View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Theoretical Computer Science 436 (2012) 35–53

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Sand piles: From physics to cellular automata models☆

G. Cattaneo*, M. Comito, D. Bianucci

Dipartimento Di Informatica, Sistemistica e Comunicazione, Università di Milano – Bicocca, Viale Sarca 336 – U14, I-20126 Milano, Italy

ARTICLE INFO

Article history:
Received 29 September 2009
Received in revised form 30 December 2011
Accepted 21 February 2012
Communicated by B. Durand

Keywords:
Sand piles
Information and thermodynamical entropies
Discrete time dynamical systems
Cellular automata

ABSTRACT

We analyze the dynamical behavior of the usual one dimensional sand pile model which actually describes the physical situation in which the pile is submitted to the uniform blow of a unidirectional wind. In the first step the Lagrangian formalism is investigated, showing that the stationary action principle does not select in a unique way the path which satisfies either the minimal or the maximal action principle. This drawback is solved making use of the information (Shannon) entropy which enables one to determine the unique path in which at any time step the entropy variation is minimal (adiabatic) or maximal (anti-adiabatic). A cellular automata (CA) model describing this sand pile behavior is introduced, and the consequent deterministic dynamic is compared with the entropy results, showing that also in this case there are some drawbacks. Moreover, it is shown that our CA local rule is a particular case of some standard CA sand pile models present in literature.

© 2012 Elsevier B.V. All rights reserved.

0. Introduction

The introduction to sand pile (SP) model and some other related generalized models are already universally accepted as paradigmatic approaches to different application domains such as the context of integer lattices, the self-organized critical situation from the physical point of view, and interesting approaches to combinatoric.

In the standard SP model, which is the argument of this paper, a sand pile of "total mass" N is defined as a finite sequence of non-increasing integers whose total sum is N. Each indexed by i component of the sequence is interpreted as the number of sand granules located in the position i of a one dimensional discrete space.

A discrete time dynamics of the SP is introduced by a horizontal rule, also called H-rule, which solves any jump from the left to the right, greater than or equal to two granules. This H-rule can be applied in a sequential or in a parallel procedure. In the first case only one jump is solved step-by-step, whereas in the latter parallel case all the jumps are solved during a unique step by a synchronous application of the H-rule.

Let us now give the logical organization of the paper by points, each of which corresponds to a section of the paper.

1. It is well known that the H-evolution rule of SP leads to a variety of possible paths depicted as a digraph $\mathcal{G}_c$ having the configuration c as "starting" (or initial) node and a unique equilibrium configuration as final node. This was the staring point of our interests, consisting in exploring some criteria which allow one to select a unique "actual" path among all the "possible" ones inside any $\mathcal{G}_c$.
2. The first approach rises from the analogy of the Hasse diagrams $\mathcal{G}_c$ of the possible SP discrete time dynamics with the possible continuous time dynamics of the variational approach to classical physics (see for instance [23, Chapter VII]). The first attempt consists then in adapting the "classical" stationary action principle to the SP case in order to see if it is possible to select a unique path satisfying this principle. In particular we introduce the SP "mechanical" Lagrangian, with

☆ This work has been supported by MIUR\PRIN project "Mathematical aspects and forthcoming applications of automata and formal languages".

* Corresponding author. Tel.: +39 02 6448 7827; fax: +39 02 6448 7839.
E-mail addresses: cattang@disco.unimib.it, cattang@live.it (G. Cattaneo).

0304-3975/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2012.02.034

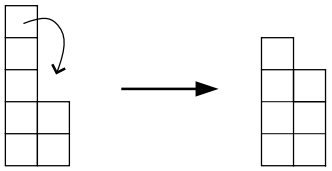


Fig. 1. Typical left-to-right granule movement of a sand pile.

associated *action* and *reduced action* functions with respect to which a local stationary action (or Hamiltonian) principle is investigated.
Unfortunately, this approach has a negative answer since it is shown by a counter-example 2.1 that the stationary action principle does not select in a unique way the path which satisfies either the minimal or the maximal local action principle.

3. On the basis of the fact that an integer partition can be considered a discrete measure distribution whose normalization by the total number N of sand granules is a discrete probability distribution, induced us to explore the information entropy approach of Shannon in order to select a unique path.
In Section 3 a positive answer has been done to this conjecture introducing, by similarity with the "mechanical" case, the SP "information" notions of Lagrangian, action and reduced action with corresponding stationary action principle which enables one to determine the unique path in which at any time step the entropy variation is minimal (adiabatic transformation) or maximal (anti-adiabatic transformation).
4. Then we explore the synchronous application of the H-rule according to the application of a 1D Sand CA local rule, whose induced global next state function produces a deterministic discrete time dynamical system which generates a unique path from any initial configuration.
In this case the conjecture was that the unique path obtained by the information Shannon entropy is fitted by the unique path generated by the Sand CA. But this conjecture is not verified as shown in Example 4.1 of Section 4.2.
5. At this point, recalling another form of entropy according to the Boltzmann approach to statistical mechanics we tried also this approach in Section 5, but also in this case with a negative answer to the attempt of obtaining a unique path fitted by the sand CA one, leaving this latter as an open problem to be further on investigated in some future papers.

1. Sand pile model under a uniform blow of unidirectional wind

The sand pile models presented in literature (see for instance [21,22], with the related references [2,26]) in general describe the real situation in which a pile of sand is submitted to the blow of a wind which is unidirectional, in general along the direction from the left to the right, with uniform intensity. This as a consequence of the general "*rule of movement*" of the single sand granule summarized/depicted in Fig. 1

Formally, a *sand pile model* (SPM) is based on the basic notion of *integer partition*. Precisely, a length l integer partition of the integer N, with l, $N \in \mathbb{N}$ fixed positive numbers, is mathematically described by a l-length finite sequence of nonnegative numbers

$$\begin{aligned} \boldsymbol{n} = (n_1\, n_2\, \ldots n_l) \quad \text{with} \quad n_i \in \mathbb{N} \quad \text{for every } i = 1, 2, \ldots, l \\ \text{under the condition} \quad \sum_{i=1}^{l} n_i = N \\ \text{and the constraints} \quad n_i \neq 0 \ \text{ for every } \ i = 1, 2, \ldots, l \end{aligned} \tag{1.1}$$

Trivially, the condition $\sum_{i=1}^{l} n_i = N$ implies that necessarily each $n_i \leq N$ and $l \leq N$. With respect to the standard SPM (see for instance [22]), we have to stress that, in order to better describe the "real" physical situation of sand piles, in the here presented model it is not required the decreasing monotonicity of the sequence formalized by the conditions $n_1 \geq n_2 \geq \cdots \geq n_l \neq 0$.

From the mathematical (or combinatorial) point of view a finite sequence $\boldsymbol{n}$ of this kind is an *integer partition* of the integer N in which each n_i is a *part* (or a *component*) of the partition. From the physical point of view any integer partition $\boldsymbol{n}$ of N can be represented as the *Ferrer diagram* consisting in a series of columns of stacked squares where the i-th column (from the left to the right) contains n_i squares each of which represents a *granule* of sand. Since one can attribute to each granule a mass m_0, assumed as the unit mass of the system, we can also interpret the number N as the *total mass* of the pile.

Two integer partitions of N granules are particularly important: the one in which only the initial cell contains all the involved granules, defined by $\boldsymbol{e} := (N)$, and the N-length *uniformly distributed* partition $\boldsymbol{n}_u = (\underbrace{1, 1, \ldots, 1}_{N\text{-times}})$.

Sometimes, it will be necessary to "extend" a length l sand pile $\boldsymbol{n}$ of total number N as a new length N (definitively 0) sand pile $\hat{\boldsymbol{n}} : \{1, 2, \ldots, N\} \to \{0, 1, \ldots, N\}$ by the law:

$$\forall i \in \{1, 2, \ldots, N\}, \quad \hat{\boldsymbol{n}}(i) := \begin{cases} n_i & \text{for } i = 1, \ldots, l \\ 0 & \text{for } i = l+1, \ldots, N. \end{cases} \tag{1.2}$$

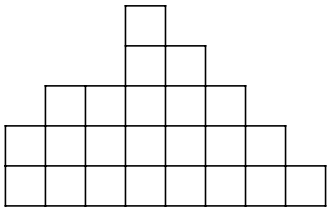


Fig. 2. Example of an equilibrium configuration of 23 granules.

Let us denote by $\Omega(N)$ the collection of all such N-length sequences obtained according to the just introduced procedure by extensions of arbitrary l-length integer partitions of N. This set is said to be the N granules sand pile *phase* (or *configuration*) *space*, and any of its element is called a N granules sand pile *configuration*.

1.1. One dimensional grid representation of the configuration space

According to Eq. (1.2) a configuration from the configuration space $\Omega(N)$ of all integer partitions of N is a mapping $\boldsymbol{n} : \{1, 2, \ldots, N\} \to \{0, 1, \ldots, N\}$, defined by the correspondence $i \to \boldsymbol{n}(i) = n_i \geq 0$. In this context the fixed set $\mathcal{L}(N) := \{1, 2, \ldots, N\}$ is considered as a one-dimensional *grid* of N cells, each of which can assume one of the value from the set of possible *states* $\mathcal{S}(N) := \{0, 1, \ldots, N\}$. A configuration $\boldsymbol{n} \in \Omega(N)$ is then a mapping assigning to any cell i of the grid $\mathcal{L}(N)$ the number of granules $\boldsymbol{n}(i) = n_i$ vertically located in this cell. The last cell l for which $n_l \neq 0$ is the length of the configuration.

Given a configuration $\boldsymbol{n} \in \Omega(N)$, when $n_i - n_{i+1} \geq 2$ holds we will say that a *right jump* is located in cell i and that a *left jump* is located in cell $i + 1$ of $\boldsymbol{n}$. In general, if it is clear from the context, we prefer to simply speak of *jump* at cell i instead of saying that at i there is a right jump.

Definition 1.1. An *equilibrium* configuration of $\Omega(N)$ is any configuration $\boldsymbol{n}^{(eq)} = \left(n_1^{(eq)}, \ldots, n_i^{(eq)}, n_{i+1}^{(eq)}, \ldots, n_l^{(eq)}, 0, \ldots, 0\right)$ without jumps. The collection of all equilibrium configurations of total mass N will be denoted by $\Omega^{(eq)}(N)$ (see Fig. 2 as an example).

