NORTH-HOLLAND

The Numerical Range of 3 × 3 Matrices

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ABSTRACT

Let A be an $n \times n$ complex matrix. Then the numerical range of A, $W(A)$, is defined to be $\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$. In this article a series of tests is given, allowing one to determine the shape of $W(A)$ for 3×3 matrices. Reconstruction of A, up to unitary similarity, from $W(A)$ is also examined. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

Let A be an $n \times n$ matrix with complex entries. The *numerical range* of A, $W(A)$, is defined as $\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$. It is well known (see [1, 2]) that $W(A)$ is a convex compact subset of $\mathbb{C}$, which contains all the

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eigenvalues of A and therefore its convex hull $\mathrm{conv}(\sigma(A))$. For A normal (that is, commuting with A^*), $W(A) = \mathrm{conv}(\sigma(A))$; the converse statement holds when $n \leqslant 4$.

For 2×2 matrices A a complete description of the numerical range $W(A)$ is well known. Namely, $W(A)$ is an ellipse with foci at the eigenvalues λ_1, λ_2 of A and a minor axis of the length

$$s = \left(\mathrm{trace}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\right)^{1/2}. \tag{1.1}$$

Of course, $s = 0$ for normal A, and the ellipse in this case degenerates into a line segment connecting λ_1 with λ_2. On the other hand, for 2×2 matrices A with coinciding eigenvalues the ellipse $W(A)$ degenerates into a disk.

For general n, the following procedure is useful: Write $A = H + iK$ with H, K Hermitian, and let

$$L_A(u, v, w) = \det(uH + vK + wI).$$

The equation $L_A(u, v, w) = 0$, with u, v, w viewed as homogeneous line coordinates, defines an algebraic curve of class n. The real part of this curve we denote by $C(A)$ and call the *associated curve* (the *randerzeugende* curve, in German terminology of [3]) of A. The n real foci of $C(A)$ correspond to the eigenvalues of A [3, 4] and $W(A) = \mathrm{conv}\, C(A)$ [3]; see [5, Sect. 3] for a detailed discussion of the connections between the polynomial L_A and the numerical range of A. Note that the usual point equation $f(x, y) = 0$ of the curve $C(A)$ also can be written down (see [6]), but for $n > 2$ it is much more complicated than the line equation.

For $n = 3$, the following classification, based on factorability of L_A, was given by Kippenhahn in [3]:

Case 1. L_A factors into three linear factors. Then $C(A)$ consists of three (not necessarily distinct) points, A is normal (and therefore *reducible*[1]), and $W(A)$ is the convex hull of its eigenvalues.

Case 2. L_A factors into a linear factor and a quadratic factor. Then $C(A)$ consists of a point λ_0 (the eigenvalue of A corresponding to the linear factor)

[1] We say that a matrix A is reducible if there exists a unitary matrix U such that $U^*AU = \mathrm{diag}[A_1, A_2]$, where both diagonal blocks are of nonzero size. For reducible A, $W(A) = \mathrm{conv}(W(A_1), W(A_2))$.

and an ellipse E. The numerical range $W(A)$ is either an ellipse (if λ_0 lies inside E) or a "cone-like" figure otherwise; in the latter case A is reducible (but not normal).

In the next two cases the polynomial L_A (and therefore the matrix A) is irreducible.

Case 3. The degree of $C(A)$ (that is, the degree of its point equation) equals 4. Then $C(A)$ has a "double tangent," and the boundary of $W(A)$ contains one flat portion but no angular points.

Case 4. The degree of $C(A)$ equals 6. Then $C(A)$ consists of two parts, one inside another; an outer part (and therefore $W(A)$) has an "ovular" shape.

This classification is complete: the same article [3] contains examples of matrices A falling in each of the above-mentioned cases.

However, it does not provide a constructive procedure that would allow one to determine, for an arbitrary given nonnormal 3×3 matrix A, the shape of its numerical range. The main purpose of our article is to offer a series of tests, in terms of a matrix A itself or its canonical unitarily equivalent forms, to determine when $W(A)$ is an ellipse, a set with a flat portion on its boundary, or an ovular set. This is done in Sections 2 and 3. The results obtained simplify dramatically for matrices with one-point spectrum; this is discussed in Section 4. Section 5 is devoted to the question of when a 3×3 matrix A can be restored (of course, up to unitary equivalence) from its numerical range.

2. $W(A)$ IS AN ELLIPSE

We begin with the following general result concerning matrices of arbitrary size whose associated curve consists of ellipses and points.

THEOREM 2.1. *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and suppose that its associated curve $C(A)$ consists of k ellipses, with minor* *axes of lengths* $s_1, s_2, \dots, s_k$, *and* $n - 2k$ *points. Then*

$$\sum_{i=1}^{k} s_i^2 = \text{trace}(A^*A) - \sum_{i=1}^{n} |\lambda_i|^2. \tag{2.1}$$

For $n = 2$ the condition imposed in Theorem 2.1 is of no restriction since $C(A)$ always is an ellipse or a pair of points; formula (2.1) in this case takes the form (1.1). In the original version of this article we considered the case $n = 3$, in which, according to Kippenhahn's classification, the conditions of Theorem 2.1 are satisfied for $C(A)$ being a union of three points (Case 1) or an ellipse and a point (Case 2). Still there is not more than one ellipse, and (2.1) in this case takes the form

$$s = \left(\text{trace}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2 - |\lambda_3|^2\right)^{1/2}. \tag{2.2}$$

The generalization to the case $n > 3$ and its proof were suggested to us by the referee.

Proof. Relabel the eigenvalues of A in such a way that $\lambda_{2i-1}, \lambda_{2i}$ become the foci of the ith ellipse ($i = 1, \dots, k$), and $\lambda_{2k+1}, \dots, \lambda_n$—the remaining points of $C(A)$.

Along with A, consider the matrix

$$B = \begin{bmatrix} \lambda_1 & s_1 \\ 0 & \lambda_2 \end{bmatrix} \oplus \begin{bmatrix} \lambda_3 & s_2 \\ 0 & \lambda_4 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \lambda_{2k-1} & s_k \\ 0 & \lambda_{2k} \end{bmatrix} \oplus \text{diag}[\lambda_{2k+1}, \dots, \lambda_n].$$

Since $C(A) = C(B)$, the polynomials L_A and L_B have to be the same.

