v \Im \lambda_i)(u \Re \lambda_j + v \Im \lambda_j) \right] - \frac{1}{4} (u^2 + v^2) \sum_{i=1}^{k} |s_i|^2.
$$

Since $L_A = L_B$, it follows from here that

$$
\sum_{i=1}^{k} |s_i|^2 = \sum_{1 \leq i < j \leq n} |a_{ij}|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 - \sum_{i=1}^{n} |a_{ii}|^2 = \text{trace}(A^*A) - \sum_{i=1}^{n} |\lambda_i|^2.
$$

Note that in the setting of Theorem 2.1 all the respective coefficients of $L_A$ and $L_B$ are equal. In particular, equating the coefficients of $u^n$, $v^n$ yields

$$
\det H = \prod_{i=1}^{k} \left( \Re \lambda_{2i-1} \Re \lambda_{2i} - \frac{1}{4} s_i^2 \right) \prod_{i=2k+1}^{n} \Re \lambda_i, \tag{2.3}
$$

$$
\det K = \prod_{i=1}^{k} \left( \Im \lambda_{2i-1} \Im \lambda_{2i} - \frac{1}{4} s_i^2 \right) \prod_{i=2k+1}^{n} \Im \lambda_i.
$$

If $n = 3$ and $A$ is in upper-triangular form

$$
A = \begin{bmatrix}
a & x & y \\
0 & b & z \\
0 & 0 & c
\end{bmatrix}, \tag{2.4}
$$

conditions (2.3) can be rewritten as $|x|^2 \Re c + |y|^2 \Re b + |z|^2 \Re a - \Re(x\Im z) = s^2 \Re \lambda_3$, $|x|^2 \Im c + |y|^2 \Im b + |z|^2 \Im a - \Im(x\Re z) = s^2 \Im \lambda_3$, or simply

$$
|x|^2 c + |y|^2 b + |z|^2 a - x\Im z = s^2 \lambda_3. \tag{2.5}
$$

Due to (2.2),

$$
s = \sqrt{|x|^2 + |y|^2 + |z|^2}. \tag{2.6}
$$
Hence, conditions (2.5), (2.6) are necessary for matrices (2.4) and

\[
B = \begin{bmatrix}
\lambda_1 & s & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]

to have the same associated curves. Surprisingly, a direct computation shows that they are also sufficient. Therefore, the following criterion holds.

**Theorem 2.2.** Let \( A \) be in upper-triangular form (2.4). Then its associated curve \( C(A) \) consists of an ellipse (possibly degenerating to a disk) and a point if and only if

1. \( d = |x|^2 + |y|^2 + |z|^2 > 0 \) and
2. the number \( \lambda = (c|x|^2 + b|y|^2 + a|z|^2 - x\bar{y}z)/d \) coincides with at least one of the eigenvalues \( a, b, c \).

*If these conditions are satisfied, then \( C(A) \) is the union of \( \lambda \) with the ellipse having its foci at two other eigenvalues of \( A \) and minor axis of length \( s = \sqrt{d} \).*

To obtain a unitary invariant form of Theorem 2.2, note that

\[
c|x|^2 + b|y|^2 + a|z|^2 - x\bar{y}z = d \text{ trace } A - \text{ trace}(A^*A^2) + \sum_{j=1}^{3} |\lambda_j|^2 \lambda_j.
\]

Therefore, Theorem 2.2 admits the following reformulation.

**Theorem 2.3.** The associated curve \( C(A) \) of a \( 3 \times 3 \) matrix \( A \) consists of an ellipse and a point if and only if

1. \( d = \text{ trace}(A^*A) - \sum_{j=1}^{3} |\lambda_j|^2 > 0 \) and
2. the number \( \lambda = \text{ trace } A + (1/d)(\sum_{j=1}^{3} |\lambda_j|^2 \lambda_j - \text{ trace}(A^*A^2)) \) coincides with at least one of the eigenvalues \( \lambda_j \) of \( A \).

*If these conditions are satisfied, then \( C(A) \) is the union of \( \lambda \) with the ellipse having its foci at two other eigenvalues of \( A \) and minor axis of length \( s = \sqrt{d} \).*

Now we are ready to formulate a necessary and sufficient condition for a \( 3 \times 3 \) matrix \( A \) to have an ellipse as its numerical range.
THEOREM 2.4. Let $A$ be a $3 \times 3$ matrix $A$ with the eigenvalues $\lambda_j$, $j = 1, 2, 3$. Then $W(A)$ is an ellipse if and only if conditions 1, 2 of Theorem 2.2 (or 2.3) hold and, in addition,

3. $(|\lambda_1 - \lambda_3| + |\lambda_2 - \lambda_3|)^2 - |\lambda_1 - \lambda_2|^2 \leq d$, where the eigenvalue coinciding with $\lambda$ is labeled $\lambda_3$.

Proof. Conditions 1, 2 are equivalent to $C(A)$ being a union of the ellipse $E$ (with the foci $\lambda_1, \lambda_2$ and minor axis of length $\sqrt{d}$) and the point $\lambda_3$. Condition 3 means that $\lambda_3$ lies inside $E$. According to Kippenhahn's classification, this is the only case when $W(A)$ is an ellipse.

The results obtained allow us to describe all $3 \times 3$ matrices for which $W(A)$ is a disk.

COROLLARY 2.5. $W(A)$ is a disk if and only if

1. $A$ has a multiple eigenvalue $\mu$ (so that its eigenvalues equal $\mu, \mu,$ and $\lambda$)
2. $2\mu \text{ trace}(A^*A) = \text{ trace}(A^*A^2) + 2|\mu|^2|\mu| + (2\mu - \lambda)|\lambda|^2$, and
3. $4|\mu - \lambda|^2 + 2|\mu|^2 + |\lambda|^2 \leq \text{ trace}(A^*A)$.

For $A$ in a triangular form (2.4), conditions 2 and 3 may be substituted by

2'. $x\bar{y}z = (\mu - \lambda)(\delta_{c, \mu}|x|^2 + \delta_{b, \mu}|y|^2 + \delta_{a, \mu}|z|^2)$, where $\delta$ is a usual Kronecker symbol, and

3'. $4|\mu - \lambda|^2 \leq |x|^2 + |y|^2 + |z|^2$.