This happens if $n_l^{(eq)} = 1$ and at least one of the following two conditions is verified for any $i = 1, 2, \ldots, l - 1$:

(Eq1) either $n_i^{(eq)} \leq n_{i+1}^{(eq)}$, and so we may have in particular that $n_i^{(eq)} = n_{i+1}^{(eq)}$;
(Eq2) or $n_i^{(eq)} = n_{i+1}^{(eq)} + 1$.

1.2. Sand pile dynamical evolution and related lattices

The *unidirectional sand pile model* (SPM) we treat, in agreement with the movement depicted in the above Fig. 1, is based on the dynamical evolution formalized by the movement of a sand grain with respect to the following rule:

– **Horizontal (H) Rule**. If $n_i - n_{i+1} \geq 2$, i.e., a jump is located at cell i, then

$$n_1, \ldots, n_i, n_{i+1}, \ldots, n_N \quad \longrightarrow \quad n_1, \ldots, n_i - 1, n_{i+1} + 1, \ldots, n_N \tag{1.3}$$

A discrete time *dynamical evolution* is defined by an initial configuration $\boldsymbol{n}_0 \in \Omega(N)$ and an evolution rule, such as the above H-rule, which says under which conditions the configuration may be changed, and which describes the new configuration one may obtain. This H-evolution rule is formulated as a local condition which implies a local modification of the current configuration.

If a configuration $\boldsymbol{n}'$ can be obtained from a configuration $\boldsymbol{n}$ after a *single* application of the H-evolution rule, we say that $\boldsymbol{n}'$ is a *successor* of $\boldsymbol{n}$, or that $\boldsymbol{n}$ is a *predecessor* of $\boldsymbol{n}'$, denoted by $\boldsymbol{n} \longrightarrow \boldsymbol{n}'$. This binary relation $\longrightarrow$ on $\Omega(N)$ is irreflexive and so, in the usual way, it gives rise to a (causal) *strict inclusion* (irreflexive and transitive, see [6]) binary relation according to

(SPO) Let $\boldsymbol{n}, \boldsymbol{m} \in \Omega(N)$. Then $\boldsymbol{n} < \boldsymbol{m}$ iff there exists a finite sequence of configurations $\boldsymbol{n}_0, \boldsymbol{n}_1, \ldots, \boldsymbol{n}_f$, with each $\boldsymbol{n}_i \in \Omega(N)$, which forms a *path* of initial configuration $\boldsymbol{n}$ and final configuration $\boldsymbol{m}$, i.e., such that

$$\boldsymbol{n}_0 = \boldsymbol{n} \longrightarrow \boldsymbol{n}_1 \longrightarrow \cdots \longrightarrow \boldsymbol{n}_f = \boldsymbol{m}$$

As usual, from a strict inclusion binary relation it is possible to induce a partial order relation on the set of all configurations defined as follows

(PO) $\boldsymbol{n} \leq \boldsymbol{m}$ iff either $\boldsymbol{n} = \boldsymbol{m}$ or $\boldsymbol{n} < \boldsymbol{m}$.

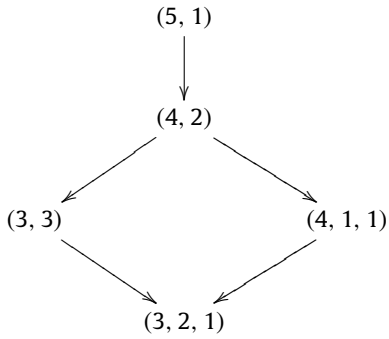


Fig. 3. Hasse diagram of the two possible dynamical evolutions with initial configuration (5, 1).

In the case of decreasing configurations this ordering is equivalent to the so-called *dominance ordering* [7]: $\boldsymbol{n} \leq \boldsymbol{m}$ iff $\forall j$, $\sum_{i=1}^{j} n_i \leq \sum_{i=1}^{j} m_j$.

Let us note that the just introduced H-evolution rule (1.3) does not assure a *uniquely determined* dynamical evolution starting from a given initial configuration. Indeed, it can be singularly applied to all the jumps of a given configuration at the time step $t \in \mathbb{N}$, leading to several configurations at the next time step $t + 1 \in \mathbb{N}$. So, in line of principle different paths may be generated, each of which is compatible with the involved H-evolution rule.

Example 1.1. For instance the initial configuration $\boldsymbol{n}(0) = (5, 1)$, representing an integer partition of the number 6, by the H-rule (1.3) uniquely leads to the time 1 configuration $\boldsymbol{n}(1) = (4, 2)$, which in its turn leads to two configurations, both compatible with the application of H-rule (1.3): $\boldsymbol{n}(2a) = (3, 3)$ and $\boldsymbol{n}(2b) = (4, 1, 1)$. Both these configurations leads to the final equilibrium configuration (3, 2, 1). This can be depicted by the digraph of Fig. 3 which shows the structure of a Hasse diagram of a (finite) lattice with the (unique) equilibrium configuration (3, 2, 1) as least element and having as greatest element the initial configuration (5, 1).

So, we can define a *possible path* (also *admissible path*) generated by the H-evolution rule (1.3) starting from a given initial configuration $\boldsymbol{n}_0$ as a $\Omega(N)$-valued finite sequence

$$\gamma_{\boldsymbol{n}_0} \equiv \big(\boldsymbol{n}(0), \boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(t_f)\big)$$

satisfying the following conditions:

(P1) The configuration at time $t = 0$ is $\boldsymbol{n}(0) = \boldsymbol{n}_0$;
(P2) the configuration $\boldsymbol{n}(t+1) \in \gamma_{\boldsymbol{n}_0}$ at time $t+1$ is obtained from the configuration $\boldsymbol{n}(t) \in \gamma_{\boldsymbol{n}_0}$ at time t by the application of the H-evolution rule (1.3) to a single jump, i.e., $\boldsymbol{n}(t+1)$ is a successor of $\boldsymbol{n}(t)$ (formally, $\boldsymbol{n}(t) \longrightarrow \boldsymbol{n}(t+1)$);
(P3) the final configuration $\boldsymbol{n}(t_f) \in \gamma_{\boldsymbol{n}_0}$ is an equilibrium configuration.

Thus, as time elapses, the configuration may change and the motion of the sand system is described by a discrete time orbit in the configuration space $\Omega(N)$, called *possible path* of the system.

Example 1.2. In the case of Example 1.1 we have two possible paths:

$$\gamma^{(a)}_{(5,1)} :\equiv (5, 1) \longrightarrow (4, 2) \longrightarrow (3, 3) \longrightarrow (3, 2, 1)$$
$$\gamma^{(b)}_{(5,1)} :\equiv (5, 1) \longrightarrow (4, 2) \longrightarrow (4, 1, 1) \longrightarrow (3, 2, 1)$$

The final equilibrium configuration (3, 2, 1) is reached in both cases after 3 time steps.

So, the framework on which we act rises up from the following steps.

1. **Levels construction**: Starting from the initial configuration (which constitutes level 0), the levels are built resolving jumps from left to right (Fig. 4);
2. **Lattice structure**: One can verify that, applying the previous building step, the resulting structure is a lattice (see [22], with the connected references [7,21], and [19] for some further generalizations);

With respect to this lattice structure, let us introduce the following

Black Hole definition: A configuration at the time step t is said to be a *black hole* iff it is the unique configuration at the level t and it admits at least a jump, i.e., it "absorbs" all the $t - 1$ configurations "producing" at least one new configuration at time $t + 1$.

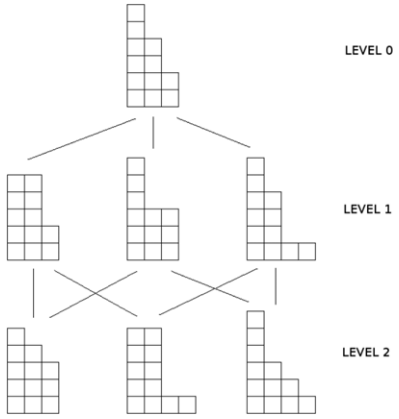
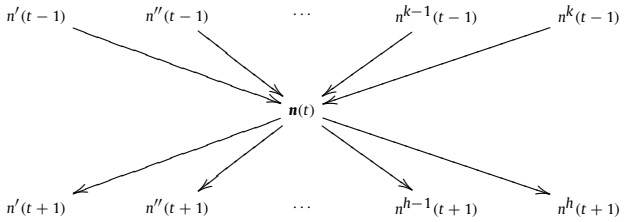


Fig. 4. Example of the first two levels construction starting from the initial configuration (6, 4, 2). Note that at the second level it happens that the same configuration is obtained by two different configurations of the first level.

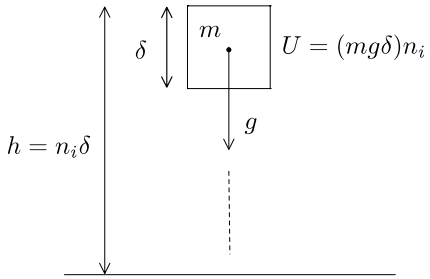


Conjecture: All the possible paths describing dynamical evolutions obtained by the H-evolution rule (1.3) starting from a given initial configuration $\boldsymbol{n}(0) \in \Omega(N)$ lead to a unique final equilibrium configuration $\boldsymbol{n}^{(eq)} \in \Omega(N)$.

The above conjecture has a positive answer for the particular condition of decreasing initial configuration without internal zeros, i.e., configuration $\boldsymbol{n} = (n_1, \ldots, n_N)$ such that $\forall i, n_i \geq n_{i+1}$ and no internal block of the type $0n_j$ with $n_j \neq 0$. This was proved in [21, Proposition 2.3] for the present case of unidirectional SPM. Indeed, given N, under the above condition, the only possible configurations different from $n^{(eq)}$ that could be potential final equilibrium configurations (i.e., such that the horizontal rule is no more applicable on these) are all non admissible. Note that some generalizations of this result can be found in [19]. Moreover, a treatment of this argument to the case of a symmetric SPM is done in [20] where the convergence of any path to a unique equilibrium configuration is proved but, as shown in some counter-example, with the possibility that different paths converge to different equilibrium configurations (non uniqueness of the final equilibrium configuration).

2. Sand piles' energy considerations

Given a configuration (integer partition) $\boldsymbol{n} \in \Omega(N)$ of a total number N of granules, the column of n_i granules at cell i can be considered as a vertical pile of massive particles, each of which of given mass m_0, submitted to the gravitational *potential energy* depending from its height h with respect to the reference system of the ground. For instance, if δ is the "linear dimension" of the single granule, the potential energy of the higher granule of the cell i of this configuration is $m_0 g(n_i\, \delta)$.



