On the Structure of Free Resolutions of Length 3

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INTRODUCTION

The sources of structure theory of finite free resolutions go back to Hilbert. In his famous paper on invariant theory he proved that every ideal of projective dimension 1 over a polynomial ring is determinantal. This theorem was then extended by Burch to arbitrary commutative rings.

There are various motivations for extending this results. One of them is a variety of applications of the Hilbert-Burch theorem in deformation theory and algebraic geometry (cf., for example, [S]). Another reason comes from Serre's conjecture. One would like to understand in terms of resolutions why the lifting property of Grothendieck does not hold.

In 1974 Buchsbaum and Eisenbud in [BE1] introduced so-called structure theorems—various factorisations which occur for acyclic complexes of arbitrary length. Then Hochster in [H] generalized the Hilbert-Burch theorem to arbitrary free complexes of length 2. His results were then improved in [Br, PW]. Still, the structure of complexes of length bigger than 2 is not understood. The first steps in this direction are the results of [B] and the construction of a generic ring for complexes of length 3 of type (1, n−1, 1) given in [PW].

In this paper I construct a ring which, I believe, is the generic ring for complexes of length 3. The construction involves new structure theorems and a new idea—the notion of defects of structure theorems.

The paper is organized as follows. Section 1 is introductory. I give there the background from representation theory and the cohomology groups of a Lie algebra. Section 2 contains the proof of the new structure theorems we need, the construction of the generic ring $R_{\text{gen}}$ and of the defect Lie algebra $L$. In Section 3 I show that genericity of $R_{\text{gen}}$ follows from vanishing of the homology of a certain family of complexes over $U(L)$—the enveloping algebra of $L$. In Section 4 I apply the results of Section 2 to study multi-
plicative structures on the resolutions of length 3. I give there the complete classification of possible multiplications modulo the maximal ideal on the finite free resolution of length 3 over a local ring.

1. Preliminaries

(A) Representation Theory

Throughout the paper we use extensively representation theory of the general linear group. Let us fix the field $\mathbb{F}$ of characteristic 0 and a vector space $F$ of dimension $n$ over $\mathbb{F}$. We denote by $GL(F)$ the group of $\mathbb{F}$-linear automorphisms of $F$.

It is well known that $GL(F)$ is linearly reductive and that the irreducible representations of $GL(F)$ are in one to one correspondence with dominant integral weights. In case of $GL(F)$ these are just the non-increasing sequences $a = (a_1, a_2, \ldots, a_n)$ of integers. We denote the corresponding representation by $S_a F$. If $a_n \geq 0$ then $a$ becomes a partition of some number $m$ and $S_a F$ is a Schur functor on $F$. The simplest way to define it is to take

$$S_a F = e(a) T^m(F),$$

where $T^m(F)$ is the $m$th tensor power of $F$ and $e(a)$ is the Young idempotent in the group ring of the symmetric group on $m$ letters corresponding to the partition $a$. The symmetric group acts on $T^m(F)$ by permuting copies of $F$. The reader not familiar with Young idempotents should consult Chapter 2 of [CD] for their definitions and basic properties. Let us single out some important special cases. For $a = (i, 0, \ldots, 0)$, $S_a F$ is the $i$th symmetric power of $F$; for $a = (1, 1, \ldots, 0)$ ($i$ ones and $n - i$ zeroes), $S_a F$ is the $i$th exterior power of $F$. Moreover, if $b = (a_1 - 1, \ldots, a_n - 1)$ then

$$S_b F = S_a F \otimes A^n F^*.$$ 

This allows us to define $S_a F$ for negative values of $a_n$. Let us also mention duality. If $b = (-a_n, \ldots, -a_1)$ then

$$S_b F^* = S_a F,$$

We will use often the Cauchy formulas. They give the decomposition of the symmetric and exterior power of a tensor product of two spaces into the Schur functors. For vector spaces $F$ and $G$

$$S_i (F \otimes G) = \sum_{a_n \geq 0, |a| = i} S_a F \otimes S_a G,$$

$$A^i (F \otimes G) = \sum_{a_n \geq 0, |a| = i} S_a F \otimes S_{a - i} G,$$
where $a^\sim$ denotes the dual partition to $a$. The simplest proof of the Cauchy formulas can be found in Chapter 1 of [McD]. The proofs of all above properties are contained in [ABW].

(B) "Depth Increasing" Problems in Commutative Algebra.

Let us fix the natural numbers $r_1, \ldots, r_n$. For $0 \leq i \leq n$ let $f_i = r_i + r_{i+1}$ ($r_0 = r_{n+1} = 0$). We consider the pairs $(R, F)$ of the free acyclic complexes

$$F : 0 \rightarrow F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots \rightarrow F_{n-1} \xrightarrow{d_{n-1}} F_n$$

over commutative rings $R$ such that rank $F_i = f_i$, rank $d_i = r_i$. We call the sequence $(r_1, \ldots, r_n)$ the type of the complex $F$. We call a pair $(R, F)$ generic (universal) if and only if for any other acyclic pair $(S, G)$ of the same type there exists a (unique) homomorphism $h : R \rightarrow S$ such that $G = F \otimes_R S$ and $F$ itself is acyclic over $R$.

By the exactness criterion of Buchsbaum and Eisenbud [BE3] the complex $F$ is acyclic if and only if $r_i + r_{i+1} = f_i$, and depth $I(d_i) \geq i$ for all $i$ ($I(d_i)$ denotes the ideal of maximal non-vanishing minors of $d_i$). My approach to the problem of finding the generic pair is to start with the generic complex and try to increase depth of ideals of minors "in a generic way." Let us make this notion more precise.

Suppose $I$ is an ideal in the ring $A$. We fix a natural number $i$. The $A$-algebra $B$ is called the generic (universal) extension of $I$ of depth $i$ if depth of $IB$ is $\geq i$ and if for any $A$-algebra $C$ with this property there exists a (unique) $A$-algebra homomorphism $h : B \rightarrow C$. There are several special cases where the construction of the generic extension is easy:

1. If $I$ has depth $i$ then we can take $B = A$.
2. If $i = 1$ we can take $B = A/J$, where $J = \bigcup \text{Ann}(I^n)$. In this case $B$ is universal because $h$ is unique. Let us notice that if $I$ is nilpotent then $B = 0$.
3. If $i = 2$ and depth $I = 1$ then we can take $B$ to be the ideal transform with respect to $I$. To see that let us assume that $I = (x_1, \ldots, x_m)$. Let us consider the following part of the Koszul complex on $x_1, \ldots, x_m$:

$$0 \rightarrow A^m A^m \rightarrow A^{m-1} A^m \rightarrow A^{m-2} A^m.$$ 

The cycles in $A^{m-1} A^m$ are the elements $a_1, \ldots, a_m$ such that $a_i x_j = a_j x_i$ for every $i, j$. This cycle is a boundary if and only if there exists $a$ in $A$ such that $a_i = a x_i$ for all $i$. We can assume that the $x_i$ are non-zerodivisors so the cycle corresponds to the collection of the elements $a_i x_i$ in the quotient ring of $A$ that are equal to each other, so it corresponds to an element of the ring

$$A_I = A[x_1^{-1}] \cap A[x_2^{-1}] \cap \cdots \cap A[x_m^{-1}].$$
This ring is called the ideal transform of $A$ with respect to $I$. If $X = \text{Spec } A$, $U = X \setminus V(I)$, then $A_I$ equals $j_\ast \mathcal{O}_U$ for $j: U \to X$ the inclusion. If $\text{depth } (I) \geq 2$ then $j_\ast \mathcal{O}_U = A$.

One should notice that $A_I$ need not be Noetherian even when $A$ is. In fact the famous Nagata counterexample of the ring of invariants of an algebraic group not being finitely generated is based on this fact (compare [DC]).

For $i \geq 3$ almost nothing is known. I hope that this paper will provide some insight to the case $i = 3$.

(C) Bott's Theorem and Kempf's Construction

Let $X_c$ be the variety of complexes of type $(r_1, \ldots, r_n)$. A point of $X_c$ is a set of $n$ matrices $d_i (1 \leq i \leq n)$ of sizes $f_i \times f_{i-1}$ satisfying the conditions $d_i d_{i+1} = 0$ and rank $d_i \leq r_i$ for all $i$. We sometimes treat the matrix $d_i$ as a map $F_i \to F_{i-1}$, where $F_i$ is a vector space of dimension $f_i$ for $0 \leq i \leq n$. Let $\text{Grass}(r, F)$ denote the Grassmannian of subspaces of $F$ of dimension $r$. We consider the product $X_c \times \Pi \text{Grass}(r_i, F_{i-1})$. Let $V_i$ denote the element of $\text{Grass}(r_{i+1}, F_i)$. We consider the subvariety $Y_c$:

$$Y_c = \{(d_1, \ldots, d_n; V_0, \ldots, V_{n-1}) | \text{Im } d_i \subset V_{i-1} \subset \text{Ker } d_{i-1} \text{ for } 0 \leq i \leq n-1\}.$$ 

Let $p: Y_c \to X_c$ and $q: Y_c \to \text{Grass} = \Pi \text{Grass}(r_i, F_{i-1})$ be the natural projections. The pair $(Y_c, p)$ is the desingularisation of $X_c$. If we denote by $R_i$ and $Q_i$ the tautological subbundle and factorbundle on $\text{Grass}(r_{i+1}, F_i)$ then the structure sheaf $\mathcal{O}_{Y_c}$ is isomorphic to the symmetric algebra on the vectorbundle $\oplus Q_i \otimes R_{i-1}^*$ (for the proofs see [PW]).

The key technique we will use throughout the paper is to use Bott's theorem to calculate higher direct images of the sheafs on $Y_c$. Here is the statement we use.

Let $F$ be a vector space of dimension $n$. Let us consider $\text{Grass}(r, F)$. We denote by $R$ and $Q$ the tautological subbundle and tautological factorbundle on $\text{Grass} (r, F)$. Let $q = n - r$. We consider two dominant integral weights $a = (a_1, \ldots, a_r)$, $b = (b_1, \ldots, b_q)$ and the corresponding vectorbundle $S_a R \otimes S_b Q$. From $a, b$ we form the sequence

$$s(a, b) = (b_1 + r + q, b_2 + r + q - 1, \ldots, b_q + r + 1, a_1 + r, \ldots, a_r + 1).$$

We distinguish two cases:

1. $s(a, b)$ has no repeated elements. Then we reorder it to get a decreasing sequence $c = (c_1, \ldots, c_n)$. We define $i(a, b)$ to be the number of exchanges (of consecutive elements) we need to get $c$ from $s(a, b)$. We also define the weight $c$ to be $(c_1 - n, c_2 - n + 1, \ldots, c_n - 1)$;
2. $s(a, b)$ has repeated entries.
Now we are ready to state Bott's theorem.

**Theorem 1.1.** (Bott). Let $X = \text{Grass}(r, F)$. The sheaf cohomology groups of $S_a R \otimes S_b Q$ are all zero in case (2) and in case (1) the only non-vanishing one is $H^{i(a, b)}(X, S_a R \otimes S_b Q) = S_c F$.

(D) **Graded Lie Algebras**

A **graded Lie algebra** is a graded vector space $L = \bigoplus L_i$ equipped with the bracket $[\cdot, \cdot] : L \otimes L \rightarrow L$ which is compatible with grading and satisfies the following properties. For $x, y, z \in L$

(a) $[x, y] = -[y, x],$
(b) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$

We notice that this definition of graded Lie algebra is different from the standard one because in (a), (b) we do not have additional signs coming from degrees of $x, y, z$.

