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On the generic and typical ranks of 3-tensors

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ABSTRACT

We study the generic and typical ranks of 3-tensors of dimension $l \times m \times n$ using results from matrices and algebraic geometry. We state a conjecture about the exact values of the generic rank of 3-tensors over the complex numbers, which is verified numerically for $l, m, n \leq 14$. We also discuss the typical ranks over the real numbers, and give an example of an infinite family of 3-tensors of the form $l = m$, $n = (m-1)^2 + 1$, $m = 3, 4, \ldots$, which have at least two typical ranks.

1. Introduction

The subject of tensors, their rank and the approximation of tensors by low rank tensors became recently a very active area of pure and applied mathematics. See [1,2,5–10,13,17–23,25,27–32]. 2-dimensional tensors, which are identified as matrices, are well understood theoretically and numerically. Tensors of dimension greater than 2, are much more complicated theoretically and numerically than matrices. Basically, matrices are strongly connected to linear operators, while tensors are strongly connected to the study of polynomial equations in several variables, which are best dealt with the tools of algebraic geometry. Indeed, there is a vast literature in algebraic geometry discussing tensors. See for example [4, Chapter 20] and references therein. Unfortunately, it is unaccessible to most researchers in applied and numerical analysis.

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The object of this paper threefold. First, we give a basic introduction to one of the most interesting topics: the rank of 3-tensors. Second, we state our conjecture for the generic tensors of 3-tensors over the complex numbers. Third, we give general results for the typical ranks of 3-tensors over the real numbers. We illustrate the strength and generality of our results by comparing them to the known results in the literature. The novelle results of this paper are obtained by using results on matrices and basic results of algebraic geometry on polynomial equations over complex and real numbers. For reader's benefit we added a short appendix on complex and real algebraic geometry. The exact references for the results in complex and real algebraic geometry used in this paper are given in the appendix.

This paper is written for the audience who has the knowledge of matrix theory and was only occasionally exposed to the study of polynomial maps in several complex variables. This paper is an expanded version of the talk I gave in *Workshop on Algorithms for Modern Massive Data Sets*, sponsored by Computer Forum of the Stanford Computer Science Department, NSF and Yahoo! Research, June 21–24, 2006, [10].

We now survey briefly the contents of this paper. Section 2 deals with the basic notions of the tensor product of three vector spaces over any field $\mathbb{F}$, 3-tensors and their rank. Theorem 2.4 gives a simple and useful characterization of the rank of a given tensor over any field $\mathbb{F}$, in terms of the minimal dimension of a subspace spanned by rank one matrices, containing a given subspace L of $\mathbb{F}^{m\times n}$. Section 3 introduces the notion of the generic rank in $\mathbb{C}^{l\times m\times n}$, denoted by $\mathrm{grank}(l, m, n)$. ($\mathrm{grank}(l, m, n)$ is a symmetric function in l, m, n.) Section 4 introduces the notion of the maximal rank in $\mathbb{C}^{l\times m\times n}$, denoted by $\mathrm{mrank}(l, m, n)$. Section 5 gives known values for $\mathrm{grank}(l, m, n)$ and states the conjectured values of $\mathrm{grank}(l, m, n)$ in the range $3 \le l \le m \le n \le (l-1)(m-1)-1$. This conjecture is verified numerically for all values of $l, m, n \le 14$. (Compare these results with the numerical results for $\mathrm{grank}(l, m, n)$ given in [7, Table 1], for the values $l \le 4, m \le 5, n \le 12$.) Section 6 shows how to apply some results on matrices to obtain bounds on $\mathrm{grank}(l, m, n)$ and $\mathrm{mrank}(l, m, n)$. Section 7 discusses the notion of typical ranks of real tensors $\mathbb{R}^{l\times m\times n}$, which are the analogs of generic rank over the complex numbers. In this case one has a finite number of typical ranks taking all the values from $\mathrm{grank}(l, m, n)$ to $\mathrm{mtrank}(l, m, n)$. The typical ranks for the case $l = 2 \le m \le n$ are known. For $m < n$ there is one typical rank which is equal to $\mathrm{grank}(2, m, n) = \min(n, 2m)$. For $2 \le m = n$ there are two typical ranks $\mathrm{grank}(2, m, m) = m$ and $\mathrm{mtrank}(2, m, m) = m + 1$. See [29] and [32]. In this paper we give another countable set of examples of the form $3 \le l = m, n = (m-1)^2 + 1, m = 3, \ldots$, where the maximal typical rank is strictly bigger than $\mathrm{grank}(m, m, (m-1)^2 + 1) = (m-1)^2 + 1$, i.e. there are at least two typical ranks in these cases. The case $m = 3$ is studied in [31]. It is shown there that $\mathrm{mtrank}(3, 3, 5) = 6$. (It is not known that if $\mathrm{mtrank}(l, m, n) \le \mathrm{grank}(l, m, n) + 1$, which holds in all known examples.) Appendix A gives a concise exposition of facts in complex and algebraic geometry needed here, with suitable references.