From this we get that the potential energy of the whole i column is

$$\begin{aligned} U_i(\boldsymbol{n}) &:= m_0 g n_i \delta + m_0 g (n_i - 1)\delta + m_0 g (n_i - 2)\delta + \cdots + m_0 g 1\delta \\ &= m_0 g\, \delta \sum_{k=0}^{n_i - 1} (n_i - k) = m_0 g\, \delta \sum_{k=1}^{n_i} k \end{aligned}$$

If, as usual in physics, we put equal to 1 the constant quantity $m_0 g \delta$, the total potential energy of the length l configuration is then

$$U(\boldsymbol{n}) = \sum_{i=1}^{l} \sum_{k=1}^{n_i} k \tag{2.1}$$

which is just the potential energy considered in [20, Section 3.1, p. 95]. Let us note that this potential energy is different from the "energy" $E_{GK}(\boldsymbol{n}) = \sum_{i=1}^{N} i\, n_i$ introduced by Goles–Kiwi (GK) in [21]. For instance, in the case of the configuration (4, 2, 1) its potential energy according to (2.1) is 13 whereas the GK energy is 11.

Fixing an initial length l_0 sand pile configuration of N granules, $\boldsymbol{n}(0) \in \Omega(N)$, the potential energy of this configuration coincide with its total energy (all granules are at rest in the initial situation, corresponding to the initial kinetic energy $T(\boldsymbol{n}(0)) = 0$), from now on denoted by:

$$E_0 = U(\boldsymbol{n}(0))$$

The application of the above H-rule of dynamical evolution may lead to a new configuration $\boldsymbol{n}(1) \in \Omega(N)$ of the invariant total number of granules N whose length is l_1, which can be either l_0 or $l_0 + 1$: i.e., $l_0 \leq l_1 \leq l_0 + 1$. In particular,

$$l_1 = \begin{cases} l_0 & \text{if } n_{l_0}(0) = 1 \\ l_0 + 1 & \text{if } n_{l_0}(0) \geq 2 \end{cases}$$

Proposition 2.1. *The potential energy of the new configuration $\boldsymbol{n}(t+1)$, obtained by the application of the H-evolution rule to a single jump of the configuration $\boldsymbol{n}(t)$, is strictly decreasing*

$$U(\boldsymbol{n}(t+1)) < U(\boldsymbol{n}(t)) \tag{2.2}$$

Proof. If we consider the transition described by the H-evolution rule (1.3) for the jump centered in the cell i, i.e., $(n_i - n_{i+1}) \geq 2$, then trivially

$$\begin{aligned} U(\boldsymbol{n}(t)) &= \big(n_i + (n_i - 1) + \cdots + 1\big) + \big(n_{i+1} + (n_{i+1} - 1) + \cdots + 1\big) + K \\ U(\boldsymbol{n}(t+1)) &= \big((n_i - 1) + (n_i - 2) + \cdots + 1\big) + \big((n_{i+1} + 1) + n_{i+1} + \cdots + 1\big) + K \end{aligned}$$

where in K we collect the common contribution to the potential energy by the two involved configurations. Then,

$$U(\boldsymbol{n}(t)) - U(\boldsymbol{n}(t+1)) = (n_i - n_{i+1}) - 1 \geq 1. \quad \square$$

Note that this result has been proved in [20, Lemma 3.2] for symmetric SPMs, in a slightly different formulation.

Owing to the assumption that the sand pile is a *conservative system*, this potential energy decreasing corresponds to a rise of a kinetic energy

$$T(\boldsymbol{n}(1)) = U(\boldsymbol{n}(0)) - U(\boldsymbol{n}(1)) = E_0 - U(\boldsymbol{n}(1))$$

in such a way that the total energy $T(\boldsymbol{n}(1)) + U(\boldsymbol{n}(1)) = E_0$ is conserved during the time transition $t = 0 \to t = 1$. Hence, and taking into account the condition $n_i - n_{i+1} \geq 2$ which assures the transition, the corresponding kinetic energy is

$$T(\boldsymbol{n}(1)) = \big(n_i - n_{i+1}\big) - 1 \geq 1$$

Let us now consider a possible path $\gamma_{\boldsymbol{n}(0)} : \mathbb{N} \to \Omega(N)$ from the initial configuration $\boldsymbol{n}(0) := \boldsymbol{n}_0$ to the (unique) equilibrium configuration $\boldsymbol{n}(t_f) = \boldsymbol{n}_f$, reached for the first time at the instant t_f:

$$\gamma_{\boldsymbol{n}(0)} \equiv \big(\boldsymbol{n}(0), \boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(t), \boldsymbol{n}(t+1), \ldots, \boldsymbol{n}(t_f)\big) \tag{2.3}$$

Making use of Eq. (2.2) we obtain the two ordered chains for the (decreasing) potential energy and the (increasing) kinetic energy, respectively

$$U(\boldsymbol{n}(0)) > U(\boldsymbol{n}(1)) > \cdots > U(\boldsymbol{n}(t)) > U(\boldsymbol{n}(t+1)) > \cdots \geq 0$$

$$0 < T(\boldsymbol{n}(1)) < \cdots < T(\boldsymbol{n}(t)) < T(\boldsymbol{n}(t+1)) < \cdots \leq E_0$$

under the *total energy conservation* (the system is conservative):

$$\forall t \in \mathbb{N}, \quad T(\boldsymbol{n}(t)) + U(\boldsymbol{n}(t)) = E_0$$

In *conservative* systems the total energy is the same along each path, but it depends on the initial configuration. Note that in the case of the initial state $\boldsymbol{u}(0) = \boldsymbol{e} = (N, 0, 0, \ldots, 0)$ we have that the conserved total energy is $E_0 = N \cdot \frac{N+1}{2}$.

Remark 2.1. At the end of the sand pile dynamical evolution, when the equilibrium configuration is reached, the total kinetic energy of the sand pile system disappears (all the granules are at rest), and so one can ask what happens from the point of view of energy conservation.

The answer is that, according to the first thermodynamics principle, taking into account the whole system "sand pile – ground surface", the disappeared kinetic energy of the sand pile subsystem is converted in heat energy of the whole system, similarly to the case of a sledge-hammer violently beaten on the anvil: the mechanical energy of the hammer is totally converted into heat energy of the sledge-hammer–anvil system by an increasing of their temperature.

Of course, these thermodynamical considerations are inessential in order to consider the pure dynamical evolution from the initial configuration to the final equilibrium one, so we disregard it in this treatment.

The *Lagrangian* at time t of this conservative dynamical system, defined in conservative systems as the difference between the kinetic energy minus the potential one (see for instance [23, §5 of chapter 1]), is then

$$L(\boldsymbol{n}(t)) = T(\boldsymbol{n}(t)) - U(\boldsymbol{n}(t)) = E_0 - 2U(\boldsymbol{n}(t)) \tag{2.4a}$$

$$= 2T(\boldsymbol{n}(t)) - E_0 \tag{2.4b}$$

where from the particular initial condition of zero kinetic energy (the system is initially at rest, i.e., $T(\boldsymbol{n}(0)) = 0$) and total energy equal to the initial potential energy ($U(0) = E_0$), we have that the initial Lagrangian is $L(\boldsymbol{n}(0)) = -E_0$.

From the strictly decreasing of the potential energy during the dynamical evolution along a potential path, it follows that the Lagrangian is a strictly increasing quantity along the same path:

$$-E_0 = L(\boldsymbol{n}(0)) < L(\boldsymbol{n}(1)) < \cdots < L(\boldsymbol{n}(t)) < L(\boldsymbol{n}(t+1)) < \cdots < E_0$$

So, if we consider the collection of all paths $\gamma_{\boldsymbol{n}_0}$ obtained by the H-evolution rule (1.3) starting from a given initial configuration $\boldsymbol{n}_0 \in \Omega(N)$ of total mass N and reaching the total mass N final configuration $\boldsymbol{n}_f \in \Omega^{(eq)}(N)$ after a number of time steps which depends on the path, each of these paths describes a *possible* dynamical evolution of the sand piles system. Besides the conservation of the total mass N, each possible path is characterized by a dynamical evolution which also preserves the total energy E_0 of the system.

In order to determine the *actual* path connecting $\boldsymbol{n}_0$ to $\boldsymbol{n}_f$ we try first of all to apply the Hamilton *stationary action principle* to this discrete time dynamical system. To this purpose, let us consider a path $\gamma_{\boldsymbol{n}_0}$ starting from the given initial configuration $\boldsymbol{n}_0$ and reaching the final equilibrium configuration $\boldsymbol{n}_f$ for the first time at the instant t_f. The corresponding discrete time version of the *action* (see [23, eq. 2.1] for the continuous time version relative to conservative system) is the quantity defined as

$$S(\gamma_{\boldsymbol{n}_0}) := \sum_{t=0}^{t_f} L(\boldsymbol{n}(t)) = 2\sum_{t=1}^{t_f} T(\boldsymbol{n}(t)) - (t_f + 1)\, E_0 \tag{2.5}$$

with associated *reduced action*

$$A(\gamma_{\boldsymbol{n}_0}) := 2\sum_{t=1}^{t_f} T(\boldsymbol{n}(t))$$

Now, we can formulate the discrete time version of

The Stationary Action or Hamilton's Principle. The *actual* path connecting the initial configuration $\boldsymbol{n}_0$ to the final one $\boldsymbol{n}_f$, is characterized by an extremum (either minimum or maximum) of the associated action (2.5) (i.e., is the one along which the action is stationary) with respect to all the other *possible* paths from $\boldsymbol{n}_0$ to $\boldsymbol{n}_f$ for which the total energy is constant and equal to the actual total energy E_0.

To this purpose, let us quote the following Landau–Lifshitz statement (in which we put "sum" instead of "integral"):

> "The [sum] (2.5) for the entire path must have an extremum, but not necessarily a minimum. This fact, however, is of no importance as regards to derivation of the equations of motion since only the extremum condition is used". [23].

From the computational point of view, in order to reach this result one must know all the possible paths (which a priori is not so easy to obtain if the total number of granules N is very large), then he must compute the action (2.5) of each of these paths in order to select the one producing the Lagrangian extremum value.

