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On triangularizability of the commutant of a single matrix

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Abstract

The main purpose of this paper is to characterize triangularizable matrices $A \in M_n(F)$ whose commutants are triangularizable, where F is an arbitrary field. More precisely, we show that the commutant of a triangularizable matrix $A \in M_n(F)$ is triangularizable if and only if for any eigenvalue λ of A, the corresponding Jordan blocks in the Jordan canonical form of A have distinct sizes.

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1. Introduction

A collection $\mathscr{A}$ of linear operators on a vector space X is called triangularizable if there exists a maximal nest $\{\mathscr{M}_\alpha\}_{\alpha\in\Lambda}$ of subspaces of X each of which is an invariant subspace of $\mathscr{A}$. In case $\mathscr{A}$ is a subcollection of bounded linear operators on a (real or complex) Banach space X, we further assume $\mathscr{M}_\alpha$ be closed for every $\alpha \in \Lambda$.

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Triangularizability of collections of matrices has been of interest to several authors some of whom have focused on the triangularizability of the commutants of certain collections.(For recent survey we may refer to [5]). Turovskii [7] showed that if $\mathscr{M}$ and $\mathscr{N}$ are triangularizable sets of compact operators on a complex Banach space X such that $\mathscr{N} \subset \mathscr{M}'$, then $\mathscr{M} \cup \mathscr{N}$ is triangularizable. (Here, $\mathscr{M}'$ denotes the commutant of $\mathscr{M}$ defined as the set of all operators commuting with every operator in $\mathscr{M}$.) This result is extended by Yahaghi [9, Corollary 2.2.12] to the case $\mathscr{M}$ and $\mathscr{N}$ are triangularizable subsets of $M_n(F)$ for a general field F. In particular, if $\mathscr{M} = \mathscr{N}'$ but no triangularizability condition is assumed on $\mathscr{M}$, Yahaghi [8] shows that the unicellularity of some $A \in \mathscr{N}$ implies the triangularizability of $\mathscr{N} \cup \mathscr{N}'$. (By a unicellular operator we mean one with a unique triangularizing chain of invariant subspaces.)

In the next section of the present paper, we show that if F is an arbitrary field and $A \in M_n(F)$ is triangularizable, then $\{A\}'$ is triangularizable if and only if the Jordan blocks corresponding to any given eigenvalue of A in its Jordan canonical form have distinct sizes. This, in conjunction with Yahaghi's extension of Turovskii's result, extends the aforemention result of Yahaghi given in [8].

Recall that $M_n(F)$ denotes the algebra of all $n \times n$ matrices with entries in a general field F. For $\lambda \in F$, the $k \times k$ Jordan block with eigenvalue λ is denoted by $J_k(\lambda)$. If W is a subspace of F^n and if a matrix $A \in M_n(F)$ leaves W invariant, then $A|_W$ will denote the restriction of A to W as an operator. The spectrum and the range of A are denoted by $\sigma(A)$ and $\mathrm{rang}(A)$, respectively.

We conclude this section with the following known lemma needed in the proof of our main results.

Lemma 1.1. *Let $A \in M_n(F)$ have the minimal polynomial $p = p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$, where p_i's are distinct irreducible polynomials in $F[X]$. Let $W_i = Ker(p_i^{l_i}(A))$ for $1 \leqslant i \leqslant k$, where A is considered as a linear transformation on F^n. Then,*

$$\{A\}' = \{A\ |_{W_1}\}' \bigoplus \{A\ |_{W_2}\}' \bigoplus \cdots \bigoplus \{A\ |_{W_k}\}'.$$

Moreover, $\{A\}'$ is triangularizable if and only if each summand on the right hand side of the above direct sum is triangularizable.

The proof of the first part is a direct consequence of the primary decomposition theorem and the fact that zero is the only solution of an equation of the form $AX = XB$ when the known matrices A and B have relatively prime minimal polynomials.(Or the fact that the invariant subspaces arising from the primary decomposition theorem are in fact hyperinvariant subspaces.)

The proof of the second part of the lemma follows from Guralnick [1] or the well-known fact that every chain of the invariant subspaces of a triangularizable collection of matrices can be imbedded into a triangularizing chain.

2. On the commutant of a single matrix

The main result of this section is a triangularizability result for the commutant of a given matrix $A \in M_n(F)$. The result will be then applied to extend a result due to Yahaghi [8].