2. Basic notions and preliminary results

In this section we let $\mathbb{F}$ be any field. Usually we denote by a bold capital letter a finite dimensional vector space $\mathbf{U}$ over $\mathbb{F}$, unless stated otherwise. A vector $\mathbf{u} \in \mathbf{U}$ is denoted by a bold face lower case letter. A matrix $A \in \mathbb{F}^{m\times n}$ denoted by a capital letter A, and we let either $A = [a_{ij}]_{i=j=1}^{m\times n}$ or simply $A = [a_{ij}]$. A *3-tensor array* $\mathcal{T} \in \mathbb{F}^{l\times m\times n}$ is denoted by a capital calligraphic letter. So either $\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{l,m,n}$ or simply $\mathcal{T} = [t_{ijk}]$.

Let $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$ be three vectors spaces. Let $m_i := \dim \mathbf{U}_i$ be the dimension of the vector space $\mathbf{U}_i$. Let $\mathbf{u}_{1,i}, \ldots, \mathbf{u}_{m_i,i}$ be a basis of $\mathbf{U}_i$ for $i = 1, 2, 3$. Then $\mathbf{U} := \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$ is the tensor product of $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$. $\mathbf{U}$ is a vector space of dimension $m_1m_2m_3$, and

$$\mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}, \quad i_j = 1, \ldots, m_j, j = 1, 2, 3, \tag{2.1}$$

is a basis of $\mathbf{U}$. For any permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ the tensor product $\mathbf{U}_{\sigma(1)} \otimes \mathbf{U}_{\sigma(2)} \otimes \mathbf{U}_{\sigma(3)}$ is isomorphic to $\mathbf{U}$. Hence it will be convenient to assume that

$$1 \le m_1 \le m_2 \le m_3, \tag{2.2}$$

unless stated otherwise. A *3-tensor* is a vector in $\mathbf{U}$. We will call 3-tensor a *tensor*, and denote it by a Greek letter. A tensor τ has the representation

$$\tau = \sum_{i_1=i_2=i_3=1}^{m_1,m_2,m_3} t_{i_1 i_2 i_3} \mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}, \tag{2.3}$$

in the basis (2.1). If the basis (2.1) is fixed then τ is identified with $\mathcal{T} = [t_{i_1 i_2 i_3}] \in \mathbb{F}^{m_1 \times m_2 \times m_3}$.

Recall that $\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$, were $\mathbf{x}_i \in \mathbf{U}_i$, $i = 1, 2, 3$, is called a *rank one* tensor, or an *indecomposable tensor*. (Usually one assumes that all $\mathbf{x}_i \neq \mathbf{0}$. Otherwise $\mathbf{0} = \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$ is called a rank zero tensor.) (2.3) is a decomposition of τ as a sum of at most $m_1 m_2 m_3$ rank one tensors, as $t_{i_1 i_2 i_3} \mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3} = (t_{i_1 i_2 i_3} \mathbf{u}_{i_1,1}) \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}$. A decomposition of $\tau \neq \mathbf{0}$ to a sum of rank one tensors is given by

$$\tau = \sum_{i=1}^{k} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i, \quad \mathbf{x}_i \in \mathbf{U}_1, \mathbf{y}_i \in \mathbf{U}_2, \mathbf{z}_i \in \mathbf{U}_3,\ i = 1, \ldots, k. \tag{2.4}$$

The minimal k for which the above equality holds is called the *rank* of the tensor τ. It is completely analogous to the rank of matrix $A = [a_{i_1 i_2}] \in \mathbb{F}^{m_1 \times m_2}$, which can be identified with 2-tensor in $\sum_{i_1=i_2=1}^{m_1,m_2} a_{i_1 i_2} \mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \in \mathbf{U}_1 \otimes \mathbf{U}_2$. It is well known that, unlike in the case of matrices, the rank of a tensor may depend on the ground field $\mathbb{F}$. In particular, by considering the algebraic closed field $\mathbb{C}$ versus $\mathbb{R}$, one may decrease the rank of the real valued tensor τ.

For $j \in \{1, 2, 3\}$ denote by $j^c := \{p, q\} = \{1, 2, 3\} \backslash \{j\}$, where $1 \leq p < q \leq 3$. Denote by $\mathbf{U}_{j^c} = \mathbf{U}_{\{p,q\}} := \mathbf{U}_p \otimes \mathbf{U}_q$. A tensor $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$ induces a linear transformation $\tau(j) : \mathbf{U}_{j^c} \to \mathbf{U}_j$ as follows. Assume that $\mathbf{u}_{1,l}, \ldots, \mathbf{u}_{m_l,l}$ is a basis in $\mathbf{U}_l$ for $l = 1, 2, 3$. Then any $\mathbf{v} \in \mathbf{U}_{j^c}$ is of the form $\mathbf{v} = \sum_{i_p=i_q=1}^{m_p,m_q} v_{i_p i_q} \mathbf{u}_{i_p,p} \otimes \mathbf{u}_{i_q,q}$. Define

$$\tau(j)\, \mathbf{v} = \sum_{i_j=1}^{m_j} \left(\sum_{i_p,i_q=1}^{m_p,m_q} t_{i_1 i_2 i_3} v_{i_p i_q} \right) \mathbf{u}_{i_j,j}. \tag{2.5}$$