Compute now the coefficients of w^{n-2} of these polynomials. When doing that, due to unitarily invariance of L_A, we may without loss of generality suppose that $A = [a_{ij}]_{i,j=1}^n$ is in upper-triangular form. The coefficient of w^{n-2} in L_A equals the sum of all 2×2 principal minors of $uH + vK$, that is,[2]

$$\sum_{1 \leqslant i < j \leqslant n} \left[(u\Re a_{ii} + v\Im a_{ii})(u\Re a_{jj} + v\Im a_{jj}) - \frac{1}{4}(u^2 + v^2)|a_{ij}|^2 \right]$$

$$= \sum_{1 \leqslant i < j \leqslant n} \left[(u\Re \lambda_i + v\Im \lambda_i)(u\Re \lambda_j + v\Im \lambda_j) - \frac{1}{4}(u^2 + v^2)|a_{ij}|^2 \right].$$

[2] Here and in what follows, we denote by $\Re z$ and $\Im z$ the real and imaginary part, respectively, of a complex number z.

Applying this formula to B (which already is in upper-triangular form) we obtain

$$\sum_{1\leqslant i<j\leqslant n}\left[(u\Re\lambda_i+v\Im\lambda_i)(u\Re\lambda_j+v\Im\lambda_j)\right]-\frac{1}{4}(u^2+v^2)\sum_{i=1}^{k}|s_i|^2.$$

Since $L_A = L_B$, it follows from here that

$$\sum_{i=1}^{k}|s_i|^2=\sum_{1\leqslant i<j\leqslant n}|a_{ij}|^2=\sum_{i,j=1}^{n}|a_{ij}|^2-\sum_{i=1}^{n}|a_{ii}|^2=\operatorname{trace}(A^*A)-\sum_{i=1}^{n}|\lambda_i|^2.$$

■

Note that in the setting of Theorem 2.1 *all* the respective coefficients of L_A and L_B are equal. In particular, equating the coefficients of u^n, v^n yields

$$\begin{aligned}\det H&=\prod_{i=1}^{k}\left(\Re\lambda_{2i-1}\Re\lambda_{2i}-\frac{1}{4}s_i^2\right)\prod_{i=2k+1}^{n}\Re\lambda_i,\\ \det K&=\prod_{i=1}^{k}\left(\Im\lambda_{2i-1}\Im\lambda_{2i}-\frac{1}{4}s_i^2\right)\prod_{i=2k+1}^{n}\Im\lambda_i.\end{aligned}\tag{2.3}$$

If $n = 3$ and A is in upper-triangular form

$$A=\begin{bmatrix}a&x&y\\0&b&z\\0&0&c\end{bmatrix},\tag{2.4}$$

conditions (2.3) can be rewritten as $|x|^2\Re c+|y|^2\Re b+|z|^2\Re a-\Re(x\bar{y}z) = s^2\Re\lambda_3$, $|x|^2\Im c+|y|^2\Im b+|z|^2\Im a-\Im(x\bar{y}z)=s^2\Im\lambda_3$, or simply

$$|x|^2c+|y|^2b+|z|^2a-x\bar{y}z=s^2\lambda_3.\tag{2.5}$$

Due to (2.2),

$$s=\sqrt{|x|^2+|y|^2+|z|^2}.\tag{2.6}$$

Hence, conditions (2.5), (2.6) are necessary for matrices (2.4) and

$$B = \begin{bmatrix} \lambda_1 & s & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

to have the same associated curves. Surprisingly, a direct computation shows that they are also sufficient. Therefore, the following criterion holds.

THEOREM 2.2. *Let A be in upper-triangular form* (2.4). *Then its associated curve $C(A)$ consists of an ellipse (possibly degenerating to a disk) and a point if and only if*

1. $d = |x|^2 + |y|^2 + |z|^2 > 0$ *and*
2. *the number* $\lambda = (c|x|^2 + b|y|^2 + a|z|^2 - x\bar{y}z)/d$ *coincides with at least one of the eigenvalues* a, b, c.

If these conditions are satisfied, then $C(A)$ is the union of λ with the ellipse having its foci at two other eigenvalues of A and minor axis of length $s = \sqrt{d}$.

To obtain a unitary invariant form of Theorem 2.2, note that

$$c|x|^2 + b|y|^2 + a|z|^2 - x\bar{y}z = d \operatorname{trace} A - \operatorname{trace}(A^*A^2) + \sum_{j=1}^{3} |\lambda_j|^2\lambda_j.$$

Therefore, Theorem 2.2 admits the following reformulation.

THEOREM 2.3. *The associated curve $C(A)$ of a 3×3 matrix A consists of an ellipse and a point if and only if*

1. $d = \operatorname{trace}(A^*A) - \sum_{j=1}^{3}|\lambda_j|^2 > 0$ *and*
2. *the number* $\lambda = \operatorname{trace} A + (1/d)(\sum_{j=1}^{3}|\lambda_j|^2\lambda_j - \operatorname{trace}(A^*A^2))$ *coincides with at least one of the eigenvalues λ_j of A.*

If these conditions are satisfied, then $C(A)$ is the union of λ with the ellipse having its foci at two other eigenvalues of A and minor axis of length $s = \sqrt{d}$.

Now we are ready to formulate a necessary and sufficient condition for a 3×3 matrix A to have an ellipse as its numerical range.

THEOREM 2.4. *Let A be a 3×3 matrix A with the eigenvalues λ_j, $j = 1, 2, 3$. Then $W(A)$ is an ellipse if and only if conditions* 1, 2 *of Theorem* 2.2 (*or* 2.3) *hold and, in addition,*

3. $(|\lambda_1 - \lambda_3| + |\lambda_2 - \lambda_3|)^2 - |\lambda_1 - \lambda_2|^2 \leqslant d$, *where the eigenvalue coinciding with λ is labeled λ_3.*

Proof. Conditions 1, 2 are equivalent to $C(A)$ being a union of the ellipse E (with the foci λ_1, λ_2 and minor axis of length $\sqrt{d}$) and the point λ_3. Condition 3 means that λ_3 lies inside E. According to Kippenhahn's classification, this is the only case when $W(A)$ is an ellipse. ■

The results obtained allow us to describe all 3×3 matrices for which $W(A)$ is a disk.

COROLLARY 2.5. *$W(A)$ is a disk if and only if*

1. *A has a multiple eigenvalue μ (so that its eigenvalues equal μ, μ, and λ)*
2. $2\mu \operatorname{trace}(A^*A) = \operatorname{trace}(A^*A^2) + 2|\mu^2|\mu + (2\mu - \lambda)|\lambda|^2$, *and*
3. $4|\mu - \lambda|^2 + 2|\mu|^2 + |\lambda|^2 \leqslant \operatorname{trace}(A^*A)$.

For A in a triangular form (2.4), *conditions* 2 *and* 3 *may be substituted by*

2′. $x\bar{y}z = (\mu - \lambda)(\delta_{c,\mu}|x|^2 + \delta_{b,\mu}|y|^2 + \delta_{a,\mu}|z|^2)$, *where δ is a usual Kronecker symbol, and*

3′. $4|\mu - \lambda|^2 \leqslant |x|^2 + |y|^2 + |z|^2$.

