Journal of Combinatorial Theory, Series A 120 (2013) 39–48

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

A characterization of metacirculants [☆]

Cai Heng Li [a,b], Shu Jiao Song [c,d], Dian Jun Wang [d]

[a] School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, PR China
[b] School of Mathematics and Statistics, The University of Western Australia, Crawley 6009 WA, Australia
[c] Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China
[d] Department of Science and Mathematics, Tsinghua University, Beijing 100084, PR China

ARTICLE INFO

Article history:
Received 1 December 2011
Available online 13 July 2012

Keywords:
Metacirculant
Edge-transitive
Cubic

ABSTRACT

Metacirculants were introduced by Alspach and Parsons in 1982 and have been a rich source of various topics since then, including the Hamiltonian path problem in metacirculants. A metacirculant has a vertex-transitive metacyclic subgroup of automorphisms, and a long-standing interesting question in the area is if the converse statement is true, namely, whether a graph with a vertex-transitive metacyclic automorphism group is a metacirculant. We shall answer this question in the negative, and then present a classification of cubic metacirculants.

1. Introduction

A *circulant* is a graph that has a cyclic vertex-transitive automorphism group. Alspach and Parsons introduced metacirculants in [2], refer to [14] for the following stated version of definition.

Definition 1.1. A graph $\Gamma = (V, E)$ is called an (m,n)-*metacirculant*, where m, n are positive integers, if Γ is of order $|V| = mn$ and has two automorphisms ρ, σ such that

(a) $\langle\rho\rangle$ is semiregular and has m orbits on V,
(b) σ cyclically permutes the m orbits of $\langle\rho\rangle$ and normalizes $\langle\rho\rangle$, and
(c) σ^m fixes at least one vertex of Γ.

☆ This work was partially supported by NSFC Grants of China and an ARC Discovery grant of Australia.

E-mail addresses: cai.heng.li@uwa.edu.au (C.H. Li), shujiao.song@gmail.com (S.J. Song), djwang@mail.tsinghua.edu.cn (D.J. Wang).

0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2012.06.010

There are lots of interesting graphs which are (m,n)-metacirculants. Circulants are of course metacirculants; the Petersen graph, and vertex-transitive generalized Petersen graphs are $(2,n)$-metacirculants; dihedrants (graphs which have a dihedral vertex-regular group of automorphisms) are $(2,n)$-metacirculants. For any $m,n \geqslant 2$, the *grids* $\mathbf{C}_n \square \mathbf{C}_m$ are (m,n)-metacirculants.

In the literature, the class of metacirculants provides a rich source of various research projects, see for instance [3,15,16,19,20]. It has been therefore studied extensively in the past 30 years. Special classes of metacirculants have been well investigated, see [1,8,10] for edge-transitive circulants; [6,14] for 2-arc-transitive dihedrants; [11,21] for half-arc-transitive metacirculants of prime-power order; [15,23] for half-arc-transitive metacirculants of valency 4.

We observe that, using the notation defined in Definition 1.1, the subgroup $\langle \rho, \sigma \rangle$ is metacyclic and transitive on the vertex set V. A long-standing interesting question is whether a graph is a metacirculant if it has a vertex-transitive metacyclic subgroup of automorphisms. For example, Marušič and Šparl in [15] called a graph a *weak metacirculant* if it has a vertex-transitive metacyclic automorphism group, and asked "if the class of weak metacirculants is indeed larger than that of metacirculants". We shall prove that the class of weak metacirculants is indeed larger than the class of metacirculants.

The 'general' metacirculants called 'weak metacirculant' are defined as follows.

Definition 1.2. A graph $\Gamma = (V, E)$ is called a *weak metacirculant* if $\mathrm{Aut}\,\Gamma$ contains a subgroup R which is metacyclic and transitive on V. To emphasis the transitive metacyclic subgroup R, this graph Γ is called a *weak metacirculant* relative to the group R.

The Petersen graph is a metacirculant relative to the Frobenius group $\mathbb{Z}_5{:}\mathbb{Z}_4$; more generally, vertex-transitive generalized Petersen graphs are metacirculants relative to $\mathbb{Z}_n{:}\mathbb{Z}_4$ or $\mathbb{Z}_n{:}\mathbb{Z}_2$. The point-hyperplane incidence graph of the projective geometry $\mathrm{PG}(d-1,q)$ is a metacirculant relative to the group $\mathbb{Z}_{(q^d-1)/(q-1)}.\mathbb{Z}_2$; the Holt graph (the smallest half-arc-transitive graph) is a metacirculant relative to the 3-group $\mathbb{Z}_9{:}\mathbb{Z}_3$ [21].

A graph $\Gamma = (V, E)$ is called *half-arc-transitive* if $\mathrm{Aut}\,\Gamma$ is transitive on both V and E but not transitive on the set of arcs. Half-arc-transitive graphs have been extensively studied, and many examples are constructed as weak metacirculants refer to [3,12,15,19,21,23]. The first theorem of this paper stated below shows that the class of weak metacirculants is strictly larger than the class of metacirculants. It clarifies the long-standing problem mentioned above.

Theorem 1.3. *Each non-split metacyclic p-group with p an odd prime has a half-arc-transitive weak metacirculant of valency* 4 *which is not a metacirculant.*

The next theorem gives a classification of cubic weak metacirculants, and it turns out that cubic weak metacirculants are all metacirculants. See Section 3 for the definitions of the graphs appeared in the theorem.

Theorem 1.4. *Let Γ be a connected cubic weak metacirculant relative to a group R. Then one of the following holds*:

(i) $\Gamma = \mathbf{M}_{2m}$, *a Möbius band of order* $2m$;
(ii) Γ *is a generalized Petersen graph* $\mathbf{P}(n,k)$, *where* $k^2 = \pm 1 \pmod n$ *or* $(n,k) = (10,2)$;
(iii) $\Gamma = \mathbf{C}(2n, 2k+1)$ *or* $\mathbf{Dih}(\ell, m)$ *with* $\ell m = 2n$, *and* $R = \mathrm{D}_{2n}$;
(iv) $\Gamma = \mathbf{MeC}_1(\ell, m, k)$ *or* $\mathbf{MeC}_2(\ell, m, k)$, *which is a cover of* $\mathbf{M}_{m/2}$ *or* $\mathbf{C}_{m/2} \square \mathrm{K}_2$, *respectively.*

We remark that $\mathbf{P}(10,2) \cong \mathbf{MeC}_1(5,4,2)$ is the dodecahedron graph; in the notation of [17], the graph $\mathbf{C}(2n, 2k+1)$ is of type $F(n,i)$, and graph $\mathbf{Dih}(l,m)$ is of type $H(n,i,j)$.

We will construct metacirculants and weak metacirculants in Section 2, provide a proof of Theorem 1.3. In Section 3, we construct various examples of cubic weak metacirculants, and then in Section 4, we classify cubic weak metacirculants, and prove Theorem 1.4.