The $\mathrm{rank_j}\tau$ *is the rank of the operator* $\tau(j)$. Equivalently, let $A(j) = [a_{l i_j}] \in \mathbb{F}^{m_p m_q \times m_j}$, where each integer $l \in [1, m_p m_q]$ corresponds to the pair (i_p, i_q), for $i_p = 1, \ldots, m_p, i_q = 1, \ldots, m_q$, and $i_j \in [1, m_j] \cap \mathbb{N}$. (For example arrange the pairs (i_p, i_q) in the lexicographical order. Then $i_p = \lceil \frac{l}{m_q} \rceil$ and $i_q = l - (i_p - 1)m_q$.) Set $a_{l i_j} = t_{i_1 i_2 i_3}$. Then $\mathrm{rank_j}\tau = \mathrm{rank}\ A(j)$. Associating a matrix $A(j)$ with the 3-tensors is called *unfolding* τ in direction j. The following proposition is straightforward.

Proposition 2.1. *Let $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$ be given by (2.3). Fix $j \in \{1, 2, 3\}, j^c = \{p, q\}$. Let $T_{i_j,j} := [t_{i_1 i_2 i_3}]_{i_p=i_q=1}^{m_p,m_q} \in \mathbb{F}^{m_p \times m_q}$, $i_j = 1, \ldots, m_j$. Then $\mathrm{rank_j}\tau$ is the dimension of subspace of $m_p \times m_q$ matrices spanned by $T_{1,j}, \ldots, T_{m_j,j}$.*

The following result is well known.

Proposition 2.2. *Let $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$. Let $r_j := \mathrm{rank_j}\tau$ for $j = 1, 2, 3$. Denote by $0 \leq R_1 \leq R_2 \leq R_3$ the rearranged values of r_1, r_2, r_3. Then $R_3 \leq \mathrm{rank}\ \tau \leq R_1 R_2$.*

Proof. We first show that $r_3 \leq \mathrm{rank}\ \tau$. Since $\mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \in \mathbf{U}_{\{1,2\}}$ it follows that the decomposition (2.3) is a decomposition of τ_3 to a sum of rank one linear operators from $\mathbf{U}_{\{1,2\}}$ to $\mathbf{U}_3$. Hence $r_3 \leq \mathrm{rank}\ \tau$. Let $j \in \{1, 2, 3\}, j^c = \{p, q\}$. Recall that $\mathbf{U}$ is isomorphic to $\mathbf{U}' := \mathbf{U}_p \otimes \mathbf{U}_q \otimes \mathbf{U}_j$. Hence $r_j \leq \mathrm{rank}\ \tau$ for $j = 1, 2$. Thus $R_3 \leq \mathrm{rank}\ \tau$.

Let $\mathbf{v}_{1,j}, \ldots, \mathbf{v}_{1,r_j}$ be the basis of $\mathbf{X}_j := \tau_j(\mathbf{U}_p \otimes \mathbf{U}_q) \subseteq \mathbf{U}_j$. It is straightforward to show that $\tau \in \mathbf{X}_1 \otimes \mathbf{X}_2 \otimes \mathbf{X}_3$. So $\tau_j : \mathbf{X}_p \otimes \mathbf{X}_q \to \mathbf{X}_j$. Assume that $R_1 = r_j$. Decompose $\tau_j = \sum_{l=1}^{R_1} \mathbf{z}_l \otimes \mathbf{x}_l$, where $\mathbf{z}_l \in \mathbf{X}_p \otimes \mathbf{X}_q, \mathbf{x}_l \in \mathbf{X}_j$ for $l = 1, \ldots, R_1$. Since $\mathbf{z}_l \in \mathbf{X}_p \otimes \mathbf{X}_q$, it follows that each $\mathbf{z}_l$ is at most a sum of R_2 rank one tensors in $\mathbf{X}_p \otimes \mathbf{X}_q$. Hence τ is a sum of at most R_1R_2 rank one tensors in $\mathbf{X}_1 \otimes \mathbf{X}_2 \otimes \mathbf{X}_3$. $\square$

The following proposition is obtained straightforward:

Proposition 2.3. *Let the assumptions and the notations of Propositions 2.1-2.2 hold. Let* $[\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{m_1,1}]$, $[\mathbf{v}_{1,2}, \ldots, \mathbf{v}_{m_2,2}]$ *be two bases in* $\mathbf{U}_1, \mathbf{U}_2$*, respectively, where*

$$[\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{m_1,1}] = [\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{m_1,1}]Q_1, \quad [\mathbf{u}_{1,2}, \ldots, \mathbf{u}_{m_2,2}] = [\mathbf{v}_{1,2}, \ldots, \mathbf{v}_{m_2,2}]Q_2,$$
$$Q_1 = [q_{pq,1}]_{p,q=1}^{m_1} \in \mathbf{GL}(m_1, \mathbb{F}), \quad Q_2 = [q_{pq,2}]_{p,q=1}^{m_2} \in \mathbf{GL}(m_2, \mathbb{F}).$$

Let

$$\tau = \sum_{i,j,k=1}^{m_1,m_2,m_3} \tilde{t}_{ijk}\mathbf{v}_{i,1} \otimes \mathbf{v}_{j,2} \otimes \mathbf{u}_{j,3}, \ \tilde{T}_{k,3} := [\tilde{t}_{ijk}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1\times m_2}, \ k = 1, \ldots, m_3.$$

Then $\tilde{T}_{k,3} = Q_1 T_{k,3} Q_2^T$ *for* $k = 1, \ldots, m_3$.