The following theorem gives a necessary and sufficient condition for the commutant of a matrix to be triangularizable. The proof reveals the structure of such commutants which is summarized in its first corollary. The explicit form of the most general matrix commuting with a Jordan block is already given in [3]. Here we need to investigate the general form of a matrix commuting with a direct sum of Jordan blocks. (See also [6], p. 28)

Theorem 2.1. *Let $A \in M_n(F)$ be triangularizable. Then $\{A\}'$ is triangularizable if and only if for any eigenvalue λ of A, the corresponding Jordan blocks in its Jordan canonical form, have distinct sizes.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of A. By Lemma 1.1, there exist invariant subspaces $W_1, W_2, \ldots, W_k$ of A such that $F^n = W_1 \bigoplus W_2 \bigoplus \cdots \bigoplus W_k$, $\sigma(A\,|_{W_j}) = \{\lambda_j\}$ $(j = 1, 2, \ldots, k)$, and $\{A\}' = \{A\,|_{W_1}\}' \bigoplus \{A\,|_{W_2}\}' \bigoplus \cdots \bigoplus \{A\,|_{W_k}\}'$. (Note that, A being triangularizable, its characteristic polynomial splits over F.)

Therefore, by Lemma 1.1, we can assume without loss of generality that $\sigma(A) = \{\lambda\}$ for some $\lambda \in F$. Also, since $\{A - \lambda I\}' = \{A\}'$, we can further reduce the problem to the case $\sigma(A) = \{0\}$; that is, we assume

$$A = \begin{bmatrix} J_{k_1}(0) & 0 & \cdots & 0 \\ 0 & J_{k_2}(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{k_m}(0) \end{bmatrix} \tag{1}$$

with respect to an appropriate direct sum $F^n = V_1 \oplus V_2 \oplus \cdots \oplus V_m$. Moreover, we assume without loss of generality that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m$.

First assume the sizes of the Jordan blocks are not distinct; i.e., $k_u = k_{u+1}$ for some $u = 1, 2, \ldots, m-1$. Define $X = [X_{ij}]$ and $Y = [Y_{ij}]$ by $X_{u,u+1} = I$, $X_{ij} = 0$ for all $(i, j) \neq (u, u+1)$, $Y_{u+1,u} = I$, and $Y_{ij} = 0$ for all $(i, j) \neq (u+1, u)$. It is easy to observe that $X, Y \in \{A\}'$, X, Y are both nilpotent and $X + Y$ is not nilpotent. This clearly implies that $\{A\}'$ is not triangularizable.

For the converse, assume $k_1 > k_2 > \cdots > k_m$. For $X \in \{A\}'$, let $X = [X_{ij}]$ be its block matrix representation with respect to $F^n = V_1 \oplus V_2 \oplus \cdots \oplus V_m$. We claim each X_{ij} is of the form T, $\begin{bmatrix} T \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & T \end{bmatrix}$ depending on whether $i = j$, $i < j$ or $i > j$, where T is a square matrix of the form

$$T = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_t \\ 0 & a_1 & a_2 & \cdots & a_{t-1} \\ 0 & 0 & a_1 & \cdots & a_{t-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{bmatrix}. \tag{2}$$

Moreover, $\sigma(X) = \{x_1, x_2, \ldots, x_m\}$, where x_i is the entry appearing on the diagonal of X_{ii} with multiplicity at least k_i $(i = 1, 2, \ldots, m)$. (Note that $x_1, x_2, \ldots, x_m$ may not be distinct.)

Since $XA = AX$ it follows that

$$X_{ij} J_{k_j}(0) = J_{k_i}(0) X_{ij} \quad (i, j = 1, 2, \ldots, m). \tag{3}$$

Fix $i, j = 1, 2, \ldots, m$ and let $t = \min\{k_i, k_j\}$. Write $X_{ij} = [a_{pq}]_{p,q}$ and deduce from (3) that $a_{p,q-1} = a_{p+1,q}$ $(p = 1, 2, \ldots, k_i;\ q = 1, 2, \ldots, k_j)$, where $a_{p0} = a_{k_i+1,q} := 0$. This proves the required form of X_{ij}.

To find the eigenvalues of X, we calculate the determinant $\det(\lambda I - X)$ by expanding it with respect to the rows $k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_m$ to obtain

$$\det(\lambda I - X) = (\lambda - x_1)(\lambda - x_2) \cdots (\lambda - x_m) \det(\lambda I - Y),$$

where Y is a matrix obtained from X by omitting the rows and the columns numbered $k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_m$. Note that $Y = [Y_{ij}]$, where each Y_{ij} is obtained from X_{ij} by omitting the last row and the last column (and, of course, ignoring the resulting empty blocks if $k_m = 1$). Now, by a finite induction, it follows that

$$\det(\lambda I - X) = (\lambda - x_1)^{k_1} (\lambda - x_2)^{k_2} \cdots (\lambda - x_m)^{k_m}.$$

Next, let X and Y be arbitrary elements of $\{A\}'$. It is routine to show that each diagonal entry of XY is precisely the product of the corresponding entries of the diagonals of X and Y. Thus, $XY - YX$ has a zero diagonal and hence is nilpotent. Therefore in view of [1], $\{A\}'$ is triangularizable. □