If these conditions are satisfied, then $W(A)$ is centered at $\mu$ and has radius $\frac{1}{2}\sqrt{\text{ trace}(A^*A) - 2|\mu|^2 - |\lambda|^2}$ ($= \frac{1}{2}\sqrt{|x|^2 + |y|^2 + |z|^2}$ in the case (2.4)).

Proof. Indeed, $W(A)$ is a disk if and only if it is an ellipse (that is, conditions of Theorem 2.4 are satisfied) and, in addition, the foci of this ellipse coincide. This means that $A$ has a multiple eigenvalue (say, $\mu$), and its third eigenvalue coincides with $\lambda$ defined by condition 2 of Theorem 2.2 or Theorem 2.3. The rest is computation.

For triangular matrices, this corollary was first proved directly by Chien and Tam, although in a very different manner [7]. Necessary and sufficient conditions for $W(A)$ to be the unit disk centered at $0$ for a $3 \times 3$ matrix $A$ were obtained earlier by N. K. Tsing (unpublished) and stated in [7]: these conditions appear as a particular case of Corollary 2.5, when we specialize $a = b = \mu = 0$, $|x|^2 + |y|^2 + |z|^2 = 4$. 
3. \( W(A) \) HAVING A FLAT PORTION ON ITS BOUNDARY

Throughout this section we assume that \( A \) is a \( 3 \times 3 \) irreducible matrix represented as \( A = H + iK \) with \( H \) and \( K \) Hermitian.

We begin by deriving a canonical form for an irreducible matrix with a flat portion on the boundary of its numerical range.

**Theorem 3.1.** Let \( A \) be a \( 3 \times 3 \) irreducible matrix. Then after unitary similarity, translation, rotation, and scaling of \( A \), \( A \) may be written in the form

\[
A = \begin{bmatrix}
i & 0 & -c_1 \\
0 & 0 & -c_2 \\
c_1 & c_2 & \zeta
\end{bmatrix},
\]

where \( c_1, c_2, \Re(\zeta) \) are positive, if and only if \( W(A) \) has a flat portion on its boundary. In this form, \( W(A) \) has a flat portion extending from 0 to \( i \) and is contained in the closed right half-plane.

**Proof.** Let \( W(A) \) have a flat portion on its boundary. After rotation, shifting, and scaling (by scaling we mean multiplication by a positive number), we may assume that a flat portion stretches from 0 to \( i \). Since \( W(A) \) is convex, it must be contained entirely in the right or the left half-plane. Applying yet another rotation and translation, if necessary, we may assume that \( W(A) \) is in the right half-plane.

Since 0 and \( i \) are in \( W(A) \), there exist \( x_1, x_2 \in \mathbb{C}^n \), \( x_1^*x_1 = x_2^*x_2 = 1 \) such that \( x_1^*Ax_1 = 0, \ x_2^*Ax_2 = i \). Let \( \mathcal{L} = \text{Span}\{x_1, x_2\} \). Since \( \mathcal{L} \) is a 2-dimensional subspace, we may represent the linear transformation of \( A \) restricted to \( \mathcal{L} \), \( A|\mathcal{L} = A' \), by a \( 2 \times 2 \) matrix. By choosing a proper basis for \( A \), \( A' \) is the leading principal submatrix of \( A \).

Now \( W(A') \) is an ellipse, as are the numerical ranges of all \( 2 \times 2 \) matrices. Since \( W(A') \) is convex, \([0, i] \subseteq W(A') \). Also, \( W(A') \subseteq W(A) \). Since \( W(A) \) does not extend into the left half-plane, the only possible ellipse \( W(A') \) can be is the degenerate ellipse \([0, i] \). This implies that \( A' \) is normal with eigenvalues 0 and \( i \). So with proper basis

\[
A' = \begin{bmatrix}i & 0 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}i & 0 & \nu_1 \\
0 & 0 & \nu_2 \\
c_1 & c_2 & \zeta
\end{bmatrix}.
\]
Since $W(A)$ is in the closed right half-plane, $H$ is positive semidefinite. A calculation shows that

$$2H = \begin{bmatrix}
0 & 0 & \nu_1 + \bar{c}_1 \\
0 & 0 & \nu_2 + \bar{c}_2 \\
\nu_1 + c_1 & \nu_2 & 2\Re(\zeta)
\end{bmatrix},$$

and therefore $\Re(\zeta) \geq 0$, $\nu_1 = -\bar{c}_1$, $\nu_2 = -\bar{c}_2$. Due to the irreducibility of $A$, however, we know $\Re(\zeta) > 0$. By a diagonal unitary similarity, we may assume that $c_1, c_2$ are nonnegative. If $c_1$ or $c_2$ are 0, then $A$ is reducible, so $c_1$ and $c_2$ are positive. We now have the form (3.1) of $A$ we had hoped for.

Now suppose that after scaling, rotation, translation, and unitary similarity, $A$ is in the form expressed in the theorem. Consider the principal submatrix $A'$ from the first two rows and columns of $A$. $W(A')$ is a line segment from 0 to $i$. Clearly, $W(A') \subseteq W(A)$. But since $H$ is positive semidefinite, $W(A)$ lies entirely in the right half-plane. So the line segment from 0 to $i$ must be on the boundary of $W(A)$.

To see that the line segment does not go beyond 0 or $i$, note that any point $y$ on that line must be pure imaginary. So if $x^*Ax - x^*Hx + ix^*Kx = y$, then $x^*Hx = 0$. Hence, $x \in \ker(H) = \text{Span}\{[1,0,0]^T, [0,1,0]^T\}$. If $\|x\| = 1$, then $x = \nu_1[1,0,0]^T + \nu_2[0,1,0]^T$ with $|\nu_1|^2 + |\nu_2|^2 = 1$, and $0 \leq x^*Kx = |\nu_1|^2 \leq 1$.