Let $[\mathbf{v}_{1,3}, \ldots, \mathbf{v}_{m_3,3}]$ *be another basis of* $\mathbf{U}_3$*, where*

$$[\mathbf{u}_{1,3}, \ldots, \mathbf{u}_{m_3,3}] = [\mathbf{v}_{1,3}, \ldots, \mathbf{v}_{m_3,3}]Q_3, \quad Q_3 = [q_{pq,3}]_{p,q=1}^{m_3} \in \mathbf{GL}(m_3, \mathbb{F}).$$

Then $\tau = \sum_{i,j,k=1}^{m_1,m_2,m_3} t'_{ijk}\mathbf{u}_{i,1} \otimes \mathbf{u}_{j,2} \otimes \mathbf{v}_{k,3}$ *and* $T'_{k,3} = [t'_{ijk}]_{i,j=1}^{m_1,m_2} = \sum_{l=1}^{k} q_{kl,3}T_l$.

Let $[\mathbf{v}_{1,i}, \ldots, \mathbf{v}_{m_i,i}]$ *be a basis in* $\mathbf{U}_i$ *such that* $\tau_i\mathbf{U}_{i^c} = \mathrm{span}(\mathbf{v}_{1,i}, \ldots, \mathbf{v}_{r_i,i})$ *for* $i = 1, 2, 3$. *Then* $\tau = \sum_{i=j=k}^{m_1,m_2,m_3} \hat{t}_{ijk}\mathbf{v}_{i,1} \otimes \mathbf{v}_{j,2} \otimes \mathbf{v}_{k,3}$ *and* $\hat{T}_{k,3} := [\hat{t}_{ijk}]_{i=j=1}^{m_1,m_2} \in \mathbb{F}^{m_1\times m_2}$ *for* $k = 1, \ldots, m_3$. *Then* $\hat{T}_{k,3} = 0$ *for* $k > r_3$ *and* $\hat{T}_{1,3}, \ldots, \hat{T}_{r_3,3}$ *are linearly independent. Furthermore, each* $\hat{T}_{k,3} = S_k \oplus 0 := \begin{bmatrix} S_k & 0 \\ 0 & 0 \end{bmatrix}$, *where* $S_k \in \mathbb{F}^{r_1\times r_2}$ *for* $k = 1, \ldots, r_3$. *Moreover, the span of* range $S_1, \ldots,$ range S_{r_3} *and the span of* range $S_1^\top, \ldots,$ range $S_{r_3}^\top$ *are* $\mathbb{F}^{r_1}$ *and* $\mathbb{F}^{r_2}$*, respectively.*

The following result is a very useful characterization of the rank of 3-tensor.

Theorem 2.4. *Let* $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$ *be given by (2.3). Fix* $j \in \{1, 2, 3\}, j^c = \{p, q\}$. *Let* $T_{i_j,j} := [t_{i_1i_2i_3}]_{i_p=i_q=1}^{m_p,m_q} \in \mathbb{F}^{m_p\times m_q}$, $i_j = 1, \ldots, m_j$. *Then* rank τ *is the minimal dimension of a subspace of* $m_p \times m_q$ *matrices spanned by rank one matrices, which contains the subspace spanned by* $T_{1,j}, \ldots, T_{m_j,j}$.

Proof. It is enough to prove the Proposition for the case $j = 3$. Proposition 2.2 and its proof yields that it is enough to consider the case where $r_3 = m_3$, i.e. $T_{1,3}, \ldots, T_{m_3,3}$ are linearly independent. Let r be the dimension of the minimal subspace of $m_1 \times m_2$ matrices spanned by rank one matrices, which contains the subspace spanned by $T_{1,3}, \ldots, T_{m_3,3}$.

