JOURNAL OF ALGEBRA **126**, 1–33 (1989)

On the Structure of Free Resolutions of Length 3

JERZY WEYMAN

Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland; and Department of Mathematics, Northeastern University, Boston, Massachusetts 02115

Communicated by David Buchsbaum

Received February 2, 1987

INTRODUCTION

The sources of structure theory of finite free resolutions go back to Hilbert. In his famous paper on invariant theory he proved that every ideal of projective dimension 1 over a polynomial ring is determinantal. This theorem was then extended by Burch to arbitrary commutative rings.

There are various motivations for extending this results. One of them is a variety of applications of the Hilbert–Burch theorem in deformation theory and algebraic geometry (cf., for example, [S]). Another reason comes from Serre's conjecture. One would like to understand in terms of resolutions why the lifting property of Grothendieck does not hold.

In 1974 Buchsbaum and Eisenbud in [BE1] introduced so-called structure theorems—various factorisations which occur for acyclic complexes of arbitrary length. Then Hochster in [H] generalized the Hilbert–Burch theorem to arbitrary free complexes of length 2. His results were then improved in [Br, PW]. Still, the structure of complexes of length bigger than 2 is not understood. The first steps in this direction are the results of [B] and the construction of a generic ring for complexes of length 3 of type $(1, n-1, 1)$ given in [PW].

In this paper I construct a ring which, I believe, is the generic ring for complexes of length 3. The construction involves new structure theorems and a new idea–the notion of defects of structure theorems.

The paper is organized as follows. Section 1 is introductory. I give there the background from representation theory and the cohomology groups of a Lie algebra. Section 2 contains the proof of the new structure theorems we need, the construction of the generic ring R_{gen} and of the defect Lie algebra $\mathbf{L}$. In Section 3 I show that genericity of R_{gen} follows from vanising of the homology of a certain family of complexes over $U(\mathbf{L})$—the enveloping algebra of $\mathbf{L}$. In Section 4 I apply the results of Section 2 to study multi-

0021-8693/89 $3.00

plicative structures on the resolutions of length 3. I give there the complete classification of possible multiplications modulo the maximal ideal on the finite free resolution of length 3 over a local ring.

1. Preliminaries

(A) *Representation Theory*

Throughout the paper we use extensively representation theory of the general linear group. Let us fix the field k of characteristic 0 and a vector space F of dimension n over k. We denote by $GL(F)$ the group of k-linear automorphisms of F.

It is well known that $GL(F)$ is linearly reductive and that the irreducible representations of $GL(F)$ are in one to one correspondence with *dominant integral weights*. In case of $GL(F)$ these are just the non-increasing sequences $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n)$ of integers. We denote the corresponding representation by $S_\mathbf{a}F$. If $\mathbf{a}_n \geqslant 0$ then $\mathbf{a}$ becomes a partition of some number m and $S_\mathbf{a}F$ is a Schur functor on F. The simplest way to define it is to take

$$S_\mathbf{a}F = e(\mathbf{a})\, T^m(F),$$

where $T^m(F)$ is the mth tensor power of F and $e(\mathbf{a})$ is the Young idempotent in the group ring of the symmetric group on m letters corresponding to the partition $\mathbf{a}$. The symmetric group acts on $T^m(F)$ by permuting copies of F. The reader not familiar with Young idempotents should consult Chapter 2 of [CD] for their definitions and basic properties. Let us single out some important special cases. For $\mathbf{a} = (i, 0, ..., 0)$, $S_\mathbf{a}F$ is the ith symmetric power of F; for $\mathbf{a} = (1, 1, ..., 0)$ (i ones and $n-i$ zeroes), $S_\mathbf{a}F$ is the ith exterior power of F. Moreover, if $\mathbf{b} = (\mathbf{a}_1 - 1, ..., \mathbf{a}_n - 1)$ then

$$S_\mathbf{b}F = S_\mathbf{a}F \otimes \Lambda^n F^*.$$

This allows us to define $S_\mathbf{a}F$ for negative values of $\mathbf{a}_n$. Let us also mention duality. If $\mathbf{b} = (-\mathbf{a}_n, ..., -\mathbf{a}_1)$ then

$$S_\mathbf{b}F^* = S_\mathbf{a}F,$$

We will use often the Cauchy formulas. They give the decomposition of the symmetric and exterior power of a tensor product of two spaces into the Schur functors. For vector spaces F and G

$$S_i(F \otimes G) = \sum_{\mathbf{a}_n \geqslant 0,\ |\mathbf{a}| = i} S_\mathbf{a}F \otimes S_\mathbf{a}G$$

$$\Lambda^i(F \otimes G) = \sum_{\mathbf{a}_n \geqslant 0,\ |\mathbf{a}| = i} S_\mathbf{a}F \otimes S_{\mathbf{a}^\sim}G,$$

where $\mathbf{a}^{\sim}$ denotes the dual partition to $\mathbf{a}$. The simplest proof of the Cauchy formulas can be found in Chapter 1 of [McD]. The proofs of all above properties are contained in [ABW].

(B) *"Depth Increasing" Problems in Commutative Algebra*.

Let us fix the natural numbers $r_1, ..., r_n$. For $0 \leqslant i \leqslant n$ let $f_i = r_i + r_{i+1}$ ($r_0 = r_{n+1} = 0$). We consider the pairs $(R, \mathbf{F})$ of the free acyclic complexes

$$\mathbf{F}: 0 \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0$$

over commutative rings R such that rank $F_i = f_i$, rank $d_i = r_i$. We call the sequence $(r_1, ..., r_n)$ the *type* of the complex $\mathbf{F}$. We call a pair $(R, \mathbf{F})$ *generic* (universal) if and only if for any other acyclic pair $(S, \mathbf{G})$ of the same type there exists a (unique) homomorphism $h: R \to S$ such that $\mathbf{G} = \mathbf{F} \otimes_R S$ and $\mathbf{F}$ itself is acyclic over R.

By the exactness criterion of Buchsbaum and Eisenbud [BE3] the complex $\mathbf{F}$ is acyclic if and only if $r_i + r_{i+1} = f_i$, and depth $I(d_i) \geqslant i$ for all i ($I(d_i)$ denotes the ideal of maximal non-vanishing minors of d_i). My approach to the problem of finding the generic pair is to start with *the generic complex* and try to increase depth of ideals of minors "in a generic way." Let us make this notion more precise.

Suppose I is an ideal in the ring A. We fix a natural number i. The A-algebra B is called *the generic* (universal) *extension of I of depth i* if depth of IB is $\geqslant i$ and if for any A-algebra C with this property there exists a (unique) A-algebra homomorphism $h: B \to C$. There are several special cases where the construction of the generic extension is easy:

(1) If I has depth i then we can take $B = A$.

(2) If $i = 1$ we can take $B = A/J$, where $J = \bigcup \operatorname{Ann}(I^n)$. In this case B is universal because h is unique. Let us notice that if I is nilpotent then $B = 0$.

(3) If $i = 2$ and depth $I = 1$ then we can take B to be *the ideal transform with respect to I*. To see that let us assume that $I = (x_1, ..., x_m)$. Let us consider the following part of the Koszul complex on $x_1, ..., x_m$:

$$0 \to \Lambda^m A^m \to \Lambda^{m-1} A^m \to \Lambda^{m-2} A^m.$$

The cycles in $\Lambda^{m-1} A^m$ are the elements $a_1, ..., a_m$ such that $a_i x_j = a_j x_i$ for every i, j. This cycle is a boundary if and only if there exists a in A such that $a_i = a x_i$ for all i. We can assume that the x_i are non-zerodivisors so the cycle corresponds to the collection of the elements a_i/x_i in the quotient ring of A that are equal to each other, so it corresponds to an element of the ring

$$A_I = A[x_1^{-1}] \cap A[x_2^{-1}] \cap \cdots \cap A[x_m^{-1}].$$

This ring is called the ideal transform of A with respect to I. If $X = \operatorname{Spec} A$, $U = X \setminus V(I)$, then A_I equals $j_* \mathcal{O}_U$ for $j\colon U \to X$ the inclusion. If depth $(I) \geqslant 2$ then $j_* \mathcal{O}_U = A$.

One should notice that A_I need not be Noetherian even when A is. In fact the famous Nagata counterexample of the ring of invariants of an algebraic group not being finitely generated is based on this fact (compare [DC]).

For $i \geqslant 3$ almost nothing is known. I hope that this paper will provide some insight to the case $i = 3$.

(C) *Bott's Theorem and Kempf's Construction*

Let X_c be the variety of complexes of type $(r_1, ..., r_n)$. A point of X_c is a set of n matrices d_i $(1 \leqslant i \leqslant n)$ of sizes $f_i \times f_{i-1}$ satisfying the conditions $d_i d_{i+1} = 0$ and rank $d_i \leqslant r_i$ for all i. We sometimes treat the matrix d_i as a map $F_i \to F_{i-1}$, where F_i is a vector space of dimension f_i for $0 \leqslant i \leqslant n$. Let $\operatorname{Grass}(r, F)$ denote the Grassmannian of subspaces of F of dimension r. We consider the product $X_c \times \Pi \operatorname{Grass}(r_i, F_{i-1})$. Let V_i denote the element of $\operatorname{Grass}(r_{i+1}, F_i)$. We consider the subvariety Y_c:

$$Y_c = \{(d_1, ..., d_n; V_0, ..., V_{n-1}) \mid \operatorname{Im} d_i \subset V_{i-1} \subset \operatorname{Ker} d_{i-1} \text{ for } 0 \leqslant i \leqslant n-1\}.$$

Let $p\colon Y_c \to X_c$ and $q\colon Y_c \to \operatorname{Grass} = \Pi \operatorname{Grass}(r_i, F_{i-1})$ be the natural projections. The pair (Y_c, p) is the desingularisation of X_c. If we denote by R_i and Q_i the tautological subbundle and factorbundle in $\operatorname{Grass}(r_{i+1}, F_i)$ then the structure sheaf $\mathcal{O}_{Y_c}$ is isomorphic to the symmetric algebra on the vectorbundle $\bigoplus Q_i \otimes R_{i-1}^*$ (for the proofs see [PW]).

The key technique we will use throughout the paper is to use Bott's theorem to calculate higher direct images of the sheafs on Y_c. Here is the statement we use.

Let F be a vector space of dimension n. Let us consider $\operatorname{Grass}(r, F)$. We denote by R and Q the tautological subbundle and tautological factorbundle on Grass (r, F). Let $q = n - r$. We consider two dominant integral weights $\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_r)$, $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_q)$ and the corresponding vectorbundle $S_{\mathbf{a}} R \otimes S_{\mathbf{b}} Q$. From $\mathbf{a}, \mathbf{b}$ we form the sequence

$$s(\mathbf{a}, \mathbf{b}) = (\mathbf{b}_1 + r + q, \mathbf{b}_2 + r + q - 1, ..., \mathbf{b}_q + r + 1, \mathbf{a}_1 + r, ..., \mathbf{a}_r + 1).$$

We distinguish two cases:

(1) $s(\mathbf{a}, \mathbf{b})$ has no repeated elements. Then we reorder it to get a decreasing sequence $c = (c_1, ..., c_n)$. We define $i(\mathbf{a}, \mathbf{b})$ to be the number of exchanges (of consecutive elements) we need to get c from $s(\mathbf{a}, \mathbf{b})$. We also define the weight $\mathbf{c}$ to be $(c_1 - n, c_2 - n + 1, ..., c_n - 1)$;

(2) $s(\mathbf{a}, \mathbf{b})$ has repeated entries.

Now we are ready to state Bott's theorem.

THEOREM 1.1. (Bott). *Let* $X = \mathrm{Grass}(r, F)$. *The sheaf cohomology groups of* $S_{\mathbf{a}}R \otimes S_{\mathbf{b}}Q$ *are all zero in case* (2) *and in case* (1) *the only non-vanishing one is* $H^{i(\mathbf{a},\mathbf{b})}(X, S_{\mathbf{a}}R \otimes S_{\mathbf{b}}Q) = S_c F$.

(D) *Graded Lie Algebras*

Graded Lie algebra is a graded vector space $\mathbf{L} = \bigoplus L_i$ equipped with the bracket $[\,,\,]: \mathbf{L} \otimes \mathbf{L} \to \mathbf{L}$ which is compatible with grading and satisfies the following properties. For $x, y, z \in \mathbf{L}$

(a) $[x, y] = -[y, x]$,

(b) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

We notice that this definition of graded Lie algebra is different from the standard one because in (a), (b) we do not have additional signs coming from degrees of x, y, z.