If these conditions are satisfied, then $W(A)$ is centered at μ and has radius $\frac{1}{2}\sqrt{\operatorname{trace}(A^*A) - 2|\mu|^2 - |\lambda|^2}$ $(= \frac{1}{2}\sqrt{|x|^2 + |y|^2 + |z|^2}$ *in the case* (2.4)).

Proof. Indeed, $W(A)$ is a disk if and only if it is an ellipse (that is, conditions of Theorem 2.4 are satisfied) and, in addition, the foci of this ellipse coincide. This means that A has a multiple eigenvalue (say, μ), and its third eigenvalue coincides with λ defined by condition 2 of Theorem 2.2 or Theorem 2.3. The rest is computation. ■

For triangular matrices, this corollary was first proved directly by Chien and Tam, although in a very different manner [7]. Necessary and sufficient conditions for $W(A)$ to be the unit disk centered at 0 for a 3×3 matrix A were obtained earlier by N. K. Tsing (unpublished) and stated in [7]; these conditions appear as a particular case of Corollary 2.5, when we specialize $a = b = \mu = 0$, $|x|^2 + |y|^2 + |z|^2 = 4$.

3. $W(A)$ HAVING A FLAT PORTION ON ITS BOUNDARY

Throughout this section we assume that A is a 3×3 irreducible matrix represented as $A = H + iK$ with H and K Hermitian.

We begin by deriving a canonical form for an irreducible matrix with a flat portion on the boundary of its numerical range.

THEOREM 3.1. *Let A be a 3×3 irreducible matrix. Then after unitary similarity, translation, rotation, and scaling of A, A may be written in the form*

$$A = \begin{bmatrix} i & 0 & -c_1 \\ 0 & 0 & -c_2 \\ c_1 & c_2 & \zeta \end{bmatrix}, \tag{3.1}$$

where $c_1, c_2, \Re(\zeta)$ are positive, if and only if $W(A)$ has a flat portion on its boundary. In this form, $W(A)$ has a flat portion extending from 0 to i and is contained in the closed right half-plane.

Proof. Let $W(A)$ have a flat portion on its boundary. After rotation, shifting, and scaling (by scaling we mean multiplication by a positive number), we may assume that a flat portion stretches from 0 to i. Since $W(A)$ is convex, it must be contained entirely in the right or the left half-plane. Applying yet another rotation and translation, if necessary, we may assume that $W(A)$ is in the right half-plane.

Since 0 and i are in $W(A)$, there exist $x_1, x_2 \in \mathbb{C}^n$, $x_1^* x_1 = x_2^* x_2 = 1$ such that $x_1^* A x_1 = 0$, $x_2^* A x_2 = i$. Let $\mathscr{L} = \text{Span}\{x_1, x_2\}$. Since $\mathscr{L}$ is a 2-dimensional subspace, we may represent the linear transformation of A restricted to $\mathscr{L}$, $A|\mathscr{L} = A'$, by a 2×2 matrix. By choosing a proper basis for A, A' is the leading principal submatrix of A.

Now $W(A')$ is an ellipse, as are the numerical ranges of all 2×2 matrices. Since $W(A')$ is convex, $[0, i] \subseteq W(A')$. Also, $W(A') \subseteq W(A)$. Since $W(A)$ does not extend into the left half-plane, the only possible ellipse $W(A')$ can be is the degenerate ellipse $[0, i]$. This implies that A' is normal with eigenvalues 0 and i. So with proper basis

$$A' = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} i & 0 & \nu_1 \\ 0 & 0 & \nu_2 \\ c_1 & c_2 & \zeta \end{bmatrix}.$$

Since $W(A)$ is in the closed right half-plane, H is positive semidefinite. A calculation shows that

$$2H = \begin{bmatrix} 0 & 0 & \nu_1 + \overline{c_1} \\ 0 & 0 & \nu_2 + \overline{c_2} \\ \overline{\nu_1} + c_1 & \overline{\nu_2} & 2\Re(\zeta) \end{bmatrix},$$

and therefore $\Re(\zeta) \geqslant 0$, $\nu_1 = -\overline{c_1}$, $\nu_2 = -\overline{c_2}$. Due to the irreducibility of A, however, we know $\Re(\zeta) > 0$. By a diagonal unitary similarity, we may assume that c_1, c_2 are nonnegative. If c_1 or c_2 are 0, then A is reducible, so c_1 and c_2 are positive. We now have the form (3.1) of A we had hoped for.

Now suppose that after scaling, rotation, translation, and unitary similarity, A is in the form expressed in the theorem. Consider the principal submatrix A' from the first two rows and columns of A. $W(A')$ is a line segment from 0 to i. Clearly, $W(A') \subseteq W(A)$. But since H is positive semidefinite, $W(A)$ lies entirely in the right half-plane. So the line segment from 0 to i must be on the boundary of $W(A)$.

To see that the line segment does not go beyond 0 or i, note that any point y on that line must be pure imaginary. So if $x^*Ax = x^*Hx + ix^*Kx = y$, then $x^*Hx = 0$. Hence, $x \in \mathrm{Ker}(H) = \mathrm{Span}\{[1,0,0]^T, [0,1,0]^T\}$. If $\|x\| = 1$, then $x = \nu_1[1,0,0]^T + \nu_2[0,1,0]^T$ with $|\nu_1|^2 + |\nu_2|^2 = 1$, and $0 \leqslant x^*Kx = |\nu_1|^2 \leqslant 1$. ■

According to Kippenhahn's classification, $W(A)$ has a flat portion on the boundary if and only if there exists a line, $ux + vy + w = 0$, tangent to $C(A)$ at two distinct points. This double tangent line corresponds to an eigenvalue $-w$ of $uH + vK$, which has multiplicity 2; since u and v are real, $uH + vK$ is Hermitian and $uH + vK + wI$ has rank 1. Conversely, if $uH + vK + wI$ has rank 1, then $-w$ is an eigenvalue of $uH + vK$ with multiplicity 2, and we get a double tangent. Observe also that, if A is irreducible, $uH + vK$ cannot have an eigenvalue of multiplicity 3 (for then the Hermitian matrix $uH + vK$ would be scalar, H and K would commute, and hence A would be normal). We summarize:

PROPOSITION 3.2. *Let $A = H + iK$ be irreducible. Then the following statements are equivalent*:

1. *$W(A)$ has a flat portion on the boundary*;
2. $\mathrm{rank}(uH + vK + wI) = 1$ *for some real u, v, w*;
3. *for some real u, v not both equal to zero, $uH + vK$ has a multiple eigenvalue.*

Under these conditions, the flat portion of the boundary lies on the line $ux + vy + w = 0$.

It also follows from Kippenhahn's classification that an irreducible 3×3 matrix can have at most one flat portion on the boundary of its numerical range.