Suppose that equality (2.4) holds. Since $r_3 = m_3$ it follows that $\mathbf{z}_1, \ldots, \mathbf{z}_k$ span $\mathbf{U}_3$. Without loss of generality we may assume that $\mathbf{z}_1, \ldots, \mathbf{z}_{m_3}$ form a basis in $\mathbf{U}_3$. For each $l > m_3$ rewrite each $\mathbf{z}_l$ as al linear combination of $\mathbf{z}_1, \ldots, \mathbf{z}_{m_3}$. Thus

$$\mathbf{z}_l = \sum_{p=1}^{m_3} b_{lj}\mathbf{z}_j, \ l = m_3 + 1, \ldots, k, \ \tau = \sum_{j=1}^{m_3}(\mathbf{x}_j \otimes \mathbf{y}_j + \sum_{l=m_3+1}^{k} b_{lj}\mathbf{x}_l \otimes \mathbf{y}_l) \otimes \mathbf{z}_j. \tag{2.6}$$

Hence

$$T_{j,3} = \mathbf{x}_j\mathbf{y}_j^\top + \sum_{l=m_3+1}^{k} b_{lj}\mathbf{x}_l\mathbf{y}_l^\top, \quad j = 1, \ldots, m_3. \tag{2.7}$$

In particular, the subspace spanned by $T_{1,3}, \ldots, T_{m_3,3}$ is contained in the subspace spanned by k rank one matrices $\mathbf{x}_1\mathbf{y}_1^\top, \ldots, \mathbf{x}_k\mathbf{y}_k^\top$. Therefore $r \le k$, hence $r \le$ rank τ.

Assume now that there exist $\mathbf{x}_i \in \mathbb{F}^{m_1}$, $\mathbf{y}_i \in \mathbb{F}^{m_2}$, $i = 1, \ldots, k$ such that $T_{p,3} = \sum_{i=1}^k a_{pi}\mathbf{x}_i\mathbf{y}_i^\top$ for $p = 1, \ldots, m_3$. View $\mathbf{x}_i\mathbf{y}_i^\top$ as $\mathbf{x}_i \otimes \mathbf{y}_i$. Then

$$\tau = \sum_{p=1}^{m_3} \left(\sum_{i=1}^k a_{pi}\mathbf{x}_i \otimes \mathbf{y}_i \right) \otimes \mathbf{z}_p = \sum_{i=1}^k \mathbf{x}_i \otimes \mathbf{y}_i \otimes \left(\sum_{p=1}^{m_3} a_{pi}\mathbf{z}_p \right). \tag{2.8}$$

Hence $k \geq \operatorname{rank} \tau$. So $\operatorname{rank} \tau = r$. $\square$

3. Generic rank

From now and $\mathbb{F}$ is either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$, unless stated otherwise. We refer the reader to Appendix A for the notations and results in algebraic geometry used in the sequel. Let $\mathbf{x}_i \in \mathbb{C}^{m_i}$, $i = 1, 2, 3$. Then a rank one tensor $\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$ is a polynomial map $\mathbf{f} : \mathbb{C}^{m_1+m_2+m_3} \to \mathbb{C}^{m_1 \times m_2 \times m_3} \equiv \mathbb{C}^{m_1m_2m_3}$, i.e. $\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$. Thus we identify a vector $\mathbf{z} = (z_1, \ldots, z_{m_1+m_2+m_3})^\top \in \mathbb{C}^{m_1+m_2+m_3}$ with $(\mathbf{x}_1^\top, \mathbf{x}_2^\top, \mathbf{x}_3^\top)^\top$, which is also denoted by $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, and a vector $\mathbf{y} \in \mathbb{C}^{m_1m_2m_3}$ with $\mathcal{T} = [t_{i_1i_2i_3}]_{i_1=i_2=i_3}^{m_1,m_2,m_3} \in \mathbb{C}^{m_1 \times m_2 \times m_3}$. (Here we arrange the three indices of $[t_{i_1i_2i_3}]$ in the lexicographical order.) Then $\mathrm{D}\mathbf{f}$, the Jacobian matrix of partial derivatives is given as

$$\mathrm{D}\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = [A_1(\mathbf{x}_2, \mathbf{x}_3)|A_2(\mathbf{x}_1, \mathbf{x}_3)|A_3(\mathbf{x}_1, \mathbf{x}_2)] \in \mathbb{C}^{m_1m_2m_3 \times (m_1+m_2+m_3)}, \tag{3.1}$$

is viewed as a block matrix, where $A_i \in \mathbb{C}^{m_1m_2m_3 \times m_i}$ for $i = 1, 2, 3$. More precisely, let

$$\mathbf{e}_{i_j,j} = (\delta_{1i_j}, \ldots, \delta_{m_ji_j})^\top, \quad i_j = 1, \ldots, m_j$$

be the standard bases in $\mathbb{C}^{m_j}$ for $j = 1, 2, 3$. Then

$$\begin{aligned}
A_1(\mathbf{x}_2, \mathbf{x}_3) &= [\mathbf{e}_{1,1} \otimes \mathbf{x}_2 \otimes \mathbf{x}_3| \cdots |\mathbf{e}_{m_1,1} \otimes \mathbf{x}_2 \otimes \mathbf{x}_3] \in \mathbb{C}^{m_1m_2m_3 \times m_1}, \\
A_2(\mathbf{x}_1, \mathbf{x}_3) &= [\mathbf{x}_1 \otimes \mathbf{e}_{1,2} \otimes \mathbf{x}_3| \cdots |\mathbf{x}_1 \otimes \mathbf{e}_{m_2,2} \otimes \mathbf{x}_3] \in \mathbb{C}^{m_1m_2m_3 \times m_2}, \\
A_3(\mathbf{x}_1, \mathbf{x}_2) &= [\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{e}_{1,3}| \cdots |\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{e}_{m_3,3}] \in \mathbb{C}^{m_1m_2m_3 \times m_3}.
\end{aligned} \tag{3.2}$$

So the pth column of $A_1(\mathbf{x}_2, \mathbf{x}_3)$ is the tensor $\mathbf{e}_{p,1} \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$. Similar statements holds for $A_2(\mathbf{x}_1, \mathbf{x}_3)$ and $A_3(\mathbf{x}_1, \mathbf{x}_2)$.