Let $L$ be any Lie algebra. We define cohomology groups $H^i(L)$ to be the $i$th cohomology group of the complex

$$0 \longrightarrow L^* \overset{d}{\longrightarrow} \Lambda^2 L^* \overset{d}{\longrightarrow} \Lambda^3 L^* \overset{d}{\longrightarrow} \cdots,$$

where for $c \in \Lambda^i L^*$, $l_1, ..., l_{i+1} \in L$,

$$dc(l_1 \wedge l_2 \wedge \cdots \wedge l_{i+1})$$

$$= \sum_{j, k} (-1)^{j+k} c([l_j, l_k] \wedge l_1 \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge \hat{l}_k \wedge \cdots \wedge l_{i+1}).$$

We will be interested mostly in the group $H^2(L)$, which has the following interpretation:

$$H^2(L) = \{\text{classes of the abelian central extensions of } L\}.$$

To describe this correspondence let us recall that the abelian central extension of $L$ comes from the sequence

$$0 \rightarrow \kappa \rightarrow L^- \rightarrow L \rightarrow 0,$$

where $\kappa$ is the one dimensional central ideal in $L^-$. The correspondence associates to the cocycle $c : \Lambda^2 L \rightarrow \kappa$ the algebra $L^- = L \oplus \kappa$ with the bracket $[(l_1, x), (l_2, y)] = ([l_1, l_2], c(l_1 \wedge l_2))$. From this correspondence it follows that there exists the universal central abelian extension $L^\wedge$ of $L \cdot L^\wedge = L \oplus H^2(L)^*$ with the bracket $[(l_1, x), (l_2, y)] = ([l_1, l_2], c'(l_1 \wedge l_2))$, where $c'$ is the dual of embedding of 2 cocycles into $\Lambda^2 L^*$. 
There is a graded version of this correspondence we will use. Let us suppose that $L$ is a graded Lie algebra. Then $H^2(L)$ carries a natural gradation and the $m$th graded component $H^2(L)_m$ corresponds to the graded extensions $L^\sim$ for which the ideal $\mathcal{I}$ is contained in the $m$th component of $L^\sim$. We also get the universal central abelian extension $L_\sim^\wedge$ with the ideal in degree $m$. Additively $L^\sim_\wedge = L \oplus H^2(L)_m^\wedge$, the bracket being defined as before. The best references for the above are [Bou, Jac].

Let us suppose now that $L = L_1 \oplus L_2 \oplus \cdots \oplus L_m$. Then we get the universal graded extension

$$0 \to H^2(L)^*_m \to L^\wedge_{m+1} \to L \to 0$$

with the ideal in degree $m + 1$. Having defined the graded Lie algebra $L_2 = L_1 \oplus L_2$ we can now repeat the above construction to get $L = L_1 \oplus L_2 \oplus \cdots$, where $L_m$ are defined by induction as

$$L_{m+1} = H^2(L_m)^*_{m+1},$$

where $L_m = L_1 \oplus L_2 \oplus \cdots \oplus L_m$. In the case when the original bracket $A^2L_1 \to L_2$ is an epimorphism the resulting graded Lie algebra $L$ can be defined as a graded Lie algebra generated by $L_1$ in degree 1 with the relations in degree 2 given by $\ker(A^2L_1 \to L_2)$.

We will use this construction to define the defect Lie algebra in Section 2.

For the Lie algebra $L$ we will denote by $U(L)$ the enveloping algebra. It is by definition the factor of the tensor algebra on $L$ by the ideal generated by the relations $x \otimes y - y \otimes x = [x, y]$. We will denote by $U(L)^*$ the graded dual of $U(L)$. It is the injective envelope of the trivial $U(L)$-module, so the functor $\text{Hom}_{U(L)}(-, U(L)^*)$ defines a duality for finite dimensional $U(L)$-modules. For those properties of $U(L)$ the reader should consult [D].

2. Structure Theorems for Resolutions of Length 3

In this section we consider the complexes of length 3 of type $(r_1, r_2, r_3)$,

$$F: 0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0,$$

i.e., rank $d_i = r_i$, rank $F_i = f_i = r_i + r_{i+1}$ ($i = 1, 2, 3, r_4 = 0$).

Buchsbaum and Eisenbud have proved in [BE1] the following results which we state for resolutions of length 3.
THEOREM 2.1. Let $F$ be an acyclic complex of type (1) over the ring $R$. Then

(a) There exist unique maps

$$a_2: A^\beta F_3 \otimes A^\beta F_2 \to A^\alpha F_1$$

$$a_1: A^\beta F_3 \otimes A^\beta F_2 \otimes A^\beta F_1 \to A^\alpha F_0$$

such that the diagrams

$$\begin{array}{ccc}
A^\alpha F_2 & \xrightarrow{A^\alpha d_2} & A^\alpha F_1 \\
\downarrow{a_2} & & \downarrow{a_1} \\
A^\beta F_3 \otimes A^\beta F_2 & \xrightarrow{A^\beta d_3} & A^\beta F_1 \\
\end{array}$$

$$\begin{array}{ccc}
A^\alpha F_1 & \xrightarrow{A^\alpha d_1} & A^\alpha F_0 \\
\downarrow{a_2} & & \downarrow{a_1} \\
A^\beta F_3 \otimes A^\beta F_2 \otimes A^\beta F_1 & & \\
\end{array}$$

are commutative with $a_3 = A^\alpha d_3$.

(b) There exists a map $b: A^\alpha +1 F_1 \otimes A^\beta F_1 \otimes A^\beta F_2 \otimes A^\beta F_3 \to F_2$ such that the diagram

$$\begin{array}{ccc}
F_2 & \xrightarrow{d_2} & F_1 \\
\downarrow{b} & & \downarrow{a_2} \\
A^\alpha +1 F_1 \otimes A^\beta F_1 \otimes A^\beta F_2 \otimes A^\beta F_3 & \xrightarrow{a_2} & A^\alpha F_1 \otimes F_1 \otimes A^\beta F_2 \otimes A^\beta F_3 \\
\end{array}$$

is commutative.

Remark 2.2. In [BE1] the authors identify $A^\beta F_i$ with $R$. For our purposes it is necessary to keep track of various powers of $A^\beta F_i$. Throughout the article we denote $A^\beta F_i$ by $M_i$. Thus the structure maps act in the following way:

$$a_2: M_3^{-1} \otimes M_2 \to A^\alpha F_1, \quad a_1: M_3 \otimes M_2^{-1} \otimes M_1 \to A^\alpha F_0$$

$$b: A^\alpha +1 F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1} \to F_2.$$

Our general goal is to construct the generic ring for the resolutions of length 3. As indicated in Section 1(B) we start from universal complex of type (1). We consider the ring

$$R_c = S \cdot (F_3 \otimes F_2^* \oplus F_2 \otimes F_1^* \oplus F_1 \otimes F_0^*)/I_c,$$

where $I_c$ is the ideal generated by the relations $d_2 d_3 = d_1 d_2 = 0$ and the $r_i + 1$ minors of $d_i$ (in our notation $d_i$ corresponds to the representation $F_i \otimes F_{i-1}^*$). Over $R_c$ we have a natural complex $F_c$ of type (1) which is clearly universal, i.e., for any complex $G$ of type (1) there exists a unique map $h: R_c \to S$ such that $G = F_c \otimes_{R_c} S$. 


Remark 2.3. We should notice here a slight abuse of notation. In the formula (4) above \( F_i \) denote vector spaces over \( k \) of dimension \( f_i \). Sometimes we will also denote the terms of the complex \( F_c \) by \( F_i \) and there they will be the free modules of rank \( f_i \) over \( R_c \). It should also be noticed that the complex \( F \) in (1) is a complex over some commutative ring \( R \) while \( F_c \) is a complex over the "concrete" ring \( R_c \).

To obtain the information about the decomposition of \( R_c \) into representations we use Kempf's construction. We consider the action of the group

\[
G = GL(F_3) \times GL(F_2) \times GL(F_1) \times GL(F_0)
\]

on \( R_c \). We consider the projection \( p: Y_c \rightarrow X_c = \text{Spec} R_c \) from Section 1(C).

By Cauchy's formula we get the decomposition

\[
\mathcal{O}_Y = \sum_{a,b,c} S_a F_3 \otimes S_a R_2^* \otimes S_b Q_2 \otimes S_b R_1^* \otimes S_c Q_1 \otimes S_c R_0^*.
\]

By Bott's theorem those bundles have cohomology concentrated in degree 0 so by Kempf's result \([K]\) we get the decomposition of \( R_c \):

\[
R_c = \sum S_{(a_1, \ldots, a_3)} F_3 \otimes S_{(b_1, \ldots, b_2, -a_3, \ldots, -a_1)} F_2 \otimes S_{(c_1, \ldots, c_2, -b_2, \ldots, -b_1)} F_1 \otimes S_{(0, \ldots, 0, -c_1, \ldots, -c_1)} F_0.
\]

We also know that \( R_c \) has many nice properties, for example, rational singularities.

Now by Theorem 2.1(a) we know that the entries of maps \( a_i \) have to exist in the generic ring so in order to get it we have to consider first the ring \( R_a \),

\[
R_a = R_c \otimes S : (A^* F_1^* \otimes M_2 \otimes M_3^{-1} \oplus A^* F_0^* \otimes M_1 \otimes M_2^{-1} \otimes M_3)/I_a,
\]

where \( I_a \) is the ideal of all polynomials in the entries of \( d_i \)'s and \( a_i \)'s which vanish when evaluated on the entries of any acyclic complex \( G \) over commutative ring \( S \). Notice that the key point here is the definition of the ideal \( I_a \). If it would happen that the maps \( a_i \) would be defined over \( R_c \) then by uniqueness of \( a_i \)'s for any \( G \) and by definition of \( I_a \) the generators we add to \( R_a \) would be equal to the "old" \( a_i \)'s modulo \( I_a \) and in such case we would have \( R_a = R_c \). To prove that it is not the case we find now the criterion for a polynomial to be in \( I_a \).

For any acyclic \( G \) over \( S \) of type (1) we can consider the multiplicative set \( T \) of non-zerodivisors in \( S \) and then \( T^{-1} S \otimes G \) splits by the exactness criterion \([BE3]\). The map \( S \rightarrow T^{-1} S \) is an injection and it follows that \( I_a \)
consists of all polynomials in the entries of $d'_i$'s and $a'_i$'s that vanish when evaluated on the entries of split acyclic complexes $G$. The split acyclic complex after some change of basis has the form

$$0 \to A_3 \to A_3 \oplus A_2 \to A_2 \oplus A_1 \to A_1 \oplus A_0,$$  \hspace{1cm} (9)

where $\dim A_i = r_i$ for $i = 1, 2, 3$ and $\dim A_0 = f_0 - r_1$. The differential $d_i$ is the identity on $A_i$. This reasoning proves

**Lemma 2.4.** A $G$-invariant set $\mathcal{X}$ of elements of

$$R_c \otimes S \cdot (A^2 F_1^* \otimes M_2 \otimes M_3^{-1} \oplus A^n F_0^* \otimes M_1 \otimes M_2^{-1} \otimes M_3)$$

belongs to $I_a$ if and only if all its elements vanish when evaluated on the entries of the complex (9).