According to Kippenhahn's classification, $W(A)$ has a flat portion on the boundary if and only if there exists a line, $ux + vy + w = 0$, tangent to $C(A)$ at two distinct points. This double tangent line corresponds to an eigenvalue $-w$ of $uH + vK$, which has multiplicity 2, since $u$ and $v$ are real, $uH + vK$ is Hermitian and $uH + vK + wI$ has rank 1. Conversely, if $uH + vK + wI$ has rank 1, then $-w$ is an eigenvalue of $uH + vK$ with multiplicity 2, and we get a double tangent. Observe also that, if $A$ is irreducible, $uH + vK$ cannot have an eigenvalue of multiplicity 3 (for then the Hermitian matrix $uH + vK$ would be scalar, $H$ and $K$ would commute, and hence $A$ would be normal). We summarize:

**Proposition 3.2.** Let $A = H + iK$ be irreducible. Then the following statements are equivalent:

1. $W(A)$ has a flat portion on the boundary;
2. $\text{rank}(uH + vK + wI) = 1$ for some real $u, v, w$;
3. for some real $u, v$ not both equal to zero, $uH + vK$ has a multiple eigenvalue.
Under these conditions, the flat portion of the boundary lies on the line
ux + vy + w = 0.

It also follows from Kippenhahn’s classification that an irreducible $3 \times 3$
matrix can have at most one flat portion on the boundary of its numerical
range.

**Corollary 3.3.** Let $A$ be irreducible and unitarily similar to a real
matrix. Then $W(A)$ has a flat portion on its boundary if and only if $H$
has a multiple eigenvalue. If $W(A)$ does have a flat portion, it is parallel to the
imaginary axis.

**Proof.** If $A$ is a real matrix, then $W(A)$ is symmetric about the real axis.
So, the (unique) flat portion of the boundary of $W(A)$ must be a vertical line.
According to Proposition 3.2, it happens if and only if the matrix $1 \cdot H + 0 \cdot K = H$
has a multiple eigenvalue. 

If $A$ is not unitarily equivalent to a real matrix, Proposition 3.2 may be
difficult to use. We now present several statements, equivalent to Proposition
3.2 (and obtained from it). As we see in Sections 4 and 5, these statements
are sometimes more suitable for application than Proposition 3.2 itself.

**Corollary 3.4.** Let an irreducible matrix $A$ be written in the form

$$A = \begin{bmatrix}
\xi & h_{12} & h_{13} \\
h_{12} & \xi & h_{23} \\
h_{13} & h_{23} & \xi
\end{bmatrix} + i \begin{bmatrix}
\eta & k_{12} & k_{13} \\
k_{12} & \eta & k_{23} \\
k_{13} & k_{23} & \eta
\end{bmatrix}. \tag{3.2}
$$

Then $W(A)$ has a flat portion on its boundary if and only if there exist real
$u,v$ not both zero such that

$$|uh_{13} + vk_{13}| = |uh_{23} + vk_{23}| = |uh_{12} + vk_{12}| \tag{3.3}$$

and

$$(uh_{12} + vk_{12})(uh_{13} + vk_{13})(uh_{23} + vk_{23}) \text{ is real.} \tag{3.4}$$
Proof. Since for any \( \zeta \in \mathbb{C} \) \( W(A) \) has a flat portion on its boundary if and only if \( W(A - \zeta I) \) has a flat portion on its boundary, we may (and do) assume that trace of \( A \) is zero: \( \xi = \eta = 0 \). The characteristic polynomial of \( uH + vK \) then equals

\[
\det(uH + vK - \lambda I) = -\lambda^3 + \left(|u h_{12}|^2 + |u h_{23}|^2 + |u \overline{h}_{13} + v \overline{k}_{13}|^2\right)\lambda \\
+ 2\Re\left((u h_{12} + v k_{12})(u h_{23} + v k_{23})(u \overline{h}_{13} + v \overline{k}_{13})\right). (3.5)
\]

According to Proposition 3.2, \( W(A) \) has a flat portion on its boundary if and only if for some real \( u, v \) not both zero \( uH + vK \) has a multiple eigenvalue; that is, the discriminant of (3.5) equals zero. For an arbitrary third-degree polynomial \( a_0 x^3 + a_1 x^2 + a_2 x + a_3 \), the discriminant is \( a_1^2 a_3^2 - 4 a_0 a_3^3 - 4 a_0^3 a_2 + 27 a_0^2 a_2^2 + 18 a_0 a_1 a_2 a_3 \); see, e.g., [8]. Due to our assumption trace \( A = 0 \) the coefficient of \( \lambda^2 \) in (3.5) vanishes, and direct computations show that the discriminant of (3.5) equals four times the expression

\[
\left(|u h_{12}|^2 + |u h_{23}|^2 + |u \overline{h}_{13} + v \overline{k}_{13}|^2\right)^3 \\
- 27\Re\left(\Re\left((u h_{12} + v k_{12})(u h_{23} + v k_{23})(u \overline{h}_{13} + v \overline{k}_{13})\right)\right)^2. (3.6)
\]

Setting

\[
x = u h_{12} + v k_{12}, \quad y = u h_{13} + v k_{13} \quad \text{and} \quad z = u h_{23} + v k_{23}, \quad (3.7)
\]

we see that (3.6) equals zero if and only if

\[
\frac{|x|^2 + |y|^2 + |z|^2}{3} = \sqrt{(\Re(xyz))^2}. (3.8)
\]

Since

\[
\frac{|x|^2 + |y|^2 + |z|^2}{3} \geq \sqrt{|x|^2|y|^2|z|^2} \geq \sqrt{(\Re(xyz))^2}. (3.9)
\]
the equality (3.8) holds exactly when both inequalities in (3.9) are actually equalities, that is, when $|x| = |y| = |z|$ (for the first inequality in (3.9)) and $xyz \in \mathbb{R}$ (for the second). The two conditions obtained are exactly the same as (3.3) and (3.4), respectively.