Proposition 3.1. *Let $\mathbf{x}_i \in \mathbb{C}^{m_i}$, $i = 1, 2, 3$, and denote by $\mathbf{f} : \mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3} \to \mathbb{C}^{m_1 \times m_2 \times m_3}$ the map $\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$. Identify $\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3}$, $\mathbb{C}^{m_1 \times m_2 \times m_3}$ with $\mathbb{C}^{m_1+m_2+m_3}$, $\mathbb{C}^{m_1m_2m_3}$, respectively. Then*

$$\operatorname{rank} \mathrm{D}\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \leq m_1 + m_2 + m_3 - 2. \tag{3.3}$$

Equality holds for any $\mathbf{x}_i \neq \mathbf{0}$ for $i = 1, 2, 3$.

Proof. Let $A_1(\mathbf{x}_2, \mathbf{x}_3), A_2(\mathbf{x}_1, \mathbf{x}_3), A_3(\mathbf{x}_1, \mathbf{x}_2)$ be defined as in (3.1). Note that

$$\sum_{i_1=1}^{m_1} x_{i_1,1}\mathbf{e}_{i_1,1} \otimes \mathbf{x}_2 \otimes \mathbf{x}_3 = \sum_{i_2=1}^{m_2} x_{i_2,2}\mathbf{x}_1 \otimes \mathbf{e}_{i_2,2} \otimes \mathbf{x}_3 = \sum_{i_3=1}^{m_3} x_{i_3,3}\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{e}_{i_3,3} = \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$$

That is, the columns of $A_1(\mathbf{x}_2, \mathbf{x}_3), A_2(\mathbf{x}_1, \mathbf{x}_3)$ and $A_3(\mathbf{x}_1, \mathbf{x}_2)$ all span the vector $\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$. Hence the inequality (3.3) holds.

Choose $\mathbf{x}_1 = \mathbf{e}_{1,1}$, $\mathbf{x}_2 = \mathbf{e}_{2,1}$, $\mathbf{x}_3 = \mathbf{e}_{1,3}$. Then in $\mathrm{D}\mathbf{f}(\mathbf{e}_{1,1}, \mathbf{e}_{1,2}, \mathbf{e}_{1,3})$ the column $\mathbf{e}_{1,1} \otimes \mathbf{e}_{1,2} \otimes \mathbf{e}_{1,3}$ appears three times. After deleting two columns $\mathbf{e}_{1,1} \otimes \mathbf{e}_{1,2} \otimes \mathbf{e}_{1,3}$, we obtain $m_1 + m_2 + m_2 - 2$ linearly independent columns, i.e. $\operatorname{rank} \mathrm{D}\mathbf{f}(\mathbf{e}_{1,1}, \mathbf{e}_{1,2}, \mathbf{e}_{1,3}) = m_1 + m_2 + m_3 - 2$. If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \neq \mathbf{0}$, then each $\mathbf{x}_i$ can be extended to a basis in C^{m_i}. Hence equality holds in (3.3). $\square$

Let k be a positive integer and consider the map $\mathbf{f}_k : (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k \to \mathbb{C}^{m_1\times m_2\times m_3}$ given by

$$\mathbf{f}_k(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_{1,3}, \ldots, \mathbf{x}_{k,1}, \mathbf{x}_{k,2}, \mathbf{x}_{k,3}) = \sum_{l=1}^{k} \mathbf{f}(\mathbf{x}_{l,1}, \mathbf{x}_{l,2}, \mathbf{x}_{l,3}) = \sum_{l=1}^{k} \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3},$$
$$\mathbf{x}_{l,j} \in \mathbb{C}^{m_j},\ j = 1, 2, 3,\ l = 1, \ldots, k. \tag{3.4}$$

In this paper the closure of a set $S \subset \mathbb{F}^n$, denoted by Closure S, is the closure in the standard topology of $\mathbb{F}^n$. Since $\mathbf{f}_k$ is a polynomial map it follows, (see Appendix A.1).

Definition 3.2. Let $Y_k \subseteq \mathbb{C}^{m_1\times m_2\times m_3}$ be the closure of $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k)$. Denote by $r(k, m_1, m_2, m_3)$ the dimension of the variety Y_k. Let $U_k \subsetneq Y_k$ be the constructible algebraic subset of Y_k, of dimension $r(k, m_1, m_2, m_3) - 1$ at most, possibly an empty set, such that $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = Y_k\backslash U_k$.