**Example.** Let $r_1 = r_2 = r_3 = 2$, $\dim F_0 = 2$. The map $\alpha_2: R \to A^2 R^4$ has entries $(\alpha_2)_{i,j}, 1 \leq i < j \leq 4$. Let us consider the Plucker relation

$$p = (a_2)_{14}(a_2)_{23} - (a_2)_{13}(a_2)_{24} + (a_2)_{12}(a_2)_{34}.$$  

We will show that $p$ belongs to $I_a$. First of all the set of multiples of $p$ forms a $G$-invariant subset because it corresponds to $A^4 F_1 \subset S_2(A^2 F_1)$. Now it is easy to see from the definitions that for complex (9) all the entries of $\alpha_2$ except one, say $(\alpha_2)_{12}$, are equal to 0. This shows that $p$ belongs to $I_a$.

Now we introduce the analogue of the Kempf construction. Let $X_a = \text{Spec } R_a$, Grass $= \text{Grass}(r_3, F_2) \times \text{Grass}(r_2, F_1) \times \text{Grass}(r_1, F_0)$.

**Definition 2.5.** $Y_a \subset X_a \times \text{Grass}$ is the set of 8-uples $(d_3, d_2, d_1, a_2, a_1, R_2, R_1, R_0)$ such that

1. $(d_3, d_2, d_1, a_2, a_1)$ belongs to $X_a$,
2. $R_i$ belongs to Grass $(r_{i+1}, F_i)$,
3. $\text{Im } a_i \subset A^n R_i$, $\text{Im } d_i \subset R_i \subset \text{Ker } d_{i-1}$,
4. for the induced maps $d'_i: Q_i = F_i/R_i \to R_{i-1}$ and $a'_i \in A^n R_i$, we have $\det d'_i = a'_i$, $\det d'_2 = a'_3 a'_2$, $\det d'_1 = a'_2 a'_1$.

Again we get the natural projections $p_a: Y_a \to X_a, q: Y_a \to \text{Grass}$. By the same method as for $R_c$ one can now show

**Proposition 2.6.** [PW]. The ring $R_a$ has the following decomposition into representations of $G$: 

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This text provides an in-depth look at the structure of free resolutions of length 3, detailing the construction of $G$-invariant sets and the Kempf construction, among other topics.

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where we sum over the set of natural numbers $a$, $b$, $c$ and partitions $a$, $b$, $c$ satisfying $a \sim < r_3$, $b \sim < r_2$, $c \sim < r_1$.

In [PW] it is also proved that $R_a$ is the ideal transform of $R_c$ with respect to $I(d_3) I(d_2) I(d_1)$ (compare Section 1(B)), and that over $R_a$ depth $I(d_3) = 2$, depth $I(d_2) = 2$, depth $I(d_1) = 1$.

This shows that to construct the hypothetical generic ring $R_{\text{gen}}$ from $R_a$ we just have to raise depth of $I(d_3)$ from 2 to 3.

The next step we have to make to construct $R_{\text{gen}}$ is to take into account Theorem 2.1(b).

Let us recall that 2.1(b) asserts the existence of a map $b$ satisfying the diagram (3). One can easily see using Lemma 2.4 that $d_1(a_2 \otimes 1) (A \otimes 1) = 0$ so existence of $b$ just says that the family of cycles $(a_2 \otimes 1) (A \otimes 1)$ has to be a boundary over $R_{\text{gen}}$. But we see that for a given complex the choice of $b$ is not unique—we can get another map $b'$ satisfying (3) by adding to $b$ any map of the form $C^uU$ for $U: Ar_1 + IF_1 + My + F_3$ because $d_2 d_3 = 0$. We introduce the ring

$$R_b = R_a \otimes S \cdot (A^{n+1} F_1 \otimes F_2^* \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1})/I_b,$$

where $I_b$ is the set of elements which vanish when evaluated at the entries of $d_1$, $a_2$, and $b$ for any acyclic complex $G$ of type (1) for any choice of $b$ for such complex. Over $R_b$ we get a natural complex $F'_b$. Now it could happen that $H_2 F'_b$ is not zero. To avoid that we introduce

$$R_b = \{\text{ideal transform of } R_b \text{ with respect to } I(d_3)\}.$$  

It turns out that there exists another map $b^*$ which plays a role similar to $b$ and is more convenient for our purposes because it is related directly to Koszul complex on $I(d_3)$. Let us describe the relation between $b$ and $b^*$.

First of all let us write $K = M_3^* \otimes A^2 F_2$ and consider the complex

$$K: 0 \to A^0 K \to A^1 K \to A^2 K \to A^3 K,$$

the initial part of Koszul complex of $I(d_3)$. We write $K_a, K_b, \ldots$ to denote the complex $K$ tensored with $R_a, R_b, \ldots$. For any acyclic complex $G$ of type (1) the corresponding part of the Koszul complex is acyclic.
Our goal is to write down explicitly the cycle in $K_a$ which corresponds to the map $b$. The tool we use is the following crucial lemma which will be used frequently in the sequel.

**Lemma 2.7.** Let $X = \text{Spec} R$, $j: U \to X$ be an open immersion. Let

$$G: 0 \to G_3 \to G_2 \to G_1 \to G_0$$

be a free complex over $X$ such that $G|U$ is acyclic. Then $H_3(G \otimes j_*\mathcal{O}_U) = 0$, $H_2(G \otimes j_*\mathcal{O}_U) = 0$ and we have an exact sequence

$$0 \to H_1(G \otimes j_*\mathcal{O}_U) \to G_3 \otimes R^1j_*\mathcal{O}_U \to G_2 \otimes R^1j_*\mathcal{O}_U.$$  \hfill (14)

**Proof.** We decompose $G \otimes \mathcal{O}_U$ into short exact sequences

$$0 \to G_3 \otimes \mathcal{O}_U \to G_2 \otimes \mathcal{O}_U \to C_2 \to 0$$

$$0 \to C_2 \to G_1 \otimes \mathcal{O}_U \to G_0 \otimes \mathcal{O}_U.$$

Now we apply functor $j_*$ to both sequences. We get

$$0 \to G_3 \otimes j_*\mathcal{O}_U \to G_2 \otimes j_*\mathcal{O}_U \to j_*C_2$$

$$0 \to j_*C_2 \to G_1 \otimes j_*\mathcal{O}_U \to G_0 \otimes j_*\mathcal{O}_U.$$

Now the lemma follows by easy diagram chase.

Next we need an application of Kempf construction. Let us denote $D_3 = V(I(d_3))$. We have a fundamental diagram:

$$
\begin{array}{c}
Y_a \setminus D_3 \xrightarrow{j_a} Y_a \xrightarrow{q_a} \text{Grass} \\
\rho_a \downarrow \makebox[0pt][l]{\hspace{1cm}} \downarrow \rho_a \\
X_a \setminus D_3 \xrightarrow{j_a} X_a
\end{array}
$$

\hfill (15)

It gives rise to two spectral sequences

$$
R^kp_a*(R^jq_a'\mathcal{O}_{Y_a \setminus D}) \Rightarrow R^{k+l}(p_a j_a)^*\mathcal{O}_{Y_a \setminus D}
$$

$$
R^k j_a^*(R^lp_a'\mathcal{O}_{Y_a \setminus D}) \Rightarrow R^{k+l}(j_a p_a')^*\mathcal{O}_{Y_a \setminus D}.
$$

$j_a$ is affine because $D_3$ is defined on $Y_a$ by one equation $a'_3 = 0$. The first spectral sequence thus degenerates and we have

$$
R^k j_a^*(R^lp_a'\mathcal{O}_{Y_a \setminus D}) \Rightarrow R^{k+l}p_{a'}(\mathcal{O}_{Y_a}[d_3^{-1}]).
$$
Moreover $X_u$ has rational singularities so

$$(R^l p_u^* \mathcal{O}_{Y_u \setminus D}) = 0 \quad \text{for} \quad l > 0 \quad \text{and} \quad p_u^*(\mathcal{O}_{X_u \setminus D}) = \mathcal{O}_{X_u \setminus D}.$$  

Finally, we get

$$R^k j_u^*(\mathcal{O}_{X_u \setminus D}) = R^k p_u^*(\mathcal{O}_{Y_u}[a'_3^{-1}]). \quad (16)$$

Now $X_u$ is affine, so $R^k p_u^*(\mathcal{O}_{Y_u}[a'_3^{-1}]) = H^k(Y_u, \mathcal{O}_{Y_u}[a'_3^{-1}]) = H^k(\text{Grass}, q_u^*(\mathcal{O}_{Y_u}[a'_3^{-1}])$ because $q_u$ is affine.

The cohomology groups over Grass are easy to calculate using Bott’s theorem using the decomposition

$$q_u^*(\mathcal{O}_{Y_u}[a'_3^{-1}]) = \sum S_a F_3 \otimes S_a R_2^1 \otimes S_b Q_2 \otimes S_b R_1^* \otimes S_c Q_1 \otimes S_c R_0^* \otimes (M_3 \otimes A'^n R_2^1)^a \otimes (M_3 \otimes A'^n R_1^*)^b \otimes (M_3 \otimes M_2 \otimes A'^n R_0^*)^c, \quad (17)$$

where we sum over all partitions $a, b, c$ satisfying $a_i < r_3, b_i < r_2, c_i < r_1$ and all integers $a, b, c$ with $b, c \geq 0$.

Now we apply Lemma 2.7 to $R = R_a, U = X_a \setminus D_3$, and $G = F_a$ and $K_a$.

We find that

$$H_1(F_a) = \text{Ker}(F_3 \otimes R^1 j_* \mathcal{O}_U \to F_2 \otimes R^1 j_* \mathcal{O}_U)$$
$$H_1(K_a) = \text{Ker}(A^0 K \otimes R^1 j_* \mathcal{O}_U \to A^1 K \otimes R^1 j_* \mathcal{O}_U)$$

because $j_* \mathcal{O}_U = \mathcal{O}_{X_u}$ (depth $I(d_3) = 2$). Now applying (17) we see that $R^1 j_* \mathcal{O}_U$ contains the crucial term $q_1^* = A'^n F_1 \otimes F_3^* M_1^{-1} \otimes M_2 \otimes M_3^{-1}$. It corresponds to $a = (1^{r_3-1}), b = (1^{r_2-1}), c = b = c = 0$. $q_1^*$ corresponds to the map

$$b^*: A'^n F_1 \otimes F_3^* M_1^{-1} \otimes M_2 \otimes M_3^{-1} \to R^1 j_* \mathcal{O}_U.$$  

It turns out that $b$ treated as an element of $F_3 \otimes R^1 j_* \mathcal{O}_U$ can be expressed in terms of $b^*$:

$$b: A'^n F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1}$$
$$\quad \to F_3 \otimes F_3^* A'^n F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1} \to F_3 \otimes R^1 j_* \mathcal{O}_U.$$  

Conversely, $b^*$ can be defined in terms of $b$:

$$b^*: F_3^* A'^n F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1}$$
$$\quad \to F_3 \otimes F_3^* R^1 j_* \mathcal{O}_U \to R^1 j_* \mathcal{O}_U.$$
The above consideration shows that to get $R_b$ from $R_b$ it is enough to kill the cycle $b^*$ in $H_1(K_a)$ and then take ideal transform with respect to $I(d_3)$.

In order to get a more concrete description of $b^*$ (not using $R_{i,j}^I \mathcal{O}_U$) we can use description of $b$ in terms of submaximal minors [BE1].

First we consider the composition

$$F_3^* \otimes A^{n+1} F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1} \rightarrow A^n F_3^* \otimes A^n M_3^{-1} F_2 \otimes A^n F_3 \otimes F_2$$

This defines the map

$$b^*: F_3^* \otimes A^{n+1} F_1 \otimes M_1^{-1} \otimes M_2 \otimes M_3^{-1} \rightarrow A^1 K$$

using submaximal minors of $d_3$.