Note that condition (3.2) is of no restriction, since it can always be obtained by a unitary similarity. To rewrite Corollary 3.4 in a unitarily equivalent form, put

$$B = \begin{bmatrix} 0 & x & y \\ \bar{x} & 0 & z \\ \bar{y} & \bar{z} & 0 \end{bmatrix} = uH + vK - \frac{1}{3} (\text{trace}(uH + vK)) I.$$

Then, in notation (3.7), $|x|^2 + |y|^2 + |z|^2 = \frac{1}{3} \text{trace} B^2$ and $\Re(x\bar{y}z) = \frac{1}{6} \text{trace} B^3$, so that condition (3.8) is equivalent to

$$\left[\text{trace}(B^3)\right]^2 = \left[\text{trace}(B^2)\right]^3 / 6. \quad (3.10)$$

**COROLLARY 3.5.** Let $A = H + iK$ be an irreducible matrix. Then $W(A)$ has a flat portion on its boundary if and only if there exist real $u, v$ not both zero so that for $B = uH + vK - \frac{1}{3} (\text{trace}(uH + vK)) I$ the equality (3.10) holds.

We see in Section 4 that Corollary 3.4 sometimes leads to explicit results in spite of the fact that it refers to the existence of $u, v$ without showing how to construct them. The criterion not using $u, v$ at all is given by the next corollary. It is applicable to matrices $A = H + iK$ with a diagonalized summand $K$.

**COROLLARY 3.6.** Let $A$ be an irreducible matrix written in the form

$$A = \begin{bmatrix} h_1 & h_{12} & h_{13} \\ \bar{h}_{12} & h_2 & h_{23} \\ \bar{h}_{13} & \bar{h}_{23} & h_3 \end{bmatrix} + i \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

$W(A)$ has a flat portion if and only if $K$ has a multiple eigenvalue or

$$h_1(k_2 - k_3) + h_2(k_3 - k_1) + h_3(k_1 - k_2)$$

$$= (k_2 - k_3) \frac{h_{12}h_{13}}{h_{23}} + (k_3 - k_1) \frac{h_{12}h_{23}}{h_{13}} + (k_1 - k_2) \frac{h_{13}h_{23}}{h_{12}}$$

$$= (k_2 - k_3) \left( \frac{h_{12}}{h_{23}} + \frac{h_{13}}{h_{12}} \right) + (k_3 - k_1) \left( \frac{h_{12}}{h_{13}} + \frac{h_{23}}{h_{12}} \right) + (k_1 - k_2) \left( \frac{h_{13}}{h_{23}} + \frac{h_{23}}{h_{13}} \right). \quad (3.11)
and

$\bar{h}_{12} \bar{h}_{13} \bar{h}_{23}$ is real. \hspace{5em} (3.12)

**Proof.** According to Proposition 3.2, $W(\Lambda)$ has a flat portion on its boundary if and only if $B = uH + vK + \omega I$ has rank 1 for some real $u, v, \omega$. If $K$ has a multiple eigenvalue $\lambda$, then the latter condition is satisfied with $u = 0, v = -1, \omega = -\lambda$. Therefore, we need to consider only the case of $K$ having distinct eigenvalues.

**Necessity.** Let rank $B = 1$. Since the eigenvalues of $K$ are all distinct, it is possible only when $u$ is nonzero. Without loss of generality we may (and will) assume that $u = 1$.

To simplify further calculations, rewrite $B$ in the form

$$B = \begin{bmatrix} h'_1 + v'k'_1 + w' & h_{12} & h_{13} \\ \bar{h}_{12} & h'_2 + v'k'_2 + w' & h_{23} \\ \bar{h}_{13} & \bar{h}_{23} & w' \end{bmatrix},$$

where $w' = w + h_3 + v'k_3, h'_i = h_i - h_3, k'_i = k_i - k_3$ ($i = 1, 2$).

Then all the off-diagonal elements $h_{12}, h_{23}, h_{13}$ are nonzero (otherwise at least two of them, located in the same row or column, equal zero, which would imply reducibility of $A$), and

$$w'/\bar{h}_{23} = h_{13}/h_{12} = h_{23}/(h'_2 + v'k'_2 + w'), \hspace{5em} (3.13)$$

$$w'/\bar{h}_{13} = h_{23}/h_{12} = h_{13}/(h'_1 + v'k'_1 + w'). \hspace{5em} (3.14)$$

Solving (3.13) with respect to $v, w'$ we find that

$$w' = \frac{h_{13} \bar{h}_{23}}{h_{12}}, \hspace{5em} v = \frac{1}{k'_2} \left( \frac{h_{12} \bar{h}_{23}}{h_{13}} - \frac{h_{13} \bar{h}_{23}}{h_{12}} - h'_2 \right). \hspace{5em} (3.15)$$

For $v, w'$ defined by (3.15) the equalities (3.14) yield, respectively, (3.12) and

$$k'_1 \left( h'_2 + \frac{h_{13} \bar{h}_{23}}{h_{12}} - \frac{h_{12} \bar{h}_{23}}{h_{13}} \right) = k'_2 \left( h'_1 + \frac{h_{13} \bar{h}_{23}}{h_{12}} - \frac{h_{13} \bar{h}_{12}}{h_{23}} \right). \hspace{5em} (3.16)$$
It is easily checked that under the restriction (3.12) the latter is equivalent to (3.11).

Sufficiency. From (3.11) it follows, in particular, that \( h_{12}h_{23}h_{13} \neq 0 \). Define \( v, w' \) by (3.15). Then, of course, (3.13) holds. Moreover, due to (3.12), \( v, w' \) are real, and

\[
\frac{w'}{h_{13}} = \frac{h_{13}h_{23}}{(h_{12}h_{13})} = \frac{h_{23}}{h_{23}h_{12}h_{13}} = \frac{h_{12}h_{13}h_{23}}{h_{12}h_{13}h_{23}} = h_{23}/h_{12} = h_{23}/h_{12}.
\]

Therefore, the first of equalities (3.14) also holds.

Finally, (3.11) and (3.12) imply (3.16), which, in turn, leads to the second of the equalities (3.14). Due to (3.13), (3.14), \( B \) is a (nonzero) matrix with collinear columns and therefore has rank 1.

Note that condition (3.11) may be written as

\[
h_1(k_2 - k_3) + h_2(k_3 - k_1) + h_3(k_1 - k_2) = \mu h_{12}h_{13}h_{23},
\]

where

\[
\mu = \frac{k_3 - k_1}{|h_{23}|^2} + \frac{k_3 - k_1}{|h_{13}|^2} + \frac{k_1 - k_2}{|h_{12}|^2}.
\]

Since \( h_i(k_2 - k_3) + h_2(k_3 - k_1) + h_3(k_1 - k_2) \) and \( \mu \) are both real, it means that condition (3.12) follows from (3.11) if \( \mu \) is nonzero.