Let $\mathbf{L}$ be any Lie algebra. We define cohomology groups $H^i(\mathbf{L})$ to be the ith cohomology group of the complex

$$0 \longrightarrow \mathbf{L}^* \xrightarrow{d} \Lambda^2\mathbf{L}^* \xrightarrow{d} \Lambda^3\mathbf{L}^* \xrightarrow{d} \cdots,$$

where for $c \in \Lambda^i\mathbf{L}^*$, $l_1, \ldots, l_{i+1} \in \mathbf{L}$,

$$dc(l_1 \wedge l_2 \wedge \cdots \wedge l_{i+1})$$
$$= \sum_{j,k} (-1)^{j+k} c([l_j, l_k] \wedge l_1 \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge \hat{l}_k \wedge \cdots \wedge l_{i+1}).$$

We will be interested mostly in the group $H^2(\mathbf{L})$, which has the following interpretation:

$$H^2(\mathbf{L}) = \{\text{classes of the abelian central extensions of } \mathbf{L}\}.$$

To describe this correspondence let us recall that the abelian central extension of $\mathbf{L}$ comes from the sequence

$$0 \to \mathscr{k} \to \mathbf{L}^{\sim} \to \mathbf{L} \to 0,$$

where $\mathscr{k}$ is the one dimensional central ideal in $\mathbf{L}^{\sim}$. The correspondence associates to the cocycle $c: \Lambda^2\mathbf{L} \to \mathscr{k}$ the algebra $\mathbf{L}^{\sim} = \mathbf{L} \oplus \mathscr{k}$ with the bracket $[(l_1, x), (l_2, y)] = ([l_1, l_2], c(l_1 \wedge l_2))$. From this correspondence it follows that there exists the universal central abelian extension $\mathbf{L}^{\wedge}$ of $\mathbf{L} \cdot \mathbf{L}^{\wedge} = \mathbf{L} \oplus H^2(\mathbf{L})^*$ with the bracket $[(l_1, x), (l_2, y)] = ([l_1, l_2], c'(l_1 \wedge l_2))$, where c' is the dual of embedding of 2 cocycles into $\Lambda^2\mathbf{L}^*$.

There is a graded version of this correspondence we will use. Let us suppose that $\mathbf{L}$ is a graded Lie algebra. Then $H^2(\mathbf{L})$ carries a natural gradation and the mth graded component $H^2(\mathbf{L})$ corresponds to the graded extensions $\mathbf{L}^{\sim}$ for which the ideal $\mathscr{k}$ is contained in the mth component of $\mathbf{L}^{\sim}$. We also get the universal central abelian extension $\mathbf{L}_m^{\sim}$ with the ideal in degree m. Additively $\mathbf{L}_m^{\wedge} = \mathbf{L} \oplus H^2(\mathbf{L})_m^*$, the bracket being defined as before. The best references for the above are [Bou, Jac].

Let us suppose now that $\mathbf{L} = L_1 \oplus L_2 \oplus \cdots \oplus L_m$. Then we get the universal graded extension

$$0 \to H^2(\mathbf{L})_{m+1}^* \to \mathbf{L}_{m+1}^{\wedge} \to \mathbf{L} \to 0$$

with the ideal in degree $m+1$. Having defined the graded Lie algebra $\mathbf{L}_2 = L_1 \oplus L_2$ we can now repeat the above construction to get $\mathbf{L} = L_1 \oplus L_2 \oplus \ldots$, where L_m are defined by induction as

$$L_{m+1} = H^2(\mathbf{L}_m)_{m+1},$$

where $\mathbf{L}_m = L_1 \oplus L_2 \oplus \cdots \oplus L_m$. In the case when the original bracket $\Lambda^2 L_1 \to L_2$ is an epimorphism the resulting graded Lie algebra $\mathbf{L}$ can be defined as a graded Lie algebra generated by L_1 in degree 1 with the relations in degree 2 given by $\operatorname{Ker}(\Lambda^2 L_1 \to L_2)$.

We will use this construction to define the defect Lie algebra in Section 2.

For the Lie algebra $\mathbf{L}$ we will denote by $U(\mathbf{L})$ the enveloping algebra. It is by definition the factor of the tensor algebra on $\mathbf{L}$ by the ideal generated by the relations $x \otimes y - y \otimes x = [x, y]$. We will denote by $U(\mathbf{L})^*$ the graded dual of $U(\mathbf{L})$. It is the injective envelope of the trivial $U(\mathbf{L})$-module, so the functor $\operatorname{Hom}_{U(\mathbf{L})}(-, U(\mathbf{L})^*)$ defines a duality for finite dimensional $U(\mathbf{L})$-modules. For those properties of $U(\mathbf{L})$ the reader should consult [D].

2. Structure Theorems for Resolutions of Length 3

In this section we consider the complexes of length 3 of type (r_1, r_2, r_3),

$$\mathbf{F}: 0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0, \tag{1}$$

i.e., rank $d_i = r_i$, rank $F_i = f_i = r_i + r_{i+1}$ $(i = 1, 2, 3, r_4 = 0)$.

Buchsbaum and Eisenbud have proved in [BE1] the following results which we state for resolutions of length 3.

THEOREM 2.1. *Let* $\mathbf{F}$ *be an acyclic complex of type* (1) *over the ring* R. *Then*

(a) *There exist unique maps*

$$a_2\colon \Lambda^{f_3}F_3^* \otimes \Lambda^{f_2}{}_2 \to \Lambda^{r_2}F_1$$
$$a_1\colon \Lambda^{f_3}F_3 \otimes \Lambda^{f_2}F_2^* \otimes \Lambda^{f_1}F_1 \to \Lambda^{r_1}F_0$$

such that the diagrams

$$\begin{tikzcd} \Lambda^{r_2}F_2 \arrow[rr, "\Lambda^{r_2}d_2"] \arrow[dr, "a_3^*"'] & & \Lambda^{r_2}F_1 \\ & \Lambda^{f_3}F_3^* \otimes \Lambda^{f_2}F_2 \arrow[ur, "a_2"'] & \end{tikzcd} \qquad \begin{tikzcd} \Lambda^{r_1}F_1 \arrow[rr, "\Lambda^{r_1}d_1"] \arrow[dr, "a_2^*"'] & & \Lambda^{r_1}F_0 \\ & \Lambda^{f_3}F_3 \otimes \Lambda^{f_2}F_2^* \otimes \Lambda^{f_1}F_1 \arrow[ur, "a_1"'] & \end{tikzcd} \tag{2}$$

are commutative with $a_3 = \Lambda^{r_3}d_3$.

(b) *There exists a map* $b\colon \Lambda^{r_1+1}F_1 \otimes \Lambda^{f_1}F_1^* \otimes \Lambda^{f_2}F_2 \otimes \Lambda^{f_3}F_3^* \to F_2$ *such that the diagram*

$$\begin{tikzcd} F_2 \arrow[rr, "d_2"] & & F_1 \\ \Lambda^{r_1+1}F_1 \otimes \Lambda^{f_1}F_1^* \otimes \Lambda^{f_2}F_2 \otimes \Lambda^{f_3}F_3^* \arrow[rr] \arrow[u, "b"] & & \Lambda^{r_2}F_1^* \otimes F_1 \otimes \Lambda^{f_2}F_2 \otimes \Lambda^{f_3}F_3^* \arrow[u, "a_2^* \otimes 1"'] \end{tikzcd} \tag{3}$$

is commutative.

Remark 2.2. In [BE1] the authors identify $\Lambda^{f_i}F_i$ with R. For our purposes it is necessary to keep track of various powers of $\Lambda^{f_i}F_i$. Throughout the article we denote $\Lambda^{f_i}F_i$ by M_i. Thus the structure maps act in the following way:

$$a_2\colon M_3^{-1} \otimes M_2 \to \Lambda^{r_2}F_1, \qquad a_1\colon M_3 \otimes M_2^{-1} \otimes M_1 \to \Lambda^{r_1}F_0$$
$$b\colon \Lambda^{r_1+1}F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1} \to F_2.$$

Our general goal is to construct the generic ring for the resolutions of length 3. As indicated in Section 1(B) we start from universal complex of type (1). We consider the ring

$$R_c = S \cdot (F_3 \otimes F_2^* \oplus F_2 \otimes F_1^* \oplus F_1 \otimes F_0^*)/I_c, \tag{4}$$

where I_c is the ideal generated by the relations $d_2 d_3 = d_1 d_2 = 0$ and the $r_i + 1$ minors of d_i (in our notation d_i corresponds to the representation $F_i \otimes F_{i-1}^*$). Over R_c we have a natural complex $\mathbf{F}_c$ of type (1) which is clearly universal, i.e., for any complex $\mathbf{G}$ of type (1) there exists a unique map $h\colon R_c \to S$ such that $\mathbf{G} = \mathbf{F}_c \otimes_{R_c} S$.

Remark 2.3. We should notice here a slight abuse of notation. In the formula (4) above F_i denote vector spaces over k of dimension f_i. Sometimes we will also denote the terms of the complex $\mathbf{F}_c$ by F_i and there they will be the free modules of rank f_i over R_c. It should also be noticed that the complex $\mathbf{F}$ in (1) is a complex over some commutative ring R while $\mathbf{F}_c$ is a complex over the "concrete" ring R_c.

To obtain the information about the decomposition of R_c into representations we use Kempf's construction. We consider the action of the group

$$\mathbf{G} = GL(F_3) \times GL(F_2) \times GL(F_1) \times GL(F_0) \tag{5}$$

on R_c. We consider the projection p: $Y_c \to X_c = \operatorname{Spec} R_c$ from Section 1(C). By Cauchy's formula we get the decomposition

$$\mathcal{O}_Y = \sum_{\mathbf{a},\mathbf{b},\mathbf{c}} S_{\mathbf{a}} F_3 \otimes S_{\mathbf{a}} R_2^* \otimes S_{\mathbf{b}} Q_2 \otimes S_{\mathbf{b}} R_1^* \otimes S_{\mathbf{c}} Q_1 \otimes S_{\mathbf{c}} R_0^*. \tag{6}$$

By Bott's theorem those bundles have cohomology concentrated in degree 0 so by Kempf's result [K] we get the decomposition of R_c:

$$\begin{aligned} R_c = \sum S_{(\mathbf{a}_1, \ldots, \mathbf{a}_{r_3})} F_3 &\otimes S_{(\mathbf{b}_1, \ldots, \mathbf{b}_{r_2}, -\mathbf{a}_{r_3}, \ldots, -\mathbf{a}_1)} F_2 \\ &\otimes S_{(\mathbf{c}_1, \ldots, \mathbf{c}_{r_1}, -\mathbf{b}_{r_2}, \ldots, -\mathbf{b}_1)} F_1 \\ &\otimes S_{(0, \ldots, 0, -\mathbf{c}_{r_1}, \ldots, -\mathbf{c}_1)} F_0. \end{aligned} \tag{7}$$

We also know that R_c has many nice properties, for example, rational singularities.

Now by Theorem 2.1(a) we know that the entries of maps a_i have to exist in the generic ring so in order to get it we have to consider first the ring R_a,

$$R_a = R_c \otimes S \cdot (\Lambda^{r_2} F_1^* \otimes M_2 \otimes M_3^{-1} \oplus \Lambda^{r_1} F_0^* \otimes M_1 \otimes M_2^{-1} \otimes M_3)/I_a, \tag{8}$$

where I_a is the ideal of all polynomials in the entries of d_i's and a_i's which vanish when evaluated on the entries of any acyclic complex $\mathbf{G}$ over commutative ring S. Notice that the key point here is the definition of the ideal I_a. If it would happen that the maps a_i would be defined over R_c then by uniqueness of a_i's for any $\mathbf{G}$ and by definition of I_a the generators we add to R_a would be equal to the "old" a_i's modulo I_a and in such case we would have $R_a = R_c$. To prove that it is not the case we find now the criterion for a polynomial to be in I_a.

For any acyclic $\mathbf{G}$ over S of type (1) we can consider the multiplicative set T of non-zerodivisors in S and then $T^{-1}S \otimes \mathbf{G}$ splits by the exactness criterion [BE3]. The map $S \to T^{-1}S$ is an injection and it follows that I_a

consists of all polynomials in the entries of d_i's and a_i's that vanish when evaluated on the entries of *split acyclic* complexes **G**. The split acyclic complex after some change of basis has the form

$$0 \to A_3 \to A_3 \oplus A_2 \to A_2 \oplus A_1 \to A_1 \oplus A_0, \tag{9}$$

where $\dim A_i = r_i$ for $i = 1, 2, 3$ and $\dim A_0 = f_0 - r_1$. The differential d_i is the identity on A_i. This reasoning proves

LEMMA 2.4. *A* **G**-*invariant set* $\mathscr{X}$ *of elements of*

$$R_c \otimes S \cdot (\Lambda^{r_2} F_1^* \otimes M_2 \otimes M_3^{-1} \oplus \Lambda^{r_1} F_0^* \otimes M_1 \otimes M_2^{-1} \otimes M_3)$$

belongs to I_a *if and only if all its elements vanish when evaluated on the entries of the complex* (9).

EXAMPLE. Let $r_1 = r_2 = r_3 = 2$, $\dim F_0 = 2$. The map $a_2 : R \to \Lambda^2 R^4$ has entries $(a_2)_{i,j}$, $1 \leqslant i < j \leqslant 4$. Let us consider the Plucker relation

$$p = (a_2)_{14}(a_2)_{23} - (a_2)_{13}(a_2)_{24} + (a_2)_{12}(a_2)_{34}.$$

We will show that p belongs to I_a. First of all the set of multiples of p forms a **G**-invariant subset because it corresponds to $\Lambda^4 F_1 \subset S_2(\Lambda^2 F_1)$. Now it is easy to see from the definitions that for complex (9) all the entries of a_2 except one, say $(a_2)_{12}$, are equal to 0. This shows that p belongs to I_a.

Now we introduce the analogue of the Kempf construction. Let $X_a = \operatorname{Spec} R_a$, $\operatorname{Grass} = \operatorname{Grass}(r_3, F_2) \times \operatorname{Grass}(r_2, F_1) \times \operatorname{Grass}(r_1, F_0)$.