Now we define the notion of defect of the structure map.

**Definition 2.8.** Let $G$ be a free complex of type (1) over some ring $S$. Suppose we want to make $G$ acyclic. For any map $x: X \rightarrow G_1$ from some free $S$-module $X$ to $G_1$ such that $d_1 x = 0$ we have a structure map $y: X \rightarrow G_2$ such that $d_2 y = x$. In this situation $y$ is not unique. To get other $y$'s we have to add to a given one a map factoring through $G_3$. The set of such maps, i.e., $\text{Hom}(X, G_3) = X^* \otimes G_3$, will be called the defect of the structure map $y$.

**Example.** The defect of $b$ is equal to the defect of $b^*$ and it equals

$$q_1 = A^{n+1} F_1^* \otimes F_3 \otimes M_1 \otimes M_2 \otimes M_3^{-1}.$$  \hfill (18)

Next we define the sequence of structure maps $p_r (n \geq 1)$. The ring $R_{\text{gen}}$ will be obtained from $R_b$ by repeating the construction we used above to obtain $R_b$. We start with the definition of defects $q_i$ of the maps $p_i$. $q_2$ is defined by the exact sequence

$$0 \leftarrow q_2 \leftarrow A^2 (F_3 \otimes A^{n+1} F_1) \otimes M_1^2 \otimes M_2^{-2} \otimes M_3^2 \times \times S_2 F_3 \otimes S_{(2^1, 1^2)} F_1^* \otimes M_1^2 \otimes M_2^{-2} \otimes M_3^2,$$

where the second map comes from embeddings

$$S_{(2^1, 1^2)} F_1^* \rightarrow S_2 (A^{n+1} F_1^*)$$

\hfill (19)
and Cauchy formulas.

The natural projection $A^2q_1 \rightarrow q_2$ defines on $L_2 = q_1 \oplus q_2$ the structure of the graded Lie algebra in the sense of Section 1(D). We define $q_n$'s inductively. Suppose we defined $L_n = q_1 \oplus \cdots \oplus q_n$. Then we define $q_{n+1}$ to be $H_2(L_n)_{n+1}$ (compare Section 1(D)) and $L_{n+1} = q_1 \oplus \cdots \oplus q_{n+1}$.

By definitions from Section 1(D) we get the exact sequences

$$0 \rightarrow q_{n+1} \xrightarrow{r} (A^2L_n)_{n+1} \xrightarrow{h} (A^3L_n)_{n+1}$$

where $h(l_1 \wedge l_2 \wedge l_3) = [l_1, l_2] \wedge l_3 - [l_1, l_3] \wedge l_2 + [l_2, l_3] \wedge l_1$.

Now we are ready to state the main result of this section.

**Theorem 2.9.** There exists a sequence of structure maps $p_j$ such that $p_1 = b^*$ and the defect of $p_j$ equals $q_j$.

**Proof.** In order to define $p_2$ let us consider the diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & A^0K & \xrightarrow{d} & A^1K & \xrightarrow{d} & A^2K & \xrightarrow{d} & A^3K \\
& & \uparrow p_2 & & \uparrow A^2b^* & & \uparrow A^3b^* & \\
& & q_2^* & \xrightarrow{r^*} & A^2q_1^* & & & \\
\end{array}
$$

We want to show that the indicated factorisation $p_2$ exists. To prove that, it is enough to show that $d(A^2p_1)r = 0$ for any acyclic $G$ of type (1). The reasoning from the proof of Lemma 2.4 shows that it is enough to check the property for split acyclic $G$ with a choice of map $b$ for it. We choose the bases $\{f_i\}, \{g_j\}, \{h_m\}, \{u_n\}$ in $F_3, F_2, F_1, F_0$, respectively. In the calculation that follows we apply the convention that the capital letters denote multiindices, for example, $J, J', J''$ denote multiindices from the set of indices $\{j\}$ corresponding to the basis of $F_2$. Similarly for the letters $i, m, n$. Moreover, $d_3(i^\times, J)$ will denote the submaximal minor of $d_3$ with the set of rows $J$ and the set of columns $\{1, \ldots, r_3\} \setminus \{i\}$. We choose our basis so that

$$(d_3)_y = \begin{cases}
\delta_y & \text{for } 1 \leq i, j \leq r_3 \\
0 & \text{otherwise}
\end{cases} \quad (*)$$

$$b_{j+r,M} = \begin{cases}
\pm 1, & M = \{r_2 + 1, \ldots, r_2 + r_1, j\}, j \in \{1, \ldots, r_2\} \\
0 & \text{otherwise.}
\end{cases}$$

Let us notice that we do not have control over $b_{j,M}$ for $1 \leq j \leq r_3$ because
of freedom of choice for $b$. Now we calculate $(dA^2b^*)(f_i \otimes h_M) \wedge (f_i \otimes h_{M'})$. We have

$$b^*(f_i \otimes h_M) = \sum_{j,J} d_3(i \wedge J) b_{j,M} g_j \wedge g_J$$

$$\times (dA^2b^*)(f_i \otimes h_M) \wedge (f_i \otimes h_{M'})$$

$$= \sum \{ d_3(j'' \cup J'') [d_3(i \wedge J) b_{j,M} d_3(i'^{\wedge}, J') b'_{j,M'}$$

$$- d_3(i^{\wedge}, J') b'_{j,M} d_3(i'^{\wedge}, J) b_{j,M} \}$$

$$\times (g_j \wedge g_J) \wedge (g_{j'} \wedge g_{J'}) \wedge (g_{j''} \wedge g_{J''}).$$

The coefficient in this sum is non-zero after the substitution (*) only when $j'' \cup J'' = \{1, \ldots, r_3\}$ and $J, J' \subseteq \{1, \ldots, r_3\}$. This means that $b_{j,M}$ and $b'_{j,M'}$ are 0's and 1's according to (*). To calculate $(A^2b^*)_r$ we start from the linear combination of $h_M \otimes h_{M'}$ which shuffles $M$ and $M'$. This means precisely (look at (19)) that $M$ and $M'$ cannot both contain \{r_2 + 1, \ldots, r_2 + r_1\}, so in each term at least one coefficient of $b$ is zero. This proves the existence of $p_2$. One should note that this is the crucial point since it explains the definition of $q_2$.

To get the existence of $p_n$'s for $n \geq 3$ let us proceed by induction. Suppose $p_1, \ldots, p_n$ are constructed. Let us consider the diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & A^0K & \longrightarrow & A^1K & \longrightarrow & A^2K & \longrightarrow & A^3K \\
\downarrow p_{n+1} & & \Sigma p_i A p_j & & \Sigma p_i A p_j A p_k & & \Sigma p_i A p_j A p_k & & \Sigma p_i A p_j A p_k \\
0 & \longrightarrow & q_{n+1}^* & \longrightarrow & (A^2L_n^*)_n+1 & \longrightarrow & (A^3L_n^*)_n+1 & & \end{array}\tag{22}$$

We want to show that the indicated factorisation exists. To do that we have to prove that the right square of the diagram commutes up to a constant. Then, for any acyclic $G$ the upper row is acyclic, the composition $d(\sum p_i \wedge p_j) r^* = (\sum p_i \wedge p_j \wedge p_k) dd = 0$, and $p_{n+1}$ exists.

To show commutativity let us consider the dual diagram

$$\begin{array}{cccccc}
A^3K^* & \longrightarrow & A^2K^* & \longrightarrow & A^1K^* & \longrightarrow & A^0K^* & \longrightarrow & 0 \\
\Sigma p_i A p_j A p_k^* & & \Sigma p_i^* A p_j^* & & \Sigma p_i^* A p_j^* & & \Sigma p_i A p_j A p_k \\
(A^3L_n^*)_n+1 & h & \longrightarrow & (A^2L_n^*)_n+1 & & & & \end{array}$$

Let us choose $u, v, w \in K^*$. For the purposes of this calculation let us denote by $x_u, x_v, x_w$ the images of $u, v, w$ under $d^*$ (we treat them as scalars):
\[(\sum p_i^* \land p_j^*) \wedge (u \land v \land w) = \sum \{ x_u(p_i^*v \land p_j^*w) - x_v(p_i^*u \land p_j^*w) + x_w(p_i^*u \land p_j^*v) \}.\]

The other compositions equals
\[h\left(\sum p_i^* \land p_j^* \land p_k^*\right)(u \land v \land w)\]
\[= \sum h(p_i^*u \land p_j^*v \land p_k^*w)\]
\[= \sum \left\{ [p_i^*u, p_j^*v] \land p_k^*w - [p_i^*u, p_j^*w] \land p_k^*v + [p_i^*v, p_j^*w] \land p_k^*u \right\}.\]

The summation above is over all \(i, j, k \geq 1\). By induction we know that
\[\left[p_i^*u, p_j^*v\right] = x_v p_i^*+j u - x_u p_i^*+j v.\]

Indeed, this is just the commutativity of the dual of the left square in (22) because the bracket in \(L_{n+1}\) is defined by the map \(r\). Putting this together we get
\[h\left(\sum p_i^* \land p_j^* \land p_k^*\right)(u \land v \land w)\]
\[= \sum (x_u p_i^*v - x_v p_j^*u) \land p_k^*w\]
\[- \sum (x_u p_i^*w - x_w p_j^*u) \land p_k^*v\]
\[+ \sum (x_v p_i^*w - x_w p_j^*v) \land p_k^*u\]
\[= 2\left( \sum x_u p_i^*v \land p_j^*w - \sum x_v p_i^*u \land p_j^*w \right)\]
\[+ \sum x_w p_i^*u \land p_j^*v.\]

Thus the desired commutativity is shown and Theorem 2.9 is proved.

We construct now our candidate for the generic ring for resolutions of length 3. For \(n \geq 1\) we define by induction
\[R_n' = R_{n-1} \otimes S \cdot (q_n^* \otimes A^\text{3}F_3 \otimes A^\text{3}F_2^*)/I_n,\]  \hspace{1cm} (23)

where \(I_n\) is the ideal of all elements which vanish when evaluated on all
acyclic complexes \( G \) of type (1) for all choices of \( p_1, \ldots, p_n \) for them. Those evaluations make sense because for each acyclic \( G \) of type (1) there exist various choices of maps \( p_1, \ldots, p_n \) and elements of \( R_{n-1} \) correspond basically to polynomials in the entries of \( p_1, \ldots, p_{n-1} \) with coefficients in \( R_n \). Next we define

\[
R_n = \{ \text{ideal transform of } R'_n \text{ with respect to } I(d_3) \}, \tag{24}
\]

One should note that whenever taking ideal transform we may run into non-Noetherian rings. However, the notion of depth of the ideal which is the key notion in all proofs can be defined over any commutative ring (compare \([N]\)) so this does not lead to difficulties.