$\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}$ has a border rank k if $\mathcal{T} \in Y_k\backslash Y_{k-1}$, where $Y_0 = \{0\}$. The border rank of $\mathcal{T}$ is denoted by brank $\mathcal{T}$. $\mathcal{T}$ is called rank ill conditioned if brank $\mathcal{T} <$ rank $\mathcal{T}$.

Clearly, $r(k, m_1, m_2, m_3)$ is a nondecreasing sequence in $k \in \mathbb{N}$. (See for more details the proof of Theorems 3.4 and 4.1.) The notion of *border rank* was introduced in [2].

Proposition 3.3. *The set of all ill conditioned tensors $\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}$ of border rank k equals to $U_k\setminus Y_{k-1}$. This set is a constructible algebraic set of dimension $r(k, m_1, m_2, m_3) - 1$ at most.*

Proof. Recall that $Y_k \setminus Y_{k-1}$ is the set of tensors of border rank k. Hence $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k)\backslash Y_{k-1}$ is the set of all tensor whose rank and border rank are k. By definition Y_k is a disjoint union of $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k)$ and U_k. Hence the set of all ill conditioned tensors of border rank k is $U_k \setminus Y_{k-1}$. Since U_k is a constructible algebraic subset of Y_k, where $\dim U_k < \dim Y_k$, and Y_{k-1} is an algebraic set, it follows from the results in Appendix A.1. that $U_k \setminus Y_{k-1}$ is a constructible algebraic set of dimension $\dim U_k$ at most. □

See [9] for related results on rank ill conditioned tensors. The following theorem is a version of what is called in literature *Terracini's* Lemma [34].

Theorem 3.4. *Let $m_1, m_2, m_3 \geq 2$ be three positive integers. Assume that $\mathbf{e}_{i_j,j} = (\delta_{1i_j}, \ldots, \delta_{m_j i_j})^\top \in \mathbb{C}^{m_j}$, $i_j = 1, \ldots, m_j$ is the standard basis in $\mathbb{C}^{m_j}$ for $j = 1, 2, 3$. Let* $\operatorname{grank}(m_1, m_2, m_3)$ *be the smallest positive integer k satisfying the following property. There exist* $3k$ *vectors $\mathbf{x}_{l,1} \in \mathbb{C}^{m_1}$, $\mathbf{x}_{l,2} \in \mathbb{C}^{m_2}$, $\mathbf{x}_{l,3} \in \mathbb{C}^{m_3}$, $l = 1, \ldots, k$ such that the following $k(m_1 + m_2 + m_3)$ tensors span $\mathbb{C}^{m_1\times m_2\times m_3}$:*

$$\mathbf{e}_{i_1,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3},\ \ \mathbf{x}_{l,1} \otimes \mathbf{e}_{i_2,2} \otimes \mathbf{x}_{l,3},\ \ \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{e}_{i_3,3}, \tag{3.5}$$
$$i_j = 1, \ldots, m_j,\ j = 1, 2, 3,\ l = 1, \ldots, k.$$

Then there exist three algebraic sets $U \subsetneq V \subseteq W \subsetneq \mathbb{C}^{m_1\times m_3\times m_3} \equiv \mathbb{C}^{m_1m_2m_3}$ such that the following holds.

(1) Any $\mathcal{T} = [t_{i_1i_2i_3}] \in \mathbb{C}^{m_1\times m_2\times m_3}\backslash U$ has rank $\operatorname{grank}(m_1, m_2, m_3)$ *at most.*
(2) Any $\mathcal{T} = [t_{i_1i_2i_3}] \in \mathbb{C}^{m_1\times m_2\times m_3}\backslash V$ has exactly rank $\operatorname{grank}(m_1, m_2, m_3)$.
(3) Let $\mathcal{T} = [t_{i_1i_2i_3}] \in \mathbb{C}^{m_1\times m_2\times m_3}\backslash W$. Then rank $\mathcal{T} = \operatorname{grank}(m_1, m_2, m_3)$. *Furthermore the set of all* $3\operatorname{grank}(m_1, m_2, m_3)$ *vectors*

$$\mathbf{x}_{l,1} \in \mathbb{C}^{m_1}, \mathbf{x}_{l,2} \in \mathbb{C}^{m_2}, \mathbf{x}_{l,3} \in \mathbb{C}^{m_3}, l = 1, \ldots, \operatorname{grank}(m_1, m_2, m_3)$$

satisfying the equality

$$\mathcal{T} = \sum_{l=1}^{\mathrm{grank}(m_1,m_2,m_3)} \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3} \tag{3.6}$$

is a union of $\deg \mathbf{f}_k$ *of pairwise disjoint varieties* $T_i(\mathcal{T}) \subsetneq (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{\mathrm{grank}(m_1,m_2,m_3)}$ *of dimension* $(m_1+m_2+m_3)\mathrm{grank}(m_1,m_2,m_3) - m_1m_2m_3$ *for* $i = 1, \ldots, \deg \mathbf{f}_k$. *View each rank one tensor* $\mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3}$ *as a point in* $(\mathbb{C}\backslash\{0\}) \times \mathbb{CP}^{m_1-1} \times \mathbb{CP}^{m_2-1} \times \mathbb{CP}^{m_3-1}$. *Then the set of all* $\mathrm{grank}(m_1,m_2,m_3)$ *rank one tensors*