DEFINITION 2.5. $Y_a \subset X_a \times \operatorname{Grass}$ is the set of 8-uples $(d_3, d_2, d_1, a_2, a_1, R_2, R_1, R_0)$ such that

(1) $(d_3, d_2, d_1, a_2, a_1)$ belongs to X_a,

(2) R_i belongs to $\operatorname{Grass}(r_{i+1}, F_1)$,

(3) $\operatorname{Im} a_i \subset \Lambda^{r_i} R_i$, $\operatorname{Im} d_i \subset R_i \subset \operatorname{Ker} d_{i-1}$,

(4) for the induced maps $d_i' : Q_i = F_i / R_i \to R_{i-1}$ and $a_i' \in \Lambda^{r_i} R_i$ we have $\det d_3' = a_3'$, $\det d_2' = a_3' a_2'$, $\det d_1' = a_2' a_1'$.

Again we get the natural projections $p_a : Y_a \to X_a$, $q : Y_a \to \operatorname{Grass}$. By the same method as for R_c one can now show

PROPOSITION 2.6. [PW]. *The ring* R_a *has the following decomposition into representations of* **G**:

$$
\begin{aligned}
R_a = \sum S_{(a-b+c+\mathbf{a}_1, \ldots, a-b+c+\mathbf{a}_{r_3-1}, a-b+c)} F_3 \\
\otimes S_{(b-c+\mathbf{b}_1, \ldots, b-c+\mathbf{b}_{r_2-1}, b-c, -a+b-c, -a+b-c-\mathbf{a}_{r_3-1}, \ldots, -a+b-c-\mathbf{a}_1)} F_2 \\
\otimes S_{(c+\mathbf{c}_1, \ldots, c+\mathbf{c}_{r_1-1}, c, c-b, c-b-\mathbf{b}_{r_2-1}, \ldots, c-b-\mathbf{b}_1)} F_1 \\
\otimes S_{(\underbrace{0,0,\ldots,0}_{r_0}, -c, -c-\mathbf{c}_{r_1-1}, \ldots, -c-\mathbf{c}_1)} F_0,
\end{aligned}
\tag{10}
$$

where we sum over the set of natural numbers a, b, c and partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ satisfying $\mathbf{a}_1^{\sim} < r_3$, $\mathbf{b}_1^{\sim} < r_2$, $\mathbf{c}_1^{\sim} < r_1$.

In [PW] it is also proved that R_a is the ideal transform of R_c with respect to $I(d_3)\, I(d_2)\, I(d_1)$ (compare Section 1(B)), and that over R_a depth $I(d_3) = 2$, depth $I(d_2) = 2$, depth $I(d_1) = 1$.

This shows that to construct the hypothetical generic ring R_{gen} from R_a we just have to raise depth of $I(d_3)$ from 2 to 3.

The next step we have to make to construct R_{gen} is to take into account Theorem 2.1(b).

Let us recall that 2.1(b) asserts the existence of a map b satisfying the diagram (3). One can easily see using Lemma 2.4 that $d_1(a_2^* \otimes 1)$ $(\Delta \otimes 1) = 0$ so existence of b just says that the family of cycles $(a_2^* \otimes 1)$ $(\Delta \otimes 1)$ has to be a boundary over R_{gen}. But we see that for a given complex the choice of b is not unique—we can get another map b' satisfying (3) by adding to b any map of the form $d_3 u$ for $u\colon \Lambda^{r_1+1} F_1 \otimes M_2 \otimes M_3^{-1} \to F_3$ because $d_2 d_3 = 0$. We introduce the ring

$$
R_b' = R_a \otimes S \cdot (\Lambda^{r_1+1} F_1 \otimes F_2^* \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1}) / I_b, \tag{11}
$$

where I_b is the set of elements which vanish when evaluated at the entries of d_i's, a_i's, and b for any acyclic complex $\mathbf{G}$ of type (1) *for any choice of b for such complex.* Over R_b' we get a natural complex $\mathbf{F}_b'$. Now it could happen that $H_2 \mathbf{F}_b'$ is not zero. To avoid that we introduce

$$
R_b = \{\text{ideal transform of } R_b' \text{ with respect to } I(d_3)\}. \tag{12}
$$

It turns out that there exists another map $b^{\#}$ which plays a role similar to b and is more convenient for our purposes because it is related directly to Koszul complex on $I(d_3)$. Let us describe the relation between b and $b^{\#}$.

First of all let us write $K = M_3^* \otimes \Lambda^{r_3} F_2$ and consider the complex

$$
\mathbf{K}\colon 0 \to \Lambda^0 K \to \Lambda^1 K \to \Lambda^2 K \to \Lambda^3 K, \tag{13}
$$

the initial part of Koszul complex of $I(d_3)$. We write $\mathbf{K}_a$, $\mathbf{K}_b$, ... to denote the complex $\mathbf{K}$ tensored with R_a, R_b, ... For any acyclic complex $\mathbf{G}$ of type (1) the corresponding part of the Koszul complex is acyclic.

Our goal is to write down explicitly the cyle in $\mathbf{K}_a$ which corresponds to the map b. The tool we use is the following crucial lemma which will be used frequently in the sequel

LEMMA 2.7. *Let* $X = \operatorname{Spec} R$, $j\colon U \to X$ *be an open immersion. Let*

$$\mathbf{G}\colon 0 \to G_3 \to G_2 \to G_1 \to G_0$$

be a free complex over X *such that* $\mathbf{G} \mid U$ *is acyclic. Then* $H_3(\mathbf{G} \otimes j_* \mathcal{O}_U) = 0$, $H_2(\mathbf{G} \otimes j_* \mathcal{O}_U) = 0$ *and we have an exact sequence*

$$0 \to H_1(\mathbf{G} \otimes j_* \mathcal{O}_U) \to G_3 \otimes R^1 j_* \mathcal{O}_U \to G_2 \otimes R^1 j_* \mathcal{O}_U. \tag{14}$$

Proof. We decompose $\mathbf{G} \otimes \mathcal{O}_U$ into short exact sequences

$$0 \to G_3 \otimes \mathcal{O}_U \to G_2 \otimes \mathcal{O}_U \to C_2 \to 0$$
$$0 \to C_2 \to G_1 \otimes \mathcal{O}_U \to G_0 \otimes \mathcal{O}_U.$$

Now we apply functor j_* to both sequences. We get

$$\begin{aligned} 0 &\to G_3 \otimes j_* \mathcal{O}_U \to G_2 \otimes j_* \mathcal{O}_U \to j_* C_2 \\ &\to G_3 \otimes R^1 j_* \mathcal{O}_U \to G_2 \otimes R^1 j_* \mathcal{O}_U, \\ 0 &\to j_* C_2 \to G_1 \otimes j_* \mathcal{O}_U \to G_0 \otimes j_* \mathcal{O}_U. \end{aligned}$$

Now the lemma follows by easy diagram chase.

Next we need an application of Kempf construction. Let us denote $D_3 = V(I(d_3))$. We have a fundamental diagram:

$$\begin{tikzcd} Y_a \backslash D_3 \arrow[r, "j'_a"] \arrow[d, "p'_a"] & Y_a \arrow[r, "q_a"] \arrow[d, "p_a"] & \text{Grass} \\ X_a \backslash D_3 \arrow[r, "j_a"] & X_a & \end{tikzcd} \tag{15}$$

It gives rise to two spectral sequences

$$R^k p_{a*}(R^l j'_{a*} \mathcal{O}_{Y_a \backslash D}) \Rightarrow R^{k+l}(p_a j'_a)_* \mathcal{O}_{Y_a \backslash D}$$
$$R^k j_{a*}(R^l p'_{a*} \mathcal{O}_{Y_a \backslash D}) \Rightarrow R^{k+l}(j_a p'_a)_* \mathcal{O}_{Y_a \backslash D}.$$

j_a is affine because D_3 is defined on Y_a by one equation $a'_3 = 0$. The first spectral sequence thus degenerates and we have

$$R^k j_{a*}(R^l p'_{a*} \mathcal{O}_{Y_a \backslash D}) \Rightarrow R^{k+l} p_{a*}(\mathcal{O}_{Y_a}[a_3'^{-1}]).$$

Moreover X_a has rational singularities so

$$(R^l p'_{a*}\mathcal{O}_{Y_a\setminus D})=0 \qquad \text{for} \quad l>0 \text{ and} \qquad p'_{a*}(\mathcal{O}_{Y_a\setminus D})=\mathcal{O}_{X_a\setminus D}.$$

Finally, we get

$$R^k j_{a*}(\mathcal{O}_{X_a\setminus D})=R^k p_{a*}(\mathcal{O}_{Y_a}[a_3'^{-1}]). \tag{16}$$

Now X_a is affine, so $R^k p_{a*}(\mathcal{O}_{Y_a}[a_3'^{-1}])=H^k(Y_a, \mathcal{O}_{Y_a}[a_3'^{-1}])=H^k(\text{Grass}, q_{a*}(\mathcal{O}_{Y_a}[a_3'^{-1}])$ because q_a is affine.

The cohomology groups over Grass are easy to calculate using Bott's theorem using the decomposition

$$\begin{aligned} q_{a*}(\mathcal{O}_{Y_a}[a_3'^{-1}])=\sum\, & S_{\mathbf{a}}F_3\otimes S_{\mathbf{a}}R_2^*\otimes S_{\mathbf{b}}Q_2\otimes S_{\mathbf{b}}R_1^* \\ & \otimes S_{\mathbf{c}}Q_1\otimes S_{\mathbf{c}}R_0^*\otimes (M_3\otimes \Lambda^{r_3}R_2^*)^a \\ & \otimes (M_3^{-1}\otimes M_2\otimes \Lambda^{r_2}R_1^*)^b \\ & \otimes (M_3\otimes M_2^{-1}\otimes M_1\otimes \Lambda^{r_1}R_0^*)^c, \end{aligned} \tag{17}$$

where we sum over all partitions $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfying $\mathbf{a}_1^\sim<r_3$, $\mathbf{b}_1^\sim<r_2$, $\mathbf{c}_1^\sim<r_1$ and all integers a, b, c with $b, c\geqslant 0$.

Now we apply Lemma 2.7 to $R=R_a$, $U=X_a\setminus D_3$, and $\mathbf{G}=\mathbf{F}_a$ and $\mathbf{K}_a$. We find that

$$\begin{aligned} H_1(\mathbf{F}_a)&=\operatorname{Ker}(F_3\otimes R^1j_*\mathcal{O}_U\to F_2\otimes R^1j_*\mathcal{O}_U) \\ H_1(\mathbf{K}_a)&=\operatorname{Ker}(\Lambda^0K\otimes R^1j_*\mathcal{O}_U\to \Lambda^1K\otimes R^1j_*\mathcal{O}_U) \end{aligned}$$

because $j_*\mathcal{O}_U=\mathcal{O}_{X_a}$ (depth $I(d_3)=2$). Now applying (17) we see that $R^1j_*\mathcal{O}_U$ contains the crucial term $\mathbf{q}_1^*=\Lambda^{r_1+1}F_1\otimes F_3^*M_1^{-1}\otimes M_2\otimes M_3^{-1}$. It corresponds to $\mathbf{a}=(1^{r_3-1})$, $\mathbf{b}=(1^{r_2-1})$, $a=-2$, $\mathbf{c}=b=c=0$ $\mathbf{q}_1^*$ corresponds to the map

$$b^{\#}\colon \Lambda^{r_1+1}F_1\otimes F_3^*\otimes M_1^{-1}\otimes M_2\otimes M_3^{-1}\to R^1j_*\mathcal{O}_U.$$

It turns out that b treated as an element of $F_3\otimes R^1j_*\mathcal{O}_U$ can be expressed in terms of $b^{\#}$:

$$\begin{aligned} b\colon & \Lambda^{r_1+1}F_1\otimes M_1^{-1}\otimes M_2\otimes M_3^{-1} \\ & \to F_3\otimes F_3^*\otimes \Lambda^{r_1+1}F_1\otimes M_1^{-1}\otimes M_2\otimes M_3^{-1}\to F_3\otimes R^1j_*\mathcal{O}_U. \end{aligned}$$

Conversely, $b^{\#}$ can be defined in terms of b:

$$\begin{aligned} b^{\#}\colon & F_3^*\otimes \Lambda^{r_1+1}F_1\otimes M_1^{-1}\otimes M_2\otimes M_3^{-1} \\ & \to F_3\otimes F_3^*\otimes R^1j_*\mathcal{O}_U\to R^1j_*\mathcal{O}_U. \end{aligned}$$

The above consideration shows that to get R_b from R_a it is enough to kill the cycle $b^{\#}$ in $H_1(\mathbf{K}_a)$ and then take ideal transform with respect to $I(d_3)$.

In order to get a more concrete description of $b^{\#}$ (not using $R^1 j_* \mathcal{O}_U$) we can use description of b in terms of submaximal minors [BE1].