On the contrary, we propose a *local* procedure which consists in evaluating, step by step, the Lagrangian of all the possible configurations at time $t + 1$ with respect to the Lagrangian of the "actual" configuration at time t. This in agreement with the remark: "it should be mentioned that [the] formulation of the principle of [stationary] action is not always valid for the entire path of the system, but only for a sufficiently short segment of the path" [23]. Precisely, we adopt the following *local procedure*: if at time t the path actually reaches the configuration $\boldsymbol{n}(t)$ and one of the possible configurations at time $t + 1$ is $\boldsymbol{n}(t + 1)$, then according to (2.4a) the "local" contribution to the action (2.5) passing from t to $t + 1$ is

$$L(\boldsymbol{n}(t)) + L(\boldsymbol{n}(t+1)) = 2E_0 - 2\big[U(\boldsymbol{n}(t)) + U(\boldsymbol{n}(t+1))\big] \tag{2.6}$$

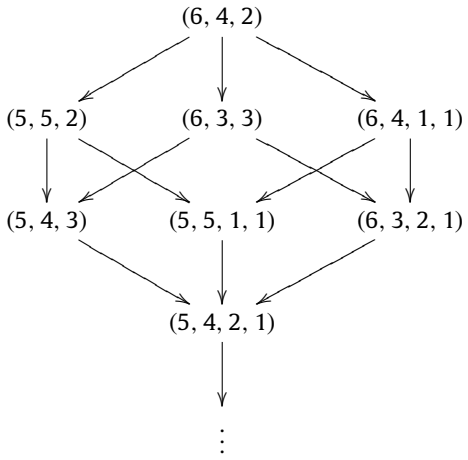


Fig. 5. The first four levels of the possible paths starting from the initial configuration (6,4,2).

So, fixing the configuration at time t with its potential energy $U(\boldsymbol{n}(t))$, if we choose the possible configuration at time $t+1$ which "maximizes" the potential energy $U(\boldsymbol{n}(t+1))$ we obtain a contribution to the action which "minimizes" its "local" value.

Summarizing, in order to select the dynamical evolutions characterized by the *"local" least action principle* (local minimization of the action), one must apply the following steps:

(LL1) Let $\boldsymbol{n}(t)$ be the configuration reached at time t starting from the initial configuration $\boldsymbol{n}(0)$ and making use of the H-rule (1.3).

(LL2) Let $\boldsymbol{n}'(t+1), \boldsymbol{n}''(t+1), \ldots, \boldsymbol{n}^k(t+1)$ be all the possible admissible next configurations at time $t+1$ obtained from $\boldsymbol{n}(t)$ by the H-rule (1.3).

(LL3) Let us compute the corresponding Lagrangian values $L(\boldsymbol{n}'(t+1)), L(\boldsymbol{n}''(t+1)), \ldots, L(\boldsymbol{n}^k(t+1))$.

(LL4) Then one chooses the next configuration of point (LL2) which maximizes the potential energy, and so minimizes the corresponding "local" action expressed by Eq. (2.6).

Of course, in a dual way it is possible to formalize the dynamical evolution generated by a "local" application of the action (2.6) maximization, choosing at time $t+1$ the possible configuration which minimizes the potential energy .

Let us apply these two local procedures to the following example.

Example 2.1. On an ensemble of total $N = 12$ sand granules, let us consider the initial configuration $\boldsymbol{n}(0) = (6, 4, 2)$, which leads by the application of the above rule (1.3) to the following three compatible next time configurations:

$$\boldsymbol{n}(1a) = (5, 5, 2) \qquad \boldsymbol{n}(1b) = (6, 3, 3) \qquad \boldsymbol{n}(1c) = (6, 4, 1, 1)$$

The initial ($t = 0$) total (i.e., potential) energy is $U(0) = 34$. The potential energies for the three possible configurations at the time $t = 1$ are all equal to 33. So the application of the local extremum action principle does not solve the indeterminacy in the selection of the actual path. Let us consider all the admissible next configurations for the successive two time steps as depicted in Fig. 5 from which it is clear that configuration (5, 4, 2, 1) is a black hole reached at time step $t = 3$ of the sandpile dynamics of initial state (6, 4, 2).

In Figs. 6 and 7 the minimal potential energy evolution possible paths (left) and the maximal potential energy evolution possible paths (right) corresponding to the four time steps dynamics of initial configuration (6, 4, 2) described in Fig. 5 are depicted.

It is clear from these Figs. 6 and 7 that the "*local*" action either maximization or minimization procedure is not able to select a unique path since the above paths turn out to remain distinct among them, whatever be the subsequent time step configuration obtained by an application of the H-rule (1.3) to the configuration (5, 4, 2, 1).

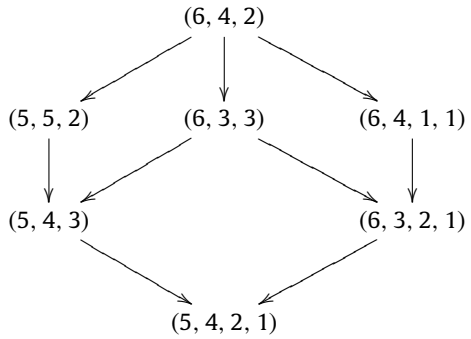


Fig. 6. Minimal Potential Energy, i.e., Maximal Action paths.

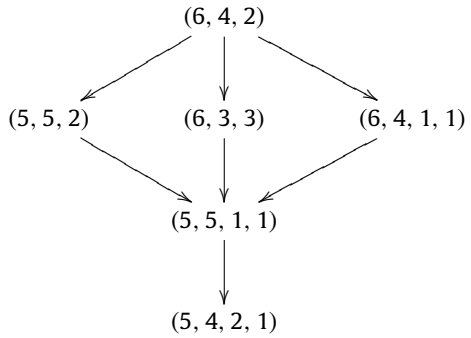
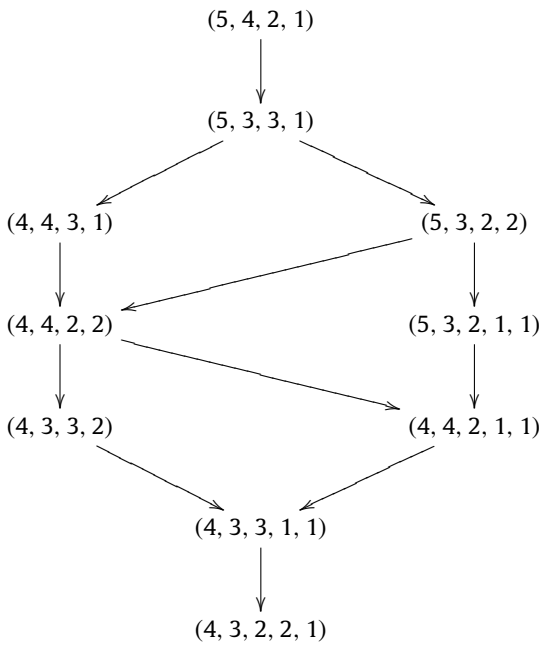


Fig. 7. Maximal Potential Energy, i.e., Minimal Action paths.

Let us now continue the discussion about the above example considering the black hole configuration (5, 4, 2, 1) as the initial one. One obtains the digraph



But in this second part of possible paths, there is no possibility to distinguish some privileged actual path with respect to the local action criterium since at any time step the potential energies are the same: $U(4, 4, 3, 1) = U(5, 3, 2, 2) = 27$, $U(4, 4, 2, 2) = U(5, 3, 2, 1, 1) = 26$, $U(4, 3, 3, 2) = U(4, 4, 2, 1, 1) = 25$.

3. Integer partitions: information (Shannon) entropy

In order to overcome the drawback seen in the Example 2.1 we will try now to make use of some entropy considerations along possible paths. Let us consider an integer partition $\boldsymbol{n} = (n_1, n_2, \ldots, n_l)$ of total mass $N = \sum_{i=1}^{l} n_i$. In discrete measure theory, this finite sequence of nonnegative integer numbers can be considered a *measure distribution* on the finite space of l abstract elementary events $\mathcal{E}_l := \{1, 2, \ldots, l\}$.

The normalized finite sequence (depending on $\boldsymbol{n}$)

$$\boldsymbol{p}(\boldsymbol{n}) = \left(p_1 = \frac{n_1}{N},\ p_2 = \frac{n_2}{N}, \ldots, p_l = \frac{n_l}{N}\right) \tag{3.1}$$

is a *probability distribution* on the same set of elementary events $\mathcal{E}_l$ since $\sum_{i=1}^{l} p_i = 1$ and for any i it is $p_i \geq 0$. On this set of events we can consider two *random variables* (RV):

(RV1) The *Information* (also *uncertainty*) *RV* based on $\boldsymbol{n}$

$$\boldsymbol{I}(\boldsymbol{n}) := \left(I_1 = -\log p_1,\ I_2 = -\log p_2,\ \ldots,\ I_l = -\log p_l\right)$$

where the single quantity $I_i = -\log p_i$ measures the uncertainty linked to the probability p_i (probability 0 means the maximum of uncertainty, probability 1 means no uncertainty, and for $p < q$ the uncertainty measure is strictly decreasing $I(p) = -\log p < -\log q = I(q)$).

(RV2) The *Granularity RV* based on $\boldsymbol{n}$

$$\boldsymbol{G}(\boldsymbol{n}) := \left(G_1 = \log n_1,\ G_2 = \log n_2,\ \ldots,\ G_l = \log n_l\right)$$

where the single quantity $G_i = \log n_i$ furnishes a measure of the granularity of the column i of the pile. A unique granule $n_i = 1$ means a minimum granularity ($G_i = 0$); the granularity measure is strictly increasing and reach its maximum value for $n_i = N$.

Of course, for any index i one has that $I_i + G_i = \log N$.

The *average uncertainty* and the *average granularity* of the integer partition $\boldsymbol{n}$ are expressed by the quantities, respectively,

$$H(\boldsymbol{n}) := \sum_{i=1}^{l} I_i \cdot p_i \quad \text{and} \quad G(\boldsymbol{n}) := \sum_{i=1}^{l} G_i \cdot p_i$$

They are called the *information uncertainty entropy* and the *information granularity entropy*, respectively [5,4,8,3]. The entropy $H(\boldsymbol{n}) = -\sum_{i=1}^{l} p_i \log p_i$ is the one introduced by Shannon in his approach to information theory [25].

Trivially, one has the relationship $H(\boldsymbol{n}) + G(\boldsymbol{n}) = \log N$ and so $G(\boldsymbol{n})$ is the *complementary* entropy (sometimes called the co-entropy) relatively to the Shannon entropy $H(\boldsymbol{n})$ with respect to the fixed quantity $\log N$.