We also denote by \( R''_n \) the subring of \( R_n \) generated over \( R_a \) by the entries of \( p_1, \ldots, p_n \). Then \( R_n \) is the ideal transform of \( R''_n \) with respect to \( I(d_3) \). One should also observe that each element of \( R_n \) can be again evaluated on an acyclic complex \( G \) of type (1) with a choice of \( p_1, \ldots, p_n \) because taking the ideal transform (24) corresponds to killing cycles in \( H_2 \). We denote \( X_n = \text{Spec } R''_n, X_a = \text{Spec } R_a. \)

**Lemma 2.10.** \( X_n \) is reduced, irreducible, and \( \dim X_n = \dim X_{n-1} + \dim q_n. \)

**Proof.** For any ring \( S \) and acyclic complex \( G \) of type (1) over \( S \) we have depth \( I(d_i) \geq i \). This implies depth \( I(d_i) \geq 1 \) over \( R''_n \). After inverting a maximal minor of \( d_3 \), \( R''_n \) becomes a polynomial ring over \( R''_{n-1} \) in \( \dim q_n \) variables. Indeed, after inverting a maximal minor of \( d_3 \) the complex (13) becomes split acyclic, so the map \( p_n \) kills a cycle which is boundary. Let \( \{ e_A \} \) be a basis of \( q_n \). If, using notation of (22), \((\sum p_i \wedge p_j) r^* e_A = d_A\) then an arbitrary \( p_n \) is of the form \( p_n(e_A) = z_A + d(w_A) \) and the elements \( w_A \) are independent. This proves our claim and it remains to show that \( R''_n \) is reduced. The group \( G \) acts on \( R''_n \) and the set of nilpotents is \( R''_n \) is obviously \( G \)-invariant. Let us choose finite dimensional \( G \)-invariant subspace \( \mathcal{X} \) in \( R''_n \) consisting of nilpotents. Thus for any \( S \), acyclic \( G \) of type (1) over \( S \), and \( h: R''_n \to S \) given by the universal property, we have \( h(\mathcal{X})^N = 0 \) for some \( N \). Let \( Z(S) \) be the set of zerodivisors of \( S, T = S \setminus Z(S), S' = T^{-1} S \). Then the natural map \( S \to S' \) is an injection and \( G \otimes S' \) splits. This shows that the map \( R''_n \to S \to S' \) factors through \( R''_n[D^{-1}], \) for some maximal minor \( D \) of \( d_3 \). But \( R''_n[D^{-1}] \) is a polynomial ring over \( R_a[D^{-1}] \) so the image of \( \mathcal{X} \) in \( S' \) is zero (\( R_a \) has no nilpotents). This shows that image of \( \mathcal{X} \) in \( S \) is zero and the proof of the lemma is finished.

The representations \( q_n \) have a nice interpretation in terms of differentials.
PROPOSITION 2.11. The module of differentials \( D_{R_{n-1}}(R_n) \) is isomorphic to \( R_n \otimes q_n \).

Proof. Let \( q_n^* \) denote free \( R_n \)-module \( q_n^* \otimes R_n \). We consider a homomorphism \( u: q_n^* \to R_n \). To define the differential \( D_u \) corresponding to \( u \) we look at the diagram

\[
0 \to A^0K \xrightarrow{d} A^1K \xrightarrow{d} A^2K \xrightarrow{d} A^3K
\]

We choose the basis \( \{e_A\} \) in \( q_n^* \). To construct \( D_u \) it suffices to describe \( D_u[p_n(e_A)] \) as a vector in \( A^1K \) (we assume \( D_u \) acts on each coordinate). Indeed, \( D_u \) is zero on \( R_n \) so it vanishes on maximal minors of \( d_3 \). Thus if \( D_u \) is defined on \( R_n \) it extends uniquely to \( R_n \) to a differential which we will also call \( D_u \). To define \( D_u[p_n(e_A)] \) let us notice that \( H^0(K) \) and \( H^1(K) \) are zero over \( R_{n-1} \) (compare Section 1(D)). Applying \( D_u \) to the coordinates of the equality \( dp_n(e_A) = p_n r^*(e_A) \) we get

\[ d[D_u(p_n(e_A))] = 0. \]

In order for this equality to be satisfied we set

\[ D_u(p_n(e_A)) = du(e_A). \quad (25) \]

This defines a derivation on \( R_n \) which extends to \( R_n \). Indeed, by 2.10 we see that (25) defines a derivation on a field of fractions of \( R_n \) and it clearly carries \( R_n \) into itself. Now \( \{e_A\} \) is a basis of \( q_n \). The derivations \( D_{e_A} \) are linearly independent over \( R_n \) because \( d: A^0K \to A^1K \) is a monomorphism over \( R_n \). Also every derivation \( D: R_n \to R_n \) has to be one of the \( D_{e_A} \)’s because, as above, \( d[D(p_n(e_A))] = 0 \) and \( H^1(K) = 0 \) over \( R_n \). The proposition is proved.

Let \( L = \oplus q_n \). \( L \) has a natural structure of Lie algebra (by Section 1(D)). We finish this section with another description of this structure.

THEOREM 2.12. \( L \) acts on \( R_{\text{gen}} = \lim R_n \) as a Lie algebra of derivations.

Proof. Let \( u \in q_n \). We extend \( D_u \) to the derivation on \( \lim R_n \) by the formula

\[ D_u(p_{n+k}(e_A)) = \sum u(e_{A_i}) p_k(e_{B_i}), \quad (26) \]

where \( e_A \) stands for a basis element of \( q_{n+k}^* \), \( r^* e_A = \sum e_{A_i} \wedge e_{B_i} \), and we use
the convention that $u(e_B^l) = 0$ for $e_B^l \in q_l^*$, $l$ not equal to $n$. To check that $D_u$ is well defined we observe that

$$dD_u(p_{n+k}(e_A)) = D_u \left( p'_{n+k} \left( \sum e_{A_i} \wedge e_{B_i} \right) \right)$$

$$= D_n \sum p_j(e_{A_i}) \wedge p_i(e_{B_i})$$

$$= \sum D_u(p_j(e_{A_i})) \wedge p_i(e_{B_i})$$

$$- \sum p_j(e_{A_i}) \wedge D_u(p_i(e_{B_i}))$$

$$= \sum u(e_{A_m}^*) p_j(e_{A_m}^*) \wedge p_i(e_{B_m}^*)$$

$$- \sum u(e_{B_m}^*) p_j(e_{A_m}^*) \wedge p_i(e_{B_m}^*),$$

where $r^*e_A^* = \sum e_{A_i} \wedge e_{B_i}^*$, $r^*e_A = \sum e_{A_m}^* \wedge e_{A_m}^*$, $r^*e_B = \sum e_{B_m}^* \wedge e_{B_m}^*$. On the other hand, we use (26) to obtain

$$dD_u(p_{n+k}(e_A)) = d \left( \sum u(e_{A_i}) p_k(e_{B_i}) \right)$$

$$= \sum u(e_{A_i}) \sum p_j(e_{B_m}^*) \wedge p_i(e_{B_m}^*).$$

Now both expressions for $dD_u(p_{n+k}(e_A))$ obtained above are equal because of commutativity of the diagram.

The last thing to check is that the commutator of derivations corresponds to the bracket in $L$. Let $u \in q_k^*$, $v \in q_l^*$. Then, using an analogous argument we have
\[(D_u D_v - D_v D_u)(p_n(e_A)) = D_u \left[ \sum v(e_{A_i}) p_{n-i}(e_{B_i}) \right] \]
\[-D_v \left[ \sum u(e_{A_i}) p_{n-k}(e_{B_i}) \right]\]
\[= \sum v(e_{A_i}) u(e_{B_m}) p_{n-k-i}(e_{B_m}) \]
\[-\sum u(e_{A_i}) v(e_{B_m}) p_{k-k-i}(e_{B_m})\]
\[=[u, v](e_{A_i}) p_{n-k-i}(e_{A_i})\]
\[= D_{[u, v]}(p_{n+k}(e_A)).\]

This concludes the proof of Theorem 2.12.

Now we can finally state our main conjecture:

Conjecture 2.13. Let \( R_{\text{gen}} = \lim R_n. \) Over \( R_{\text{gen}} \), we have the naturally defined complexes \( F_{\text{gen}} = F \otimes R_{\text{gen}}. \) The pair \( (R_{\text{gen}}, F_{\text{gen}}) \) is the generic pair for the acyclic complexes of type (1) in the sense of Section 1(B).

Remark 2.14. The action of \( L \) on \( R_{\text{gen}} \) extends to the action of the enveloping algebra \( U(L) \). We will use this action heavily in the next section.

Remark 2.15. The analogous construction of the defect Lie algebra can be carried out in general, i.e., when we increase the depth of some ideal from 2 to 3. What seems to be special for our situation is that the resulting Lie algebra \( L \) is defined over \( k \). If one could find an independent proof of this fact, one would prove our Main Conjecture 2.13.

3. THE SPECTRAL SEQUENCE

In this section we will show that to prove Conjecture 2.13 it is enough to show that homology of the family of certain complexes over \( U(L) \) is zero.

These complexes come from analysis of the Kempf construction for \( X_n. \) We define \( Y_n \subset X_n \times_X Y_\alpha \) to be the irreducible component of \( X_n \times_X Y_\alpha \) which is the closure of \( (Y_n \setminus D_\lambda) \times \text{Spec } S \cdot (q_1^* \oplus \cdots \oplus q_n^*(\lambda)). \) We denote the corresponding projections by \( p_n \) and \( q_n. \) They form a diagram

\[
\begin{array}{ccc}
Y_n & \xrightarrow{p_n} & X_n \\
\downarrow{q_n} & & \\
\text{Grass} & & \\
\end{array}
\]
Conjecture 2.13 says that $F_{\text{gen}}$ defined over $R_{\text{gen}}$ is acyclic. We now that $F_n$ is acyclic on $X_n \setminus D_3$. Now we apply Lemma 2.7 to $X = X_n$, $U = X_n \setminus D_3$ and $G$ equal to $F$ and $K$. We have $j_*\mathcal{O}_U = R_n$. By 2.7, $H_1(G \otimes R_n) = 0$ if $R^{1}j_*\mathcal{O}_U = 0$. We calculate $R^1j_*\mathcal{O}_U$ using the varieties $Y_n$. We start with the diagram

$$
\begin{array}{ccc}
Y_n \setminus D_3 & \xrightarrow{j_n} & Y_n \\
\downarrow \rho_n & & \downarrow \rho_n \\
X_n \setminus D_3 & \xrightarrow{j_n} & X_n
\end{array}
$$

It gives two spectral sequences corresponding to two compositions. Now we can apply the reasoning following (15). We notice that $\rho'_n = p_n' \otimes 1$ if we identify $\mathcal{O}_{Y_n[a_3^{-1}]}$ with $\mathcal{O}_{Y_n[a_3^{-1}]} \otimes S \cdot (q_n' \oplus \cdots \oplus q^*_a)$. Our conclusion (analogue of (16)) is that

$$R^{k}j_*(\mathcal{O}_{X_n \setminus D_3}) = R^{k}p_n*(\mathcal{O}_{Y_n[a_3^{-1}]})$$

$$= H^k(\text{Grass}, q_n*\mathcal{O}_{Y_n[a_3^{-1}]}) \quad (28)$$

In the sequel we will denote $q_n*\mathcal{O}_{Y_n[a_3^{-1}]}$ by $\mathcal{F}_n$ and $q^*_a*\mathcal{O}_{Y_n[a_3^{-1}]}$ by $\mathcal{F}_a$. We know from Section 2 that $\mathcal{F}_a$ has decomposition (17).

Now we calculate the cohomology of $\mathcal{F}_n$ in the following five steps.

1. We construct the filtration of $\mathcal{F}_n$ such that the cohomology of the associated graded object $\mathcal{G}_n$ can be easily calculated using (17).

2. The cohomology of $\mathcal{F}_n$ is the limit of the spectral sequence with the first term equal to cohomology of $\mathcal{G}_n$. We have to analyse which terms in the cohomology of $\mathcal{G}_n$ disappear in the spectral sequence.