First we consider the composition

$$\begin{aligned} &F_3^* \otimes \Lambda^{r_1+1} F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1} \\ &\quad \to \Lambda^{r_3} F_3^* \otimes \Lambda^{r_3-1} F_3 \otimes \Lambda^{r_3-1} F_2^* \otimes \Lambda^{r_3-1} F_2 \otimes F_2 \\ &\quad \to \Lambda^{r_3} F_3^* \otimes \Lambda^{r_3-1} F_3 \otimes \Lambda^{r_3-1} F_2^* \otimes \Lambda^{r_3} F_2. \end{aligned}$$

This defines the map

$$\begin{aligned} b^{\#}: F_3^* \otimes \Lambda^{r_1+1} F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1} &\to \Lambda^1 K \\ &= \Lambda^{r_3} F_3^* \otimes \Lambda^{r_3} F_2 \end{aligned}$$

using submaximal minors of d_3.

Now we define the notion of defect of the structure map.

DEFINITION 2.8. Let $\mathbf{G}$ be a free complex of type (1) over some ring S. Suppose we want to make $\mathbf{G}$ acyclic. For any map $x: X \to G_1$ from some free S-module X to G_1 such that $d_1 x = 0$ we have a structure map $y: X \to G_2$ such that $d_2 y = x$. In this situation y is not unique. To get other y's we have to add to a given one a map factoring through G_3. The set of such maps, i.e., $\mathrm{Hom}(X, G_3) = X^* \otimes G_3$, will be called *the defect of the structure map* y.

EXAMPLE. The defect of b is equal to the defect of $b^{\#}$ and it equals

$$\mathbf{q_1} = \Lambda^{r_1+1} F_1^* \otimes F_3 \otimes M_1 \otimes M_2 \otimes M_3^{-1}. \tag{18}$$

Next we define the sequence of structure maps p_n $(n \geqslant 1)$. The ring R_{gen} will be obtained from R_a by repeating the construction we used above to obtain R_b. We start with the definition of defects $\mathbf{q_i}$ of the maps p_i, $\mathbf{q_2}$ is defined by the exact sequence

$$\begin{aligned} 0 \leftarrow \mathbf{q_2} \leftarrow &\Lambda^2(F_3 \otimes \Lambda^{r_1+1} F_1) \otimes M_1^2 \otimes M_2^{-2} \otimes M_3^2 \\ \leftarrow &\Lambda^2 F_3 \otimes S_{(2^{r_1+1})} F_1^* \otimes M_1^2 \otimes M_2^{-2} \otimes M_3^2 \\ &\otimes S_2 F_3 \otimes S_{(2^{r_1}, 1^2)} F_1^* \otimes M_1^2 \otimes M_2^{-2} \otimes M_3^2, \end{aligned} \tag{19}$$

where the second map comes from embeddings

$$S_{(2^{r_1+1})} F_1^* \hookrightarrow S_2(\Lambda^{r_1+1} F_1^*)$$

and

$$S_{(2^{r_1},1^2)}F_1^* \circlearrowleft \Lambda^2(\Lambda^{r_1+1}F_1^*)$$

and Cauchy formulas.

The natural projection $\Lambda^2\mathbf{q}_1 \to^r \mathbf{q}_2$ defines on $\mathbf{L}_2 = \mathbf{q}_1 \oplus \mathbf{q}_2$ the structure of the graded Lie algebra in the sense of Section 1(D). We define $\mathbf{q}_n$'s inductively. Suppose we defined $\mathbf{L}_n = \mathbf{q}_1 \oplus \cdots \oplus \mathbf{q}_n$. Then we define $\mathbf{q}_{n+1}$ to be $H_2(\mathbf{L}_n)_{n+1}$ (compare Section 1(D)) and $\mathbf{L}_{n+1} = \mathbf{q}_1 \oplus \cdots \oplus \mathbf{q}_{n+1}$.

By definitions from Section 1(D) we get the exact sequences

$$0 \longleftarrow \mathbf{q}_{n+1} \xleftarrow{r} (\Lambda^2\mathbf{L}_n)_{n+1} \xleftarrow{h} (\Lambda^3\mathbf{L}_n)_{n+1}, \tag{20}$$

where $h(l_1 \wedge l_2 \wedge l_3) = [l_1, l_2] \wedge l_3 - [l_1\ l_3] \wedge l_2 + [l_2, l_3] \wedge l_1$.

Now we are ready to state the main result of this section.

THEOREM 2.9. *There exists a sequence of structure maps p_j such that $p_1 = b^\#$ and the defect of p_j equals $\mathbf{q}_j$.*

Proof. In order to define p_2 let us consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Lambda^0 K \arrow[r] & \Lambda^1 K \arrow[r, "d"] & \Lambda^2 K \arrow[r, "d"] & \Lambda^3 K \\ & & \mathbf{q}_2^* \arrow[u, "p_2"] \arrow[r, "r^*"] & \Lambda^2\mathbf{q}_1^* \arrow[u, "\Lambda^2 b^\#"'] & \end{tikzcd} \tag{21}$$

We want to show that the indicated factorisation p_2 exists. To prove that, it is enough to show that $d(\Lambda^2 p_1)r = 0$ for any acyclic $\mathbf{G}$ of type (1). The reasoning from the proof of Lemma 2.4 shows that it is enough to check the property for split acyclic $\mathbf{G}$ with a choice of map b for it. We choose the bases $\{f_i\}$, $\{g_j\}$, $\{h_m\}$, $\{u_n\}$ in F_3, F_2, F_1, F_0, respectively. In the calculation that follows we apply the convention that the capital letters denote multiindices, for example, J, J', J'' denote multiindices from the set of indices $\{j\}$ corresponding to the basis of F_2. Similarly for the letters i, m, n. Moreover, $d_3(i^\wedge, J)$ will denote the submaximal minor of d_3 with the set of rows J and the set of columns $\{1, ..., r_3\} \setminus \{i\}$. We choose our basis so that

$$\begin{aligned} (d_3)_{ij} &= \begin{cases} \delta_{ij} & \text{for } 1 \leqslant i, j \leqslant r_3 \\ 0 & \text{otherwise} \end{cases} \\ b_{j+r,M} &= \begin{cases} \pm 1, & M = \{r_2+1, ..., r_2+r_1, j\}, j \in \{1, ..., r_2\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{$*$}$$

Let us notice that we do not have control over $b_{j,M}$ for $1 \leqslant j \leqslant r_3$ because

of freedom of choice for b. Now we calculate $(d\Lambda^2 b^{\#})[(f_i \otimes h_M) \wedge (f_{i'} \otimes h_{M'})]$. We have

$$\begin{aligned} b^{\#}(f_i \otimes h_M) &= \sum_{j,J} d_3(i^\wedge, J)\, b_{j,M}\, g_j \wedge g_J \\ &\quad \times (d\Lambda^2 b^{\#})[(f_i \otimes h_M) \wedge (f_{i'} \otimes h_{M'})] \\ &= \sum \{d_3(j'' \cup J'')[d_3(i^\wedge J)\, b_{j,M} d_3(i'^\wedge, J')\, b_{j',M'} \\ &\quad - d_3(i^\wedge, J')\, b_{j',M} d_3(i'^\wedge, J)\, b_{j,M'}]\} \\ &\quad \times (g_j \wedge g_J) \wedge (g_{j'} \wedge g_{J'}) \wedge (g_{j''} \wedge g_{J''}). \end{aligned}$$

The coefficient in this sum is non-zero after the substitution $(*)$ only when $j'' \cup J'' = \{1, ..., r_3\}$ and $J, J' \subset \{1, ..., r_3\}$. This means that $b_{j,M}$ and $b_{j',M}$ are 0's and 1's according to $(*)$. To calculate $d(\Lambda^2 b^{\#})r$ we start from the linear combination of $h_M \otimes h_{M'}$ which shuffles M and M'. This means precisely (look at (19)) that M and M' cannot both contain $\{r_2+1, ..., r_2+r_1\}$, so in each term at least one coefficient of b is zero. This proves the existence of p_2. One should note that this is the crucial point since it explains the definition of $\mathbf{q}_2$.

To get the existence of p_n's for $n \geqslant 3$ let us proceed by induction. Suppose $p_1, ..., p_n$ are constructed. Let us consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Lambda^0 K \arrow[r, "d"] & \Lambda^1 K \arrow[r, "d"] & \Lambda^2 K \arrow[r, "d"] & \Lambda^3 K \\ & 0 \arrow[r] & \mathbf{q}^*_{n+1} \arrow[r, "r^*"] \arrow[u, "p_{n+1}"] & (\Lambda^2 \mathbf{L}^*_n)_{n+1} \arrow[r, "h^*"] \arrow[u, "\sum p_i \Lambda p_j"] & (\Lambda^3 \mathbf{L}^*_n)_{n+1} \arrow[u, "\sum p_i \Lambda p_j \Lambda p_k"] \end{tikzcd} \tag{22}$$

We want to show that the indicated factorisation exists. To do that we have to prove that the right square of the diagram commutes up to a constant. Then, for any acyclic $\mathbf{G}$ the upper row is acyclic, the composition $d(\sum p_i \wedge p_j)r^* = (\sum p_i \wedge p_j \wedge p_k)\, dd = 0$, and p_{n+1} exists.

To show commutativity let us consider the dual diagram

$$\begin{tikzcd} \Lambda^3 K^* \arrow[r] \arrow[d, "\sum p_i^* \Lambda p_j^* \Lambda p_k^*"] & \Lambda^2 K^* \arrow[r] \arrow[d, "\sum p_i^* \Lambda p_j^*"] & \Lambda^1 K^* \arrow[r] & \Lambda^0 K^* \arrow[r] & 0 \\ (\Lambda^3 \mathbf{L}_n)_{n+1} \arrow[r, "h"] & (\Lambda^2 \mathbf{L}_n)_{n+1}. & & & \end{tikzcd}$$

Let us choose $u, v, w \in K^*$. For the purposes of this calculation let us denote by x_u, x_v, x_w the images of u, v, w under d^* (we treat them as scalars):

$$\left(\sum p_i^* \wedge p_j^*\right) d^*(u \wedge v \wedge w) = \sum [x_u(p_i^* v \wedge p_j^* w) - x_v(p_i^* u \wedge p_j^* w) + x_w(p_i^* u \wedge p_j^* v)].$$

The other compositions equals

$$h\left(\sum p_i^* \wedge p_j^* \wedge p_k^*\right)(u \wedge v \wedge w)$$

$$= \sum h(p_i^* u \wedge p_j^* v \wedge p_k^* w)$$

$$= \sum \{[p_i^* u, p_j^* v] \wedge p_k^* w - [p_i^* u, p_j^* w] \wedge p_k^* v + [p_i^* v, p_j^* w] \wedge p_k^* u\}.$$

The summation above is over all $i, j, k \geqslant 1$. By induction we know that

$$[p_i^* u, p_j^* v] = x_v p_{i+j}^* u - x_u p_{i+j}^* v.$$

Indeed, this is just the commutativity of the dual of the left square in (22) because the bracket in $\mathbf{L}_{n+1}$ is defined by the map r. Putting this together we get

$$h\left(\sum p_i^* \wedge p_j^* \wedge p_k^*\right)(u \wedge v \wedge w)$$

$$= \sum (x_u p_i^* v - x_v p_j^* u) \wedge p_k^* w$$

$$- \sum (x_u p_i^* w - x_w p_j^* u) \wedge p_k^* v$$

$$+ \sum (x_v p_i^* w - x_w p_j^* v) \wedge p_k^* u$$

$$= 2\left(\sum x_u p_i^* v \wedge p_j^* w - \sum x_v p_i^* u \wedge p_j^* w + \sum x_w p_i^* u \wedge p_j^* v\right).$$

Thus the desired commutativity is shown and Theorem 2.9 is proved.

We construct now our candidate for the generic ring for resolutions of length 3. For $n \geqslant 1$ we define by induction

$$R_n' = R_{n-1} \otimes S \cdot (\mathbf{q}_n^* \otimes \Lambda^{r_3} F_3 \otimes \Lambda^{r_3} F_2^*)/I_n, \tag{23}$$

where I_n is the ideal of all elements which vanish when evaluated on all

acyclic complexes **G** of type (1) for all choices of $p_1, ..., p_n$ for them. Those evaluations make sense because for each acyclic **G** of type (1) there exist various choices of maps $p_1, ..., p_n$ and elements of R_{n-1} correspond basically to polynomials in the entries of $p_1, ..., p_{n-1}$ with coefficients in R_a. Next we define

$$R_n = \{\text{ideal transform of } R'_n \text{ with respect to } I(d_3)\}, \tag{24}$$

One should note that whenever taking ideal transform we may run into non-Noetherian rings. However, the notion of depth of the ideal which is the key notion in all proofs can be defined over any commutative ring (compare [N]) so this does not lead to difficulties.

We also denote by R''_n the subring of R_n generated over R_a by the entries of $p_1, ..., p_n$. Then R_n is the ideal transform of R''_n with respect to $I(d_3)$. One should also observe that each element of R_n can be again evaluated on an acyclic complex **G** of type (1) with a choice of $p_1, ..., p_n$ because taking the ideal transform (24) corresponds to killing cycles in H_2. We denote $X_n = \operatorname{Spec} R''_n$, $X_a = \operatorname{Spec} R_a$.

LEMMA 2.10. *X_n is reduced, irreducible, and* $\dim X_n = \dim X_{n-1} + \dim \mathbf{q_n}$.