Corollary 2.2. *Let $A \in M_n(F)$ be triangularizable. If for each $\lambda \in \sigma(A)$ the corresponding Jordan blocks of A have distinct sizes, then $\{A\}'$ has a simultaneous block decomposition with respect to which every block X_{ij} of any $X \in \{A\}'$ is of the form T, $\begin{bmatrix} T \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & T \end{bmatrix}$ depending on whether $i = j$, $i < j$ or $i > j$, where T is as in (2). Moreover, the eigenvalues of X appear on the diagonals of X_{11}, X_{22}, etc.*

Corollary 2.3. *Let $A \in M_n(F)$ satisfy $A^k = 0$, where $k(k + 1) < 2n$. Then $\{A\}'$ is not triangularizable.*

Proof. $A^k = 0$ implies that the sizes of the Jordan blocks of A are less than $k + 1$. Moreover, the inequality $k(k + 1) < 2n$ implies that these sizes are not distinct. The rest of the proof follows from the theorem. □

Corollary 2.4. *Let A be a nonzero nilpotent element of $M_n(F)$, such that $A^{n-2} \neq 0$. Then $\{A\}'$ is triangularizable.*

Proof. $A^{n-2} \neq 0$ implies that the Jordan blocks of A have distinct sizes. □

Remark. Notice that the distinctness of the sizes of the Jordan blocks corresponding to a fixed eigenvalue can be rephrased in terms of the ranks of consecutive powers of A. That is [2, p. 131], for an eigenvalue λ of A, the number of Jordan blocks corresponding to λ with size greater than k, in the Jordan canonical form of A, is equal to

$$\operatorname{rank}(A-\lambda I)^k - \operatorname{rank}(A-\lambda I)^{k+1}.$$

Thus, the sizes of the Jordan blocks corresponding to λ are distinct if and only if for any $k \in \mathbb{N}$,

$$\operatorname{rank}(A-\lambda I)^{k-1} + \operatorname{rank}(A-\lambda I)^{k+1} - 2\operatorname{rank}(A-\lambda I)^k \leqslant 1. \tag{4}$$

Corollary 2.5. *Assume $A \in M_n(F)$ satisfy (4) for any eigenvalue λ and any $k \in \mathbb{N}$. Then, there exist $A_1, A_2, \ldots, A_n \in \{A\}'$ such that* $\operatorname{rank}(A_k) = k$ $(1 \leqslant k \leqslant n)$, *and* $\operatorname{rang}(A_1) \subset \operatorname{rang}(A_2) \subset \cdots \subset \operatorname{rang}(A_n)$.

Proof. By the theorem and the above remark, $\{A\}'$ is triangularizable. Let $V_1 \subset V_2 \subset \cdots \subset V_n$ be a triangularizing chain of the invariant subspaces of $\{A\}'$. By [4], every invariant subspace of $\{A\}'$ is the range of some element in $\{A\}'$. □

Theorem 2.1 also helps us to get triangularizability results for the commutant of a collection of matrices. Let $\mathscr{F}$ be a triangularizable collection of matrices . If $\mathscr{F}$ contains a matrix A which satisfies the conditions of Theorem 2.1, then $\{A\}'$ and, hence, $\mathscr{F}'$ are triangularizable. Now by Yahaghi's extension of Turovskii's result, $\mathscr{F} \cup \mathscr{F}'$ is triangularizable. Summing up, we have shown the following corollary.

Corollary 2.6. *Let $\mathscr{F}$ be a triangularizable collection of matrices in $M_n(F)$ which contains an element whose Jordan blocks corresponding to any fixed eigenvalue have distinct sizes. Then $\mathscr{F} \cup \mathscr{F}'$ is triangularizable.*

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References

[1] R.M. Guralnick, Triangularization of sets of matrices, Linear Mult. Algebra 9 (1980) 133–140.
[2] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, NY, 1985.
[3] N.H. McCoy, On quasi-commutative matrices, Trans. Amer. Math. Soc. 36 (1934) 327–340.

[4] S.-C. Ong, Remarks on invariant subspaces for finite dimensional operators, Linear Algebra Appl. 42 (1982) 99–101.
[5] H. Radjavi, P. Rosenthal, Simultaneous Triangularization, Springer-Verlag, New York, 2000.
[6] D.A. Suprunenko, R.I. Tyshkevich, Commutative Matrices, Academic Press, New York, 1968.
[7] Yu.Y. Turovskii, Spectral properties of certain Lie subalgebras and the spectral radius of subsets of a Banach algebra, Spectral Theory of Operators and its Applications, vol. 6, Elm, Baku, 1985, pp. 144–181 (in Russian).
[8] R. Yahaghi, On Simultaneous triangularization of commutants, Acta Sci. Math. (Szeged) 66 (2000) 711–718.
[9] B.R. Yahaghi, Reducibility results on operator semigroups, Ph.D. thesis, Dalhousie University, Halifax, N.S., Canada, 2002.