3. The terms of cohomology of $\mathcal{G}_n$ are representations of $G$. It turns out that the boundary maps in the spectral sequence multiply the representation of $F_2$ by the appropriate power of $M_2$ and preserve the representation of $F_0$. We fix the representations of $F_2$ and $F_0$ and look at the part of the spectral sequence corresponding to them. This part has very few terms. We get one term in $H^0$, one term in $H^1$, and two terms in $H^2$. Let us call those terms $K_0$, $K_1$, $K_2$, and $K_{1,1}$, respectively. Those terms considered as $U(L)$-modules are duals of certain free modules. The connecting maps from the spectral sequence between our terms are thus uniquely determined up to scalar and the piece of $H^1$ (corresponding to the choice of representations of $F_2$ and $F_0$) consists of the homology of the complex

$$K_0 \to K_1 \to K_2 \oplus K_{1,1}.$$
scalars are non-zero) we notice that those maps involve only the entries of $p_1$, so it is enough to check it for $n = 1$.

5. We introduce the other filtration on $F_1$ and using it we finally prove that the connecting maps are non-zero.

Now we perform the steps indicated above.

Step 1. We order $n$-tuples $(u_1, \ldots, u_n)$ by reverse lexicographic order, i.e., $(u_1, \ldots, u_n) < (v_1, \ldots, v_n)$ if $u_i < v_i$ for the biggest $i$ for which $u_i \neq v_i$. By construction $F_n$ has the filtration $\{F_n(u_1, \ldots, u_n)\}$ of $U(L)$-submodules

$$F_n(u_1, \ldots, u_n) = \{ f \in F_n | \text{(deg}_1 f, \ldots, \text{deg}_n f) \leq (u_1, \ldots, u_n) \},$$

where $\text{deg}_i f$ denotes the degree of $f$ with respect to the elements of $q_i^*$. Let

$$F_n(N) = \bigcup_{\sum u_i \leq N} F_n(u_1, \ldots, u_n)$$

and

$$G_n(N) = F_n(N)/F_n(N-1).$$

Then we have

$$G_n(N) = F_n \otimes U(L_n)^*$$

if we put $\text{deg} q_i^* = -i$. We define

$$G_n(u_1, \ldots, u_n) = F_n(u_1, \ldots, u_n)/\sum_{(v_1, \ldots, v_n) < (u_1, \ldots, u_n)} F_n(v_1, \ldots, v_n).$$

Then we have

$$G_n(u_1, \ldots, u_n) = F_n \otimes S_{u_1}(q_1^*) \otimes \cdots \otimes S_{u_n}(q_n^*).$$

We know by construction that the action of $L_n$ on $F_n$ sends $q_i \otimes F_n(u_1, \ldots, u_n)$ into $F_n(u_1, \ldots, u_i-1, \ldots, u_n)$. The induced action on $G_n$ sends $q_i \otimes G_n(N)$ into $G_n(N-i)$. It follows that

$$\bigoplus G_n(N) = F_n \otimes U(L_n)^*$$

as $U(L_n)$-modules.
Using Bott's theorem and the decomposition (17) of $\mathcal{F}_a$ we find

$$H^*(\text{Grass}, \mathcal{F}_a)$$

$$= \sum S_{a-b+c+a_1, \ldots, a-b+c+a_{r_3-1}, a-b+c} F_3$$

$$\otimes S_{b-c+b_1, \ldots, b-c+b_{r_2-1}, b-c, -a+b-c, -a+b-c, -a+b-c+a_{r_2-1}, \ldots, -a+b-c-a_1} F_2$$

$$\otimes S_{c+c_1, \ldots, c+c_{r_1-1}, c-c-b, c-c-b, -b_{r_2-1}, \ldots, -b-b_1} F_1$$

$$\otimes S_{0,0, \ldots, 0, -c, -c-c-c_{r_1-1}, \ldots, -c-c_1} F_0. \quad (30)$$

Here the right side has to be read according to Bott's theorem, i.e., for given $a, b, c, a, b, c$ we apply Theorem 1.1 to the sequences above. If any of the sequences corresponds to case (2) of 1.1 then the term does not contribute to cohomology. If all sequences correspond to case (1) of 1.1 and we need $j$ exchanges to order them then the term contributes to $H^j(\text{Grass}, \mathcal{F}_a)$. We notice that the only sequence in (30) which needs reordering is the sequence corresponding to $F_3$, for $a < 0$ (then $b-c = -a+b-c$).

All representations in $U(L)$ are induced from $X_n$, so by the projection formula

$$H^j(\text{Grass}, \mathcal{F}_n) = H^j(\text{Grass}, \mathcal{F}_a) \otimes U(L_n)^* \quad (31)$$

**Step 2.** We get the usual spectral sequence converging to $H^*(\text{Grass}, \mathcal{F}_a)$ with the first term $H^*(\text{Grass}, \mathcal{F}_a)$ so we have to determine which representations disappear in $E$. We should keep in mind that our spectral sequence consists of $U(L_n)$-modules.

**Step 3.** The spectral sequence we got above is the $G$-invariant spectral sequence of $U(L_n)$-modules. The representations in $U(L_n)$ do not contain $F_0$ and they contain only the maximal exterior powers of $F_2$. Thus in the spectral sequence two terms can cancel (i.e., some boundary map carries one to another) only when they have the same partition corresponding to $F_0$ and the partitions corresponding to $F_2$ differ by some power of $M_2$. In the sequel we denote the term in (30) corresponding to $a, b, c, a, b, c$ by $(a, b, c, a, b, c)$. Since we are interested in $H^1$, from now on we look at the terms in (30) contributing to $H^0, H^1, H^2$.

Let us consider the term $(a, b, c, a, b, c)$ contributing to $H^0$ and the term $(a', b', c', a', b', c')$ contributing to $H^1$ having the same representation corresponding to $F_0$ and representations corresponding to $F_2$ differing by the powers of $M_2$. This means that $c = c'$, $c = c'$ and there exists $t$ such that
(b - c + b_1, ..., b - c + b_{r_2 - 1}, b - c, -a + b - c, -a + b - c - a_{r_2 - 1}, ..., -a + b - c - a_1)

(32)

(b' - c' + b'_1 - t, ..., b' - c' + b'_{r_2 - 1} - t, -a' + b' - c' - 1 - t, b' - c' + 1 - t, -a' + b' - c' - a'_{r_2 - 1} - t, ..., -a' + b' - c' - a_1 - t)

The possible cancellation between both terms can occur only if we can get (a', b', c', a', b', c') inside of (a, b, c, a, b, c) \( \otimes U(L_n)^* \). Let us look at the corresponding partitions of \( F_3 \). They are

(a - b + c + a_1, ..., a - b + c + a_{r_3 - 1}, a - b + c) and

(a' - b' + c' + a'_1 + t, ..., a' - b' + c' + a'_{r_3 - 1} + t, a' - b' + c' + t).

In those sequences the first \( r_3 - 1 \) terms are equal by (32). The only possibility is to get (a', b', c', a', b', c') in \( S_r F_3^* \otimes S_r (A'^{r+1} F_1) \otimes M_3^{-1} \otimes M_2^{'-1} \otimes M_1^{-1} \) because we can add boxes only in the last row. From (32) we get also

\[ b_i - b_{i+1} = b'_i - b'_{i+1}, \quad a_i - a_{i+1} = a'_i - a'_{i+1} \]

\[ b - b' = b'_i - b_{i+1} - t, \quad b_{r_2 - 1} = b'_{r_2 - 1} + a' + 1, \quad a = -a' - 2. \]

(33)

We also see that \( a' - b' + c' + t = a - b + c \) which gives \( t = a + 1 \). Looking at the sequences corresponding to \( F_1 \) we see that multiplying by \( S_r (A'^{r+1} F_1) \) occurs just by adding boxes to the last \( r_1 + 1 \) rows. Putting this all together gives

\[ a = a' \]

\[ a = -1, \quad a' = -1 - t \]

\[ b_i = b_i + t \]

\[ b = b', \quad c = c', \quad c = c'. \]

(34)

We see that both terms determine each other and \( t \). Also (a', b', c', a', b', c') occurs in \( (a, b, c, a, b, c) \otimes U(L_n)^* \) with multiplicity one.

Let us look now at the possible cancellations between \( H^1 \) and \( H^2 \). We choose (a, b, c, a, b, c) contributing to \( H^1 \) and (a', b', c', a', b', c') contributing to \( H^2 \). Again we assume that both terms have the same representation corresponding to \( F_0 \) and that the representations corresponding to \( F_2 \) differ
STRUCTURE OF FREE RESOLUTIONS OF LENGTH 3

by the power of $M_2$. This means we have $c = c'$, $c = c'$. Now we have two possibilities of exchanges in Bott's algorithm:

(i) there exists $s$ such that

\[(b - c + b_1, ..., b - c + b_{r_2 - 1}, -a + b - c - 1, b - c + 1, -a + b - c - a_{r_3 - 1}, ..., -a + b - c - a_1)\]

\[(b' - c' + b'_1 - s, ..., b' - c' + b'_{r_2 - 1} - s, -a' + b' - c' - 1 - s, -a' + b' - c' - a'_{r_3 - 1} - 1 - s, b' - c' + 2 - s, ..., -a' + b' - c' - a'_1 - s);\]

(ii) there exists $u$ such that

\[(b - c + b_1, ..., b - c + b_{r_2 - 1}, -a + b - c - 1, b - c + 1, -a + b - c - a_{r_3 - 1}, ..., -a + b - c - a_1)\]

\[(b' - c' + b'_1 - u, ..., -a' + b' - c' - 2 - u, b' - c' + b'_{r_2 - 1} + 1 - u, b' - c' + 1 - u, -a' + b' - c' - a'_{r_3 - 1} - u, ..., -a' + b' - c' - a'_1 - u).\]

Let us look at case (i). We want to get $(a', b', c', a', b', c')$ inside of $(a, b, c, a, b, c) \otimes U(L_n)^*$. Now we look at the corresponding representations of $F_3$. They are

\[(a - b + c + a_1, ..., a - b + c + a_{r_3 - 1}, a - b + c)\]

and

\[(a' - b' + c' + a'_1 + s, ..., a' - b' + c' + a'_{r_3 - 1} + s, a' - b' + c' + s).\]

These sequences are equal except at the $r_3 - 1$st place. This means we can add boxes only in the $r_3 - 1$st row and $(a', b', c', a', b', c')$ has to occur inside of $(a, b, c, a, b, c) \otimes S_3 F_3^* \otimes S_3 (A^{r_1 + 1} F_1) \otimes M_2^{-1} \otimes M_2^* \otimes M_1^{-1}$. From (35i) we get also

\[b_i - b_{i+1} = b'_i - b'_{i+1}, \quad a_i - a_{i+1} = a'_i - a'_{i+1}\]

for $i + 1 \leq r_3 - 2$,

\[b_{r_2 - 1} + a + 1 = b'_{r_2 - 1} + a' + 1, \quad -a - 2 = a'_{r_3 - 1}, \quad 1 + a + a_{r_3 - 1}\]

\[= -a' - a'_{r_3 - 1} - 3 - a_{r_3 - 1} + a_{r_3 - 2} = a'_{r_3 - 2} + 2 + a'.\]

Now, looking at the partitions corresponding to $F_1$ we see that multiplying
by $S_j(A^{r_1+1}F_1)$ consists of adding $s$ boxes to all of the $r_1 + 1$ last rows. Putting this all together gives

\[ b_i = b'_i - s, \quad a = a' - s, \quad a_{r_3 - 1} = -a' - 2 \]

\[ s = 2 + a + a_{r_3 - 1}, \quad a_i = a'_i \quad \text{for} \quad i \leq r_3 - 2 \quad (37i) \]

\[ s = -2 - a' - a_{r_3 - 1}, \quad a'_{r_3 - 1} = -a - 2. \]

This shows that both terms determine each other and any of them determines $s$.