2. Metacirculants and AP-metacirculants

We study here the relation between metacirculants and weak metacirculants, and prove Theorem 1.3.

We first quote some properties of metacyclic groups. Let R be a *metacyclic* group, that is, R has a cyclic normal subgroup M such that R/M is also cyclic. Then R is an extension of the cyclic subgroup $M = \mathbb{Z}_m$ by the cyclic group $R/M = \mathbb{Z}_n$, and we write $R \cong \mathbb{Z}_m.\mathbb{Z}_n$. Thus, R has a generating set $\{a, b\}$ such that

$$a^m = b^l = 1, \qquad b^{l/n} = a^r, \quad \text{and} \quad a^b = a^s,$$

where m, n, l, r, s are integers. If the extension $\mathbb{Z}_m.\mathbb{Z}_n = \mathbb{Z}_m{:}\mathbb{Z}_n$ is split, then R is called a *split metacyclic group*.

Finite metacyclic p-groups have been classified [7,22]. For an odd prime p, these groups satisfy the following lemma.

Lemma 2.1. *(See [22].) Let $p > 2$ be an odd prime, and let R be a metacyclic p-group. Then R has a presentation:*

$$R = \langle a, b \mid a^{p^{r+s+u}} = 1,\ b^{p^{r+s+t}} = a^{p^{r+s}},\ b^{-1}ab = a^{1+p^r} \rangle.$$

Further, R is split if and only if $stu = 0$.

In terms of metacyclic groups, we have a criterion for a weak metacirculant to be a metacirculant.

Lemma 2.2. *Each metacirculant has a split metacyclic group of automorphisms.*

Proof. We observe that, using the notation defined in Definition 1.1, ρ and σ generate a subgroup $R := \langle \rho, \sigma \rangle \leqslant \mathrm{Aut}\,\Gamma$, such that $\langle \rho \rangle$ is normal. Hence R is metacyclic, and transitive on the vertex set. Since σ^m fixes at least one vertex, either $\sigma^m = 1$, or $\langle \sigma^m \rangle$ is core-free in R. For the later, $\langle \sigma^m \rangle \cap \langle \rho \rangle = 1$. So, in either case, $R = \langle \rho \rangle{:}\langle \sigma \rangle$ is split. □

For a weak metacirculant graph Γ, this lemma tells us that, if $\mathrm{Aut}\,\Gamma$ does not contain split vertex-transitive metacyclic subgroups, then Γ is not a metacirculant. Because of this, a weak metacirculant which has a vertex-transitive split metacyclic group of automorphisms sometimes will be called a *split metacirculant*. Otherwise, a weak metacirculant Γ is said to be a *non-split metacirculant* if $\mathrm{Aut}\,\Gamma$ does not contain any split metacyclic subgroup which is vertex-transitive.

A large special class of weak metacirculants are Cayley graphs on metacyclic groups.

Definition 2.3. For a group R and a subset S, the *Cayley digraph* $\mathrm{Cay}(R, S)$ is the digraph with vertex set R and arc set (x, y) with $yx^{-1} \in S$. If further $S = S^{-1} = \{s^{-1} \mid s \in S\}$, then $\mathrm{Cay}(R, S)$ is undirected, simply called a *Cayley graph*. For convenience, an undirected edge $\{x, y\}$ is called a *g-edge* if $yx^{-1} = g \in S$.

A Cayley graph $\mathrm{Cay}(R, S)$ has a subgroup of automorphisms which is isomorphic to R and regular on the vertex set, that is, the right regular multiplication group $\hat{R}$. Thus, if R is metacyclic, then $\mathrm{Cay}(R, S)$ is a weak metacirculant. However, not all weak metacirculants are Cayley graphs. For example, the Petersen graph is a metacirculant relative to $\mathbb{Z}_5{:}\mathbb{Z}_4$, but it is not a Cayley graph. A weak metacirculant is called a *metacirculant Cayley graph* if it is a Cayley graph of a metacyclic group. We do not know whether it is true that a weak metacirculant which is a Cayley graph must be a weak metacirculant Cayley graph.

Suppose that $\Gamma = \mathrm{Cay}(R, S)$ is G-edge-transitive such that $\hat{R} \leqslant G \leqslant \mathrm{Aut}\,\Gamma$. If $\hat{R}$ is normal in G, then Γ is called a *normal edge-transitive* Cayley graph. We will call a G-edge-transitive weak metacirculant Γ a *normal edge-transitive weak metacirculant* if G contains a normal transitive metacyclic subgroup. In particular, if $\mathrm{Aut}\,\Gamma = \hat{R}$ and Γ is undirected, then Γ is called a *graphical regular representation* of R, or simply called a *GRR*. If Γ is directed and $\mathsf{Aut}\,\Gamma = \hat{R}$, then Γ is called a directed regular representation of R, or *DRR* for short.

In the following we will construct weak metacirculants which are not metacirculants.

Example 2.4. Let P be a generalized quaternion group of order at least 16, that is, for some $n \geqslant 2$,

$$P = \langle a, b \mid a^{2^n} = b^2,\ b^4 = 1,\ a^b = a^{-1} \rangle.$$

Let $S = \{a, b\}$, and let $\Gamma = \mathsf{Cay}(P, S)$. Then Γ is connected, and $\mathsf{Aut}\,\Gamma = \hat{P}$. Since P is metacyclic, Γ is a weak metacirculant. Now $\langle \hat{a} \rangle$ has index 2 in $\hat{P}$, and it thus has exactly two orbits Δ_1 and Δ_2 on the vertex set. However, $\mathsf{Aut}\,\Gamma$ has only one involution $\hat{a}^{2^{n-1}}$ which fixes Δ_1 and Δ_2 but has no fixed point. Thus, $\mathsf{Aut}\,\Gamma$ has no element g which interchanges Δ_1 and Δ_2 such that g^2 fixes a vertex. So Γ is not a metacirculant. $\square$

In the example, an important property is that the Cayley digraph Γ is a DRR of P. It follows from Lemma 2.2 that GRRs and DRRs of a non-split metacyclic group are non-split weak metacirculants. In the early 1980s, the finite groups which have GRR and DRR were determined. By Babai [4], except for the quaternion group Q_8, every non-split metacyclic p-group with p prime has a DRR. Thus, we have the following conclusion.

Proposition 2.5. *Except for* Q_8, *every generalized quaternion group has at least one non-split directed weak metacirculant.*

Undirected non-split weak metacirculants can be constructed from metacyclic p-groups for odd prime p.