In particular, if one consider the mass N *uniform* configuration $\boldsymbol{n}_u = (\underbrace{1, 1, \ldots, 1}_{N\text{-times}})$, and the *certain* configuration $\boldsymbol{n}_c = (N)$, then for any configuration $\boldsymbol{n}$ of total mass N (whatever be its length l) we have the boundaries

$$H(\boldsymbol{n}_c) = 0 \leq H(\boldsymbol{n}) \leq \log N = H(\boldsymbol{n}_u) \tag{3.2}$$

with $H(\boldsymbol{n}) = 0$ iff $\boldsymbol{n} = \boldsymbol{n}_c$ and $H(\boldsymbol{n}) = \log N$ iff $\boldsymbol{n} = \boldsymbol{n}_u$.

The interpretation of the extreme cases is the following one:

(E1) The "certain" configuration $\boldsymbol{n}_c$, in which all the n granules are in a unique column, corresponds to the maximum of certainty (probability distribution $\boldsymbol{p}(\boldsymbol{u}_c) = (1)$) and can be considered as a situation of *maximum order* of the pile.

(E2) The "uniform" configuration $\boldsymbol{n}_u$, in which we have N columns each containing a unique granule, corresponds to a maximum of uncertainty (uniform probability distribution) and can be considered as a situation of *maximum disorder*.

Hence, we can conclude that:

(E3) All the other configurations, according to (3.2), present an intermediate situation of disorder quantified by its information entropy.

From this point of view we assume that the uncertainty entropy is a quantity that expresses a *degree of disorder* of the configuration: the higher the uncertainty entropy of a configuration the higher its disorder. From the dual point of view, the granularity entropy expresses a *degree of order*.

Note that there can exist two different configurations with the same entropy value, but this fact cannot happen in the flow from a configuration to another by the H-rule iteration, i.e., along a possible path, as showed in the following proposition where we make use of the new "normalized" granularity entropy:

$$\hat{G}(\boldsymbol{n}) := N \cdot G(\boldsymbol{n}) = \sum_{i=1}^{l} n_i \log n_i \tag{3.3}$$

Proposition 3.1. *The granularity entropy of the configuration $\boldsymbol{n}(t+1)$, obtained by the application of the H-evolution rule to a jump centered at the cell i of the configuration $\boldsymbol{n}(t)$, is strictly decreasing $\hat{G}(\boldsymbol{n}(t)) > \hat{G}(\boldsymbol{n}(t+1))$. So, owing to (3.3), also*

$$G(\boldsymbol{n}(t)) > G(\boldsymbol{n}(t+1)) \quad \textit{and} \quad H(\boldsymbol{n}(t)) < H(\boldsymbol{n}(t+1))$$

Proof. Let i be the center of the jump of the configuration $\boldsymbol{n}(t)$ and let us set for simplicity $n = n_i$ and $m = n_{i+1}$, with $n - m \geq 2$. Then we have to prove that

$$n \log n + m \log m > (n-1)\log(n-1) + (m+1)\log(m+1)$$

This is equivalent to prove

$$\log(n^n\, m^m) > \log\left[(n-1)^{(n-1)}\,(m+1)^{(m+1)}\right]$$

That is, that

$$n^n\, m^m > (n-1)^{n-1}\,(m+1)^{m+1}$$

Equivalently that

$$\frac{n^n}{(n-1)^{n-1}} > \frac{(m+1)^{m+1}}{m^m}$$

Putting $n = m + k + 1$, with $k \geq 1$, this should lead to prove

$$\frac{[(m+k)+1]^{(m+k)+1}}{(m+k)^{m+k}} > \frac{(m+1)^{m+1}}{m^m} \tag{*}$$

Let us now consider the auxiliary function of real variable $x \geq 1$ defined as

$$f(x) = \frac{(x+1)^{x+1}}{x^x}$$

For $x = 1$ we have $f(1) = 4 \geq 0$, i.e., it is strictly positive; moreover, its derivative is

$$f'(x) = \frac{x^x(x+1)^{x+1}\log\frac{x+1}{x}}{\left[x^x(\log x + 1)\right]^2} > 0 \quad \text{for } x \geq 1$$

Therefore, the auxiliary function is increasing on the interval $x \geq 1$. Since for every integer number $k \geq 1$ it is $(m+k)+1 > (m+1)$, this means that the thesis is true. □

The following result is very important.

Proposition 3.2. *Let $\boldsymbol{n}(t)$ be a decreasing configuration and let $\boldsymbol{n}_i(t+1)$ and $\boldsymbol{n}_j(t+1)$, with $i < j$, be two configurations obtained by the application of the H-evolution rule to the jumps centered, respectively, at the cell i and at the cell j of the same "father" configuration $\boldsymbol{n}(t)$. Then, the two respective granularity entropies are different.*

In particular, we have that the former is strictly greater than the latter, i.e.

$$G(\boldsymbol{n}_i(t+1)) > G(\boldsymbol{n}_j(t+1))$$

Proof. Let i and j (with $i < j$) be the centers of two possible jumps of the configuration $\boldsymbol{n}(t)$. Then, we have to prove that

$$(n_i - 1)\log(n_i - 1) + (n_{i+1} + 1)\log(n_{i+1} + 1) > (n_j - 1)\log(n_j - 1) + (n_{j+1} + 1)\log(n_{j+1} + 1)$$

The above inequality follows straightforward just noticing that $\boldsymbol{n}(t)$ is a decreasing sequence and log is an increasing function. □

This result does not exclude that during the dynamical evolution there should be two configurations which assume the same granularity entropy, but at two different time instants.

Example 3.1. Let us consider the initial configuration $\boldsymbol{n}_0 = (6, 2, 2)$, which leads, by the re-iterate application of the H-evolution rule, to the following two possible paths:

$$\gamma^{(1)}_{\boldsymbol{n}_0} \equiv (6,2,2) \longrightarrow (6,2,1,1) \longrightarrow \cdots.$$
$$\gamma^{(2)}_{\boldsymbol{n}_0} \equiv (6,2,2) \longrightarrow (5,3,2) \longrightarrow (4,4,2) \longrightarrow (4,3,3) \longrightarrow \cdots.$$

It is straightforward to show that the configurations $(6, 2, 2, 1)$ and $(4, 3, 3)$ admit the same granularity entropy, but they are reached at two different levels (times) of the dynamical evolution. $G(6, 2, 1, 1) = G(4, 3, 3) = 8\log 2 + 6\log 3$. Trivially, $\gamma^{(1)}_{\boldsymbol{n}(0)}(1) = (6, 2, 2, 1)$ and $\gamma^{(2)}_{\boldsymbol{n}(0)}(3) = (4, 3, 3)$.

Along an admissible path (2.3) one starts from an initial configuration $\boldsymbol{n}_0$ of mass N characterized by a maximal order (minimal information entropy) and according to Proposition 3.1 evolves (preserving the total energy of the system) towards a maximal disorder (maximal information entropy) of the final configuration $\boldsymbol{n}_f$, quantified by the strictly increasing of the information uncertainty entropy according to the following chain (where we also present in the second line the information granularity entropy strictly decreasing behavior):

$$H(\boldsymbol{n}(0)) < H(\boldsymbol{n}(1)) < \cdots < H(\boldsymbol{n}(t)) < H(\boldsymbol{n}(t+1)) < \ldots \leq \log N$$

$$G(\boldsymbol{n}(0)) > G(\boldsymbol{n}(1)) > \cdots > G(\boldsymbol{n}(t)) > G(\boldsymbol{n}(t+1)) > \quad \ldots \geq 0$$

under the *total entropy conservation* (the system is isolated):

$$\forall t \in \mathbb{N}, \quad H(\boldsymbol{n}(t)) + G(\boldsymbol{n}(t)) = \log N$$

Similarly to the case of energy Lagrangian (2.4) we can introduce an *entropy Lagrangian* at time t along a possible path $\boldsymbol{n}$ as

$$\mathcal{L}(\boldsymbol{n}(t)) = H(\boldsymbol{n}(t)) - G(\boldsymbol{n}(t)) = \log N - 2G(\boldsymbol{n}(t)) \tag{3.4a}$$

$$= 2H(\boldsymbol{n}(t)) - \log N \tag{3.4b}$$

in which, by analogy, the term $H(\boldsymbol{n}(t))$ plays the role of *kinetic entropy* and $G(\boldsymbol{n}(t))$ the one of *potential entropy*. This entropy Lagrangian is a strictly increasing quantity during the dynamical evolution:

$$\mathcal{L}(\boldsymbol{n}(0)) < \mathcal{L}(\boldsymbol{n}(1)) < \cdots < \mathcal{L}(\boldsymbol{n}(t)) < \mathcal{L}(\boldsymbol{n}(t+1)) < \cdots \leq \log N$$

Always in analogy with the (2.5) we can introduce the *entropy action* along a possible path $\gamma_{\boldsymbol{n}_0}$

$$\mathcal{S}(\gamma_{\boldsymbol{n}_0}) := \sum_{t=0}^{t_f} \mathcal{L}(\boldsymbol{n}(t)) = 2 \sum_{t=0}^{t_f} H(\boldsymbol{n}(t)) - t_f \ \log N$$

with associated *reduced entropy action*

$$\mathcal{A}(\gamma_{\boldsymbol{n}_0}) := 2 \sum_{t=0}^{t_f} H(\boldsymbol{n}(t)) \tag{3.5}$$

So, along a path the dynamical evolution is characterized by a tendency towards the increase of disorder (increase of uncertainty entropy or, equivalently, decrease of granularity entropy). But we can make two different assumptions: the *adiabatic evolution* and the *anti-adiabatic evolution* towards the increase of disorder (uncertainty entropy). Let us start discussing the *step-by-step* adiabatic case.

Adiabatic Evolution. The *actual* path connecting the initial configuration $\boldsymbol{n}_0$ to the final configuration $\boldsymbol{n}_f$ is the one whose configuration at any time step t, among all the possible ones, minimizes (resp., maximizes) the uncertainty (resp., granularity) entropy, i.e., the disorder (resp., order) degree.

In other words, in this case the dynamics of the sand system evolves from ordered configurations to the disorder, but in an adiabatic way, i.e., the tendency towards the disorder is step-by-step the slower possible. This means that at any time step the evolution of the system is subject to a slow variation of the uncertainty entropy or, according to (3.4b), of the Lagrangian. We adopted this term in analogy with quantum mechanics in which it denotes "the evolution of systems subject to slowly varying Hamiltonian" [24].