The above corollary also works with \( H \) and not \( K \) diagonal. To see this, multiply \( A \) by \( i \). This makes \( H' = -K \) and \( K' = H \). Clearly \( W(iA) \) has a flat portion on its boundary if and only if \( W(A) \) has a flat portion.

4. \( W(A) \) FOR MATRICES WITH A TRIPLE EIGENVALUE

In this section we apply our results to the special case of matrices with a triple eigenvalue. In their triangular form (2.4), of course, all diagonal
elements coincide:

\[ A = \begin{bmatrix} p & x & y \\ 0 & p & z \\ 0 & 0 & p \end{bmatrix}, \quad \text{(4.1)} \]

with \( p, x, y, z \) complex. Note that \( W(A) \) cannot be a noncircular ellipse since such an ellipse requires two distinct foci (eigenvalues of \( A \)) of the associated curve.

**Theorem 4.1.** Let \( A \) be in the form (4.1). Then:

1. \( W(A) \) is a disk if and only if \( xyz = 0 \); in this case the disk has radius \( \frac{1}{2} \left( |x|^2 + |y|^2 + |z|^2 \right) \) with center \( p \).

2. \( W(A) \) has a flat portion on its boundary if and only if \( |x| = |y| = |z| > 0 \); in this case \( C(A) \) is a cardioid.

3. \( W(A) \) is of the ovoidal shape if and only if \( xyz \neq 0 \) and \( |x|, |y|, |z| \) are not all equal.

A version of Part 1 of this theorem for the nilpotent case was first shown by Marcus and Pesce, who also developed a unitarily invariant form of this condition [9].

**Proof.** Part 1 follows easily from Corollary 2.5. In the rest of the proof we may therefore suppose that \( xyz \neq 0 \), so that \( A \) is irreducible.

Part 2: To simplify further calculations, consider the matrix \( 2A \) instead of \( A \):

\[
2A = \begin{bmatrix} 2 \Re(p) & x & y \\ \bar{x} & 2 \Re(p) & z \\ \bar{y} & \bar{z} & 2 \Re(p) \end{bmatrix} + i \begin{bmatrix} 2 \Im(p) & -ix & -iy \\ ix & 2 \Im(p) & -iz \\ iy & iz & 2 \Im(p) \end{bmatrix}.
\]

By Corollary 3.4, \( W(A) \) has a flat portion if and only if there exist real \( u, v \) not both zero such that

\[
|ux + v(-ix)| = |uy + v(-iy)| = |uz + v(-iz)|,
\]

\[
\arg(ux - ivx) + \arg(uz - ivz) = \arg(uy - ivy).
\]
From the first equation we see that we must have

$$|u - iv||x| = |u - iv||y| = |u - iv||z|,$$

which implies that we must have $|x| = |y| = |z|$ since $u - iv \neq 0$. The second equation becomes

$$\arg(u - iv) = \arg(y) - \arg(x) - \arg(z).$$

We can easily choose $u, v$ so that this is true. And so the only condition we have is $|x| = |y| = |z|$.

We now prove that under this condition $C(A)$ is a cardioid. Using unitary transformations $A \rightarrow U^*A U$ with $U = \text{diag}[e^{i\nu_1}, e^{i\nu_2}, e^{i\nu_3}]$ (which do not change $C(A)$) and multiplying $A$ by scalars (which rotate and dilate $C(A)$) we may reduce the general case to $x = y = z = 1$. Shifting then $A$ by $\lambda I$ (which shifts $C(A)$ by $\lambda$), we may suppose also that its eigenvalue is $\frac{1}{3}$ (such a choice of the eigenvalue ensures that the cusp of the cardioid would be at the origin). In other words, without loss of generality

$$A = \begin{bmatrix}
\frac{1}{3} & 1 & 1 \\
0 & \frac{1}{3} & 1 \\
0 & 0 & \frac{1}{3}
\end{bmatrix}.$$  

Using Fiedler’s formula (see [6]) for the point equation of $C(A)$ and transforming to polar coordinates we find

$$3r^2(-3r - 2 + 2\cos \theta)(-3r + 2 + 2\cos \theta) = 0.$$  

The factor of $3r^2$ is redundant since $r = 0$ is a solution to the other two factors. The other two factors actually define the same curve. This is because if one replaces $r$ with $-r$ and $\theta$ with $\theta + \pi$, the factors are identical within a scalar multiple. In polar coordinates, this means that the factors trace the same curve. The equation therefore simplifies to $r = \frac{2}{3}(1 - \cos \theta)$, which is the equation of a cardioid [10].

Part 3: Since $W(A)$ cannot be an ellipse without being a disk, the ovular shape is the only case left.
A computer image of $W(A)$, where

$$ A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, $$

was given in [9]. By Theorem 4.1, $W(A)$ is the convex hull of a cardioid.

As in Sections 2 and 3, the unitarily invariant version exists:

**Theorem 4.2.** Let $A$ be a matrix with triple eigenvalue $p$, $\Psi = \text{trace}(A^*A) - 3|p|^2$, and $\Omega = \text{trace}(A^*A^2) - 2p(\text{trace}(A^*A)) + 3p|p|^2$. Then

1. $W(A)$ is a disk centered at $p$ if and only if $\Omega = 0$. In this case it has radius $\frac{1}{2}\sqrt{\Psi}$.

2. $W(A)$ has a flat portion on its boundary if and only if $\Psi = 3\sqrt{|\Omega|^2} > 0$.

3. $W(A)$ has an ovular shape if and only if $\Psi \neq 3\sqrt{|\Omega|^2} > 0$.

*Proof.* It suffices to consider $A$ in the form (4.1). The direct computation then shows that $\Psi = |x|^2 + |y|^2 + |z|^2$ and $\Omega = xy\bar{z}$. Obviously, condition 1 is equivalent to $xyz = 0$. Due to the case of equality between arithmetic and geometric means of $|x|^2, |y|^2, |z|^2$, 2 holds if and only if $|x| = |y| = |z| > 0$. Finally, 3 is the only logically possible case left.