Lemma 2.6. *Let P be a non-split metacyclic p-group with $p \geqslant 3$, and let Γ be a Cayley graph of P. Suppose that $X := \mathsf{Aut}\,\Gamma = \hat{P}X_v$ such that the vertex stabilizer X_v is a 2-group. Then Γ is a non-split weak metacirculant.*

Proof. We observe that a Sylow p-subgroup of $X = \mathsf{Aut}\,\Gamma$ is isomorphic to P and regular on the vertex set. Let R be a transitive metacyclic subgroup of X. Then R contains a Sylow p-subgroup, we assume that $R \geqslant P$. Suppose that R is split. Then $R = \langle a \rangle{:}\langle b \rangle$. Since Γ is of order a power of p, $\langle a \rangle$ is a p-group. Let b_p be the p-part of b, that is, $\langle b_p \rangle$ is a Sylow p-subgroup of $\langle b \rangle$. Then $R_p = \langle a \rangle{:}\langle b_p \rangle$ is a Sylow p-subgroup of X. This is a contradiction since we assume that $P \cong R_p$ is non-split. So $X = \mathsf{Aut}\,\Gamma$ does not have a transitive split metacyclic subgroup, and thus Γ is a non-split weak metacirculant. $\square$

By Lemma 2.1, there are a lot of non-split metacyclic p-groups for odd primes p, which provides resources for the construction of non-split weak metacirculants.

Now we are ready to prove our first theorem.

Proof of Theorem 1.3. Let p be an odd prime, and let

$$P = \langle a, b \mid a^{p^{r+s+u}} = 1,\ b^{p^{r+s+t}} = a^{p^{r+s}},\ b^{-1}ab = a^{1+p^r} \rangle \quad \text{with } stu \neq 0.$$

Then P is a non-split metacyclic p-group of order $p^{2r+2s+t+u}$. It is easily shown that P has an automorphism σ such that

$$a^\sigma = a^{-1}, \qquad b^\sigma = b.$$

Let

$$S = \{ab, (ab)^{-1}, a^{-1}b, (a^{-1}b)^{-1}\},$$

and let $\Gamma = \mathsf{Cay}(P, S)$. Then S generates P, and Γ is connected. Let $X = \mathsf{Aut}\,\Gamma$. Then by [11], $X = \hat{P}{:}X_v$, and X_v is a 2-group. Thus, Lemma 2.6 tells us that Γ is a non-split weak metacirculant. So Γ is not a metacirculant.

Moreover, $(ab)^\sigma = a^{-1}b$, and $((ab)^{-1})^\sigma = (a^{-1}b)^{-1}$. It follows that σ induces an automorphism of Γ, and maps all (ab)-edges to $(a^{-1}b)$-edges. Thus, Γ is edge-transitive. Further, by [11], we have that $X_v \cong \mathbb{Z}_2$. Hence Γ is a half-arc-transitive graph of valency 4. This completes the proof of Theorem 1.3. □

3. Examples of cubic weak metacirculants

Here we study examples of weak metacirculants. It is easily shown that a connected cubic circulant is a Möbius band or a prism, defined below. A *Möbius band* $\mathbf{M}_{2n}$ is a graph with vertex set $\{\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}\}$, and edge set $\{\{\alpha_i, \alpha_{i+1}\}, \{\alpha_i, \alpha_{i+n}\} \mid 0 \leqslant i \leqslant 2n-1\}$, reading the subscripts modulo $2n$. Möbius bands are all vertex-transitive, and among them, $\mathbf{M}_4 \cong \mathsf{K}_4$ and $\mathbf{M}_6 \cong \mathsf{K}_{3,3}$ are the only edge-transitive graphs. A *prism* $\mathbf{C}_n \square \mathsf{K}_2$ is a graph with vertex set $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \cup \{\beta_0, \beta_1, \ldots, \beta_{n-1}\}$ and edge set $\{\{\alpha_i, \alpha_{i+1}\}, \{\alpha_i, \beta_i\}, \{\beta_i, \beta_{i+1}\} \mid 0 \leqslant i \leqslant n-1\}$. Prisms are all vertex-transitive, and among them, the only edge-transitive one is $\mathbf{C}_4 \square \mathsf{K}_2 \cong \mathsf{Q}_3$, which is the *cube*.

A natural generalization of prisms are generalized Petersen graphs. A *generalized Petersen graph* $\mathbf{P}(n,k)$ is a graph with vertex set and edge set as follows

$$\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \cup \{\beta_0, \beta_1, \ldots, \beta_{n-1}\},$$
$$\bigl\{\{\alpha_i, \alpha_{i+1}\}, \{\alpha_i, \beta_i\}, \{\beta_i, \beta_{i+k}\} \bigm| 0 \leqslant i \leqslant n-1\bigr\}$$

reading the subscripts modulo n, and $k \leqslant \frac{n}{2}$. Obviously, $\mathbf{P}(n,1) \cong \mathbf{C}_n \square \mathsf{K}_2$ is an n-prism. It is known that $\mathbf{P}(n,k)$ is vertex-transitive if and only if $(n,k) = (5,2)$, or $k^2 \equiv \pm 1 \pmod n$. In particular, $\mathbf{P}(5,2)$ is the Petersen graph.

As mentioned before, a graph $\Gamma = (V, E)$ is a dihedrant if $\mathrm{Aut}\,\Gamma$ contains a dihedral subgroup which is regular on the vertex set V. The next construction produces two classes of cubic dihedrants.

Construction 3.1.

(a) Let n be a positive integer and $1 \leqslant k \leqslant n-1$, and define a graph $\mathbf{C}(2n, 2k+1)$ which has vertices $\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}$, edges $\{\alpha_i, \alpha_{i+1}\}$, and $\{\alpha_{2i}, \alpha_{2i+2k+1}\}$, reading the subscripts modulo $2n$.
(b) Let m, n be two integers, and define a graph $\mathbf{Dih}(m,n)$ with vertex set V and edge set E, where

$$V = \{\alpha_{ij}, \beta_{ij} \mid 0 \leqslant i \leqslant m-1,\ 0 \leqslant j \leqslant n-1\},$$
$$E = \bigl\{\{\alpha_{ij}, \beta_{ij}\}, \{\beta_{ij}, \alpha_{i,j+1}\}, \{\beta_{ij}, \alpha_{i+1,j}\} \bigm| 0 \leqslant i \leqslant m-1,\ 0 \leqslant j \leqslant n-1\bigr\}.$$

We next define a family of cubic weak metacirculants as Cayley graphs.