COROLLARY 3.3. *Let A be irreducible and unitarily similar to a real matrix. Then $W(A)$ has a flat portion on its boundary if and only if H has a multiple eigenvalue. If $W(A)$ does have a flat portion, it is parallel to the imaginary axis.*

Proof. If A is a real matrix, then $W(A)$ is symmetric about the real axis. So, the (unique) flat portion of the boundary of $W(A)$ must be a vertical line. According to Proposition 3.2, it happens if and only if the matrix $1 \cdot H + 0 \cdot K = H$ has a multiple eigenvalue. ■

if A is not unitarily equivalent to a real matrix, Proposition 3.2 may be difficult to use. We now present several statements, equivalent to Proposition 3.2 (and obtained from it). As we see in Sections 4 and 5, these statements are sometimes more suitable for application than Proposition 3.2 itself.

COROLLARY 3.4. *Let an irreducible matrix A be written in the form*

$$A = \begin{bmatrix} \xi & h_{12} & h_{13} \\ \overline{h_{12}} & \xi & h_{23} \\ \overline{h_{13}} & \overline{h_{23}} & \xi \end{bmatrix} + i \begin{bmatrix} \eta & k_{12} & k_{13} \\ \overline{k_{12}} & \eta & k_{23} \\ \overline{k_{13}} & \overline{k^{23}} & \eta \end{bmatrix}. \tag{3.2}$$

Then $W(A)$ has a flat portion on its boundary if and only if there exist real u,v not both zero such that

$$|uh_{13} + vk_{13}| = |uh_{23} + vk_{23}| = |uh_{12} + vk_{12}| \tag{3.3}$$

and

$$(uh_{12} + vk_{12})\,\overline{(uh_{13} + vk_{13})}\,(uh_{23} + vk_{23}) \text{ is real.} \tag{3.4}$$

Proof. Since for any $\zeta \in \mathbb{C}$ $W(A)$ has a flat portion on its boundary if and only if $W(A - \zeta I)$ has a flat portion on its boundary, we may (and do) assume that trace of A is zero: $\xi = \eta = 0$. The characteristic polynomial of $uH + vK$ then equals

$$\begin{aligned}\det(uH + vK - \lambda I)\\ &= -\lambda^3 + \left(|uh_{12} + vk_{12}|^2 + |uh_{23} + vk_{23}|^2 + |u\overline{h_{13}} + v\overline{k_{13}}|^2\right)\lambda\\ &\quad + 2\Re\left((uh_{12} + vk_{12})(uh_{23} + vk_{23})\left(u\overline{h_{13}} + v\overline{k_{13}}\right)\right). \qquad (3.5)\end{aligned}$$

According to Proposition 3.2, $W(A)$ has a flat portion on its boundary if and only if for some real u, v not both zero $uH + vK$ has a multiple eigenvalue; that is, the discriminant of (3.5) equals zero. For an arbitrary third-degree polynomial $a_0x^3 + a_1x^2 + a_2x + a_3$, the discriminant is $a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3$; see, e.g., [8]. Due to our assumption trace $A = 0$ the coefficient of λ^2 in (3.5) vanishes, and direct computations show that the discriminant of (3.5) equals four times the expression

$$\begin{aligned}&\left(|uh_{12} + vk_{12}|^2 + |uh_{23} + vk_{23}|^2 + |u\overline{h_{13}} + v\overline{k_{13}}|^2\right)^3\\ &\quad - 27\left(\Re\left((uh_{12} + vk_{12})(uh_{23} + vk_{23})\left(u\overline{h_{13}} + v\overline{k_{13}}\right)\right)\right)^2. \qquad (3.6)\end{aligned}$$

Setting

$$x = uh_{12} + vk_{12},\ y = uh_{13} + vk_{13} \quad \text{and} \quad z = uh_{23} + vk_{23}, \qquad (3.7)$$

we see that (3.6) equals zero if and only if

$$\frac{|x|^2 + |y|^2 + |z|^2}{3} = \sqrt[3]{(\Re(x\bar{y}z))^2}. \qquad (3.8)$$

Since

$$\frac{|x|^2 + |y|^2 + |z|^2}{3} \geqslant \sqrt[3]{|x|^2|y|^2|z|^2} \geqslant \sqrt[3]{(\Re(x\bar{y}z))^2}, \qquad (3.9)$$

the equality (3.8) holds exactly when both inequalities in (3.9) are actually equalities, that is, when $|x| = |y| = |z|$ (for the first inequality in (3.9)) and $x\bar{y}z \in \mathbb{R}$ (for the second). The two conditions obtained are exactly the same as (3.3) and (3.4), respectively. ■

Note that condition (3.2) is of no restriction, since it can always be obtained by a unitary similarity. To rewrite Corollary 3.4 in a unitarily equivalent form, put

$$B = \begin{bmatrix} 0 & x & y \\ \bar{x} & 0 & z \\ \bar{y} & \bar{z} & 0 \end{bmatrix} = uH + vK - \frac{1}{3}(\operatorname{trace}(uH + vK))I.$$

Then, in notation (3.7), $|x|^2 + |y|^2 + |z|^2 = \frac{1}{2}\operatorname{trace} B^2$ and $\Re(x\bar{y}z) = \frac{1}{6}\operatorname{trace} B^3$, so that condition (3.8) is equivalent to

$$\left[\operatorname{trace}(B^3)\right]^2 = \left[\operatorname{trace}(B^2)\right]^3/6. \tag{3.10}$$

COROLLARY 3.5. *Let $A = H + iK$ be an irreducible matrix. Then $W(A)$ has a flat portion on its boundary if and only if there exist real u, v not both zero so that for $B = uH + vK - \frac{1}{3}(\operatorname{trace}(uH + vK))I$ the equality* (3.10) *holds.*

We see in Section 4 that Corollary 3.4 sometimes leads to explicit results in spite of the fact that it refers to the existence of u, v without showing how to construct them. The criterion not using u, v at all is given by the next corollary. It is applicable to matrices $A = H + iK$ with a diagonalized summand K.

COROLLARY 3.6. *Let A be an irreducible matrix written in the form*

$$A = \begin{bmatrix} h_1 & h_{12} & h_{13} \\ \overline{h_{12}} & h_2 & h_{23} \\ \overline{h_{13}} & \overline{h_{23}} & h_3 \end{bmatrix} + i \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

$W(A)$ has a flat portion if and only if K has a multiple eigenvalue or

$$h_1(k_2 - k_3) + h_2(k_3 - k_1) + h_3(k_1 - k_2)$$
$$= (k_2 - k_3)\frac{h_{12}\overline{h_{13}}}{\overline{h_{23}}} + (k_3 - k_1)\frac{h_{12}h_{23}}{h_{13}} + (k_1 - k_2)\frac{\overline{h_{13}}h_{23}}{\overline{h_{12}}} \tag{3.11}$$

and

$$h_{12}\overline{h_{13}}h_{23} \text{ is real}. \tag{3.12}$$

Proof. According to Proposition 3.2, $W(A)$ has a flat portion on its boundary if and only if $B = uH + vK + wI$ has rank 1 for some real u, v, w. If K has a multiple eigenvalue λ, then the latter condition is satisfied with $u = 0$, $v = 1$, $w = -\lambda$. Therefore, we need to consider only the case of K having distinct eigenvalues.