Proof. For any ring S and acyclic complex **G** of type (1) over S we have depth $I(d_i) \geqslant i$. This implies depth $I(d_i) \geqslant 1$ over R''_n. After inverting a maximal minor of d_3, R''_n becomes a polynomial ring over R''_{n-1} in dim $\mathbf{q_n}$ variables. Indeed, after inverting a maximal minor of d_3 the complex (13) becomes split acyclic, so the map p_n kills a cycle which is boundary. Let $\{e_A\}$ be a basis of $\mathbf{q_n}$. If, using notation of (22), $(\sum p_i \wedge p_j)\, r^* e_A = dz_A$ then an arbitrary p_n is of the form $p_n(e_A) = z_A + d(w_A)$ and the elements w_A are independent. This proves our claim and it remains to show that R''_n is reduced. The group **G** acts on R''_n and the set of nilpotents is R''_n is obviously **G**-invariant. Let us choose finite dimensional **G**-invariant subspace $\mathscr{X}$ in R''_n consisting of nilpotents. Thus for any S, acyclic **G** of type (1) over S, and $h: R''_n \to S$ given by the universal property, we have $h(\mathscr{X})^N = 0$ for some N. Let $Z(S)$ be the set of zerodivisors of S, $T = S \backslash Z(S)$, $S' = T^{-1}S$. Then the natural map $S \to S'$ is an injection and $\mathrm{G} \otimes S'$ splits. This shows that the map $R''_n \to S \to S'$ factors through $R''_n[D^{-1}]$, for some maximal minor D of d_3. But $R''_n[D^{-1}]$ is a polynomial ring over $R_a[D^{-1}]$ so the image of $\mathscr{X}$ in S' is zero (R_a has no nilpotents). This shows that image of $\mathscr{X}$ in S is zero and the proof of the lemma is finished.

The representations $\mathbf{q_n}$ have a nice interpretation in terms of differentials.

PROPOSITION 2.11. *The module of differentials $D_{R_{n-1}}(R_n)$ is isomorphic to $R_n \otimes \mathbf{q_n}$.*

Proof. Let $\mathbf{q_n^*}$ denote free R_n-module $\mathbf{q_n^*} \otimes R_n$. We consider a homomorphism $u\colon \mathbf{q_n^*} \to R_n$. To define the differential D_u corresponding to u we look at the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^0 K \arrow[r, "d"] & \Lambda^1 K \arrow[r, "d"] & \Lambda^2 K \arrow[r, "d"] & \Lambda^3 K \\
 & 0 \arrow[r] & \mathbf{q_n^*} \arrow[r, "r^*"] \arrow[u, "p_n"] & (\Lambda^2 \mathbf{L}^*_{\mathbf{n}-1})_n \arrow[r, "h^*"] \arrow[u, "p'_n"] & (\Lambda^3 \mathbf{L}^*_{\mathbf{n}-1})_n \arrow[u, "p''_n"]
\end{tikzcd}$$

We choose the basis $\{e_A\}$ in $\mathbf{q_n^*}$. To construct D_u it suffices to describe $D_u[p_n(e_A)]$ as a vector in $\Lambda^1 K$ (we assume D_u acts on each coordinate). Indeed, D_u is zero on R_a so it vanishes on maximal minors of d_3. Thus if D_u is defined on R'_n it extends uniquely to R_n to a differential which we will also call D_u. To define $D_u[p_n(e_A)]$ let us notice that $H^0(\mathbf{K})$ and $H^1(\mathbf{K})$ are zero over R_{n-1} (compare Section 1(D)). Applying D_u to the coordinates of the equality $dp_n(e_A) = p'_n r^*(e_A)$ we get

$$d[D_u(p_n(e_A))] = 0.$$

In order for this equality to be satisfied we set

$$D_u(p_n(e_A)) = du(e_A). \tag{25}$$

This defines a derivation on R'_n which extends to R_n. Indeed, by 2.10 we see that (25) defines a derivation on a field of fractions of R'_n and it clearly carries R'_n into itself. Now $\{e_A^*\}$ is a basis of $\mathbf{q_n}$. The derivations $D_{e_A^*}$ are linearly independent over R_n because $d\colon \Lambda^0 K \to \Lambda^1 K$ is a monomorphism over R_n. Also every derivation $D\colon R_n \to R_n$ has to be one of the D_u's because, as above, $d[D(p_n(e_A))] = 0$ and $H^1(\mathbf{K}) = 0$ over R_n. The proposition is proved.

Let $\mathbf{L} = \oplus \mathbf{q_n}$. $\mathbf{L}$ has a natural structure of Lie algebra (by Section 1(D)). We finish this section with another description of this structure.

THEOREM 2.12. $\mathbf{L}$ *acts on $R_{\text{gen}} = \varinjlim R_n$ as a Lie algebra of derivations.*

Proof. Let $u \in \mathbf{q_n}$. We extend D_u to the derivation on $\lim R_n$ by the formula

$$D_u(p_{n+k}(e_A)) = \sum u(e_{A_i})\, p_k(e_{B_i}), \tag{26}$$

where e_A stands for a basis element of $\mathbf{q}^*_{\mathbf{n}+\mathbf{k}}$, $r^* e_A = \sum e_{A_i} \wedge e_{B_i}$, and we use

the convention that $u(e_B)=0$ for $e_B \in \mathbf{q}_l^*$, l not equal to n. To check that D_u is well defined we observe that

$$\begin{aligned} dD_u(p_{n+k}(e_A)) &= D_u\left(p'_{n+k}\left(\sum e_{A_i} \wedge e_{B_i}\right)\right) \\ &= D_u \sum p_j(e_{A_i}) \wedge p_l(e_{B_i}) \\ &= \sum D_u(p_j(e_{A_i})) \wedge p_l(e_{B_i}) \\ &\quad - \sum p_j(e_{A_i}) \wedge D_u(p_l(e_{B_i})) \\ &= \sum u(e_{A'_{im}})\, p_s(e_{A''_{im}}) \wedge p_l(e_{B_i}) \\ &\quad - \sum u(e_{B'_{im}})\, p_j(e_{A_i}) \wedge p_s(e_{B''_{im}}), \end{aligned}$$

where $r^*e_A = \sum e_{A_i} \wedge e_{B_i}$, $r^*e_{A_i} = \sum e_{A'_{im}} \wedge e_{A''_{im}}$, $r^*e_{B_i} = \sum e_{B'_{im}} \wedge e_{B''_{im}}$. On the other hand, we use (26) to obtain

$$\begin{aligned} dD_u(p_{n+k}(e_A)) &= d\left(\sum u(e_{A_i})\, p_k(e_{B_i})\right) \\ &= \sum u(e_{A_i}) \sum p_j(e_{B'_{im}}) \wedge p_l(e_{B''_{im}}). \end{aligned}$$

Now both expressions for $dD_u(p_{n+k}(e_A))$ obtained above are equal because of commutativity of the diagram

$$\begin{tikzcd} & \mathbf{q}^*_{\mathbf{n+j}} \otimes \mathbf{q}_l^* \oplus \mathbf{q}^*_{\mathbf{n+l}} \otimes \mathbf{q}_j^* \arrow[dr, "r^* \otimes 1"] & \\ \mathbf{q}^*_{\mathbf{n+k}} \arrow[ur, "r^*"] \arrow[dr, "r^*"'] & & \mathbf{q}_n^* \otimes \mathbf{q}_j^* \otimes \mathbf{q}_l^* \\ & \mathbf{q}_n^* \otimes \mathbf{q}^*_{\mathbf{j+l}} \arrow[ur, "1 \otimes r^*"'] & \end{tikzcd}$$

The last thing to check is that the commutator of derivations corresponds to the bracket in **L**. Let $u \in \mathbf{q}_\mathbf{k}$, $v \in \mathbf{q}_\mathbf{l}$. Then, using an analogous argument we have

$$
\begin{aligned}
(D_u D_v - D_v D_u)(p_n(e_A)) &= D_u\left[\sum v(e_{A_i})\, p_{n-l}(e_{B_i})\right] \\
&\quad - D_v\left[\sum u(e_{A_i})\, p_{n-k}(e_{B_i})\right] \\
&= \sum v(e_{A_i})\, u(e_{B'_{im}})\, p_{n-k-l}(e_{B''_{im}}) \\
&\quad - \sum u(e_{A_i})\, v(e_{B'_{im}})\, p_{k-k-l}(e_{B''_{im}}) \\
&= [u, v](e_{A'_i})\, p_{n-k-l}(e_{A''_i}) \\
&= D_{[u,v]}(p_{n+k}(e_A)).
\end{aligned}
$$

This concludes the proof of Theorem 2.12.

Now we can finally state our main conjecture:

Conjecture 2.13. Let $R_{\text{gen}} = \varprojlim R_n$. Over R_{gen} we have the naturally defined comples $\mathbf{F}_{\text{gen}} = \mathbf{F}_c \otimes R_{\text{gen}}$. The pair $(R_{\text{gen}}, \mathbf{F}_{\text{gen}})$ is the generic pair for the acyclic complexes of type (1) in the sense of Section 1(B).

Remark 2.14. The action of $\mathbf{L}$ on R_{gen} extends to the action of the enveloping algebra $U(\mathbf{L})$. We will use this action heavily in the next section.

Remark 2.15. The analogous construction of the defect Lie algebra can be carried out in general, i.e., when we increase the depth of some ideal from 2 to 3. What seems to be special for our situation is that the resulting Lie algebra $\mathbf{L}$ is defined over $\mathscr{k}$. If one could find an independent proof of this fact, one would prove our Main Conjecture 2.13.

3. The Spectral Sequence

In this section we will show that to prove Conjecture 2,13 it is enough to show that homology of the family of certain complexes over $U(\mathbf{L})$ is zero.

These complexes come from analysis of the Kempf construction for X_n. We define $Y_n \subset X_n \times_X Y_a$ to be the irreducible component of $X_n \times_X Y_a$ which is the closure of $(Y_a \backslash D_3) \times \operatorname{Spec} S \cdot (\mathbf{q}_1^* \oplus \cdots \oplus \mathbf{q}_n^*)$. We denote the corresponding projections by p_n and q_n. They form a diagram

$$
\begin{tikzcd}
Y_n \arrow[r, "p_n"] \arrow[d, "q_n"] & X_n \\
\text{Grass} &
\end{tikzcd}
$$

Conjecture 2.13 says that $\mathbf{F}_{\text{gen}}$ defined over R_{gen} is acyclic. We now that $\mathbf{F}_n$ is acyclic on $X_n \backslash D_3$. Now we apply Lemma 2.7 to $X = X_n$, $U = X_n \backslash D_3$ and $\mathbf{G}$ equal to $\mathbf{F}$ and $\mathbf{K}$. We have $j_* \mathcal{O}_U = R_n$. By 2.7, $H_1(\mathbf{G} \otimes R_n) = 0$ if $R^1 j_* \mathcal{O}_U = 0$. We calculate $R^1 j_* \mathcal{O}_U$ using the varieties Y_n. We start with the diagram

$$\begin{tikzcd} Y_n \backslash D_3 \arrow[r, "j_n'"] \arrow[d, "p_n'"] & Y_n \arrow[r, "q_n"] \arrow[d, "p_n"] & \text{Grass} \\ X_n \backslash D_3 \arrow[r, "j_n"] & X_n & \end{tikzcd} \tag{27}$$

It gives two spectral sequences corresponding to two compositions. Now we can apply the reasoning following (15). We notice that $p_n' = p_a' \otimes 1$ if we identify $\mathcal{O}_{Y_n}[a_3'^{-1}]$ with $\mathcal{O}_{Y_a}[a_3'^{-1}] \otimes S \cdot (\mathbf{q}_1^* \oplus \cdots \oplus \mathbf{q}_n^*)$. Our conclusion (analogue of (16)) is that

$$\begin{aligned} R^k j_{n*}(\mathcal{O}_{X_n \backslash D_3}) &= R^k p_{n*}(\mathcal{O}_{Y_n}[a_3'^{-1}]) \\ &= H^k(\text{Grass}, q_{n*} \mathcal{O}_{Y_n}[a_3'^{-1}]). \end{aligned} \tag{28}$$

In the sequel we will denote $q_{n*} \mathcal{O}_{Y_n}[a_3'^{-1}]$ by $\mathcal{F}_n$ and $q_{a*} \mathcal{O}_{Y_a}[a_3'^{-1}]$ by $\mathcal{F}_a$. We know from Section 2 that $\mathcal{F}_a$ has decomposition (17).

Now we calculate the cohomology of $\mathcal{F}_n$ in the following five steps.

1. We construct the filtration of $\mathcal{F}_n$ such that the cohomology of the associated graded object $\mathcal{G}_n$ can be easily calculated using (17).

2. The cohomology of $\mathcal{F}_n$ is the limit of the spectral sequence with the first term equal to cohomology of $\mathcal{G}_n$. We have to analyse which terms in the cohomology of $\mathcal{G}_n$ disappear in the spectral sequence.

3. The terms of cohomology of $\mathcal{G}_n$ are representations of $\mathbf{G}$. It turns out that the boundary maps in the spectral sequence multiply the representation of F_2 by the appropriate power of M_2 and preserve the representation of F_0. We fix the representations of F_2 and F_0 and look at the part of the spectral sequence corresponding to them. This part has very few terms. We get one term in H^0, one term in H^1, and two terms in H^2. Let us call those terms K_0, K_1, K_2, and $K_{1,1}$, respectively. Those terms considered as $U(\mathbf{L})$-modules are duals of certain free modules. The connecting maps from the spectral sequence between our terms are thus uniquely determined up to scalar and the piece of H^1 (corresponding to the chjoice of representations of F_2 and F_0) consists of the homology of the complex

$$K_0 \to K_1 \to K_2 \oplus K_{1,1}.$$

4. To check that the connecting maps are non-zero (i.e., corresponding

scalars are non-zero) we notice that those maps involve only the entries of p_1, so it is enough to check it for $n=1$.