Construction 3.2. Let $G = \langle a \rangle{:}\langle b \rangle \cong \mathbb{Z}_l{:}\mathbb{Z}_m$ be such that $a^b = a^k$ for some integer k. Assume further that the order $o(b)$ is even, and one of $o(a)$ and $m/2$ is odd. Let $z \in \langle b \rangle$ be an involution, and let i be coprime to $o(b)$. Define graphs

$$\mathbf{MeC}_1(l,m,k) = \mathsf{Cay}(G, S_i), \quad \text{where } S_i = \bigl\{ab^i, \bigl(ab^i\bigr)^{-1}, z\bigr\}, \quad \text{and}$$
$$\mathbf{MeC}_2(l,m,k) = \mathsf{Cay}(G, T_i), \quad \text{where } T_i = \bigl\{ab^{2i}, \bigl(ab^{2i}\bigr)^{-1}, z\bigr\}, \text{ with } m/2 \text{ odd}.$$

The concepts of quotient and cover is important in the study of transitive graphs, refer to [5,13,18]. Let $\Gamma = (V, E)$ be a G-vertex-transitive graph, where $G \leqslant \mathrm{Aut}\,\Gamma$. Let $N \lhd G$ be intransitive on V, and let V_N be the set of N-orbits on V. The *normal quotient* Γ_N of Γ induced by N is defined as the graph with vertex set V_N such that $B, C \in V_N$ are adjacent if some vertices $\beta \in B$ and $\gamma \in C$ are adjacent in Γ. If the valency of Γ_N equals the valency of Γ, then Γ is a *cover* of Γ_N, that is, each edge of Γ_N is the image of a perfect matching of Γ, see [13,18].

Lemma 3.3. *The graph $\mathbf{MeC}_1(l,m,k)$ is a normal cover of $\mathbf{M}_{m/2}$, and $\mathbf{MeC}_2(l,m,k)$ is a normal cover of $\mathbf{C}_{m/2} \square \mathsf{K}_2$.*

Proof. Let $\overline{G} = G/\langle a\rangle$, and let $\overline{S}_i$ and $\overline{T}_i$ be the images of S_i, T_i in $G/\langle a\rangle$, respectively. Then $\overline{G} \cong \mathbb{Z}_m$, $\mathsf{Cay}(\overline{G}, \overline{S}_i) \cong \mathbf{M}_{m/2}$, and $\mathsf{Cay}(\overline{G}, \overline{T}_i) \cong \mathbf{C}_{m/2}\Box\mathrm{K}_2$. Thus, $\mathbf{MeC}_1(l,m,k) = \mathsf{Cay}(G, S_i)$ is a cover of $\mathbf{M}_{m/2}$, and $\mathbf{MeC}_2(l,m,k) = \mathsf{Cay}(G, T_i)$ is a cover of $\mathbf{C}_{m/2}\Box\mathrm{K}_2$. $\Box$

More general representations for vertex-transitive graphs are coset graphs. Let G be a finite group, and let H be a subgroup of G. Let $[G:H] = \{Hx \mid x \in G\}$, the set of right cosets of H in G. For a subset $S \subset G \setminus H$, define a *coset graph* $\Gamma = \mathsf{Cos}(G, H, HSH)$ to be the graph with vertex set $[G:H]$ such that $\{Hx, Hy\}$ is an edge if and only if $yx^{-1} \in HSH$. The following statements for coset graphs are well known.

(a) Γ is undirected if and only if $HSH = HS^{-1}H$.
(b) G acts transitively on the vertex set $[G:H]$ with kernel being the core of H in G; so G is a subgroup of $\mathsf{Aut}\,\Gamma$ if and only if H is core-free in G.
(c) Γ is connected if and only if $\langle H, S\rangle = G$.
(d) Γ is G-arc-transitive if and only if $HSH = HgH$ where $g \in G$ such that $g^2 \in H$.

Construction 3.4. Let $G = \langle a\rangle{:}\langle b\rangle = \mathbb{Z}_n{:}\mathbb{Z}_4$ such that $a^{b^2} = a^{-1}$. Let $H = \langle b^2\rangle$ and $S = \{a, a^{-1}, b\}$. Then the coset graph $\Gamma = \mathsf{Cos}(G, H, HSH)$ is connected and cubic.

Lemma 3.5. *The graphs constructed in Construction 3.4 are generalized Petersen graphs.*

Proof. For the group $G = \langle a\rangle{:}\langle b\rangle$ as in Construction 3.4, there exists an integer k such that $bab^{-1} = a^k$ and $b^{-2}ab^2 = a^{-1}$. Then $k^2 \equiv -1 \pmod n$. Label $\alpha_i = Ha^i$ and $\beta_i = Hba^i$, where $0 \leqslant i \leqslant n-1$, which give rise to all vertices of Γ. Then $\{\alpha_i, \alpha_{i+1}\}$ and $\{\alpha_i, \beta_i\}$ are edges. Moreover, $\beta_i = Hba^i$ and $\beta_j = Hba^j$ are adjacent if and only if $a^{(j-i)k} = ba^{j-i}b^{-1} = ba^j(ba^i)^{-1}$ equals a or a^{-1}, if and only if $(j-i)k = \pm 1 \pmod n$. This is equivalent to $j = i \pm k \pmod n$. Therefore, this coset graph Γ is isomorphic to the generalized Petersen graph $\mathbf{P}(n,k)$. $\Box$

4. Proof of Theorem 1.4

In this section, we classify cubic weak metacirculants, by proving Theorem 1.4. Let $\Gamma = (V, E)$ be a connected cubic weak metacirculant. Let $R \leqslant G = \mathsf{Aut}\,\Gamma$ be a vertex-transitive metacyclic group. Let $R = \langle a, b\rangle = \langle a\rangle.\langle \bar{b}\rangle$. Then the quotient graph $\Gamma_{\langle a\rangle}$ is a circulant. If $\Gamma_{\langle a\rangle}$ has only two vertices, then Γ is a *bicirculant*, see [17]. Based on several previous known results, cubic bicirculants were classified by Pisanski [17]. (The authors are grateful to a referee for pointing out this.) Employing Pisanski's classification, the proofs of the next two lemmas can be slightly shortened. However, for the completeness, we present here a different, direct, independent proof. Moreover, this argument can be developed to treat metacirculants of larger valencies.

Since the vertex stabilizer R_α is core-free, it follows that R_α is cyclic. There are two different cases: $R_\alpha = 1$, or $R_\alpha \neq 1$, where $\alpha \in V$.