The same thing happens in case (ii) except that there adding boxes for $F_3$ occurs in the $r_3$th row and multiplying by $S_n(A^{r_1+1}F_1)$ occurs by adding $u$ boxes to the $-1 + r_3 - th, 1 + r_3 - th, ..., r_1 + r_2 - th$ row. We omit the details because this case is analogous to case (i).

The above reasoning shows that after fixing the representations of $F_2$ (up to the powers of $M_2$) and $F_0$ the spectral sequence decomposes into complexes of the form

\[ 0 \to K_0 \to K_1 \to K_{1,1} \oplus K_2, \]

where each term is a sum of copies of $U(L_n)^*$ and the maps between terms are uniquely defined up to scalar. Now we dualize those complexes to get free modules over $U(L_n)$. Each piece we get is (after disregarding $F_2$ and $F_0$) a complex over $U(L_n)$ of the form

\[ S_{(s_1 + s_2, ..., s_{r_3})} F_3 \]

\[ \otimes S_{(t_1 + s_2 + s_3, ..., t_{r_3})} F_3 \]

\[ S_{(s_1 + t, s_2, ..., s_{r_3})} F_3 \otimes S_{(t_1 + t_2, ..., t_{r_3})} F_1^* \]

\[ S_{(s_1, ..., s_{r_3})} F_3 \otimes S_{(t_1 + t_2, ..., t_{r_3})} F_1^*, \quad (38) \]

where $u$ and $s$ are determined by $s$, $t$, and $t$ in the following way:

\[ s_2 + s = s_1 + 1 \quad \text{and} \quad t_{r_1 + 2} + u = t_{r_1 + 1} + 1. \]

We call the complex defined by (38) $K(s, t, t)$.

**Step 4.** So far we have determined that all possible cancellations are of the form (38). Now we have to check that all of them really occur. This means that we have to show that the maps in (38) are non-zero. Let us
notice that the maps in (38) are the maps of free $U(L_n)$-modules so they are determined by their images on generators. All maps come from multiplying by the representations $S_i F_i^\otimes S_j (A^{n+1} F_1)$ so in fact they involve only the representation $q_*^\otimes$. So it is enough to look at $\mathcal F_i$ and to show that the representations giving the generators of terms in (38) disappear in $H^*(\text{Grass}, \mathcal F_i)$.

**Step 5.** To prove this last statement we introduce the other filtration on $q_1 \otimes \mathcal O_{Y_1} \setminus D_3$. The structure map $b$ satisfies, as we know, the diagram (3) and is described in terms of submaximal minors of $d_2$ [BE1]. This second description has the following interpretation in terms of bundles on Grass. We start with the extension

$$0 \to Q_2^\otimes A^{n-1} R_1^\otimes M_3^{-1} \otimes M_2 \to \mathcal F \to R_2^\otimes A^{n-1} F_1 \otimes M_3^{-1} \otimes M_2 \to 0 \quad (39)$$

which is a factor of $F_2^\otimes A^{n-1} F_1^\otimes M_3^{-1} \otimes M_2$. This means that $\mathcal F$ is induced by the extension $0 \to Q_2^\otimes \to F_2^\otimes \to R_2^\otimes \to 0$. Now $\mathcal O_{Y_1}$ is constructed from $\mathcal F$. Indeed, let us observe that

$$Q_2^\otimes A^{n-1} R_1^\otimes M_3^{-1} \otimes M_2 = A^{n-1} Q_2 \otimes A^{n-1} R_1^\otimes M_3^{-1} \otimes A'^{-1} R_2 \in \mathcal O_{Y_1}.$$ 

It is obvious by construction that $\mathcal O_{Y_1}$ has a filtration over Grass whose associated graded object is $\mathcal O_{Y_1} \otimes S \cdot (R_2^\otimes A^{n-1} F_1^\otimes M_3^{-1} \otimes M_2)$. The boundary maps in the spectral sequence that evolves are induced by (39). We have to show that the maps in this spectral sequence connecting the terms $(a, b, c, a, b, c)$ and $(a', b', c', a', b', c')$ satisfying (34), (37i) and its analogue for the case (ii) (which we did not state) are non-zero.

Let us first look at the case (34), i.e., the cancellations between $H^0$ and $H^1$. We want to cancel out $(a', b', c', a', b', c')$ inside of $(a, b, c, a, b, c) \otimes (Q_2^\otimes A^{n-1} R_1^\otimes M_3^{-1} \otimes M_2)$ with $(a, b, c, a, b, c) \otimes S_i (R_2^\otimes A^{n-1} R_1^\otimes M_3^{-1} \otimes M_2)$. The $F_2$ components of the two terms are

$$S_{(b_1, \ldots, b_{n-1})} Q_2 \otimes S_{(1-t, 1-a_{n-1}, \ldots, 1-a_1)} R_2^\otimes M_2$$

and

$$S_{(b_1 + t, \ldots, b_{n-1} + t)} Q_2 \otimes S_{(1 + t, 1 + t - a_{n-1}, \ldots, 1 + t - a_1)} R_2^\otimes.$$ 

We have to show that the connecting homomorphisms coming from the extension $0 \to Q_2^\otimes \to F_2^\otimes \to R_2^\otimes \to 0$ cancel out those terms.

Let us look at a more general situation. Let $\text{Grass}(r, n)$ be the grassmanian of $r$-subspaces of vector space $F$ of dimension $n$. Let $R, Q$ denote the tautological subbundle and tautological quotientbundle, respectively.
consider two partitions \( s = (s_1, \ldots, s_q), \ t = (t_1, \ldots, t_r) \). Let \( t_1 = s_q + 1 \). Then we claim that for \( 1 \leq t \leq t_1 - t_2 \) the terms

\[
S_{(s_1 + t, \ldots, s_q + t, t_1, t_2 + t, \ldots, t_r + t)} \in H^0(S_s Q \otimes S_t R \otimes S_r R^* \otimes M)
\]

and

\[
S_{(s_1 + t, \ldots, s_q + t, t_1 + t, t_2 + t, \ldots, t_r + t)} \in H^1(S_s Q \otimes S_t R \otimes (A^q - 1)Q^t \otimes (A'R)^t)
\]

cancel out in the cohomology of the filtered bundle with the associated graded object \( \sum S_s Q \otimes S_t R \otimes (A'R)^t \otimes S_{t-j} R^* \otimes S_j Q^t \otimes (A^q Q)^t \). Indeed, the bundle in question is just \( S_s Q \otimes S_t R \otimes (A^q Q)^t \otimes (A'R)^t \otimes S_r F^* \) so it does not have cohomology because \( t_1 = s_q + 1 \! \)!

To complete the argument we notice that in the case of our application all pieces for intermediate \( j \) sit inside of \( q_1 \cdot \mathcal{O}_{Y_1} \cdot D_3 \), so we are really looking at "the piece" of this bundle corresponding to those representations.

The same thing happens for the cancellations between \( H^1 \) and \( H^2 \). Let us look at \( (a, b, c, a', b', c') \) satisfying (37i) (we leave case (ii) to the reader). As above the statement follows from the following claim.

We consider the terms

\[
S_{(s_1 + t, \ldots, s_q + t, t_1 + t, t_2 + t, \ldots, t_r + t)} \in H^1(S_s Q \otimes S_t R \otimes S_r R^* \otimes M)
\]

and

\[
S_{(s_1 + t, \ldots, s_q, t_1 + t, \ldots, t_r + t)} \in H^2(S_s Q \otimes S_t R \otimes (A^q - 1)Q^t \otimes (A'R)^t).
\]

Let us assume that \( t_1 = s_q + 2 \). Then those terms cancel out in the cohomology of the filtered bundle whose associated graded object is \( \sum S_s Q \otimes S_t R \otimes S_{t-j} R^* \otimes (A'R)^t \otimes S_j Q^t \) with filtration coming from the tautological extension.

The claim follows instantly from the observation that our "filtered bundle" is nothing but \( S_s Q \otimes S_t R \otimes (A'F)^t \otimes S_r F^* \) so it has no cohomology.

This completes Step 5 of our proof, so we can state our theorem.

**Theorem 3.1.** To prove Conjecture 2.13 it is enough to show that the complexes \( K(s, t, t) \) defined by (38) have no homology in the middle term.

4. THE MULTIPlicative STRUCTURE ON \( \text{Tor}^R(R/I, \mathcal{A}) \)

Let \( R \) be a local ring with maximal ideal \( m \) and quotient field \( \mathcal{A} \). Let \( I \) be the ideal in \( R \) such that the projective dimension of \( R/I \) equals 3. We apply the structure theorems to classify all possible multiplications on the graded algebra \( \text{Tor}^R(R/I, \mathcal{A}) \).
The only fact we will need in this section is the existence of structure map \( p_2 \) so the results of this section are in fact characteristic free.

Let

\[
0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R
\]

be a minimal free resolution of \( R/I \). The graded algebra \( \text{Tor}^R(R/I, \mathcal{E}) \) has components equal to \( \mathcal{E}, F_1 \otimes \mathcal{E}, F_2 \otimes \mathcal{E}, F_3 \otimes \mathcal{E} \). In the sequel we will denote those components by \( \mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \). The multiplication on \( \text{Tor}^R(R/I, \mathcal{E}) \) is given by the maps \( b': A^2 F_1 \rightarrow \mathcal{F}_2, c': \mathcal{F}_1 \otimes \mathcal{F}_2 \rightarrow \mathcal{F}_3, b' \) equals \( b \) modulo \( m \) and \( c' \) equals modulo \( m \), where \( c \) is defined by means of the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & F_3 \\
& & \downarrow{d_3} \\
& \rightarrow & F_2 \\
& & \downarrow{d_2} \\
& \rightarrow & F_1 \\
& & \downarrow{d_1} \\
\end{array}
\]

\[
\begin{array}{ccc}
& & F_2 \otimes F_1 \\
& & \downarrow{d'} \\
\mathcal{F}_1 & \rightarrow & A^2 F_1 \\
\end{array}
\]

\( c \) satisfies the equality \( d_3 c + bd' = d_1 \otimes 1 \) so it exists by acyclicity of the first row of (41).

Let us consider the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & A^2 F_3 \\
& & \downarrow{A^2 c} \\
& \rightarrow & F_3 \otimes F_2 \\
& & \downarrow{bc} \\
& \rightarrow & S_2 F_2 \\
& & \downarrow{p_2} \\
S_2 F_2 \otimes A^2 F_1 & \rightarrow & F_2 \otimes A^3 F_1 \\
& & \downarrow{s_2 b} \\
& \rightarrow & A^4 F_1 \\
\end{array}
\]

The map \( bc \) is the composition \( F_2 \otimes A^3 F_1 \rightarrow F_2 \otimes F_1 \otimes A^2 F_1 \rightarrow c \otimes b \ F_3 \otimes F_2 \) and \( A^2 c_2 \) equals \( S_2 F_2 \otimes A^2 F_1 \rightarrow A^2(F_2 \otimes F_1) \rightarrow A^2 F_1 \).

To check that (42) is commutative we first do it for the splitting \( F \) and then apply the argument from the proof of Lemma 2.4. Checking commutativity for splitting \( F \) (with a choice of \( b \)) is straightforward so we omit it.