Example 3.2. Making reference to the Example 2.1, the initial configuration has the normalized granularity entropy in the reduced form (3.3) $\hat{G}(6, 4, 2) = 17.68$.
The previously described local procedure of minimizing the uncertainty entropy can be applied step-by-step leading to the actual unique adiabatic dynamics represented by the Table 1. It corresponds to a step by step choice of maximal granularity entropy (equivalently minimal uncertainty entropy) with a corresponding step by step maximal action.

Let us note (see Fig. 6) that this adiabatic evolution towards disordered is one of the possible paths which satisfies the maximum action principle.

Unfortunately, the step-by-step adiabatic approach does not always guarantee the minimum of the *global* reduced entropy action. This global minimum is obtained computing the (3.5) for any possible path and then choosing the path which furnishes the minimum value. In the following example we show a case in which this global reduced entropy action is different from the one obtained using the step-by-step procedure.

Example 3.3. Starting from a pile of 12 elements with initial configuration $\boldsymbol{n}_0 = (12)$ and applying rule (1.3), the final configuration will be $\boldsymbol{n}_f = (4, 3, 2, 2, 1)$. Let us consider the adiabatic approach from configuration (8, 3, 1): it generates the subpath

$$(8, 3, 1) \longrightarrow (8, 2, 2) \longrightarrow (7, 3, 2) \longrightarrow (6, 4, 2)$$

whose reduced entropy action is 7.325.

Table 1
Sandpile actual path from the initial configuration (6, 4, 2) with respect to an adiabatic evolution towards disorder, ending at the equilibrium configuration (4, 3, 2, 2, 1) with (5, 4, 2, 1) as a black hole configuration.

Dynamics	U	G
(6, 4, 2)	34	17.682
(5, 5, 2)	33	17.480
(5, 4, 3)	31	16.888
(5, 4, 2, 1)	29	14.978
(5, 3, 3, 1)	28	14.638
(4, 4, 3, 1)	27	14.386
(4, 4, 2, 2)	26	13.862
(4, 3, 3, 2)	25	13.523
(4, 3, 3, 1, 1)	24	12.136
(4, 3, 2, 2, 1)	23	11.613

Table 2
Sandpile anti-adiabatic actual path from the initial configuration (6, 4, 2) towards disorder (local choice of minimal granularity entropy).

Dynamics	U	G
(6, 4, 2)	34	17.682
(6, 4, 1, 1)	33	16.295
(6, 3, 2, 1)	31	16.888
(5, 4, 2, 1)	29	14.978
(5, 3, 3, 1)	28	14.638
(5. 3, 2, 2)	27	14.115
(5, 3, 2, 1, 1)	26	12.729
(4, 4, 2, 1, 1)	25	12.476
(4, 3, 3, 1, 1)	24	12.136
(4, 3, 2, 2, 1	23	11.613

On the other hand, starting from the same configuration (8, 3, 1) the subpath corresponding to the minimum of the global reduced entropy action is

$$(8, 3, 1) \longrightarrow (7, 4, 1) \longrightarrow (6, 5, 1) \longrightarrow (6, 4, 2)$$

with corresponding value 7.283.

In a dual way we can introduce the

Anti-Adiabatic Evolution: The *actual* path connecting the initial configuration $\boldsymbol{n}_0$ to the final configuration $\boldsymbol{n}_f$ is the one whose configuration at any time step t, among all the possible ones, maximizes (resp., minimizes) the uncertainty (resp., granularity) entropy, i.e., the disorder (resp., order) degree.

In other words, the dynamics of the sand system evolves from ordered configurations to the disorder, but in a non-adiabatic way, i.e., step by step the tendency towards the disorder is the stronger possible.

Example 3.4. Making reference to the Example 3.2, we have seen that the initial configuration has normalized granularity entropy $\hat{G}(6, 4, 2) = 17.682$. The corresponding local anti-adiabatic evolution can be summarized by the (unique) dynamical evolution represented by the Table 2, which also in this case is one of the possible paths satisfying the principle of maximal action (see Fig. 7).

As previously illustrated, starting from a certain initial state, the anti-adiabatic (from another point of view, the Greedy) approach chooses at each step the configuration with the maximum information (Shannon) entropy (minimum granularity). A question arises at this point:

- whether the resulting anti-adiabatic path corresponds to the path which maximizes the global information (Shannon) entropy, i.e., to the path for which the sum of all the entropies of its configurations (equivalently, the reduced entropy action of Eq. (3.5)) is the maximum among all the possible paths.

Unfortunately, the answer is no: this anti-adiabatic approach does not always guarantee the maximum of the reduced entropy action, as illustrated in the following example.

Example 3.5. Let us start from a pile of 12 elements with initial configuration $\boldsymbol{n}_0 = (12)$, with its normal lattice $\Omega_{(12)}(12)$, and let us apply the rule (1.3) arriving to the final configuration $\boldsymbol{n}_f = (4, 3, 2, 2, 1)$.

In the present case we will have a difference between the anti-adiabatic path and the maximum reduced entropy action path during the transitions from the state (8, 3, 1) to the state (7, 3, 1, 1) only. Indeed, the anti-adiabatic approach from the configuration (8, 3, 1) generates the subpath

$$(8, 3, 1) \longrightarrow (7, 4, 1) \longrightarrow (7, 3, 2) \longrightarrow (7, 3, 1, 1)$$

whose reduced entropy action is 7.493. On the other hand, starting from (8, 3, 1) the subpath corresponding to the maximum of reduced action entropy is

$$(8, 3, 1) \longrightarrow (8, 2, 2) \longrightarrow (8, 2, 1, 1) \longrightarrow (7, 3, 1, 1)$$

with corresponding value 7.499.

4. 1D Sand CA model under a uniform blow of unidirectional wind

Formally, the cellular automata (CA) version of the total mass N sand pile model is based on a one-dimensional (1D) grid of N *cells* $\mathcal{L}(N) := \{1, 2, \ldots, N\}$ where every integer $i \in \mathcal{L}(N)$ represents the cell of position i. The set of *states* of the single CA cell is $\mathcal{S}(N) := \{0, 1, \ldots, N\}$ and a configuration is any map $n : \mathcal{L} \to \mathcal{S}(N)$ assigning to any cell of the lattice $i \in \mathcal{L}(N)$ the *number* $n_i \in \mathcal{S}(n)$ of granules in the position i of the pile, under the *total mass* N condition $\sum_{i\in\mathcal{L}(N)} n_i = N$ and the 0-*finiteness condition* $\exists k \in \mathcal{L}(N)$ s.t. $n_i > 0$ for every $i \leq k$ and $n_j = 0$ for every $j > k$:

n_1	…	n_i	…	n_k	0	…	0

(4.1)

The collection consisting of all such mass N *configurations* as usual will be denoted by $\Omega(N) \subset \mathcal{S}(N)^{\mathcal{L}(N)}$ and called the *phase space* of the CA.

A particular 1D sand-CA is characterized by a *radius* r *local rule* $f : \mathcal{S}(N)^{2r+1} \to \mathcal{S}(N)$ under the 0-*quiescent* condition $f(0, \ldots, 0, \ldots 0) = 0$. This local rule generates a *discrete time dynamical system* (DTDS) $\left(\Omega(N), F\right)$ based on the phase space $\Omega(N)$ of all possible one-dimensional definitively zero configurations and the global *next state* mapping $F : \Omega(N) \to \Omega_(N)$ which describes the phase space transition from the configuration $\boldsymbol{n} \in \Omega(N)$ to the next time configuration $F(\boldsymbol{n}) \in \Omega(N)$, defined by the *synchronous* (i.e., parallel) application to any cell i of the lattice of the local rule by the law

$$\forall i \in \mathbb{Z}, \quad [F(\boldsymbol{n})]_i := f(n_{i-r}, \ldots, n_i, \ldots, n_{i+r})$$

From the dynamical point of view, for any initial configuration $\boldsymbol{n}_0 \in \Omega(N)$ it is possible to obtain the (discrete time) *positive motion* as the time depending sequence $t \in \mathbb{N} \to \boldsymbol{n}(t) \in \Omega(N)$ defined as

$$\forall t \in \mathbb{N}, \quad \boldsymbol{n}(t) = F^t(\boldsymbol{n}_0)$$

This positive motion is the (unique) solution of the Cauchy problem for difference equations

$$\begin{cases} \boldsymbol{n}(t+1) = F(\boldsymbol{n}(t)) \\ \boldsymbol{n}(0) = \boldsymbol{n}_0 \end{cases}$$

Hence, the DTDS induced in this way from a CA is *autonomous* in the sense that the states of a positive motion are obtained by the iteration of the same (unchanged) next state function F.

4.1. Sand CA heuristic and a formal approach

In a first approach let us consider a radius $r = 1$ sand CA. The first requirement on the local rule is that it is 0-*quiescent*, i.e., $f(0, 0, 0) = 0$ in order to assure that there is no "creation" of granules with an increasing of the total mass N of the sand pile.

Let us discuss the heuristic of a "parallel" (but deterministic) application of the movement rule depicted in Fig. 1 to the particular initial configuration of the sandpile dynamics discussed in Example 2.1 according to the following Fig. 8.
Let us discuss the adopted heuristic:

Step 1 In the initial configuration (0|6, 4, 2, 0, 0) the cell $i = 1$ is the center of a right jump and so the local rule f applied to the input triple (0, 6, 4) performs the transformation $(0, 6, 4) \xrightarrow{f} 5$, corresponding to the decrease of the total number of granules located at $i = 1$ according to $6 \to 5$. On the other hand, the cell $i = 2$ is the center of a left and a right jump corresponding to the acceptance of a granule from the cell $i = 0$ and the assignment of a granule to the cell $i = 2$ maintaining in this way the same number 4 of granules with the corresponding local rule transition $(6, 4, 2) \xrightarrow{f} 4$. Also the cell $i = 2$ is the center of a right and a left jump corresponding to a transition of the local rule $(2, 0, 0) \xrightarrow{f} 1$. And finally, we have the cell $i = 3$ which is the center of a right jump with corresponding transition $(4, 2, 0) \xrightarrow{f} 1$, with an increase from 0 to 1 of the number of granules located in this cell.

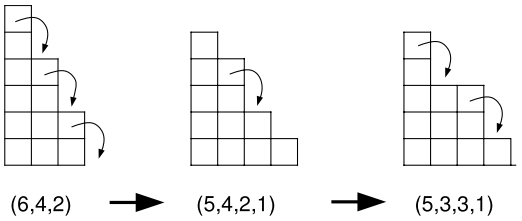


Fig. 8. The first three iterations of the CA local rule according to the heuristic of a parallel application of the H-evolution rule.