Lemma 4.1. *If $R_\alpha \neq 1$, and Γ is not R-arc-transitive, then $R_\alpha = \mathbb{Z}_2$, and Γ is a Möbius band or a generalized Petersen graph.*

Proof. The stabilizer R_α is core-free in R, and hence $R_\alpha \cap \langle a\rangle = 1$. Since R is metacyclic, $R_\alpha = \langle z\rangle$ is cyclic. As Γ is not R-arc-transitive, $\Gamma(v) = \{\beta_1, \beta_2, \gamma\}$ such that $\{\beta_1, \beta_2\}$ is an orbit of R_α, and γ is fixed by R_α. Then z^2 fixes every vertex which is adjacent to α. It easily follows that z^2 fixes $\Gamma(\beta_1)$ and $\Gamma(\beta_2)$ pointwise. If z^2 does not fix $\Gamma(\gamma) \setminus \{\alpha\}$ pointwise, then the induced action of $\langle z\rangle$ on $\Gamma(\gamma)$ is isomorphic to $\mathbb{Z}_4$, which is not possible. Thus, z^2 fixes each vertex in $\Gamma(\gamma)$, and so fixes all vertices which are at distance up to 2 from α. In particular, $z^2 \in G_{\beta_i} = \langle x_i\rangle \cong \langle z\rangle$, where $i = 1$ or 2. Arguing as above with x_i in the place of z, we conclude that z^2 fixes all vertices which are at distance up to 2 from β_i. Since Γ is connected, it easily follows that z^2 fixes all vertices of Γ. So $z^2 = 1$, and $R_\alpha = \langle z\rangle = \mathbb{Z}_2$. As $R_\alpha = \langle z\rangle$ is core-free in R, $\langle z\rangle \cap \langle a\rangle = 1$. So $z \notin \langle a\rangle$.

Let $s, t \in R$ be such that $(\alpha, \beta_2)^s = (\beta_1, \alpha)$ and $(\alpha, \gamma)^t = (\gamma, \alpha)$. Then $\alpha^{s^{-1}} = \beta_2$, and $\alpha^{sz} = \beta_1^z = \beta_2 = \alpha^{s^{-1}}$. Hence $\alpha^{szs} = \alpha$, and so $szs = 1$ or z. And it's obvious that $o(s) \neq 2$. Suppose that $szs = 1$. Then $z = s^2$, and so s normalizes $R_\alpha = \langle z \rangle$. Since t interchanges α and γ, we have that t normalizes $R_{\alpha\gamma} = R_\alpha$. Thus, $R_\alpha \lhd \langle R_\alpha, s, t \rangle = R$, which is not possible. So $szs = z$, or equivalently, as z is an involution, $s^z = s^{-1}$.

Observe that $\alpha^{tz} = \gamma^z = \gamma = \alpha^t$, and so $tzt^{-1} = 1$ or z. Since tzt^{-1} is of order equal to $o(z) = 2$, we have $tzt^{-1} = z$, and hence $tz = zt$. Because Γ is connected, we have that $\langle s, t, z \rangle = R$. Since $s^z = s^{-1}$ and $t^z = t$, the metacyclic group R has a normal subgroup $\langle s, t \rangle$. Moreover, we claim that $s \in \langle a \rangle$. Since $s^z = s^{-1}$, and $R/\langle a \rangle$ is cyclic, if $s \notin \langle a \rangle$, then the image $\bar{s}$ of s under $R\langle a \rangle$ is of order 2. It follows that $zs \in \langle a \rangle$. Further, $zszs = 1$. Thus $zs = a^{n/2}$, which implies that $o(s) = 2$. This is a contradiction occurs and we conclude that $s \in \langle a \rangle$.

Assume first that $z \notin \langle s, t \rangle$. Then $R = \langle s, t \rangle{:}\langle z \rangle$. Since $o(z) = o(t) = o(tz) = 2$, and $z \notin \langle a \rangle$, it follows that t or zt lies in $\langle a \rangle$. On the other hand, as we mentioned before, $s \in \langle a \rangle$. Thus, $\langle s, t \rangle$ or $\langle s, zt \rangle$ is a subgroup of $\langle a \rangle$, and we conclude that $\langle s, t \rangle = \langle a \rangle$ or $\langle s, zt \rangle = \langle a \rangle$, say $\langle s, t \rangle = \langle a \rangle$. Further, either $\langle s, t \rangle = \langle s \rangle$, or $o(s)$ is odd and $\langle s, t \rangle = \langle s \rangle \times \langle t \rangle$. In either case, $\langle st \rangle = \langle a \rangle$, $(st)^z = (st)^{-1}$, and $R = \langle s, t, z \rangle \cong \mathrm{D}_{2n}$ with $n = o(st) = o(a)$. If $\langle s, t \rangle = \langle s \rangle$, then $\Gamma = \mathbf{M}_{n/2}$, while if $\langle s, t \rangle = \langle s \rangle \times \langle t \rangle$, then $\Gamma = \mathbf{C}_{n/2} \Box \mathrm{K}_2 = \mathbf{P}(n/2, 1)$.

Assume now that $z \in \langle s, t \rangle$. Since $\alpha^{t^2} = (\alpha^t)^t = \gamma^t = \alpha$, we have that $t^2 \in R_\alpha$. Thus, $t^2 = 1$ or z. Since $R_\alpha = \langle z \rangle$ is core-free, $z \notin \langle a \rangle$, and thus the image $\bar{z}$ of z in $R/\langle a \rangle$ is an involution. If $t^2 = 1$, then as $z \in \langle s, t \rangle$ and $s \in \langle a \rangle$, we conclude that $t \notin \langle a \rangle$, and hence the image $\bar{t}$ equals $\bar{z}$. It follows that $R = \mathrm{D}_{2n}$, and in this case, $\Gamma = \mathbf{C}_{n/2} \Box \mathrm{K}_2$. If $t^2 = z$, then $o(t) = 4$, and since $s^z = s^{-1}$, we conclude that $R = \langle s \rangle{:}\langle t \rangle = \mathbb{Z}_n{:}\mathbb{Z}_4$ such that the center $\mathbf{Z}(R) \leqslant \mathbb{Z}_2$. In this case, Γ is a generalized Petersen graph $\mathbf{P}(n, k)$, where $k^2 \equiv 1 \pmod{n}$. □

Lemma 4.2. *$R_\alpha \neq 1$ and Γ is R-arc-transitive, then $R = \mathbb{Z}_n{:}\mathbb{Z}_6 = \mathrm{D}_{2n}{:}\mathbb{Z}_3$ and Γ is a Cayley graph of D_{2n}.*

Proof. Let $N = \langle a \rangle \lhd R$, and let K be the kernel of R on V_N. Then $N \lhd K$, and so $R/K \cong (R/N)/(K/N)$ is cyclic. Let $\overline{R} = R/K$. Then $\overline{R} \leqslant \mathrm{Aut}\,\Gamma_N$, and $\overline{R}_{\overline{\alpha}} = 1$. Thus $\Gamma_N \cong \mathrm{K}_2$, $K = \langle a \rangle{:}K_\alpha \cong \mathbb{Z}_n{:}\mathbb{Z}_3$, and $R/K \cong \mathbb{Z}_2$.