Necessity. Let rank $B = 1$. Since the eigenvalues of K are all distinct, it is possible only when u is nonzero. Without loss of generality we may (and will) assume that $u = 1$.

To simplify further calculations, rewrite B in the form

$$B = \begin{bmatrix} h_1' + vk_1' + w' & h_{12} & h_{13} \\ \overline{h_{12}} & h_2' + vk_2' + w' & h_{23} \\ \overline{h_{13}} & \overline{h_{23}} & w' \end{bmatrix},$$

where $w' = w + h_3 + vk_3$, $h_i' = h_i - h_3$, $k_i' = k_i - k_3$ $(i = 1, 2)$.

Then all the off-diagonal elements h_{12}, h_{23}, h_{13} are nonzero (otherwise at least two of them, located in the same row or column, equal zero, which would imply reducibility of A), and

$$w'/\overline{h_{23}} = h_{13}/h_{12} = h_{23}/(h_2' + vk_2' + w'), \tag{3.13}$$

$$w'/\overline{h_{13}} = h_{23}/\overline{h_{12}} = h_{13}/(h_1' + vk_1' + w'). \tag{3.14}$$

Solving (3.13) with respect to v, w' we find that

$$w' = \frac{h_{13}\overline{h_{23}}}{h_{12}}, \qquad v = \frac{1}{k_2'}\left(\frac{h_{12}h_{23}}{h_{13}} - \frac{h_{13}\overline{h_{23}}}{h_{12}} - h_2'\right), \tag{3.15}$$

For v, w' defined by (3.15) the equalities (3.14) yield, respectively, (3.12) and

$$k_1'\left(h_2' + \frac{h_{13}\overline{h_{23}}}{h_{12}} - \frac{h_{12}h_{23}}{h_{13}}\right) = k_2'\left(h_1' + \frac{h_{13}\overline{h_{23}}}{h_{12}} - \frac{h_{13}\overline{h_{12}}}{h_{23}}\right). \tag{3.16}$$

It is easily checked that under the restriction (3.12) the latter is equivalent to (3.11).

Sufficiency. From (3.11) it follows, in particular, that $h_{12}h_{23}h_{13} \neq 0$. Define v, w' by (3.15). Then, of course, (3.13) holds. Moreover, due to (3.12), v, w' are real, and

$$w'/\overline{h_{13}} = h_{13}\overline{h_{23}}/(h_{12}\overline{h_{13}})$$

$$= \frac{h_{23}\overline{h_{12}}h_{13}\overline{h_{23}}}{h_{23}\overline{h_{12}}h_{12}\overline{h_{13}}} = \frac{\overline{h_{12}}h_{13}\overline{h_{23}}}{h_{12}\overline{h_{13}}h_{23}} \cdot h_{23}/\overline{h_{12}} = h_{23}/\overline{h_{12}}\,.$$

Therefore, the first of equalities (3.14) also holds.

Finally, (3.11) and (3.12) imply (3.16), which, in turn, leads to the second of the equalities (3.14). Due to (3.13), (3.14), B is a (nonzero) matrix with collinear columns and therefore has rank 1. ■

Note that condition (3.11) may be written as

$$h_1(k_2 - k_3) + h_2(k_3 - k_1) + h_3(k_1 - k_2) = \mu h_{12}\overline{h_{13}}h_{23},$$

where

$$\mu = \frac{k_2 - k_3}{|h_{23}|^2} + \frac{k_3 - k_1}{|h_{13}|^2} + \frac{k_1 - k_2}{|h_{12}|^2}\,.$$

Since $h_1(k_2 - k_3) + h_2(k_3 - k_1) + h_3(k_1 - k_2)$ and μ are both real, it means that condition (3.12) follows from (3.11) if μ is nonzero.

The above corollary also works with H and not K diagonal. To see this, multiply A by i. This makes $H' = -K$ and $K' = H$. Clearly $W(iA)$ has a flat portion on its boundary if and only if $W(A)$ has a flat portion.

4. W(A) FOR MATRICES WITH A TRIPLE EIGENVALUE

In this section we apply our results to the special case of matrices with a triple eigenvalue. In their triangular form (2.4), of course, all diagonal elements coincide:

$$A = \begin{bmatrix} p & x & y \\ 0 & p & z \\ 0 & 0 & p \end{bmatrix}, \tag{4.1}$$

with p, x, y, z complex. Note that $W(A)$ cannot be a noncircular ellipse since such an ellipse requires two distinct foci (eigenvalues of A) of the associated curve.

THEOREM 4.1. *Let A be in the form* (4.1). *Then*:

(1) *$W(A)$ is a disk if and only if $xyz = 0$; in this case the disk has radius $\frac{1}{2}\sqrt{|x|^2 + |y|^2 + |z|^2}$ with center p.*

(2) *$W(A)$ has a flat portion on its boundary if and only if $|x| = |y| = |z| > 0$; in this case $C(A)$ is a cardioid.*

(3) *$W(A)$ is of the ovular shape if and only if $xyz \neq 0$ and $|x|, |y|, |z|$ are not all equal.*

A version of Part 1 of this theorem for the nilpotent case was first shown by Marcus and Pesce, who also developed a unitarily invariant form of this condition [9].

Proof. Part 1 follows easily from Corollary 2.5. In the rest of the proof we may therefore suppose that $xyz \neq 0$, so that A is irreducible.

Part 2: To simplify further calculations, consider the matrix $2A$ instead of A:

$$2A = \begin{bmatrix} 2\Re(p) & x & y \\ \bar{x} & 2\Re(p) & z \\ \bar{y} & \bar{z} & 2\Re(p) \end{bmatrix} + i\begin{bmatrix} 2\Im(p) & -ix & -iy \\ i\bar{x} & 2\Im(p) & -iz \\ i\bar{y} & i\bar{z} & 2\Im(p) \end{bmatrix}.$$

By Corollary 3.4, $W(A)$ has a flat portion if and only if there exist real u, v not both zero such that

$$|ux + v(-ix)| = |uy + v(-iy)| = |uz + v(-iz)|,$$

$$\arg(ux - ivx) + \arg(uz - ivz) = \arg(uy - ivy).$$

From the first equation we see that we must have

$$|u - iv||x| = |u - iv||y| = |u - iv||z|,$$

which implies that we must have $|x| = |y| = |z|$ since $u - iv \neq 0$. The second equation becomes

$$\arg(u - iv) = \arg(y) - \arg(x) - \arg(z).$$

We can easily choose u, v so that this is true. And so the only condition we have is $|x| = |y| = |z|$.