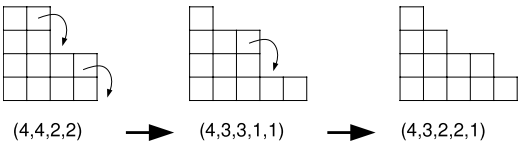


Fig. 9. The last iterations of the CA local rule according to the heuristic of a parallel application of the H-evolution rule leading to the equilibrium configuration (4, 3, 2, 2, 1).

Step 2 The time step 1 configuration $(0|5, 4, 2, 1, 0)$ is such that the cell $i = 1$ maintains the same number 5 of granules under the application of the local rule f according to the transition $(0, 5, 4) \xrightarrow{f} 5$, whereas the cell $i = 2$ is submitted to the local rule transition $(5, 4, 2) \xrightarrow{f} 3$ decreasing from 4 to 3 the total number of granules located in it. The cell $i = 3$ as center of a left jump receives a granule from the cell $i = 2$ passing from a number of 2 to a number of 3 granules according to the local rule transition $(4, 2, 1) \xrightarrow{f} 3$. And finally the cells $i = 4$ and $i = 5$ remain with the same number of granules according to the local rule transitions $(2, 1, 0) \xrightarrow{f} 1$ and $(1, 0, 0) \xrightarrow{f} 0$.

The further application of the CA local rule leads to the transitions in Fig. 9 reaching the equilibrium configuration (4, 3, 2, 2, 1).

Comparing this CA deterministic evolution with the Table 1, it is clear that the CA dynamics is in agreement with the corresponding adiabatic evolution towards disorder.

Note that all the above CA heuristic can be formalized by the following local rule formalization:

$$f(u,x,y) = \begin{cases} x+1 & \text{if} \quad x \le u-2 \quad \text{and} \quad x \le y+1 \\ x-1 & \text{if} \quad x \ge y+2 \quad \text{and} \quad x \ge u-1 \\ x & \text{otherwise} \end{cases} \tag{4.2}$$

Of course, the "left directed" information formalized by the local rule is the origin of the global action of a wind which uniformly blows from the left to the right direction. This corresponds to a very particular physical situation.

Let us recall that in [11–13] a general notion of sand automata (SA) which permits the definition of more rules than the horizontal H one exposed here is given (for some complementary information on SA see [14–16], and for some results on CA see for instance [9,10,1,17,18]). Obviously, the present approach formalized by the local rule (4.2) can be obtained as a particular case of this latter. In fact, according to SA formalism, and in the *size* 1 context (which is a very particular case of their approach defined in the more general context of arbitrary size or precision $l \in \mathbb{N}$, see for instance [11]) one considers first of all the so-called *measuring device* β_1^x of side 1 and *reference height* x as a function from $\mathbb{Z}$ to $\{-\infty, -1, 0, 1, +\infty\}$ defined in the following way:

$$\beta_1^x(z) = \begin{cases} +\infty & \text{if} \quad x+1 < z \\ -\infty & \text{if} \quad x-1 > z \\ z-x & \text{otherwise} \end{cases} \tag{4.3}$$

Note that in particular $\beta_1^x(x-1) = -1$, $\beta_1^x(x) = 0$, and $\beta_1^x(x+1) = 1$.

The SA considered here can be defined as the pair $\langle 1, \lambda \rangle$ where $l = 1$ is the side and $\lambda : (\{-\infty, -1, 0, 1, \infty\})^2 \to (\{-\infty, 0, -1, 0, 1, \infty\})$ is an *auxiliary function*, improperly called local rule, with the help of which one can define the "true" local rule of a CA defined as follows:

$$\tilde{f}(u,x,y) = \begin{cases} x & \text{if} \quad x = \pm\infty \\ x + \lambda(\beta_1^x(u), \beta_1^x(y)) & \text{otherwise} \end{cases} \tag{4.4}$$

Let $\tilde{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, +\infty\}$, then the induced global next state function defined on the configuration space $\tilde{\mathbb{Z}}^{\mathbb{Z}}$ assumes, for any $\underline{x} \in \tilde{\mathbb{Z}}^{\mathbb{Z}}$ and every cell $i \in \mathbb{Z}$, the form

$$[\tilde{F}(\underline{x})]_i = \begin{cases} x_i & \text{if} \quad x_i = \pm\infty \\ x_i + \lambda(\beta_1^{x_i}(x_{i-1}), \beta_1^{x_i}(x_{i+1})) & \text{otherwise} \end{cases} \tag{4.5}$$

As pointed out in Example 1 of [11], the SA version of the SPM is obtained by the following peculiar version of the auxiliary function:

$$\lambda(a,b) = \begin{cases} +1 & \text{if } a = +\infty \text{ and } b \neq -\infty \\ -1 & \text{if } a \neq +\infty \text{ and } b = -\infty \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

leading, according to (4.4), to the following local rule for SPM:

$$\tilde{f}(u,x,y) = \begin{cases} x+1 & \text{if } \beta_1^x(u) = +\infty \text{ and } \beta_1^x(y) \neq -\infty \\ x-1 & \text{if } \beta_1^x(u) \neq +\infty \text{ and } \beta_1^x(y) = -\infty \\ x & \text{otherwise} \end{cases} \tag{4.7}$$

Looking back to the Eq. (4.3), one can immediately observe that the condition $\beta_1^x(u) = +\infty$ is equivalent to say that $u \geq x+2$, i.e., a grain can fall from pile with reference u to the pile x. On the other hand, the condition $\beta_1^x(y) = -\infty$ can be expressed as $y \leq x-2$, so a grain can fall on y from pile with reference x. Starting from these facts, one can rewrite Eq. (4.7) into the form of Eq. (4.2).

4.2. An interesting counter-example

In the previous Section 4.1 we have discussed a particular example of the CA dynamics with initial configuration (6, 4, 2), in which the CA dynamics fits with one of the two sand pile dynamics induced by the information entropy. This could lead to the conjecture that this behavior happens for any sand pile dynamics. This is false, as the following example shows.

Example 4.1. Let us consider the initial configuration $\boldsymbol{n}_0 = (12, 0)$. Both adiabatic and anti-adiabatic dynamics converge towards configuration (8, 3, 1).

From this configuration onwards, the adiabatic evolution is

$$\begin{aligned} &(8,3,1) \longrightarrow (8,2,2) \longrightarrow (7,3,2) \longrightarrow (6,4,2) \longrightarrow (5,5,2) \longrightarrow \\ &(5,4,3) \longrightarrow (5,4,2,1) \longrightarrow (5,3,3,1) \longrightarrow (4,4,3,1) \longrightarrow \\ &(4,4,2,2) \longrightarrow (4,3,3,2) \longrightarrow (4,3,3,1,1) \longrightarrow (4,3,2,2,1) \end{aligned}$$

where from configuration (6, 4, 2) onward it is the same dynamics shown in Table 1. On the other hand, the anti-adiabatic evolution from the configuration (8, 3, 1) onwards is

$$\begin{aligned} &(8,3,1) \longrightarrow (7,4,1) \longrightarrow (7,3,2) \longrightarrow (7,3,1,1) \longrightarrow (6,4,1,1) \longrightarrow \\ &(6,3,2,1) \longrightarrow (5,4,2,1) \longrightarrow (5,3,3,1) \longrightarrow (5,3,2,2) \longrightarrow \\ &(5,3,2,1,1) \longrightarrow (4,4,2,1,1) \longrightarrow (4,3,3,1,1) \longrightarrow (4,3,2,2,1) \end{aligned}$$

where from the black hole configuration (5, 4, 2, 1) onward it is the same dynamics shown in Table 2.

The CA dynamics of initial configuration (8, 3, 1) is

$$\begin{aligned} &(8,3,1) \longrightarrow (7,3,2) \longrightarrow \boxed{(6,4,1,1)} \longrightarrow (5,4,2,1) \longrightarrow (5,3,3,1) \longrightarrow \\ &\boxed{(4,4,2,2)} \longrightarrow (4,3,3,1,1) \longrightarrow (4,3,2,2,1) \end{aligned}$$

which does not fit any of the two granularity entropy dynamics. In particular, one can observe that the adiabatic path excludes the configuration (6, 4, 1, 1) of the CA dynamics while the anti-adiabatic one does not admit the configuration (4, 4, 2, 2) of this CA dynamics. But note that, anyway, the CA dynamics fits with one of the paths satisfying the principle of maximal action (see Fig. 7 for the partial path of initial configuration (6, 4, 1, 1)).

5. Integer partitions: thermodynamics (Boltzmann) entropy

The drawback showed in the previous Example 4.1 should lead to the investigation of some other form of entropy with respect to which there is an agreement between the CA dynamics and the evolution characterizing the general behavior of this new entropy. Let us consider here as potential candidate the thermodynamical Boltzmann entropy.

Given a numerical partition $\boldsymbol{n} = (n_1, n_2, \ldots, n_N)$ of total mass N, the Boltzmann entropy is defined as

$$S(\boldsymbol{n}) = \log W_N(\boldsymbol{n})$$

where $W_N(\boldsymbol{n})$, also called *thermodynamics probability*, is the following quantity

$$W_N(\boldsymbol{n}) = \frac{N!}{n_1! n_2! \ldots n_N!}$$

The following proposition gives us a formula that allows us to calculate step by step the Boltzmann entropy of a configuration. Moreover, it will be the basis to discover an analogy with the case of information entropy.

Proposition 5.1. *Let* $\log W_N(\boldsymbol{n}(t))$ *be the Boltzmann entropy at time* t*. Then, at time* $t+1$*, i.e., after a single jump at a generic position* i *according to the H-rule, we have that*

$$W_N(\boldsymbol{n}(t+1)) = \eta_i(t) \cdot W_N(\boldsymbol{n}(t))$$

where

$$\eta_i(t) := \frac{n_i(t)}{n_{i+1}(t)+1}$$

This quantity represents the growth factor *of jump* i *at time* t*.*

Proof. If we consider the generic jump at position i, we have:
at time t

$$W_N(\boldsymbol{n}(t)) = \frac{N!}{n_1(t)!n_2(t)!\ldots n_i(t)!n_{i+1}(t)!\ldots n_N(t)!};$$

and at time $t+1$

$$\begin{aligned} W_N(\boldsymbol{n}(t+1)) &= \frac{N!}{n_1(t+1)!n_2(t+1)!\ldots (n_i)(t+1)!(n_{i+1})(t+1)!\ldots n_N(t+1)!} \\ &= \frac{N!}{n_1(t)!n_2(t)!\ldots (n_i(t)-1)!(n_{i+1}(t)+1)!\ldots n_N(t)!} \end{aligned}$$

Hence,

$$\frac{W_N(\boldsymbol{n}(t+1))}{W_N(\boldsymbol{n}(t))} = \frac{n_i(t)}{n_{i+1}(t)+1} = \eta_i(t) \quad \square$$

Let us observe that, omitting the subscript i of the jump site in the growth factor, in general we obtain:

$$W_N(\boldsymbol{n}(t+k)) = \prod_{h=0}^{k-1} \eta(t+h) \cdot W_N(\boldsymbol{n}(t))$$

This new entropy falls in the same situation of the of Example 4.1.