5. We introduce the other filtration on F_1 and using it we finally prove that the connecting maps are non-zero.

Now we perform the steps indicated above.

Step 1. We order n-tuples $(u_1, ..., u_n)$ by reverse lexicographic order, i.e., $(u_1, ..., u_n) < (v_1, ..., v_n)$ if $u_i < v_i$ for the biggest i for which $u_i \neq v_i$. By construction $\mathscr{F}_n$ has the filtration $\{\mathscr{F}_n(u_1, ..., u_n)\}$ of $U(\mathbf{L})$-submodules

$$\mathscr{F}_n(u_1, ..., u_n) = \{f \in \mathscr{F}_n \mid (\deg_1 f, ..., \deg_n f) \leqslant (u_1, ..., u_n)\},$$

where $\deg_i f$ denotes the degree of f with respect to the elements of $\mathbf{q}_i^*$.

Let

$$\mathscr{F}_n(N) = \bigcup_{\sum iu_i \leqslant N} \mathscr{F}_n(u_1, ..., u_n)$$

and

$$\mathscr{G}_n(N) = \mathscr{F}_n(N)/\mathscr{F}_n(N-1).$$

Then we have

$$\mathscr{G}_n(N) = \mathscr{F}_a \otimes U(\mathbf{L_n})_N^*$$

if we put $\deg \mathbf{q}_\mathbf{i}^* = -i$. We define

$$\mathscr{G}_n(u_1, ..., u_n) = \mathscr{F}_n(u_1, ..., u_n) / \sum_{(v_1, ..., v_n) < (u_1, ..., u_n)} \mathscr{F}_n(v_1, ..., v_n).$$

Then we have

$$\mathscr{G}_n(u_1, ..., u_n) = \mathscr{F}_a \otimes S_{u_1}(\mathbf{q}_\mathbf{1}^*) \otimes \cdots \otimes S_{u_n}(\mathbf{q}_\mathbf{n}^*).$$

We know by construction that the action of $\mathbf{L_n}$ on $\mathscr{F}_n$ sends $\mathbf{q_i} \otimes \mathscr{F}_n(u_1, ..., u_n)$ into $\mathscr{F}_n(u_1, ..., u_i - 1, ..., u_n)$. The induced action on $\mathscr{G}$'s sends $\mathbf{q_i} \otimes \mathscr{G}_n(N)$ into $\mathscr{G}_n(N-i)$. It follows that

$$\bigoplus \mathscr{G}_n(N) = \mathscr{F}_a \otimes U(\mathbf{L_n})^* \tag{29}$$

as $U(\mathbf{L_n})$-modules.

Using Bott's theorem and the decomposition (17) of $\mathscr{F}_a$ we find

$$
\begin{aligned}
&H^*(\text{Grass}, \mathscr{F}_a) \\
&\quad = \sum S_{(a-b+c+\mathbf{a}_1, \ldots, a-b+c+\mathbf{a}_{r_3-1}, a-b+c)} F_3 \\
&\qquad \otimes S_{(b-c+\mathbf{b}_1, \ldots, b-c+\mathbf{b}_{r_2-1}, b-c, -a+b-c, -a+b-c-\mathbf{a}_{r_3-1}, \ldots, -a+b-c-\mathbf{a}_1)} F_2 \\
&\qquad \otimes S_{(c+\mathbf{c}_1, \ldots, c+\mathbf{c}_{r_1-1}, c, c-b, c-b-\mathbf{b}_{r_2-1}, \ldots, c-b-\mathbf{b}_1)} F_1 \\
&\qquad \otimes S_{(0,0, \ldots 0, -c, -c-\mathbf{c}_{r_1-1}, \ldots, -c-\mathbf{c}_1)} F_0. \qquad (30)
\end{aligned}
$$

Here the right side has to be read according to Bott's theorem, i.e., for given $\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c$ we apply Theorem 1.1 to the sequences above. If any of the sequences corresponds to case (2) of 1.1 then the term does not contribute to cohomology. If all sequences correspond to case (1) of 1.1 and we need j exchanges to order them then the term contributes to $H^j(\text{Grass}, \mathscr{F}_a)$. We notice that the only sequence in (30) which needs reordering is the sequence corresponding to F_2 for $a<0$ (then $b-c<-a+b-c$). All representations in $U(\mathbf{L})$ are induced from X_n, so by the projection formula

$$H^i(\text{Grass}, \mathscr{G}_n) = H^i(\text{Grass}, \mathscr{F}_a) \otimes U(\mathbf{L_n})^*. \qquad (31)$$

Step 2. We get the usual spectral sequence converging to $H^*(\text{Grass}, \mathscr{F}_n)$ with the first term $H^*(\text{Grass}, \mathscr{G}_n)$ so we have to determine which representations disappear in E. We should keep in mind that our spectral sequence consists of $U(\mathbf{L_n})$-modules.

Step 3. The spectral sequence we got above is the $\mathbf{G}$-invariant spectral sequence of $U(\mathbf{L_n})$-modules. The representations in $U(\mathbf{L_n})$ do not contain F_0 and they contain only the maximal exterior powers of F_2. Thus in the spectral sequence two terms can cancel (i.e., some boundary map carries one to another) only when they have the same partition corresponding to F_0 and the partitions corresponding to F_2 differ by some power of M_2. In the sequel we denote the term in (30) corresponding to $\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c$ by $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c)$. Since we are interested in H^1, from now on we look at the terms in (30) contributing to H^0, H^1, H^2.

Let us consider the term $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c)$ contributing to H^0 and the term $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ contributing to H^1 having the same representation corresponding to F_0 and representations corresponding to F_2 differing by the powers of M_2. This means that $\mathbf{c}=\mathbf{c}'$, $c=c'$ and there exists t such that

$$\begin{gathered}(b-c+\mathbf{b}_1, \ldots, b-c+\mathbf{b}_{r_2-1}, b-c, -a+b-c, \\ -a+b-c-\mathbf{a}_{r_3-1}, \ldots, -a+b-c-\mathbf{a}_1) \\ \| \\ (b'-c'+\mathbf{b}'_1-t, \ldots, b'-c'+\mathbf{b}'_{r_2-1}-t, -a'+b'-c'-1-t, \\ b'-c'+1-t, -a'+b'-c'-\mathbf{a}'_{r_3-1}-t, \ldots, -a'+b'-c'-\mathbf{a}_1-t)\end{gathered} \tag{32}$$

The possible cancellation between both terms can occur only if we can get $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ inside of $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c) \otimes U(\mathbf{L}_n)^*_{-t}$. Let us look at the corresponding partitions of F_3. They are

$$(a-b+c+\mathbf{a}_1, \ldots, a-b+c+\mathbf{a}_{r_3-1}, a-b+c)$$

and

$$(a'-b'+c'+\mathbf{a}'_1+t, \ldots, a'-b'+c'+\mathbf{a}'_{r_3-1}+t, a'-b'+c'+t).$$

In those sequences the first r_3-1 terms are equal by (32). The only possibility is to get $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ in $S_t F_3^* \otimes S_t(\Lambda^{r_1+1}F_1) \otimes M_3^{-t} \otimes M_2^t \otimes M_1^{-t}$ because we can add boxes only in the last row. From (32) we get also

$$\begin{aligned} &\mathbf{b}_i - \mathbf{b}_{i+1} = \mathbf{b}'_i - \mathbf{b}'_{i+1}, \qquad \mathbf{a}_i - \mathbf{a}_{i+1} = \mathbf{a}'_i - \mathbf{a}'_{i+1} \\ &b - b' = \mathbf{b}'_i - \mathbf{b}_1 - t, \qquad \mathbf{b}_{r_2-1} = \mathbf{b}'_{r_2-1} + a' + 1, \qquad a = -a' - 2. \end{aligned} \tag{33}$$

We also see that $a'-b'+c'+t = a-b+c$ which gives $t = a+1$. Looking at the sequences corresponding to F_1 we see that multiplying by $S_t(\Lambda^{r_1+1}F_1)$ occurs just by adding boxes to the last r_1+1 rows. Putting this all together gives

$$\begin{aligned} &\mathbf{a} = \mathbf{a}' \\ &a = -1, \qquad a' = -1-t \\ &\mathbf{b}'_i = \mathbf{b}_i + t \\ &b = b', \qquad c = c', \qquad \mathbf{c} = \mathbf{c}'. \end{aligned} \tag{34}$$

We see that both terms *determine each other and t*. Also $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ occurs in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c) \otimes U(\mathbf{L}_n)^*_{-t}$ with multiplicity one.

Let us look now at the possible cancellations between H^1 and H^2. We choose $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c)$ contributing to H^1 and $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ contributing to H^2. Again we assume that both terms have the same representation corresponding to F_0 and that the representations corresponding to F_2 differ

by the power of M_2. This means we have $\mathbf{c}=\mathbf{c}'$, $c=c'$. Now we have two possibilities of exchanges in Bott's algorithm:

(i) there exists s such that

$$\begin{gathered}(b-c+\mathbf{b}_1, \ldots, b-c+\mathbf{b}_{r_2-1}, -a+b-c-1, b-c+1,\\ -a+b-c-\mathbf{a}_{r_3-1}, \ldots, -a+b-c-\mathbf{a}_1)\\ \Vert \qquad\qquad (35\mathrm{i})\\ (b'-c'+\mathbf{b}'_1-s, \ldots, b'-c'+\mathbf{b}'_{r_2-1}-s, -a'+b'-c'-1-s,\\ -a'+b'-c'-\mathbf{a}'_{r_3-1}-1-s, b'-c'+2-s, \ldots, -a'+b'-c'-\mathbf{a}'_1-s);\end{gathered}$$

(ii) there exists u such that

$$\begin{gathered}(b-c+\mathbf{b}_1, \ldots, b-c+\mathbf{b}_{r_2-1}, -a+b-c-1, b-c+1,\\ -a+b-c-\mathbf{a}_{r_3-1}, \ldots, -a+b-c-\mathbf{a}_1)\\ \Vert \qquad\qquad (35\mathrm{ii})\\ (b'-c'+\mathbf{b}'_1-u, \ldots, -a'+b'-c'-2-u, b'-c'+\mathbf{b}'_{r_2-1}+1-u,\\ b'-c'+1-u, -a'+b'-c'-\mathbf{a}'_{r_3-1}-u, \ldots, -a'+b'-c'-\mathbf{a}'_1-u).\end{gathered}$$

Let us look at case (i). We want to get $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ inside of $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c)\otimes U(\mathbf{L}_n)^*_{-s}$. Now we look at the corresponding representations of F_3. They are

$$(a-b+c+\mathbf{a}_1, \ldots, a-b+c+\mathbf{a}_{r_3-1}, a-b+c)$$

and

$$(a'-b'+c'+\mathbf{a}'_1+s, \ldots, a'-b'+c'+\mathbf{a}'_{r_3-1}+s, a'-b'+c'+s).$$

These sequences are equal except at the r_3-1st place. This means we can add boxes only in the r_3-1st row and $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ has to occur inside of $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c)\otimes S_sF_3^*\otimes S_s(\Lambda^{r_1+1}F_1)\otimes M_3^{-s}\otimes M_2^s\otimes M_1^{-s}$. From (35i) we get also

$$\begin{gathered}\mathbf{b}_i-\mathbf{b}_{i+1}=\mathbf{b}'_i-\mathbf{b}'_{i+1}, \qquad \mathbf{a}_i-\mathbf{a}_{i+1}=\mathbf{a}'_i-\mathbf{a}'_{i+1}\\ \text{for} \quad i+1\leqslant r_3-2,\\ \mathbf{b}_{r_2-1}+a+1=\mathbf{b}'_{r_2-1}+a'+1, \qquad -a-2=\mathbf{a}'_{r_3-1}, \qquad 1+a+\mathbf{a}_{r_3-1}\\ =-a'-\mathbf{a}'_{r_3-1}-3-\mathbf{a}_{r_3-1}+\mathbf{a}_{r_3-2}=\mathbf{a}'_{r_3-2}+2+a'.\end{gathered} \tag{36i}$$

Now, looking at the partitions corresponding to F_1 we see that multiplying

by $S_s(\Lambda^{r_1+1}F_1)$ consists of adding s boxes to all of the r_1+1 last rows. Putting this all together gives

$$\begin{aligned} &\mathbf{b}_i = \mathbf{b}'_i - s, \qquad a = a' - s, \qquad \mathbf{a}_{r_3-1} = -a' - 2 \\ &s = 2 + a + \mathbf{a}_{r_3-1}, \qquad \mathbf{a}_i = \mathbf{a}'_i \qquad \text{for} \quad i \leqslant r_3 - 2 \\ &s = -2 - a' - \mathbf{a}'_{r_3-1}, \qquad \mathbf{a}'_{r_3-1} = -a - 2. \end{aligned} \tag{37i}$$

This shows that both terms *determine each other and any of them determines s.*

The same thing happens in case (ii) except that there adding boxes for F_3 occurs in the r_3th row and multiplying by $S_u(\Lambda^{r_1+1}F_1)$ occurs by adding u boxes to the $-1+r_2-$th, $1+r_2-$th, ..., r_1+r_2-th row. We omit the details because this case is analogous to case (i).