We know from Section 2 that the factorisation \( p_2 \) in (42) exists. The top row of (42) is acyclic so the map of complexes given by \( A^2 c, bc, S_2 b \) is homotopically equivalent to 0, so the factorisation \( p_2 \) satisfying \( bc = dp_2^2 + p_2 d \) and \( A^2 c = p_2^2 d \) exists over \( R \). Now we pass modulo \( m \). All horizontal maps in (42) become 0, to the above equalities show that \( A^2 c', b' c', S_2 b' \) are all zero.

To classify all possible multiplications on \( \text{Tor}^R(R/I, \mathcal{E}) \) we look at the space \( Z \) of pairs \( (b', c') \) satisfying the relations \( A^2 c' = b' c' = S_2 b' = 0 \). \( b' \) is the element of \( A^2 \mathcal{F}_1 \otimes \mathcal{F}_2^* \) and \( c' \) belongs to \( \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 \) so we can treat \( Z \) as Spec \( T \), where \( T \) is the commutative ring

\[
T = S \cdot (A^2 \mathcal{F}_1 \otimes \mathcal{F}_2^*) \otimes S \cdot (\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3^*) / J,
\]

(43)
where \( J \) is the ideal generated by the following representations:
\[
\begin{align*}
A^4\mathcal{F}_1 \otimes S_2\mathcal{F}_2 & \text{ in degree } (2, 0) \text{ (corresponding to } A^2\mathcal{c}') , \\
A^3\mathcal{F}_1 \otimes \mathcal{F}_2^2 \otimes \mathcal{F}_2 \otimes \mathcal{F}_1 & \text{ in degree } (1, 1) \text{ (corresponding to } b'\mathcal{c}') , \\
A^2\mathcal{F}_1 \otimes S_2\mathcal{F}_2 \otimes A^2\mathcal{F}_1 & \text{ in degree } (0, 2) \text{ (corresponding to } S_2b') , \\
S_{21}\mathcal{F}_1 \otimes \mathcal{F}_2 & \text{ in degree } (1, 1) \text{ (corresponding to associativity) .}
\end{align*}
\]

The group \( G = GL(\mathcal{F}_3) \times GL(\mathcal{F}_2) \times GL(\mathcal{F}_1) \) acts on \( Z \). The key result that allows us to classify the multiplications is the following

**Theorem 4.1.** Let us assume that \( \dim F_1 \geq 4 \). Then \( Z \) has finitely many \( G \)-orbits listed below.

**Proof.** Let us fix the basis \( \{e_i\}, \{f_j\}, \{g_k\} \) of \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \), respectively. We want to reduce pairs \((b', c')\) to the canonical form. We start the analysis by considering the map \( b' \) alone. The relations in degree \((2, 0)\) mean that the image of any \( f \in \mathcal{F}_2 \) under \( b' : \mathcal{F}_2 \rightarrow A^2 \mathcal{F}_1 \) is decomposable. This gives us two possibilities:

(a) \( b' \) is decomposable, i.e., there exists \( h : \mathcal{F}_2^* \rightarrow \mathcal{F}_1^* \) such that for some \( e^* \in \mathcal{F}_1^* \) we have \( b'(f^*) = e^* \land h(f^*) \).

(b) \( \text{Im } b' \subset A^2 H^* \), where \( \dim H = 3 \).

This shows that \( b' \) has one of the following canonical forms:

\((A_p)\) \( e_1 \cdot e_{i+1} = f_i \) (\( 1 \leq i \leq p \)), all other products are zero,

\((B)\) \( e_1 \cdot e_2 = f_3 \), \( e_1 \cdot e_3 = -f_2 \), \( e_2 \cdot e_3 = f_1 \), all other products are zero.

The index \( p \) has the range \( 0 \leq p \leq \min(\dim \mathcal{F}_1, \dim \mathcal{F}_2) \).

Now we consider the possibilities for \( c' \). Let us assume that \( b' \) has form \( A_p \) with \( p \geq 2 \). Then the relations in degree \((1, 1)\) force
\[
\begin{align*}
 f_j \cdot e_i & = 0 \quad \text{for } i \geq 2, \\
f_j \cdot e_1 & = 0 \quad \text{for } j \leq p.
\end{align*}
\]

The only possibilities of non-zero products are \( f_j \cdot e_1 \) for \( j > p \). Thus for \( 0 \leq q \leq \min(\dim \mathcal{F}_3, \dim \mathcal{F}_2 - p) \) we get the orbit

\((A_{p,q})\) \( e_1 \cdot e_{i+1} = f_i \) (\( 1 \leq i \leq p \)), \( e_1 \cdot f_{p+j} = g_j \) (\( 1 \leq j \leq q \)), all other products are zero.

Let \( b' \) be of type \( A_1 \). Then the relations in degree \((1, 1)\) force \( e_i \cdot f_j = 0 \) for \( i > 2 \). Also \( f_1 \cdot e_i = 0 \) for any \( i \). The possible non-zero products are
\[
e_1 \cdot f_j, e_2 \cdot f_j \quad \text{for } j > 1.
\]

The relations in degree \((0, 2)\) mean that if we treat \( c' \) as a map \( \mathcal{F}_2 \rightarrow \)
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Then for any \( f \in \mathcal{F}_2 \), \( c'(f) \) is decomposable. This leaves us with the following possibilities:

\[
(A_{1,q}) \quad e_1 \cdot e_2 = f_1, \quad e_2 \cdot f_j = 0, \quad e_1 \cdot f_j +_1 = g_j \quad (1 \leq j \leq q), \quad \text{all other products zero, for } 0 \leq q \leq \min(\dim \mathcal{F}_3, \dim \mathcal{F}_2 - 1),
\]

\[
(A_{2,q}) \quad e_1 \cdot e_2 = f_1, \quad e_1 \cdot f_2 = g_1, \quad e_2 \cdot f_3 = g_1, \quad \text{all other products zero.}
\]

Let us assume now that \( b' \) has type \( A_0 \). This means \( b' \) equals zero. Again the relations in degree \((0, 2)\) mean that for every \( f \in \mathcal{F}_2 \), \( c'(f) \) is decomposable in \( \mathcal{F}_3 \otimes \mathcal{F}_1^* \). This leads to the following orbits:

\[
(A_{0,q}) \quad e_1 \cdot f_j = g_j \quad (1 \leq j \leq q), \quad \text{all other products zero, for } 0 \leq q \leq \min(\dim \mathcal{F}_3, \dim \mathcal{F}_2),
\]

\[
(A_{1,q}) \quad e_i \cdot f_i = g_1 \quad (1 \leq i \leq q), \quad \text{all other products zero, for } 2 \leq q \leq \min(\dim \mathcal{F}_1, \dim \mathcal{F}_2).
\]

Finally, let us assume that \( b' \) has type \( B \). Relations in degree \((1, 1)\) give in this case \( e_i \cdot f_j = 0 \) for all \( i, j \). This means we get one orbit:

\[
(B) \quad e_1 \cdot e_2 = f_3, \quad e_1 \cdot e_3 = -f_2, \quad e_2 \cdot e_3 = f_1, \quad \text{all other products are zero.}
\]

Now the list of our orbits is complete and Theorem 4.1 is proved.

**Remark 4.2.** The fact that for \( b' \) of type \( B \), \( c' \) has to be zero is not in contradiction with the existence of the Koszul complex. We assumed above that \( \dim F_1 \geq 4 \). For \( \dim F_1 \leq 3 \) we do not have the diagram (42). For \( \dim F_1 = 3 \) it is known [Br] that the generic ring is the polynomial ring on the entries of \( d_3 \) and \( b \). Similar analysis as above gives in this case four orbits for multiplication, among those the orbit of the exterior algebra in three variables.

**Remark 4.3.** We saw that the restrictions on multiplication were determined by the existence of \( p_2 \). The higher \( p_n \) give other operations on \( \text{Tor}^R(R/I, \ell) \). It would be interesting to get a general framework for this sort of structure, for example, in terms of higher Massey products.

**Remark 4.4.** A similar result is possible for other resolutions of length 3 (\( r_1 > 1 \)). Then for the map \( b: A^{r_1+1}F_1 \to F_2 \) we get two components for \( b' = b \mod m \). Let \( b'^*: \mathcal{F}_2^* \to A^{r_1+1}\mathcal{F}_1^* \) be the dual map. Then we have two possibilities:

\[
(a) \quad \text{Im } b'^* \subset A^{r_1+1}H^* \quad \text{dim } H^* = r_1 + 2,
\]

\[
(b) \quad b'^* \text{ is decomposable, i.e., there exists a map } h: \mathcal{F}_2^* \to \mathcal{F}_1^* \text{ such that } b'^*(f) = h(f) \wedge e_1^* \wedge \cdots \wedge e_r^* \text{ for some } e_1, \ldots, e_r \in \mathcal{F}_1.
\]

In this case we also get finitely many \( G \)-orbits for \( b' \).
COROLLARY 4.5. Let $R$ be a regular local ring. Let $I$ be an ideal in $R$ such that $\text{pd}_R(R/I) \leq 3$. Then $R/I$ has rational Poincaré series.

Proof. For $\text{pd}_R(R/I) \leq 2$ the result is well known, so we assume that $\text{pd}_R(R/I) = 3$. Every resolution of length 3 has an associative and commutative multiplication by [BE2]. By a theorem of Avramov [Av] it is enough to check that $\text{Tor}_R^k(R/I, \mathcal{E})$ has rational Poincaré series. To prove the corollary we have to show that all algebras listed in Theorem 4.1 have rational Poincaré series. First of all all the algebras of type $A_{p,q}$ can be expressed as a sum

$$A_{p,q} := \text{Tor}_k^R(R/I, \mathcal{E}) = A'_{p,q} \oplus I_{p,q},$$

where $A'_{p,q}$ is the span of $1, e_1, \ldots, e_{p+1}, f_1, \ldots, f_{p+q}, g_1, \ldots, g_q$ and $I_{p,q}$ is the span of other $e$'s, $f$'s, and $g$'s. $I_{p,q}$ is the ideal in our algebra and $A_{p,q} = A'_{p,q} \rtimes I_{p,q}$ in the sense of [G]. It follows that the Poincaré series of $A_{p,q}$ is rational if and only if the Poincaré series of $A'_{p,q}$ is. Next we observe that $A'_{p,q}$ is a tensor product

$$A'_{p,q} = \left\{ k1 \oplus ke_1 \right\} \otimes \left\{ k1 \oplus \sum_{i \geq 2} ke_i \oplus \sum_{j \geq 1} kf_{p+j} \right\}.$$

Now it is easy to check that the Poincaré resolution for $A'_{p,q}$ is the tensor product of the tensor algebras $T(ke_1)$ and $T(\sum ke_i \oplus \sum kf_{p+j})$. We conclude that

$$P(A'_{p,q})(z) = (1 - z^2)^{-1} (1 - pz^2 - qz^3)^{-1}.$$

We have to consider the algebras of type $A_{0,q}^\star, A_{1,2}^\star, B$ and the case of the exterior algebra on three elements. In each case we can apply the result [G] for the ideal similar to $I_{p,q}$. Thus we can assume in case $B$ that our algebra is the span of $1, e_1, e_2, e_3, f_1, f_2, f_3$. The Poincaré series of the exterior algebra is rational by [J]. The result for the algebra of type $B$ follows from that by the result of Avramov [Av]. In the case $A_{0,q}^\star$ we can assume that our algebra is a span of $1, e_1, \ldots, e_q, f_1, \ldots, f_q, g_1$. This algebra is Gorenstein and after dividing by the ideal generated by socle $kg_1$ we get an algebra with rational Poincaré series. Thus we can apply again the same result of Avramov. The case $A_{1,2}^\star$ can be checked directly by calculating the Poincaré resolution.

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