Suppose that K_α centralizes a Sylow subgroup of $\langle a \rangle$. Then we have $K = \langle a_p \rangle \times (\langle a_{p'} \rangle{:}K_\alpha)$, where $a = a_p a_{p'}$ such that $\langle a_p \rangle$ is a Sylow p-subgroup of $\langle a \rangle$ and $\langle a_{p'} \rangle$ is the Hall p'-subgroup of $\langle a \rangle$. Hence $M := \langle a_{p'} \rangle{:}K_\alpha$ is normal in R, and as $M_\alpha = K_\alpha \cong \mathbb{Z}_3$, the normal quotient Γ_M is of order $2o(a_p)$ and of valency 1, which is not possible. Moreover, since $\mathrm{Aut}(\langle a \rangle)$ is abelian, we have

$$R = \big(\langle a \rangle{:}K_\alpha\big).\mathbb{Z}_2 = \big(\langle a \rangle.\mathbb{Z}_2\big){:}K_\alpha.$$

Suppose that $\langle a \rangle.\mathbb{Z}_2$ centralizes a Sylow p-subgroup of $\langle a \rangle$. Then $\langle a \rangle.\mathbb{Z}_2 = \langle a_p \rangle \times (\langle a_{p'} \rangle.\mathbb{Z}_2)$, where $\langle a_{p'} \rangle$ is the Hall p'-subgroup of $\langle a \rangle$, and the normal quotient of Γ induced by $\langle a_{p'} \rangle.\mathbb{Z}_2$ is $\langle a_{p'} \rangle{:}\mathbb{Z}_3$-arc-transitive unless $\langle a_{p'} \rangle = \mathbb{Z}_2$. Thus, we conclude that $\langle a \rangle.\mathbb{Z}_2 \cong \mathrm{D}_{2n}$, and so $R = \mathbb{Z}_n{:}\mathbb{Z}_6$ and Γ is a Cayley graph of D_{2n}. □

Next we consider weak metacirculant Cayley graphs. A connected cubic weak metacirculant on an abelian group is a Möbius band or a prism. We thus focus on arc-transitive dihedrants. Arc-transitive dihedrants were studied in [6] for 2-arc-transitive case, and [9] for dihedrants admitting arc-regular groups actions. A rough characterization of cubic arc-transitive dihedrants was given in [17]. Here we present a simple classification of cubic arc-transitive dihedrants.

Lemma 4.3. *Let Γ be a connected cubic dihedrant of order $2n$. Then Γ is $\mathbf{M}_n$, $\mathbf{C}_n \Box \mathrm{K}_2$, $\mathbf{C}(2n, 2k+1)$, or $\mathbf{Dih}(l, m)$ with $lm = 2n$.*

Proof. Let $G = \langle a \rangle{:}\langle b \rangle \cong \mathrm{D}_{2n}$ be regular on the vertex set of Γ. Then a connected cubic Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is such that $\langle S \rangle = G$, where either $S = \{x, x^{-1}, y\}$ with $o(x) \geqslant 3$, or $S = \{x, y, z\}$ consisting of three involutions. If $S = \{x, x^{-1}, y\}$, then since $\langle S \rangle = G$, we have $o(x) = o(a) = n$, and

$\Gamma \cong \mathrm{Cay}(G, \{a, a^{-1}, b\}) \cong \mathbf{C}_n \Box \mathrm{K}_2$. Thus, we next assume that S consists of three involutions x, y and z. There are two different cases.

Suppose that two of the three involutions generate G, say $\langle x, y\rangle = G$. We may assume that $a = xy$, and $z = a^k$ or $a^k x$. If $z = a^k$, since $o(z) = 2$, $z = a^{n/2}$. Then $\Gamma = \mathbf{M}_{2n}$. Assume that $z = a^k x$ for some integer k. Label $\alpha_{2i} = (xy)^i$ and $\alpha_{2i+1} = (xy)^i x$, where $0 \leqslant i \leqslant n-1$. Then $\{\alpha_i, \alpha_{i+1}\}$, and $\{\alpha_{2i}, \alpha_{2i-(2(k+1))}\}$ are edges, and hence $\Gamma \cong \mathbf{C}(2n, 2k+1)$.

Finally, assume that any two of the three involutions generate a proper subgroup of G. Let $c = xy$, and label $\alpha_{ij} = (zy)^i c^j$ and $\beta_{ij} = y(zy)^i c^j$. Then $\alpha_{i,j+1}(\beta_{i,j})^{-1} = (zy)^i c (zy)^{-i} y^{-1} = c^{(zy)^{-1}} y$. Since there are three involutions, without loss of generality, suppose $zy \in \langle a\rangle$. Then $\alpha_{i,j+1}(\beta_{i,j})^{-1} = cy = x$. So there is an edge between $\alpha_{i,j+1}$ and $\beta_{i,j}$. Also, it is easily to shown that $(\beta_{i,j}, \alpha_{i,j}), (\beta_{i,j}, \alpha_{i+1,j}) \in E$. So the graph $\Gamma = \mathbf{Dih}(l, m)$, where $l = o(zy)$ and $m = o(xy)$. $\Box$

Next, we consider the case where R is regular on the vertex set of Γ.

Lemma 4.4. *If R is a 2-group, then either R is cyclic, or $R \cong \mathbb{Z}_n{:}\mathbb{Z}_2$.*

Proof. If R is abelian, then since $R = \langle x, y, z\rangle$ contains an involution, either R is cyclic, or $R = \mathbb{Z}_n \times \mathbb{Z}_2$.

Suppose that R is not abelian. We prove the lemma by induction. Let $\langle a\rangle \lhd R$ be such that $R/\langle a\rangle$ is cyclic. Let $N = \langle a_0\rangle \lhd R$, where $a_0 \in \langle a\rangle$ is an involution. If Γ_N is of valency 1, then since R is not abelian, it follows that $\Gamma = \mathrm{K}_4$ and $R = \mathrm{D}_8$. Suppose Γ_N is of valency 2. Then $S = \{x, y, z\}$ consists three involutions, with $z = a_0$ and $\bar{S} = \{\bar{x}, \bar{y}\}$. Thus $\bar{R} = \mathbb{Z}_2^2$ or is a dihedral so that $R = \mathbb{Z}_2.\bar{R}$. As R is metacyclic, we have $R = \mathrm{D}_8$ or $\mathbb{Z}_n{:}\mathbb{Z}_2$.

Thus, assume that Γ_N is cubic. Recall that $S = \{x, y, z\}$, where either x, y, z are all involutions, or $y = x^{-1}$ has order at least 4 and $o(z) = 2$. Inductively, we may assume that R/N is cyclic or has the form $\mathbb{Z}_m{:}\mathbb{Z}_2$. If R/N is cyclic, then R is abelian, which is a contradiction. Suppose that R/N is not cyclic. Then $R/N = \langle\bar{a}\rangle.\langle\bar{b}\rangle \cong \mathbb{Z}_m{:}\mathbb{Z}_2$. Assume that $o(\bar{a}) \geqslant 4$. Then a is an element of R of order $\frac{1}{2}|R|$. Since Γ_N is cubic, $z \notin \langle a\rangle$ and hence $R = \langle a\rangle{:}\langle z\rangle$. Assume that $o(\bar{a}) = 2$. Then $\langle a\rangle = \mathbb{Z}_4$, and so $a^b = a$ or a^{-1}. Since R is not abelian, $a^b = a^{-1}$. Let $z = a^i b^j$, where i, j are positive integers.