The above reasoning shows that after fixing the representations of F_2 (up to the powers of M_2) and F_0 the spectral sequence decomposes into complexes of the form

$$0 \to K_0 \to K_1 \to K_{1,1} \oplus K_2,$$

where each term is a sum of copies of $U(\mathbf{L}_n)^*$ and the maps between terms are uniquely defined up to scalar. Now we dualize those complexes to get free modules over $U(\mathbf{L}_n)$. Each piece we get is (after disregarding F_2 and F_0) a complex over $U(\mathbf{L}_n)$ of the form

$$\begin{array}{c} S_{(\mathbf{s}_1+t+u,\, \mathbf{s}_2,\, \ldots,\, \mathbf{s}_{r_3})} F_3 \\ \otimes\, S_{(\mathbf{t}_1+t+u,\, \ldots,\, \mathbf{t}_{r_1}+t+u,\, \mathbf{t}_{r_1+1}+t,\, \mathbf{t}_{r_1+2}+u,\, \ldots,\, \mathbf{t}_{r_1+r_2})} F_1^* \\ S_{(\mathbf{s}_1+t+s,\, \mathbf{s}_2+s,\, \mathbf{s}_3,\, \ldots,\, \mathbf{s}_{r_3})} F_3 \\ \otimes\, S_{(\mathbf{t}_1+t+s,\, \ldots,\, \mathbf{t}_{r_1+1}+t+s,\, \mathbf{t}_{r_1+2},\, \ldots,\, \mathbf{t}_{r_1+r_2})} F_1^* \\ \downarrow \\ S_{(\mathbf{s}_1+t,\, \mathbf{s}_2,\, \ldots,\, \mathbf{s}_{r_3})} F_3 \otimes S_{(\mathbf{t}_1+t,\, \ldots,\, \mathbf{t}_{r_1+1}+t,\, \mathbf{t}_{r_1+2},\, \ldots,\, \mathbf{t}_{r_1+r_2})} F_1^* \\ \downarrow \\ S_{(\mathbf{s}_1,\, \ldots,\, \mathbf{s}_{r_3})} F_3 \otimes S_{(\mathbf{t}_1,\, \ldots,\, \mathbf{t}_{r_1+r_2})} F_1^*, \end{array} \tag{38}$$

where u and s are determined by $\mathbf{s}, \mathbf{t}$, and t in the following way: $\mathbf{s}_2 + s = \mathbf{s}_1 + 1$ and $\mathbf{t}_{r_1+2} + u = \mathbf{t}_{r_1+1} + 1$. We call the complex defined by (38) $K(\mathbf{s}, \mathbf{t}, t)$.

Step 4. So far we have determined that all possible cancellations are of the form (38). Now we have to check that all of them really occur. This means that we have to show that the maps in (38) are non-zero. Let us

notice that the maps in (38) are the maps of free $U(\mathbf{L}_n)$-modules so they are determined by their images on generators. All maps come from multiplying by the representations $S_t F_3^* \otimes S_t(\Lambda^{r_1+1} F_1)$ so in fact they involve only the representation $\mathbf{q}_1^*$. So it is enough to look at $\mathscr{F}_1$ and to show that the representations giving the generators of terms in (38) disappear in $H^*(\text{Grass}, \mathscr{F}_1)$.

Step 5. To prove this last statement we introduce the other filtration on $q_{1*}\mathcal{O}_{Y_1}\backslash D_3$. The structure map b satisfies, as we know, the diagram (3) and is described in terms of submaximal minors of d_2 [BE1]. This second description has the following interpretation in terms of bundles on Grass. We start with the extension

$$\begin{aligned} 0 &\to Q_2^* \otimes \Lambda^{r_2-1} R_1^* \otimes M_3^{-1} \otimes M_2 \\ &\to \mathscr{T} \to R_2^* \otimes \Lambda^{r_2-1} F_1 \otimes M_3^{-1} \otimes M_2 \to 0 \end{aligned} \tag{39}$$

which is a factor of $F_2^* \otimes \Lambda^{r_2-1} F_1^* \otimes M_3^{-1} \otimes M_2$. Ths means that $\mathscr{T}$ is induced by the extension $0 \to Q_2^* \to F_2^* \to R_2^* \to 0$. Now $\mathcal{O}_{Y_1}$ is constructed from $\mathscr{T}$. Indeed, let us observe that

$$\begin{aligned} &Q_2^* \otimes \Lambda^{r_2-1} R_1^* \otimes M_3^{-1} \otimes M_2 \\ &\quad = \Lambda^{r_2-1} Q_2 \otimes \Lambda^{r_2-1} R_1^* \otimes M_3^{-1} \otimes \Lambda^{r_3} R_2 \in \mathcal{O}_{Y_a}. \end{aligned}$$

It is obvious by construction that $\mathcal{O}_{Y_1}$ has a filtration over Grass whose associated graded object is $\mathcal{O}_{Y_a} \otimes S \cdot (R_2^* \otimes \Lambda^{r_2-1} F_1^* \otimes M_3^{-1} \otimes M_2)$. The boundary maps in the spectral sequence that evolves are induced by (39). We have to show that the maps in this spectral sequence connecting the terms $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c)$ and $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ satisfying (34), (37i) and its analogue for the case (ii) (which we did not state) are non-zero.

Let us first look at the case (34), i.e., the cancellations between H^0 and H^1. We want to cancel out $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ inside of $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c) \otimes (Q_2^* \otimes \Lambda^{r_2-1} R_1^* \otimes M_3^{-1} \otimes M_2)^t$ with $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c) \otimes S_t(R_2^* \otimes \Lambda^{r_2-1} R_1^*) \otimes M_3^{-t} \otimes M_2^t$. The F_2 components of the two terms are

$$S_{(\mathbf{b}_1, \ldots, \mathbf{b}_{r_2-1})} Q_2 \otimes S_{(1-t, 1-\mathbf{a}_{r_3-1}, \ldots, 1-\mathbf{a}_1)} R_2^* \otimes M_2$$

and

$$S_{(\mathbf{b}_1+t, \ldots, \mathbf{b}_{r_2-1}+t)} Q_2 \otimes S_{(1+t, 1+t-\mathbf{a}_{r_3-1}, \ldots, 1+t-\mathbf{a}_1)} R_2^*.$$

We have to show that the connecting homomorphisms coming from the extension $0 \to Q_2^* \to F_2^* \to R_2^* \to 0$ cancel out those terms.

Let us look at a more general situation. Let $\text{Grass}(r, n)$ be the grassmannian of r-subspaces of vector space F of dimension n. Let R, Q denote the tautological subbundle and tautological quotientbundle, respectively. We

consider two partitions $\mathbf{s}=(\mathbf{s}_1, ..., \mathbf{s}_q)$, $\mathbf{t}=(\mathbf{t}_1, ..., \mathbf{t}_r)$. Let $\mathbf{t}_1=\mathbf{s}_q+1$. Then we claim that for $1 \leqslant t \leqslant \mathbf{t}_1-\mathbf{t}_2$ the terms

$$S_{(\mathbf{s}_1+t, ..., \mathbf{s}_q+t, \mathbf{t}_1, \mathbf{t}_2+t, ..., \mathbf{t}_r+t)}F \in H^0(S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes S_t R^* \otimes M)$$

and

$$S_{(\mathbf{s}_1+t, ..., \mathbf{s}_{q-1}+t, \mathbf{s}_q, \mathbf{t}_1+t, \mathbf{t}_2+t, ..., \mathbf{t}_r+t)}F \in H^1(S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes (\Lambda^{q-1}Q)^t \otimes (\Lambda^r R)^t)$$

cancel out in the cohomology of the filtered bundle with the associated graded object $\sum S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes (\Lambda^r R)^t \otimes S_{t-j}R^* \otimes S_j Q^* \otimes (\Lambda^q Q)^t$. Indeed, the bundle in question is just $S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes (\Lambda^q Q)^t \otimes (\Lambda^r R)^t \otimes S_t F^*$ so it does not have cohomology because $\mathbf{t}_1=\mathbf{s}_q+1$!

To complete the argument we notice that in the case of our application all pieces for intermediate j sit inside of $q_{1*}\mathcal{O}_{Y_1}\backslash D_3$, so we are really looking at "the piece" of this bundle corresponding to those representations.

The same thing happens for the cancellations between H^1 and H^2. Let us look at $(\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b, c)$, $(\mathbf{a}', \mathbf{b}', \mathbf{c}', a', b', c')$ satisfying (37i) (we leave case (ii) to the reader). As above the statement follows from the following claim.

We consider the terms

$$S_{(\mathbf{s}_1+t, ..., \mathbf{s}_q+t, \mathbf{t}_1+t, \mathbf{t}_2, \mathbf{t}_3+t, ..., \mathbf{t}_r+t)}F \in H^1(S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes S_t R^* \otimes M)$$

and

$$S_{(\mathbf{s}_1+t, ..., \mathbf{s}_q, \mathbf{t}_1+t, ..., \mathbf{t}_r+t)}F \in H^2(S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes (\Lambda^{q-1}Q)^t \otimes (\Lambda^r R)^t).$$

Let us assume that $\mathbf{t}_2=\mathbf{s}_q+2$. Then those terms cancel out in the cohomology of the filtered bundle whose associated graded object is $\sum S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes S_{t-j}R^* \otimes (\Lambda^r R)^t \otimes (\Lambda^q Q)^t \otimes S_j Q^*$ with filtration coming from the tautological extension.

The claim follows instantly from the observation that our "filtered bundle" is nothing but $S_{\mathbf{s}}Q \otimes S_{\mathbf{t}}R \otimes (\Lambda^f F)^t \otimes S_t F^*$ so it has no cohomology.

This completes Step 5 of our proof, so we can state our theorem.

THEOREM 3.1. *To prove Conjecture* 2.13 *it is enough to show that the complexes* $K(\mathbf{s}, \mathbf{t}, t)$ *defined by* (38) *have no homology in the middle term.*

4. THE MULTIPLICATIVE STRUCTURE ON $\mathrm{Tor}^R(R/I, k)$

Let R be a local ring with maximal ideal m and quotient field k. Let I be the ideal in R such that the projective dimension of R/I equals 3. We apply the structure theorems to classify all possible multiplications on the graded algebra $\mathrm{Tor}^R(R/I, k)$.

The only fact we will need in this section is *the existence of structure map* p_2 so the results of this section are in fact characteristic free.

Let

$$0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \tag{40}$$

be a minimal free resolution of R/I. The graded algebra $\mathrm{Tor}^R(R/I, k)$ has components equal to k, $F_1 \otimes k$, $F_2 \otimes k$, $F_3 \otimes k$. In the sequel we will denote those components by k, $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$. The multiplication on $\mathrm{Tor}^R(R/J, k)$ is given by the maps $b'\colon \Lambda^2\mathscr{F}_1 \to \mathscr{F}_2$, $c'\colon \mathscr{F}_1 \otimes \mathscr{F}_2 \to \mathscr{F}_3$, b' equals b modulo m and c' equals modulo m, where c is defined by means of the diagram

$$\begin{tikzcd}
0 \arrow[r] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & R \\
 & & F_2 \otimes F_1 \arrow[ul, "c"] \arrow[u, "d_1 \otimes 1"] \arrow[r, "d'"] & \Lambda^2 F_1 \arrow[ul, "b"] \arrow[u, "a"] &
\end{tikzcd} \tag{41}$$

c satisfies the equality $d_3 c + bd' = d_1 \otimes 1$ so it exists by acyclicity of the first row of (41).

Let us consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^2 F_3 \arrow[r] & F_3 \otimes F_2 \arrow[r] & S_2 F_2 \arrow[r] & S_2 F_1 \\
 & S_2 F_2 \otimes \Lambda^2 F_1 \arrow[u, "\Lambda^2 c"] \arrow[r] \arrow[ur, "p_2'"] & F_2 \otimes \Lambda^3 F_1 \arrow[u, "bc"] \arrow[r] \arrow[ur, "p_2"] & \Lambda^4 F_1 \arrow[u, "S_2 b"] &
\end{tikzcd} \tag{42}$$

The map bc is the composition $F_2 \otimes \Lambda^3 F_1 \to F_2 \otimes F_1 \otimes \Lambda^2 F_1 \to^{c \otimes b} F_3 \otimes F_2$ and $\Lambda^2 c_2$ equals $S_2 F_2 \otimes \Lambda^2 F_1 \to \Lambda^2(F_2 \otimes F_1) \to \Lambda^2 F_3$.

To check that (42) is commutative we first do it for the splitting **F** and then apply the argument from the proof of Lemma 2.4. Checking commutativity for splitting **F** (with a choice of b) is straightforward so we omit it.

We know from Section 2 that the factorisation p_2 in (42) exists. The top row of (42) is acyclic so the map of complexes given by $\Lambda^2 c$, bc, $S_2 b$ is homotopically equivalent to 0, so the factorisation p_2' satisfying $bc = dp_2' + p_2 d$ and $\Lambda^2 c = p_2' d$ exists over R. Now we pass modulo m. All horizontal maps in (42) become 0, to the above equalities show that $\Lambda^2 c'$, $b'c'$, $S_2 b'$ are all zero.