Example 5.1. Let us reconsider the initial configuration $\boldsymbol{n}_0 = (8, 3, 1)$.
With respect the Boltzmann entropy, the adiabatic evolution is

$$\begin{aligned} &(8,3,1) \longrightarrow (8,2,2) \longrightarrow (8,2,1,1) \longrightarrow (7,3,1,1) \longrightarrow \boxed{(6,4,1,1)} \longrightarrow \\ &(5,5,1,1) \longrightarrow (5,4,2,1) \longrightarrow (5,3,3,1) \longrightarrow (4,4,3,1) \longrightarrow \\ &(4,4,2,2) \longrightarrow (4,3,3,2) \longrightarrow (4,3,3,1,1) \longrightarrow (4,3,2,2,1) \end{aligned}$$

On the other hand the anti-adiabatic evolution is

$$\begin{aligned} &(8,3,1) \longrightarrow (7,4,1) \longrightarrow \boxed{(7,3,2)} \longrightarrow (7,3,1,1) \longrightarrow (7,2,2,1) \longrightarrow \\ &(6,3,2,1) \longrightarrow (5,4,2,1) \longrightarrow (5,3,3,1) \longrightarrow (5,3,2,2) \longrightarrow \\ &(5,3,2,1,1) \longrightarrow (4,4,2,1,1) \longrightarrow (4,3,3,1,1) \longrightarrow (4,3,2,2,1) \end{aligned}$$

Of course, the CA dynamics is the same of the Example 4.1. Also in this case it does not fit any of the two Boltzmann entropy dynamics. In particular, one can observe that the adiabatic path excludes the CA dynamics configuration $(7, 3, 2)$ while the anti-adiabatic one does not admit the configuration $(6, 4, 1, 1)$.

Moreover, the Boltzmann entropy has another significant drawback: it sometimes can be affected by the lack of determinism as the following example explains.

Example 5.2. Let us consider the initial configuration $\boldsymbol{n}_0 = (6, 3, 1)$ which leads, by the application of the H-evolution rule, to the following two next time configurations (Fig. 10):

$$\boldsymbol{n}(1a) = (5, 4, 1) \qquad \boldsymbol{n}(1b) = (6, 2, 2)$$

These two derived configurations result to have the same Boltzmann entropy, indeed

$$W(5,4,1) = \frac{10!}{5!\,4!} = 1260 \quad \text{and} \quad W(6,2,2) = \frac{10!}{6!\,2!\,2!} = 1260$$

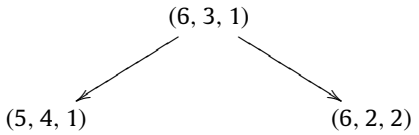


Fig. 10. First iteration starting from (6, 3, 1).

6. Comments and conclusions

The intrinsical non-deterministic dynamics of sand piles by the standard H-rule of evolution is analyzed from the deterministic point of view using the Shannon entropy approach to information systems applied to integer partitions. The use of the information entropy allows one to select a unique path generated by the sand pile rule according to either the min or the max entropy variation during each single time step.

This approach by entropy is compared with the parallel synchronous CA deterministic one in which any jump of a given configuration is solved by the simultaneous application of the H-rule in a unique time step. In some example this CA approach fits with the entropy one, but it is possible to give a counter-example in which this behavior does not happens. This leads to some possible further investigation in which, maintaining the parallel H-rule CA model, it will be important to find some other form of entropy with respect to which the fit of the two deterministic dynamics is proved be theorem. Of course, another possible approach is the one in which the Shannon entropy deterministic dynamics is tried to fit with some other CA model of sand pile.

On the other hand, in all the examples treated in this paper (and some others investigated by us but not published here) the deterministic CA sand dynamics agree with the Hamiltonian max action principle, and the possible general result induced from these finite number of cases could be an open problem for a possible future investigation.

Another possible development is the alternate bi-CA local rule in which in a time step one apply in a parallel way the left-to-right (LR) local evolution rule to any jump of the configuration and then to the so obtained new configuration a parallel local right-to-left (RL) evolution rule. This is very interesting from the physical point of view since it corresponds to a model without a privileged direction of blowing wind. The obtained discrete time dynamical system (DTDS) is no more autonomous, but the global next state function varies during time according to a fixed alternate sequential application of the two local rules RL and LR. It will be the argument of a forthcoming paper the question of introducing an autonomous DTDS generated by a fixed CA local rule which in some sense should be able to simulate the same "absence of wind" situation, called the *bi-CA sand model*.

References

[1] L. Acerbi, A. Dennunzio, E. Formenti, Conservation of some dynamical properties for operations on cellular automata, Theoretical Computer Science 410 (2009) 3685–3693.
[2] R. Anderson, L. Lovasz, P. Shor, E. Tardos, S. Winograd, Disks, ball and walls: analysis of a combinatorial game, American Mathematical Monthly 96 (1989) 481–493.
[3] D. Bianucci, G. Cattaneo, Information entropy and granulation co–entropy of partitions and coverings: a summary, in: J.F. Peter, A. Skowron (Eds.), Special Issue on Foundations of Rough Sets, in: Transactions on Rough Sets (TRS), vol. 5656, Springer–Verlag, 2009, pp. 15–66.
[4] D. Bianucci, G. Cattaneo, D. Ciucci, Entropies and co–entropies for incomplete information systems, in: Lecture Notes in Artificial Intelligence, vol. 4481, Springer–Verlag, 2007, pp. 84–92.
[5] D. Bianucci, G. Cattaneo, D. Ciucci, Entropies and co–entropies of coverings with application to incomplete information systems, Fundamenta Informaticae 75 (2007) 77–105.
[6] G. Birkhoff, Lattice theory, in: American Mathematical Society Colloquium Publication, vol. XXV, third ed., American Mathematical Society, Providence, Rhode Island, 1967, first edition 1940, second (revisited) edition 1948.
[7] T. Brylawski, The lattice of integer partitions, Discrete Mathematics 6 (1973) 201–219.
[8] G. Cattaneo, D. Ciucci, D. Bianucci, Entropy and co-entropy of partitions and coverings with applications to roughness theory, in: R. Bello, R. Falcon, W. Pedrycz, J. Kacprzyk (Eds.), Granular Computing: At the Junction of Fuzzy Sets and Rough Sets, in: Studies in Fuzziness and Soft Computing, vol. 224, Springer–Verlag, 2008, pp. 55–77.
[9] G. Cattaneo, A. Dennunzio, L. Margara, Chaotic subshifts and related languages: Applications to one-dimensional cellular automata, Fundamenta Informaticae 52 (2002) 39–80.
[10] G. Cattaneo, A. Dennunzio, L. Margara, Solution of some conjectures about topological properties of linear cellular automata, Theoretical Computer Science 325 (2004) 249–271.
[11] J. Cervelle, E. Formenti, On sand automata, in: STACS '03: Proceedings of the 20th Annual Symposium on Theoretical Aspects of Computer Science, London, UK, Springer-Verlag, 2003, pp. 642–653.
[12] J. Cervelle, E. Formenti, B. Masson, Basic properies for sand automata, in: 30th International Symposium of Mathematical Foundations of Computer Science, in: Lecture Notes in Computer Science, no. 3618, Springer-Verlag, MFCS'05, Berlin, 2005, pp. 192–211.
[13] J. Cervelle, E. Formenti, B. Masson, Fundamental study: from sandpiles to sand automata, Theoretical Computer Science 381 (2007) 1–28.
[14] A. Dennunzio, P. Guillon, B. Masson, Stable dynamics of sand automata, in: TCS '08: Proceedings of the 5th IFIP International Conference on Theoretical Computer Science, London, UK, Springer-Verlag, 2008, pp. 157–169.
[15] A. Dennunzio, P. Guillon, B. Masson, Topological properties of sand automata as cellular automata, in: JAC '08: 1th Symposium on Cellular Automata, 2008, pp. 216–227.
[16] A. Dennunzio, P. Guillon, B. Masson, Sand automata as cellular automata, Theoretical Computer Science 410 (2009) 3962–3974.
[17] A. Dennunzio, Di Lena, E. Formenti, L. Margara, On the directional dynamics of additive Cellular Automata, Theoretical Computer Science 410 (2009) 4823–4833.
[18] F. Farina, A. Dennunzio, A predator–prey cellular automaton with parasitic interactions and environmental effects, Fundamenta Informaticae 83 (2008) 337–352.
[19] E. Formenti, B. Masson, On computing fixed points for generalized sandpiles, Journal of Unconventional Computing 2 (2005) 51–72.

[20] E. Formenti, B. Masson, T. Pisokas, Advanced in symmetric sand piles, Fundamenta Informaticae 76 (2007) 91–112.
[21] E. Goles, M.A. Kiwi, Games on line graphs and sand pile automata, Theoretical Computer Science 115 (1993) 321–349.
[22] E. Goles, M. Morvan, H.D. Phan, Sandpiles and ordered structures of integer partitions, Discrete Applied Mathematics 17 (2002) 51–64.
[23] L.D. Landau, E.M. Lifshitz, Mechanics, in: Course of Theoretical Physics, vol. 1, Butterworth–Einemann, Oxford, 1999, Translated from the 3rd revised and enlarged edition of Mekanika, Nauka, Moskow 1993.
[24] M.S. Sarandy, L.A. Wu, D.A. Lidar, Consistency of the adiabatic theorem, Quantum Information Processing 3 (2004) 331–349.
[25] C.E. Shannon, A mathematical theory of communication, The Bell System Technical Journal 27 (1948) 379–423. 623–656.
[26] J. Spencer, Balancing vectors in the max norms, Combinatorica 6 (1986) 55–65.