We now prove that under this condition $C(A)$ is a cardioid. Using unitary transformations $A \mapsto U^*AU$ with $U = \operatorname{diag}[e^{i\nu_1}, e^{i\nu_2}, e^{i\nu_3}]$ (which do not change $C(A)$) and multiplying A by scalars (which rotate and dilate $C(A)$) we may reduce the general case to $x = y = z = 1$. Shifting then A by λI (which shifts $C(A)$ by λ), we may suppose also that its eigenvalue is $\frac{1}{3}$ (such a choice of the eigenvalue ensures that the cusp of the cardioid would be at the origin). In other words, without loss of generality

$$A = \begin{bmatrix} \frac{1}{3} & 1 & 1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Using Fiedler's formula (see [6]) for the point equation of $C(A)$ and transforming to polar coordinates we find

$$3r^2(-3r - 2 + 2\cos\theta)(-3r + 2 + 2\cos\theta) = 0.$$

The factor of $3r^2$ is redundant since $r = 0$ is a solution to the other two factors. The other two factors actually define the same curve. This is because if one replaces r with $-r$ and θ with $\theta + \pi$, the factors are identical within a scalar multiple. In polar coordinates, this means that the factors trace the same curve. The equation therefore simplifies to $r = \frac{2}{3}(1 - \cos\theta)$, which is the equation of a cardioid [10].

Part 3: Since $W(A)$ cannot be an ellipse without being a disk, the ovular shape is the only case left. ■

A computer image of $W(A)$, where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

was given in [9]. By Theorem 4.1, $W(A)$ is the convex hull of a cardioid.

As in Sections 2 and 3, the unitarily invariant version exists:

THEOREM 4.2. *Let A be a matrix with triple eigenvalue p, $\Psi = \operatorname{trace}(A^*A) - 3|p|^2$, and $\Omega = \operatorname{trace}(A^*A^2) - 2p(\operatorname{trace}(A^*A)) + 3p|p|^2$. Then*

1. *$W(A)$ is a disk centered at p if and only if $\Omega = 0$. In this case it has radius $\frac{1}{2}\sqrt{\Psi}$.*

2. *$W(A)$ has a flat portion on its boundary if and only if $\Psi = 3\sqrt[3]{|\Omega|^2} > 0$.*

3. *$W(A)$ has an ovular shape if and only if $\Psi \neq 3\sqrt[3]{|\Omega|^2} > 0$.*

Proof. It suffices to consider A in the form (4.1). The direct computation then shows that $\Psi = |x|^2 + |y|^2 + |z|^2$ and $\Omega = x\bar{y}z$. Obviously, condition 1 is equivalent to $xyz = 0$. Due to the case of equality between arithmetic and geometric means of $|x|^2, |y|^2, |z|^2$, 2 holds if and only if $|x| = |y| = |z| > 0$. Finally, 3 is the only logically possible case left. ■

5. RESTORATION OF A FROM $W(A)$

An inverse problem concerning numerical ranges may be formulated. Given a numerical range $W(A)$ for some A, can one reconstruct A? Since $W(A) = W(U^*AU)$, we cannot restore A uniquely (with the exception of $W(A)$ being a single point), but we can sometimes find a unique unitary equivalence class that generates $W(A)$. The latter is always the case for 2×2 matrices (see, e.g., [1]).

In the remainder of this section, we deal with 3×3 matrices. For reducible A, it can be easily seen that in this case A cannot always be restored from $W(A)$, but it can be restored from $W(A)$ and the trace of A, or equivalently $C(A)$. We show later that in the irreducible case, A cannot always be restored, even if $C(A)$ is known.

Unexpectedly, there is a case of a $W(A)$ arising from an irreducible matrix, which allows A to be restored up to unitary similarity:

THEOREM 5.1. *Let $W(A)$ be a 2-dimensional shape with only one flat portion on its boundary. Then A is an irreducible matrix, which can be restored up to unitary similarity.*

Proof. Having only one flat portion on the boundary of its numerical range, A belongs to Case 4 of Kippenhahn's classification and is therefore irreducible. After scaling, rotation, and shifting of $W(A)$, we can have the flat portion as the line segment $[0, i]$ and $W(A)$ lies entirely in the right half-plane. We restore A in this case. After the restoration, one can obtain the original A by reversing the scaling, rotation, and shifting.

According to Theorem 3.1, A must be unitarily similar to (3.1). Let us assume A is in that form. The real part H of A is then $\operatorname{diag}(0, 0, \Re(\zeta))$, with $\Re(\zeta)$ positive. Since $W(\Re(A))$ is the projection of $W(A)$ onto the real axis, which is a line segment from 0 to $\Re(\zeta)$, we can determine $\Re(\zeta)$.

Since there is only one flat portion, the real part of any point on that portion is 0. So ζ is not on that portion. Because $\Re(\zeta)$ is an endpoint of the projection of $W(A)$ onto the real axis and ζ is not on the flat portion, ζ is uniquely determined as the point on the boundary of $W(A)$ having a maximum real part, namely $\Re(\zeta)$. So $\Im(\zeta)$ is also determined.

The imaginary part K of A is

$$\begin{bmatrix} 1 & 0 & ic_1 \\ 0 & 0 & ic_2 \\ -ic_1 & -ic_2 & \Im(\zeta) \end{bmatrix}.$$

Since $W(K)$ is a line segment, which is a projection of $W(A)$ onto the imaginary axis, we know two of the eigenvalues λ_1, λ_2 of K, namely the endpoints of the line segment. Calculating the characteristic polynomial of K and substituting in λ_1, λ_2 give us the system of linear equations in c_1^2, c_2^2

$$\begin{aligned} c_1^2(-\lambda_1) + c_2^2(-\lambda_1 + 1) &= -\lambda_1^3 + \lambda_1^2(1 + \Im(\zeta)) - \lambda_1(\Im(\zeta)) \\ c_1^2(-\lambda_2) + c_2^2(-\lambda_2 + 1) &= -\lambda_2^3 + \lambda_2^2(1 + \Im(\zeta)) - \lambda_2(\Im(\zeta)). \end{aligned} \tag{5.1}$$

The determinant of this system is $\lambda_2 - \lambda_1$, which is nonzero since the flat portion is of nonzero length, causing the projection of $W(A)$ onto the imaginary axis to be of nonzero length. So the system (5.1) has a unique solution:

$$c_1^2 = -(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 + \lambda_2 - \Im(\zeta))$$

$$c_2^2 = \lambda_1\lambda_2(\lambda_1 + \lambda_2 - 1 - \Im(\zeta)).$$

Since c_1, c_2 are positive, we thus have unique values for them. Therefore we know all the elements of A in this canonical form, which determines A up to unitary similarity. ■

As it turns out, in cases of other shapes of $W(A)$ for an irreducible A, the matrix A cannot be uniquely (up to unitary similarity) restored by $W(A)$. We summarize all these cases, as well as the cases of a reducible A, in the following theorem.