To classify all possible multiplications on $\mathrm{Tor}^R(R/I, k)$ we look at the space Z of pairs (b', c') satisfying the relations $\Lambda^2 c' = b'c' = S_2 b' = 0$. b' is the element of $\Lambda^2 \mathscr{F}_1 \otimes \mathscr{F}_2^*$ and c' belongs to $\mathscr{F}_1^* \otimes \mathscr{F}_2^* \otimes \mathscr{F}_3$ so we can treat Z as Spec T, where T is the commutative ring

$$T = S \cdot (\Lambda^2 \mathscr{F}_1 \otimes \mathscr{F}_2^*) \otimes S \cdot (\mathscr{F}_1 \otimes \mathscr{F}_2 \otimes \mathscr{F}_3^*)/J, \tag{43}$$

where J is the ideal generated by the following representations:

$\Lambda^4\mathscr{F}_1 \otimes S_2\mathscr{F}_2^*$ in degree $(2, 0)$ (corresponding to $\Lambda^2 c'$),

$\Lambda^3\mathscr{F}_1 \otimes \mathscr{F}_2^* \otimes \mathscr{F}_2 \otimes \mathscr{F}_3^*$ in degree $(1, 1)$ (corresponding to $b'c'$),

$\Lambda^2\mathscr{F}_1 \otimes S_2\mathscr{F}_2 \otimes \Lambda^2\mathscr{F}_3^*$ in degree $(0, 2)$ (corresponding to $S_2 b'$),

$S_{21}\mathscr{F}_1 \otimes \mathscr{F}_3^*$ in degree $(1, 1)$ (corresponding to associativity).

The group $\mathbf{G} = GL(\mathscr{F}_3) \times GL(\mathscr{F}_2) \times GL(\mathscr{F}_1)$ acts on Z. The key result that allows us to classify the multiplications is the following

THEOREM 4.1. *Let us assume that* $\dim F_1 \geqslant 4$. *Then Z has finitely many* $\mathbf{G}$*-orbits listed below.*

Proof. Let us fix the basis $\{e_i\}$, $\{f_j\}$, $\{g_k\}$ of $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$, respectively. We want to reduce pairs (b', c') to the canonical form. We start the analysis by considering the map b' alone. The relations in degree $(2, 0)$ mean that the image of any $f^* \in \mathscr{F}_2^*$ under $b'^*: \mathscr{F}_2^* \to \Lambda^2\mathscr{F}_1^*$ is decomposable. This gives us two possibilities:

(a) b'^* is decomposable, i.e., there exists $h: \mathscr{F}_2^* \to \mathscr{F}_1^*$ such that for some $e^* \in \mathscr{F}_1^*$ we have $b'^*(f^*) = e^* \wedge h(f^*)$,

(b) $\operatorname{Im} b'^* \subset \Lambda^2 H^*$, where $\dim H = 3$.

This shows that b' has one of the following canonical forms:

(A_p) $e_1 \cdot e_{i+1} = f_i$ $(1 \leqslant i \leqslant p)$, all other products are zero,

(B) $e_1 \cdot e_2 = f_3$, $e_1 \cdot e_3 = -f_2$, $e_2 \cdot e_3 = f_1$, all other products are zero.

The index p has the range $0 \leqslant p \leqslant \min(\dim \mathscr{F}_1, \dim \mathscr{F}_2)$.

Now we consider the possibilities for c'. Let us assume that b' has form A_p with $p \geqslant 2$. Then the relations in degree $(1, 1)$ force

$$f_j \cdot e_i = 0 \quad \text{for } i \geqslant 2, \qquad f_j \cdot e_1 = 0 \quad \text{for } j \leqslant p.$$

The only possibilities of non-zero products are $f_j \cdot e_1$ for $j > p$. Thus for $0 \leqslant q \leqslant \min(\dim \mathscr{F}_3, \dim \mathscr{F}_2 - p)$ we get the orbit

$(A_{p,q})$ $e_1 \cdot e_{i+1} = f_i$ $(1 \leqslant i \leqslant p)$, $e_1 \cdot f_{p+j} = g_j$ $(1 \leqslant j \leqslant q)$, all other products are zero.

Let b' be of type A_1. Then the relations in degree $(1, 1)$ force $e_i \cdot f_j = 0$ for $i > 2$. Also $f_1 \cdot e_i = 0$ for any i. The possible non-zero products are

$$e_1 \cdot f_j, e_2 \cdot f_j \qquad \text{for} \quad j > 1.$$

The relations in degree $(0, 2)$ mean that if we treat c' as a map $\mathscr{F}_2 \to$

$\mathscr{F}_3 \otimes \mathscr{F}_1^*$ then for any $f \in \mathscr{F}_2$, $c'(f)$ is decomposable. This leaves us with the following possibilities:

$(A_{1,q})$ $e_1 \cdot e_2 = f_1$, $e_2 \cdot f_j = 0$, $e_1 \cdot f_{j+1} = g_j$ $(1 \leqslant j \leqslant q)$, all other products zero, for $0 \leqslant q \leqslant \min(\dim \mathscr{F}_3, \dim \mathscr{F}_2 - 1)$,

$(A_{1,2}^*)$ $e_1 \cdot e_2 = f_1$, $e_1 \cdot f_2 = g_1$, $e_2 \cdot f_3 = g_1$, all other products zero.

Let us assume now that b' has type A_0. This means b' equals zero. Again the relations in degree $(0, 2)$ mean that for every $f \in \mathscr{F}_2$, $c'(f)$ is decomposable in $\mathscr{F}_3 \otimes \mathscr{F}_1^*$. This leads to the following orbits:

$(A_{0,q})$ $e_1 \cdot f_j = g_j$ $(1 \leqslant j \leqslant q)$, all other products zero, for $0 \leqslant q \leqslant \min(\dim \mathscr{F}_3, \dim \mathscr{F}_2)$,

$(A_{0,q}^*)$ $e_i \cdot f_i = g_1$ $(1 \leqslant i \leqslant q)$, all other products zero, for $2 \leqslant q \leqslant \min(\dim \mathscr{F}_1, \dim \mathscr{F}_2)$.

Finally, let us assume that b' has type B. Relations in degree $(1, 1)$ give in this case $e_i \cdot f_j = 0$ for all i, j. This means we get one orbit

(B) $e_1 \cdot e_2 = f_3$, $e_1 \cdot e_3 = -f_2$, $e_2 \cdot e_3 = f_1$, all other products are zero.

Now the list of our orbits is complete and Theorem 4.1 is proved.

Remark 4.2. The fact that for b' of type B, c' has to be zero is not in contradiction with the existence of the Koszul complex. We assumed above that $\dim F_1 \geqslant 4$. For $\dim F_1 \leqslant 3$ we do not have the diagram (42). For $\dim F_1 = 3$ it is known [Br] that the generic ring is the polynomial ring on the entries of d_3 and b. Similar analysis as above gives in this case four orbits for multiplication, among those the orbit of the exterior algebra in three variables.

Remark 4.3. We saw that the restrictions on multiplication were determined by the existence of p_2. The higher p_n give other operations on $\operatorname{Tor}^R(R/I, k)$. It would be interesting to get a general framework for this sort of structure, for example, in terms of higher Massey products.

Remark 4.4. A similar result is possible for other resolutions of length 3 $(r_1 > 1)$. Then for the map $b: \Lambda^{r_1+1} F_1 \to F_2$ we get two components for $b' = b \bmod m$. Let $b'^*: \mathscr{F}_2^* \to \Lambda^{r_1+1} \mathscr{F}_1^*$ be the dual map. Then we have two possibilities:

(a) $\operatorname{Im} b'^* \subset \Lambda^{r_1+1} H^*$ $\dim H^* = r_1 + 2$,

(b) b'^* is decomposable, i.e., there exists a map $h: \mathscr{F}_2^* \to \mathscr{F}_1^*$ such that $b'^*(f) = h(f) \wedge e_1^* \wedge \cdots \wedge e_r^*$ for some $e_1, \ldots, e_r \in \mathscr{F}_1$.

In this case we also get finitely many **G**-orbits for b'.

COROLLARY 4.5. *Let R be a regular local ring. Let I be an ideal in R such that* $\mathrm{pd}_R(R/I) \leqslant 3$. *Then R/I has rational Poincaré series.*

Proof. For $\mathrm{pd}_R(R/I) \leqslant 2$ the result is well known, so we assume that $\mathrm{pd}_R(R/I) = 3$. Every resolution of length 3 has an associative and commutative multiplication by [BE2]. By a theorem of Avramov [Av] it is enough to check that $\mathrm{Tor}^R(R/I, k)$ has rational Poincaré series. To prove the corollary we have to show that all algebras listed in Theorem 4.1 have rational Poincaré series. First of all the algebras of type $A_{p,q}$ can be expressed as a sum

$$A_{p,q} := \mathrm{Tor}^R(R/I, k) = A'_{p,q} \oplus I_{p,q},$$

where $A'_{p,q}$ is the span of $1, e_1, ..., e_{p+1}, f_1, ..., f_{p+q}, g_1, ..., g_q$ and $I_{p,q}$ is the span of other e's, f's, and g's. $I_{p,q}$ is the ideal in our algebra and $A_{p,q} = A'_{p,q} \ltimes I_{p,q}$ in the sense of [G]. It follows that the Poincaré series of $A_{p,q}$ is rational if and only if the Poincaré series of $A'_{p,q}$ is. Next we observe that $A'_{p,q}$ is a tensor product

$$A'_{p,q} = \{k1 \oplus ke_1\} \otimes \left\{k1 \oplus \sum_{i \geqslant 2} ke_i \oplus \sum_{j \geqslant 1} kf_{p+j}\right\}.$$

Now it is easy to check that the Poincaré resolution for $A'_{p,q}$ is the tensor product of the tensor algebras $T(ke_1)$ and $T(\sum ke_i \oplus \sum kf_{p+j})$. We conclude that

$$P(A'_{p,q})(z) = (1 - z^2)^{-1} (1 - pz^2 - qz^3)^{-1}.$$

We have to consider the algebras of type $A^*_{0,q}$, $A^*_{1,2}$, B and the case of the exterior algebra on three elements. In each case we can apply the result [G] for the ideal similar to $I_{p,q}$. Thus we can assume in case B that our algebra is the span of $1, e_1, e_2, e_3, f_1, f_2, f_3$. The Poincaré series of the exterior algebra is rational by [J]. The result for the algebra of type B follows from that by the result of Avramov [Av]. In the case $A^*_{0,q}$ we can assume that our algebra is a span of $1, e_1, ..., e_q, f_1, ..., f_q, g_1$. This algebra is Gorenstein and after dividing by the ideal generated by socle kg_1 we get an algebra with rational Poincaré series. Thus we can apply again the same result of Avramov. The case $A^*_{1,2}$ can be checked directly by calculating the Poincaré resolution.

ACKNOWLEDGMENTS

I thank David Buchsbaum for many helpful discussions of the material of this paper. I am also indebted to Luchezar Avramov, Andrew Kustin, and Matthew Miller for pointing out various errors in the first version of this paper.

REFERENCES

[ABW] K. AKIN, D. BUCHSBAUM, AND J. WEYMAN, Schur functors and Schur complexes, *Adv. in Math.* **44** (1982), 207–278.

[Av] L. AVRAMOV, Small homomorphisms of local rings, *J. Algebra* **50** (1978), 400–453.

[Bou] N. BOURBAKI, "Groupes et algebres de Lie," Chaps. I–III, p. 72, Hermann, Paris, 1971.

[B] A. BROWN, Thesis, Brandeis University, 1982.

[Br] W. BRUNS, Divisors on varieties of complexes, *Math. Ann.* **264** (1983), 53–71.

[BE1] D. BUCHSBAUM AND D. EISENBUD, Some structure theorems for finite free resolutions, *Adv. in Math.* **1** (1974), 84–139.

[BE2] D. BUCHSBAUM AND D. EISENBUD, Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3, *Amer. J. Math.* **99** (1977), 447–485.

[BE3] D. BUCHSBAUM AND D. EISENBUD, What makes a complex exact, *J. Algebra* **25** (1973), 259–268.

[D] J. DIXMIER, Algèbres Enveloppantes, Cahiers Scientifiques XXXVII, Gauthier-Villars, Paris, 1974.

[DC] J. DIEUDONNÉ AND J. CARREL, "Invariant Theory, Old and New," Academic Press, New York, 1971.

[G] T. GULLIKSEN, Massey operations and Poincaré series of certain local rings, *J. Algebra* **22** (1972), 223–232.

[H] M. HOCHSTER, *CBMS Regional Conf. Ser. in Math.* **24** (1975).

[Jac] N. JACOBSON, "Lie Algebras," Dover, New York, 1962.

[J] T. JÓZEFIAK, Tate resolutions for commutative graded algebras over local ring, *Fund. Math.* **74** (1972), 209–231.

[K] G. KEMPF, On the collapsing of homogeneous bundles, *Invent. Math.* **37** (1976), 229–239.

[McD] I. MACDONALD, "Symmetric Functions and Hall Polynomials," Clarendon, Oxford, 1979.

[N] D. G. NORTHCOTT, "Finite Free Resolutions," Cambridge Univ. Press, London/New York, 1976.

[PW] P. PRAGACZ AND J. WEYMAN, On the generic free resolutions, *J. Algebra*, in press.

[S] M. SCHAPS, Thesis, MIT, 1972.