5. **RESTORATION OF A FROM W(A)**

An inverse problem concerning numerical ranges may be formulated. Given a numerical range $W(A)$ for some $A$, can one reconstruct $A$? Since $W(A) = W(U^*AU)$, we cannot restore $A$ uniquely (with the exception of $W(A)$ being a single point), but we can sometimes find a unique unitary equivalence class that generates $W(A)$. The latter is always the case for $2 \times 2$ matrices (see, e.g., [11]).

In the remainder of this section, we deal with $3 \times 3$ matrices. For reducible $A$, it can be easily seen that in this case $A$ cannot always be restored from $W(A)$, but it can be restored from $W(A)$ and the trace of $A$, or equivalently $C(A)$. We show later that in the irreducible case, $A$ cannot always be restored, even if $C(A)$ is known.

Unexpectedly, there is a case of a $W(A)$ arising from an irreducible matrix, which allows $A$ to be restored up to unitary similarity:
THEOREM 5.1. Let \( W(A) \) be a 2-dimensional shape with only one flat portion on its boundary. Then \( A \) is an irreducible matrix, which can be restored up to unitary similarity.

Proof. Having only one flat portion on the boundary of its numerical range, \( A \) belongs to Case 4 of Kippenhahn's classification and is therefore irreducible. After scaling, rotation, and shifting of \( W(A) \), we can have the flat portion as the line segment \([0, i]\) and \( W(A) \) lies entirely in the right half-plane. We restore \( A \) in this case. After the restoration, one can obtain the original \( A \) by reversing the scaling, rotation, and shifting.

According to Theorem 3.1, \( A \) must be unitarily similar to (3.1). Let us assume \( A \) is in that form. The real part \( H \) of \( A \) is then \( \text{diag}(0, 0, \Re(\xi)) \), with \( \Re(\xi) \) positive. Since \( W(\Re(A)) \) is the projection of \( W(A) \) onto the real axis, which is a line segment from 0 to \( \Re(\xi) \), we can determine \( \Re(\xi) \).

Since there is only one flat portion, the real part of any point on that portion is 0. So \( \xi \) is not on that portion. Because \( \Re(\xi) \) is an endpoint of the projection of \( W(A) \) onto the real axis and \( \xi \) is not on the flat portion, \( \xi \) is uniquely determined as the point on the boundary of \( W(A) \) having a maximum real part, namely \( \Re(\xi) \). So \( \Im(\xi) \) is also determined.

The imaginary part \( K \) of \( A \) is

\[
\begin{bmatrix}
1 & 0 & ic_1 \\
0 & 0 & ic_2 \\
-ic_1 & -ic_2 & \Im(\xi)
\end{bmatrix}
\]

Since \( W(K) \) is a line segment, which is a projection of \( W(A) \) onto the imaginary axis, we know two of the eigenvalues \( \lambda_1, \lambda_2 \) of \( K \), namely the endpoints of the line segment. Calculating the characteristic polynomial of \( K \) and substituting in \( \lambda_1, \lambda_2 \) give us the system of linear equations in \( c_1^2, c_2^2 \)

\[
c_1^2(-\lambda_1) + c_2^2(-\lambda_1 + 1) = -\lambda_1^3 + \lambda_1^2(1 + \Im(\xi)) - \lambda_1(\Im(\xi))
\]

\[
c_1^2(-\lambda_2) + c_2^2(-\lambda_2 + 1) = -\lambda_2^3 + \lambda_2^2(1 + \Im(\xi)) - \lambda_2(\Im(\xi)).
\]

The determinant of this system is \( \lambda_2 - \lambda_1 \), which is nonzero since the flat portion is of nonzero length, causing the projection of \( W(A) \) onto the imaginary axis to be of nonzero length. So the system (5.1) has a unique
solution:

\[ c_1^2 = - (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 + \lambda_2 - 2) \]
\[ c_2^2 = \lambda_1 \lambda_2 (\lambda_1 + \lambda_2 - 1 - 2\Im(\xi)). \]

Since \( c_1, c_2 \) are positive, we thus have unique values for them. Therefore we know all the elements of \( A \) in this canonical form, which determines \( A \) up to unitary similarity.

As it turns out, in cases of other shapes of \( W(A) \) for an irreducible \( A \), the matrix \( A \) cannot be uniquely (up to unitary similarity) restored by \( W(A) \). We summarize all these cases, as well as the cases of a reducible \( A \), in the following theorem.

**Theorem 5.2.** A \( 3 \times 3 \) matrix \( A \) can be restored (up to unitary similarity) from \( W(A) \) if and only if \( W(A) \) is one of the following: (1) a point, (2) a triangle, (3) the convex hull of an ellipse and a point outside the ellipse, (4) a 2-dimensional shape with only one flat portion on its boundary.

In the other cases, that of (5) a line segment, (6) an ellipse, and (7) an ovular shape, the matrix cannot be restored. In the cases 5–7 there is a continuum of nonunitarily equivalent matrices whose numerical range is \( W(A) \).

**Proof.** Cases 1–3 are well known; case 4 was discussed in Theorem 5.1.

In case 5 \( A \) is normal, with at least two distinct eigenvalues and all three eigenvalues collinear. The eigenvalues corresponding to the endpoints can be determined, but the third eigenvalue cannot. There is a continuum choice for this third eigenvalue.

If \( W(A) \) is an ellipse and \( A \) is reducible, \( A \) cannot be restored since the point defined by its \( 1 \times 1 \) block may be anywhere within the ellipse defined by the \( 2 \times 2 \) block. Again, there is a continuum of choices for the \( 1 \times 1 \) block.

The proof in the remaining two situations (\( W(A) \) is an ellipse produced by an irreducible \( A \) or an ovular shape) is based on a series of lemmas and is therefore relegated to the end of this section.

One might ask whether a matrix \( A \) can be restored (up to unitary similarity) from \( W(A) \) and the trace of \( A \). In this respect we note that for a \( 3 \times 3 \) matrix \( A \) each of the following pieces of information completely
determines two others: (1) $W(A)$ and the trace of $A$; (2) $W(A)$ and the eigenvalues of $A$; (3) $C(A)$.