THEOREM 5.2. *A 3×3 matrix A can be restored (up to unitary similarity) from $W(A)$ if and only if $W(A)$ is one of the following*: (1) *a point*, (2) *a triangle*, (3) *the convex hull of an ellipse and a point outside the ellipse*, (4) *a 2-dimensional shape with only one flat portion on its boundary.*

In the other cases, that of (5) *a line segment*, (6) *an ellipse, and* (7) *an ovular shape, the matrix cannot be restored. In the cases* 5–7 *there is a continuum of nonunitarily equivalent matrices whose numerical range is $W(A)$.*

Proof. Cases 1–3 are well known; case 4 was discussed in Theorem 5.1.

In case 5 A is normal, with at least two distinct eigenvalues and all three eigenvalues collinear. The eigenvalues corresponding to the endpoints can be determined, but the third eigenvalue cannot. There is a continuum choice for this third eigenvalue.

If $W(A)$ is an ellipse and A is reducible, A cannot be restored since the point defined by its 1×1 block may be anywhere within the ellipse defined by the 2×2 block. Again, there is a continuum of choices for the 1×1 block.

The proof in the remaining two situations ($W(A)$ is an ellipse produced by an irreducible A or an ovular shape) is based on a series of lemmas and is therefore relegated to the end of this section. ■

One might ask whether a matrix A can be restored (up to unitary similarity) from $W(A)$ and the trace of A. In this respect we note that for a 3×3 matrix A each of the following pieces of information completely determines two others: (1) $W(A)$ and the trace of A; (2) $W(A)$ and the eigenvalues of A; (3) $C(A)$.

Indeed, $C(A)$ determines uniquely $W(A)$ (because $W(A)$ is the convex hull of $C(A)$ and the eigenvalues of A (because there are the foci of $C(A)$). On the other hand, if $W(A)$ is known then the maximal and minimal eigenvalues of every linear combination $H\cos\xi + K\sin\xi$ (here $A = H + iK$ with Hermitian H and K and ξ is a real number) are determined by using the orthogonal projection of $W(e^{-i\xi}A)$ onto the real axis; note that $H\cos\xi + K\sin\xi$ is the real part of $e^{-i\xi}A$. If, in addition, the trace of A is known, then all eigenvalues of $H\cos\xi + K\sin\xi$ are known, and therefore the polynomial $\det(uH + vK + wI)$ is known, which determines $C(A)$.

It will be clear from the proof of Theorem 5.2 that, in addition to the cases when a 3×3 matrix A can be restored from $W(A)$, such a matrix can be restored from $C(A)$ (or equivalently from $W(A)$ and the trace of A) if $W(A)$ is a line segment. On the contrary, if $W(A)$ is ovular or $W(A)$ is an ellipse (without any information concerning the reducibility of A), then there are uncountably many unitarily inequivalent matrices B such that $C(B) = C(A)$. However, if $W(A)$ is an ellipse and it is known that A is reducible, then A can be restored (up to unitary similarity) from $C(A)$.

The rest of this section is devoted to completion of the proof of Theorem 5.2.

We use the two matrices

$$A = H + iK = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} + i\begin{bmatrix} \beta_1 & e & f \\ e & \beta_2 & g \\ f & \bar{g} & \beta_3 \end{bmatrix}, \tag{5.2}$$

$$B = H' + iK' = \begin{bmatrix} \alpha_1' & 0 & 0 \\ 0 & \alpha_2' & 0 \\ 0 & 0 & \alpha_3' \end{bmatrix} + i\begin{bmatrix} \beta_1' & e' & f' \\ e' & \beta_2' & g' \\ f' & \overline{g'} & \beta_3' \end{bmatrix}, \tag{5.3}$$

with $\alpha_1 > \alpha_2 > \alpha_3$, $\alpha_1' > \alpha_2' > \alpha_3'$, β_i, β_i' real, and off-diagonal elements such that

$$e, f, e', f' \geqslant 0; \qquad \text{if } ef = 0, \text{ then } g \geqslant 0; \qquad \text{if } e'f' = 0, \text{ then } g' \geqslant 0. \tag{5.4}$$

LEMMA 5.3. *Let A and B be written in the form* (5.2) *and* (5.3). *Then* $L_A = L_B$ *if and only if*

1. *All the diagonal elements are equal*: $\alpha_j = \alpha_j'$, $\beta_j = \beta_j'$ *for* $j = 1, 2, 3$.
2. $e' = \sqrt{e^2 + ((\alpha_1 - \alpha_2)/(\alpha_2 - \alpha_3))(|g'|^2 - |g|^2)}$
3. $f' = \sqrt{f^2 - ((\alpha_1 - \alpha_3)/(\alpha_2 - \alpha_3))(|g'|^2 - |g|^2)}$
4. $e'f'(g' + \overline{g'}) = ef(g + \bar{g}) + (|g|^2 - |g'|^2)((\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2))/(\alpha_2 - \alpha_3))$

Proof. For L_A and L_B to coincide it is necessary, in particular, that A and B have the same sets of eigenvalues. Then their traces also are the same. Without loss of generality we may assume that they equal zero. Assuming that, redenote $\alpha_1 = a$, $\alpha_2 = b$, $\beta_1 = c$ and $\beta_2 = d$; then, of course, $\alpha_3 = -a - b$, $\beta_3 = -c - d$. Analogously, the diagonal elements of H' and K' are now $a', b', -a' - b'$ and $c', d', -c' - d'$, respectively.

From the equality $L_A(u, v, w) = L_B(u, v, w)$ for $u = 1$, $v = 0$ it follows that H and H' have the same eigenvalues. Since in both matrices the eigenvalues are ordered, it means that $H' = H$ and thus $a' = a$, $b' = b$.

Calculation shows that $\det(uH + vK + wI) =$

$$\begin{aligned}
&(ab^2 - a^2b)u^3 + (-2abd - a^2d - 2abc - cb^2)u^2v \\
&\quad + (-a^2 - ab - b^2)u^2w \\
&\quad + \left(-2bcd + a\left(e^2 - |g|^2\right) - bc^2 - 2acd - ad^2 + b\left(e^2 - f^2\right)\right)uv^2 \\
&\quad + (c(-b - 2a) + d(-a - 2b))uvw \\
&\quad + \left(c\left(e^2 - |g|^2\right) - c^2d + ef(g + \bar{g}) + d(e^2 - f^2) - cd^2\right)v^3 \\
&\quad + \left(-d^2 - cd - f^2 - c^2 - |g|^2 - e^2\right)v^2w + w^3.
\end{aligned}$$

Comparing this to the equation for $\det(uH' + vK' + wI)$ we see from the coefficients of u^2v and uvw that we must have

$$(-2ab - b^2)c' + (-2ab - a^2)d' = (-2ab - b^2)c + (-2ab - a^2)d$$

$$(-2a - b)c' + (-a - 2b)d' = (-2a - b)c + (-a - 2b)d.$$

Considering this as a linear system of equations in c', d' of the form $Cx = y$, $\det(C) = (-2ab - b^2)(-a - 2b) - (-2ab - a^2)(-2a - b) = (-a - 2b)(a - b)(2a + b) \neq 0$ by our assumption that $a > b > -a - b$. The system therefore has a unique solution, which is obviously $c' = c$, $d' = d$.