If j is odd then $1 = z^2 = a^i b^j a^i b^j = a^i (b^j a^i b^{-j}) b^{2j} = b^{2j}$, and hence b is an involution, and $R = \langle a, b\rangle = \mathrm{D}_8$.

Suppose that j is even. Then $1 = z^2 = a^{2i} b^{2j}$, and $(a^i)^2 = b^{-2j}$. If i is odd, then $a^2 \in \langle b\rangle$, and $\langle b\rangle$ has index 2 in R, and thus $\langle b\rangle$ is normal. It follows that R is a generalized quaternion group, which is not possible. If both i and j are even, then $z = a^i b^j$ lies in the center of R, and hence $R = \langle x, y\rangle \times \langle z\rangle$ or $\langle x, y\rangle$. The former is not possible as R is metacyclic, and so $R = \langle x, y\rangle$ is dihedral. $\Box$

Lemma 4.5. *Assume that $R = \langle a\rangle.\langle b\rangle$ is non-abelian and regular on V, and assume further that R is not dihedral. Then $\Gamma = \mathrm{Cay}(R, S)$ is a Cayley graph of R, and one of the following statements is true*:

(i) *$R = \mathbb{Z}_n{:}\mathbb{Z}_2$, and Γ is a generalized Petersen graph.*
(ii) *$\Gamma = \mathbf{MeC}_1(l, m, k)$ or $\mathbf{MeC}_2(l, m, k)$, where $l = o(a)$, $m = o(b)$ and k is such that $a^b = a^k$.*

Proof. Since R is regular on V, Γ is a Cayley graph of R, and there exists a subset $S \subset R$ such that $\Gamma = \mathrm{Cay}(R, S)$. Since Γ is cubic, $S = \{x, y, z\}$ such that either x, y, z are all involutions, or $o(x) > 2$, $z = x^{-1}$, and $o(y) = 2$.

Suppose that x, y, z are all involutions. Since $\langle a\rangle$ contains at most one involution, at least two of x, y, z do not lie in $\langle a\rangle$, say $x, y \notin \langle a\rangle$. Now $R/\langle a\rangle$ is cyclic and generated by $\bar{x}, \bar{y}, \bar{z}$. Thus $R/\langle a\rangle \cong \mathbb{Z}_2$, and so $R = \langle a\rangle{:}\langle x\rangle = \mathbb{Z}_n{:}\mathbb{Z}_2$. Since R is not dihedral, any two of x, y, z do not generate R. Hence $z \in \langle a\rangle$ and $R = \langle x, y\rangle \times \langle z\rangle = \mathrm{D}_n \times \mathbb{Z}_2$. Since R is metacyclic, $n/2$ is odd, and so $R = \mathrm{D}_{2n}$, which is a contradiction.

Thus, we have $o(x) = n > 2$ and $z = x^{-1}$.

Suppose that $\Gamma_{\langle a\rangle}$ is a cycle of size at least three. Since $R/\langle a\rangle$ is cyclic and vertex-transitive on $\Gamma_{\langle a\rangle}$, it is edge-transitive on $\Gamma_{\langle a\rangle}$. Thus, $\Gamma_{\langle a\rangle} = \mathrm{Cay}(R/\langle a\rangle, \{\bar{x}, \bar{x}^{-1}\})$, and so $\bar{y} = 1$. Thus, $y \in \langle a\rangle$, a unique involution of $\langle a\rangle$, and $\langle y\rangle \lhd R$. Then the quotient graph $\Gamma_{\langle y\rangle}$ is a cycle, and so $R = \mathbb{Z}_2.\mathbb{Z}_n$ or $\mathbb{Z}_2.\mathrm{D}_n$. It follows that $R = \mathbb{Z}_2 \times \mathbb{Z}_n$, $\mathbb{Z}_{2n}$ or $\mathbb{Z}_n{:}\mathbb{Z}_2$. The first two cases are not possible by our assumption. For the last case, either R is dihedral, or $\langle S\rangle \neq R$ so it is not possible either. Thus, $\Gamma_{\langle a\rangle}$ is of valency 1 or 3.

Assume first that $\Gamma_{\langle a\rangle} \cong \mathbf{K}_2$. Then $x \in \langle a\rangle$ and $R = \langle a\rangle{:}\langle y\rangle = \langle x\rangle{:}\langle y\rangle$. Let $x^y = x^k$ for some positive integer k. As $o(y) = 2$, $k^2 \equiv 1 \pmod n$. Label the vertices $\alpha_i = x^i$ and $\beta_i = yx^i$, where $0 \leqslant i \leqslant n-1$. Then $\{\alpha_i, \alpha_{i+1}\}$ and $\{\alpha_i, \beta_i\}$ are edges of Γ. Moreover, $\{\beta_i, \beta_j\}$ is an edge if and only if $x^{k(j-i)} = yx^jx^{-i}y = \beta_j\beta_i^{-1}$ equals x or x^{-1}, and if and only if $j = i \pm k \pmod n$. Thus, $\Gamma = \mathbf{P}(n,k)$ is a generalized Petersen graph.