Indeed, $C(A)$ determines uniquely $W(A)$ (because $W(A)$ is the convex hull of $C(A)$ and the eigenvalues of $A$ (because there are the foci of $C(A)$). On the other hand, if $W(A)$ is known then the maximal and minimal eigenvalues of every linear combination $H \cos \xi + K \sin \xi$ (here $A = H + iK$ with Hermitian $H$ and $K$ and $\xi$ is a real number) are determined by using the orthogonal projection of $W(e^{-i\xi}A)$ onto the real axis; note that $H \cos \xi + K \sin \xi$ is the real part of $e^{-i\xi}A$. If, in addition, the trace of $A$ is known, then all eigenvalues of $H \cos \xi + K \sin \xi$ are known, and therefore the polynomial $\det(uH + vK + wI)$ is known, which determines $C(A)$.

It will be clear from the proof of Theorem 5.2 that, in addition to the cases when a $3 \times 3$ matrix $A$ can be restored from $W(A)$, such a matrix can be restored from $C(A)$ (or equivalently from $W(A)$ and the trace of $A$) if $W(A)$ is a line segment. On the contrary, if $W(A)$ is ovular or $W(A)$ is an ellipse (without any information concerning the reducibility of $A$), then there are uncountably many unitarily inequivalent matrices $B$ such that $C(B) = C(A)$. However, if $W(A)$ is an ellipse and it is known that $A$ is reducible, then $A$ can be restored (up to unitary similarity) from $C(A)$.

The rest of this section is devoted to completion of the proof of Theorem 5.2.

We use the two matrices

$$A = H + iK = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} + i \begin{bmatrix} \beta_1 & e & f \\ e & \beta_2 & g \\ f & \bar{g} & \beta_3 \end{bmatrix}, \quad (5.2)$$

$$B = H' + iK' = \begin{bmatrix} \alpha_1' & 0 & 0 \\ 0 & \alpha_2' & 0 \\ 0 & 0 & \alpha_3' \end{bmatrix} + i \begin{bmatrix} \beta_1' & e' & f' \\ e' & \beta_2' & g' \\ f' & \bar{g}' & \beta_3' \end{bmatrix}, \quad (5.3)$$

with $\alpha_1 > \alpha_2 > \alpha_3, \alpha_1' > \alpha_2' > \alpha_3', \beta_i, \beta_i'$ real, and off-diagonal elements such that

$$e, f, e', f' \geq 0; \quad \text{if } ef = 0, \text{ then } g \geq 0; \quad \text{if } e'f' = 0, \text{ then } g' \geq 0. \quad (5.4)$$
Lemma 5.3. Let $A$ and $B$ be written in the form (5.2) and (5.3). Then $L_A = L_B$ if and only if

1. All the diagonal elements are equal: $\alpha_j = \alpha'_j$, $\beta_j = \beta'_j$ for $j = 1, 2, 3$.

2. $e' = \sqrt{e^2 + ((\alpha_1 - \alpha_2)/(\alpha_2 - \alpha_3))(|g'|^2 - |g|^2)}$

3. $f' = \sqrt{f^2 - ((\alpha_1 - \alpha_3)/(\alpha_2 - \alpha_3))(|g'|^2 - |g|^2)}$

4. $e'f'(g' + \overline{g'}) = ef(g + \overline{g}) + (|g'|^2 - |g|^2)\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2)/(\alpha_2 - \alpha_3)$

Proof. For $L_A$ and $L_B$ to coincide it is necessary, in particular, that $A$ and $B$ have the same sets of eigenvalues. Then their traces also are the same.

Without loss of generality we may assume that they equal zero. Assuming that, redenote $\alpha_1 = a$, $\alpha_2 = b$, $\beta_1 = c$ and $\beta_2 = d$; then, of course, $\alpha_3 = -a - b$, $\beta_3 = -c - d$. Analogously, the diagonal elements of $H'$ and $K'$ are now $a'$, $b'$, $-a' - b'$ and $c'$, $d'$, $-c' - d'$, respectively.

From the equality $L_A(u, v, w) = L_B(u, v, w)$ for $u = 1$, $v = 0$ it follows that $H$ and $H'$ have the same eigenvalues. Since in both matrices the eigenvalues are ordered, it means that $H' = H$ and thus $a' = a$, $b' = b$.

Calculation shows that

$$
(ab^2 - a^2b)u^3 + (-2abd - a^2d - 2abc - cb^2)u^2v
$$

$$
+ (-a^2 - ab - b^2)u^2w
$$

$$
+ (-2bcd + a(e^2 - |g|^2) - bc^2 - 2acd - ad^2 + b(e^2 - f^2))uv^2
$$

$$
+ (c(-b - 2a) + d(-a - 2b))uvw
$$

$$
+ (c(e^2 - |g|^2) - c^2d + ef(g + \overline{g}) + d(e^2 - f^2) - cd^2)v^3
$$

$$
+ (-d^2 - cd - f^2 - e^2 - |g|^2 - e^2)v^2w + w^3.
$$

Comparing this to the equation for $det(uH' + vK' + wI)$ we see from the coefficients of $u^2v$ and $uvw$ that we must have

$$( -2ab - b^2)c' + (-2ab - a^2)d' = (-2ab - b^2)c + (-2ab - a^2)d$$

$$( -2a - b)c' + (-a - 2b)d' = (-2a - b)c + (-a - 2b)d.$$
Considering this as a linear system of equations in \( c', d' \) of the form \( Cx = y \),

\[
\det(C) = (-2ab - b^2)(-a - 2b) - (-2ab - a^2)(-2a - b) = (-a - 2b)(a - b)(2a + b) 
\]

is non-zero by our assumption that \( a > b > -a - b \). The system therefore has a unique solution, which is obviously \( c' = c, d' = d \).

We can now conclude that all the diagonal elements of \( A \) and \( B \) are the same.