We can now conclude that all the diagonal elements of A and B are the same.

From the coefficients of v^2w and v^2u we have, after elimination of identical terms,

$$e'^2 + f'^2 = e^2 + f^2 + |g|^2 - |g'|^2$$

$$(a + b)e'^2 - bf'^2 = (a + b)e^2 - bf^2 - a(|g|^2 - |g'|^2),$$

which can be viewed as a linear system of equations in e'^2, f'^2. By solving the system and using our assumption that e', f' are nonnegative, we obtain

$$e' = \sqrt{e^2 + \frac{a - b}{a + 2b}(|g'|^2 - |g|^2)}$$

$$f' = \sqrt{f^2 - \frac{2a + b}{a + 2b}(|g'|^2 - |g|^2)}\,.$$

Finally from the coefficient of v^3 we have

$$\begin{aligned} e'f'(g' + \overline{g'}) &= ef(g + \bar{g}) + c(|g'|^2 - |g|^2) \\ &\quad + (c + d)(e^2 - e'^2) - d(f' - f'^2) \\ &= ef(g + \bar{g}) + (|g'|^2 - |g|^2) \\ &\quad \times \left[c - (c + d)\frac{a - b}{a + 2b} - d\frac{2a + b}{a + 2b}\right] \\ &= ef(g + \bar{g}) + (|g'|^2 - |g|^2)\frac{3(bc - ad)}{a + 2b}. \end{aligned}$$

Substituting our definitions of a, b, c, d into the above equations gives us the equations stated in the lemma. ∎

LEMMA 5.4. *Let A and B be in the forms* (5.2) *and* (5.3). *Then A is unitarily similar to B if and only if $A = B$.*

Proof. As in the proof of the preceding lemma, we assume that A and B have zero trace. Suppose $A = U^*BU$ for some unitary U. Then obviously $C(A) = C(B)$, and therefore $H = H'$ by Lemma 5.3. Now $H = U^*HU$, which implies that U must be diagonal. Using the condition (5.4), the equality $K' = U^*KU$ implies $K' = K$. ■

LEMMA 5.5. *Let A be an irreducible matrix in the form* (5.2). *If g is not real, or condition*

$$efg(\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2)) = (\alpha_2 - \alpha_3)ef + (\alpha_3 - \alpha_1)eg + (\alpha_1 - \alpha_2)fg \quad (5.5)$$

is not satisfied, there exists a continuum of unitary equivalences classes of matrices with the same associated curve $C(A)$.

Proof. We construct matrices B of the form (5.3) with $C(A) = C(B)$. Let $\rho = |g'|^2 - |g|^2$,

$$q = -\frac{\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2)}{\alpha_2 - \alpha_3}$$

$$\mu = \frac{\alpha_1 - \alpha_2}{\alpha_2 - \alpha_3}, \qquad \tau = \frac{\alpha_1 - \alpha_3}{\alpha_2 - \alpha_3}. \quad (5.6)$$

Note that $\mu, \tau > 0$.

Then by Lemma 5.3, for $C(B) = C(A)$ we must have

$$e' = \sqrt{e^2 + \mu\rho} \quad (5.7)$$

$$f' = \sqrt{f^2 - \tau\rho} \quad (5.8)$$

$$e'f'(g' + \overline{g'}) = ef(g + \bar{g}) + \rho q. \quad (5.9)$$

To satisfy (5.7), (5.8), let us choose $\rho \in I = (-e^2/\mu, f^2/\tau)$. Note that the length of I is positive, because otherwise $f = e = 0$, and A would be reducible. The last equation (5.9) is then equivalent to

$$\Re g' = \pm \frac{2df \Re g + \rho q}{2\sqrt{(e^2 + \mu\rho)(f^2 - \tau\rho)}}. \tag{5.10}$$

For number $g' \in \mathbb{C}$ with $|g'|^2 = \rho + |g|^2$ and $\Re g'$ given by (5.10) to exist, it is necessary and sufficient that

$$\frac{(2ef \Re g + \rho q)^2}{4(e^2 + \mu\rho)(f^2 - \tau\rho)} \leqslant \rho + |g|^2.$$

The latter inequality can be rewritten as $\mathscr{F}(\rho) \geqslant 0$, if we denote

$$\mathscr{F}(\rho) = (e^2 + \mu\rho)(f^2 - \tau\rho)(4\rho + 4|g|^2) - (2ef\Re g + \rho q)^2.$$

It g is not real, then $\mathscr{F}(0) > 0$, so that there is an $\epsilon > 0$ such that $\mathscr{F}(\rho) \geqslant 0$ for $|\rho| < \epsilon$. Every $\rho \in I \cap (-\epsilon, \epsilon)$ generates a matrix (5.3) with $C(B) = C(A)$. Different values of ρ correspond to different matrices B, and none of them are unitarily similar due to Lemma 5.4.

If g is real, then $\mathscr{F}(0) = 0$, and

$$\left.\frac{d\mathscr{F}}{d\rho}\right|_{\rho=0} = 4\mu f^2 g^2 - 4\tau e^2 g^2 + 4e^2 f^2 - 4efgq.$$

Substituting in the values of μ, τ, and q from (5.6) we see that (5.5) is equivalent to $d\mathscr{F}/d\rho|_{\rho=0} = 0$. Hence, if (5.5) does not hold, there is a one-sided neighborhood N of zero such that $\mathscr{F}(\rho) \geqslant 0$ for $\rho \in N$. Observe that this neighborhood is positive if $e = 0$ and negative if $f = 0$, so that in any case $N \cap I$ is a continuum. All $\rho \in N \cap I$ generate matrices with the same associated curve as $C(A)$, and, as above, all these matrices belong to different unitarily equivalence classes. ■

We now complete the proof of Theorem 5.2. Consider an irreducible matrix A. Without loss of generality we may suppose that it is in the form (5.2). Corollary 3.6 implies that $W(A)$ contains a flat portion on its boundary if and only if g is real and (5.5) holds. From here and Lemma 5.5 it follows that in all other cases (that is, when $W(A)$ is an ellipse or has an ovular shape) there is a continuum of unitary equivalence classes of matrices with the same numerical range $W(A)$ (and even the same associated curve $C(A)$). ■

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