Finally, assume that $\Gamma_{\langle a\rangle}$ is cubic. Then $x^2, y \notin \langle a\rangle$, and Γ is a cover of $\Gamma_{\langle a\rangle}$. Since $R/\langle a\rangle$ is cyclic, $\Gamma_{\langle a\rangle}$ is a cubic circulant, and so $\Gamma_{\langle a\rangle} = \mathbf{M}_m$ or $\mathbf{C}_m\square\mathrm{K}_2$. Let P be a Sylow p-subgroup of R with p odd which is normalized by y. Suppose that $P = \langle s,t\rangle = \langle s\rangle.\langle \bar t\rangle$ is non-split metacyclic. Then $\langle P, y\rangle = \langle s\rangle.\langle \bar t y\rangle$ and $s^y = s^{-1}$. Let $Q = \langle s\rangle \cap \langle t\rangle = \langle c\rangle$, where $c = s^i = t^j$. Then $c^y = c^{-1}$ and $P/Q = (\langle s\rangle/Q){:}(\langle t\rangle/Q) = \langle s^*\rangle{:}\langle t^*\rangle$. Further, $\langle \bar t\rangle = \langle s\rangle\langle t\rangle/\langle s\rangle \cong \langle t\rangle/\langle s\rangle \cap \langle t\rangle \cong \langle t^*\rangle$. It follows that y centralizes t^*, and so y centralizes t, which is not possible since $(t^j)^y = c^y = c^{-1} = t^{-j}$. Thus, every Sylow subgroup of R of odd order is split, and letting H be a Hall $2'$-subgroup of R, we have $H = \langle a_1\rangle{:}\langle b_1\rangle$. Since R is metacyclic, it follows that $H \lhd R$ and $R = H{:}R_2$, where R_2 is a Sylow 2-subgroup. If the quotient Γ_H is of valency 1 or 2, then R_2 is cyclic or dihedral, while if Γ_H is cubic, then by Lemma 4.4, R_2 is cyclic or has the form $\mathbb{Z}_m{:}\mathbb{Z}_2$. Hence, in either case, $R_2 = \langle b_2\rangle$ is cyclic, or $R_2 = \langle a_2\rangle{:}\langle b_2\rangle$, where $o(b_2) = 2$. Thus, $R = \langle a\rangle{:}\langle b\rangle$, where either $a = a_1$ and $b = b_1b_2$, or $a = a_1a_2$ and $b = b_1b_2$. Since $\langle \bar x, \bar y\rangle = R/\langle a\rangle$, we have $\bar x = \bar b^i$ and $y = b^{o(b)/2}$, $(i, o(b)) = 1$. Thus, $x = a^jb^i$ and $y = a^{j'}b^{o(b)/2}$. Since $\langle x, y\rangle = R$, we conclude that $(j, o(a)) = 1$. Since all elements of $\langle a\rangle$ of order $o(a)$ are conjugate in $\mathrm{Aut}(R)$ and all involutions are conjugate by $\langle a\rangle$, we may assume that $x = ab^i$ and $y = z$. Then $\Gamma = \mathbf{MeC}_1(l,m,k)$ or $\mathbf{MeC}_2(l,m,k)$, where $l = o(a)$, $m = o(b)$ and k is such that $a^b = a^k$. □

Lemma 4.6. *If R is abelian and regular on V, then R is cyclic or $\mathbb{Z}_n \times \mathbb{Z}_2$, and $\Gamma = \mathbf{C}_n\square\mathrm{K}_2$ is a prism.*

Proof. Since R is regular on V, Γ is a Cayley graph of R and there is a subset S of R such that $\Gamma = \mathrm{Cay}(R,S)$. Because $S = \{x,y,z\}$, S contains three involutions, or $o(x) > 2$, $z = x^{-1}$, and $o(y) = 2$.

Suppose x, y, z are all involutions. Since $\langle S\rangle = R$ and R is abelian, we have $R = \mathbb{Z}_2^3$ which contradicts the fact that R is metacyclic. So $o(x) > 2$, $z = x^{-1}$, and $o(y) = 2$. Thus $R = \langle x, y\rangle$ is cyclic or $\mathbb{Z}_n \times \mathbb{Z}_2$. And it easily shown that in either case $\Gamma = \mathbf{C}_n\square\mathrm{K}_2$ is a prism. □

Finally, we prove Theorem 1.4.

Proof of Theorem 1.4. Let $\Gamma = (V,E)$ be a connected weak cubic metacirculant of a metacyclic group R of valency 3. Let $\alpha \in V$ be a vertex.

If Γ is R-arc-transitive, then by Lemma 4.2, Γ is a dihedrant, and then by Lemma 4.3, Γ satisfies Theorem 1.4.

We thus assume that Γ is not R-arc-transitive. If $R_\alpha \neq 1$, then by Lemma 4.1, Γ is a Möbius band or a generalized Petersen graph, as in Theorem 1.4.

Hence, we further assume that $R_\alpha = 1$, so Γ is a Cayley graph of R. By Lemmas 4.5 and 4.6, Γ satisfies Theorem 1.4. □

References

[1] B. Alspach, M. Conder, D. Marušič, M.Y. Xu, A classification of 2-arc-transitive circulants, J. Algebraic Combin. 5 (1996) 83–86.
[2] B. Alspach, T.D. Parsons, A construction for vertex-transitive graphs, Canad. J. Math. 34 (1982) 307–318.
[3] B. Alspach, D. Marušič, L. Nowitz, Constructing graphs which are 1/2-transitive, J. Aust. Math. Soc. Ser. A 56 (1994) 391–402.
[4] L. Babai, Finite digraphs with given regular automorphism groups, Period. Math. Hungar. 11 (1980) 257–270.
[5] N. Biggs, Algebraic Graph Theory, 2nd ed., Cambridge University Press, New York, 1993.
[6] S.F. Du, A. Malnič, D. Marušič, Classification of 2-arc-transitive dihedrants, J. Combin. Theory Ser. B 98 (2008) 1349–1372.
[7] B.W. King, Presentations of metacyclic groups, Bull. Aust. Math. Soc. 8 (1973) 103–131.
[8] I. Kovacs, Classifying arc-transitive circulants, J. Algebraic Combin. 20 (2004) 353–358.

[9] I. Kovács, D. Marušič, M.E. Muzychuk, On dihedrants admitting arc-regular group actions, J. Algebraic Combin. 33 (2011) 409–426.
[10] C.H. Li, Permutation groups with a cyclic regular subgroup and arc transitive circulants, J. Algebraic Combin. 21 (2005) 131–136.
[11] C.H. Li, H. Sims, On half-transitive metacirculant graphs of prime-power order, J. Combin. Theory Ser. B 81 (2001) 45–57.
[12] C.H. Li, H.S. Sim, Automorphism groups of metacyclic Cayley graphs of prime-power order, J. Aust. Math. Soc. 71 (2001) 223–231.
[13] A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182 (1998) 203–218.
[14] D. Marušič, On 2-arc-transitivity of Cayley graphs, J. Combin. Theory Ser. B 87 (2003) 162–196;
D. Marušič, J. Combin. Theory Ser. B 96 (2006) 761–764.
[15] D. Marušič, P. Šparl, On quartic half-arc-transitive metacirculants, J. Algebraic Combin. 28 (2008) 365–395.
[16] J. Morris, M. Šajna, Brian Alspach and his work, Discrete Math. 299 (2005) 269–287.
[17] T. Pisanski, A classification of cubic bicirculants, Discrete Math. 307 (2007) 567–578.
[18] C.E. Praeger, Imprimitive symmetric graphs, Ars Combin. 19 (1985) 149–163.
[19] M. Šajna, Half-transitivity of some metacirculants, Discrete Math. 185 (1998) 117–136.
[20] N.D. Tan, Non-Cayley tetravalent metacirculant graphs and their Hamiltonicity, J. Graph Theory 23 (1996) 273–287.
[21] M.Y. Xu, Half-transitive graphs of prime-cube order, J. Algebraic Combin. 1 (1992) 275–282.
[22] M.Y. Xu, Q.H. Zhang, A classification of metacyclic 2-groups, Algebra Colloq. 13 (2006) 25–34.
[23] C.X. Zhou, Y.Q. Feng, An infinite family of tetravalent half-arc-transitive graphs, Discrete Math. 306 (2006) 2205–2211.