From the coefficients of \( v^2w \) and \( v^2u \) we have, after elimination of identical terms,

\[
e^{e^2} + f^{f^2} = e^2 + f^2 + |g|^2 - |g'|^2
\]

\[(a + b)e^{e^2} - bf^{f^2} = (a + b)e^2 - bf^2 - a(|g|^2 - |g'|^2),\]

which can be viewed as a linear system of equations in \( e^{e^2}, f^{f^2} \). By solving the system and using our assumption that \( e', f' \) are nonnegative, we obtain

\[
e' = \sqrt{e^2 + \frac{a - b}{a + 2b}(|g|^2 - |g'|^2)}
\]

\[
f' = \sqrt{f^2 - \frac{2a + b}{a + 2b}(|g'|^2 - |g|^2)}.\]

Finally from the coefficient of \( v^3 \) we have

\[
e'f'(g' + \overline{g'}) = ef(g + \overline{g}) + c(|g'|^2 - |g|^2)
\]

\[+(c + d)(e^2 - e^{e^2}) - d(f^2 - f^{f^2})\]

\[= ef(g + \overline{g}) + (|g'|^2 - |g|^2)\]

\[\times \left[ c - (c + d) \frac{a - b}{a + 2b} - d \frac{2a + b}{a + 2b} \right] \]

\[= ef(g + \overline{g}) + (|g'|^2 - |g|^2) \frac{3(bc - ad)}{a + 2b} .\]

Substituting our definitions of \( a, b, c, d \) into the above equations gives us the equations stated in the lemma.
**Lemma 5.4.** Let $A$ and $B$ be in the forms (5.2) and (5.3). Then $A$ is unitarily similar to $B$ if and only if $A = B$.

**Proof.** As in the proof of the preceding lemma, we assume that $A$ and $B$ have zero trace. Suppose $A = U^*BU$ for some unitary $U$. Then obviously $C(A) = C(B)$, and therefore $H = H'$ by Lemma 5.3. Now $H = U^*HU$, which implies that $U$ must be diagonal. Using the condition (5.4), the equality $K' = U^*KU$ implies $K' = K$.

**Lemma 5.5.** Let $A$ be an irreducible matrix in the form (5.2). If $g$ is not real, or condition

\[ efg(\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2)) \]

\[ = (\alpha_2 - \alpha_3)ef + (\alpha_3 - \alpha_1)eg + (\alpha_1 - \alpha_2)fg \quad (5.5) \]

is not satisfied, there exists a continuum of unitary equivalences classes of matrices with the same associated curve $C(A)$.

**Proof.** We construct matrices $B$ of the form (5.3) with $C(A) = C(B)$. Let $\rho = |g'|^2 - |g|^2$, 

\[ q = -\frac{\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2)}{\alpha_2 - \alpha_3} \quad (5.6) \]

\[ \mu = \frac{\alpha_1 - \alpha_2}{\alpha_2 - \alpha_3}, \quad \tau = \frac{\alpha_1 - \alpha_3}{\alpha_2 - \alpha_3}. \]

Note that $\mu, \tau > 0$.

Then by Lemma 5.3, for $C(B) = C(A)$ we must have

\[ e' = \sqrt{e^2 + \mu \rho} \quad (5.7) \]

\[ f' = \sqrt{f^2 - \tau \rho} \quad (5.8) \]

\[ e'f'(g' + \bar{g}') = ef(g + \bar{g}) + \rho q. \quad (5.9) \]

To satisfy (5.7), (5.8), let us choose $\rho \in I = (-e^2/\mu, f^2/\tau)$. Note that the length of $I$ is positive, because otherwise $f = e = 0$, and $A$ would be
reducible. The last equation (5.9) is then equivalent to
\[ \Re g' = \pm \frac{2df \Re g + \rho q}{2\sqrt{(e^2 + \mu \rho)(f^2 - \tau \rho)}}. \tag{5.10} \]

For number \( g' \in \mathbb{C} \) with \( |g'|^2 = \rho + |g|^2 \) and \( \Re g' \) given by (5.10) to exist, it is necessary and sufficient that
\[ \frac{(2ef \Re g + \rho q)^2}{4(e^2 + \mu \rho)(f^2 - \tau \rho)} \leq \rho + |g|^2. \]

The latter inequality can be rewritten as \( \mathcal{F}(\rho) \geq 0 \), if we denote
\[ \mathcal{F}(\rho) = (e^2 + \mu \rho)(f^2 - \tau \rho)(4\rho + 4|g|^2) - (2ef \Re g + \rho q)^2. \]

If \( g \) is not real, then \( \mathcal{F}(0) > 0 \), so that there is an \( \epsilon > 0 \) such that \( \mathcal{F}(\rho) \geq 0 \) for \( |\rho| < \epsilon \). Every \( \rho \in I \cap (-\epsilon, \epsilon) \) generates a matrix (5.3) with \( C(B) = C(A) \). Different values of \( \rho \) correspond to different matrices \( B \), and none of them are unitarily similar due to Lemma 5.4.

If \( g \) is real, then \( \mathcal{F}(0) = 0 \), and
\[ \frac{d\mathcal{F}}{d\rho}_{\rho=0} = 4\mu f^2 g^2 - 4\tau e^2 g^2 + 4e^2 f^2 - 4efgq. \]

Substituting in the values of \( \mu, \tau, \) and \( q \) from (5.6) we see that (5.5) is equivalent to \( d\mathcal{F}/d\rho|_{\rho=0} = 0 \). Hence, if (5.5) does not hold, there is a one-sided neighborhood \( N \) of zero such that \( \mathcal{F}(\rho) \geq 0 \) for \( \rho \in N \). Observe that this neighborhood is positive if \( e = 0 \) and negative if \( f = 0 \), so that in any case \( N \cap I \) is a continuum. All \( \rho \in N \cap I \) generate matrices with the same associated curve as \( C(A) \), and, as above, all these matrices belong to different unitarily equivalence classes.

We now complete the proof of Theorem 5.2. Consider an irreducible matrix \( A \). Without loss of generality we may suppose that it is in the form (5.2). Corollary 3.6 implies that \( W(A) \) contains a flat portion on its boundary if and only if \( g \) is real and (5.5) holds. From here and Lemma 5.5 it follows that in all other cases (that is, when \( W(A) \) is an ellipse or has an ovular shape) there is a continuum of unitary equivalence classes of matrices with the same numerical range \( W(A) \) (and even the same associated curve \( C(A) \)).
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REFERENCES


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