$$(\mathbf{x}_{1,1} \otimes \mathbf{x}_{1,2} \otimes \mathbf{x}_{1,3}, \ldots, \mathbf{x}_{\mathrm{grank}(m_1,m_2,m_3),1} \otimes \mathbf{x}_{\mathrm{grank}(m_1,m_2,m_3),2} \otimes \mathbf{x}_{\mathrm{grank}(m_1,m_2,m_3),3})$$

in $(\mathbb{C}\backslash\{0\}) \times \mathbb{PC}^{m_1-1} \times \mathbb{PC}^{m_2-1} \times \mathbb{PC}^{m_3-1}$ *satisfying (3.6) is a disjoint union of* $\deg \mathbf{f}_k$ *varieties each of dimension* $(m_1+m_2+m_3-2)\mathrm{grank}(m_1,m_2,m_3) - m_1m_2m_3$.

Proof. (3.1) yields that

$$\begin{aligned} \mathrm{D}\mathbf{f}_k(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3}) = [&A_1(\mathbf{x}_{1,2}, \mathbf{x}_{1,3})|A_2(\mathbf{x}_{1,1}, \mathbf{x}_{1,3})|A_3(\mathbf{x}_{1,1}, \mathbf{x}_{1,2})| \ldots \\ &|A_1(\mathbf{x}_{k,2}, \mathbf{x}_{k,3})|A_2(\mathbf{x}_{k,1}, \mathbf{x}_{k,3})|A_3(\mathbf{x}_{k,1}, \mathbf{x}_{k,2})]. \end{aligned} \tag{3.7}$$

Moreover the column space of $\mathrm{D}\mathbf{f}_k$ is spanned by the vectors (3.5). As in the proof of the Proposition 3.1, generically the rank of $\mathrm{D}\mathbf{f}_k(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3})$ is equal to $r(k, m_1, m_2, m_3)$. (See Appendix A.1., top of page 495, for the definition of the term *generically*.) Thus, there exists a strict algebraic set $X_k \subsetneq (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k$ such rank $\mathrm{D}\mathbf{f}_k(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3}) = r(k, m_1, m_2, m_3)$ for any $(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3}) \in (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k \backslash X_k$ and rank $\mathrm{D}\mathbf{f}_k(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3}) < r(k, m_1, m_2, m_3)$ for any $(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3}) \in X_k$.

Let $k = 1$. Then Proposition 3.1 yields that generically rank $\mathrm{D}\mathbf{f}_1(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_{1,3}) = m_1+m_2+m_3-2$. Hence $f_1(\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})$ is a constructible algebraic set of dimension $m_1+m_2+m_3-2$. (In this case it is straightforward to show that $f_1(\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})$ is a variety.) If $m_1+m_2+m_3-2 = m_1m_2m_3$ then $f_1(\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3}) = \mathbb{C}^{m_1 \times m_2 \times m_3}$, $\mathrm{grank}(m_1, m_2, m_3) = 1$ and the theorem is trivial in this case. That is every tensor $\mathcal{T}$ is either rank one or rank zero tensor.

Assume now that $m_1m_2m_3 > m_1+m_2+m_3-2$. Then $f_1(\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3}) \subsetneq \mathbb{C}^{m_1 \times m_2 \times m_3}$ is a strict subvariety of tensors of rank 1 at most. Since $\mathbf{f}_k(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3}) = \mathbf{f}_{k+1}(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{k,3}, \mathbf{0}, \mathbf{0}, \mathbf{0})$, it follows

$$\begin{gathered} \mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) \subseteq \mathbf{f}_{k+1}((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k+1}), \ k = 1, \ldots \\ \text{and } \mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = \mathbb{C}^{m_1 \times m_2 \times m_3} \text{ for } k \geq m_1m_2m_3. \end{gathered} \tag{3.8}$$

In particular

$$\begin{aligned} &r(k, m_1, m_2, m_3), k = 1, \ldots \text{ a nondecreasing sequence,} \\ &r(\mathrm{grank}(m_1, m_2, m_3) - 1, m_1, m_2, m_3) < m_1m_2m_3, \\ &r(k, m_1, m_2, m_3) = m_1m_2m_3 \text{ for } k \geq \mathrm{grank}(m_1, m_2, m_3). \end{aligned} \tag{3.9}$$

So $1 < \mathrm{grank}(m_1, m_2, m_3) \leq m_1m_2m_3$. Furthermore, $Y_{\mathrm{grank}(m_1,m_2,m_3)-1}$ is a strict subvariety of $\mathbb{C}^{m_1 \times m_2 \times m_3}$. Since $\mathbb{C}^{m_1 \times m_2 \times m_3}$ is the only variety of dimension $m_1m_2m_3$ in $\mathbb{C}^{m_1 \times m_2 \times m_3}$ it follows that $Y_k = \mathbb{C}^{m_1 \times m_2 \times m_3}$ for $k \geq \mathrm{grank}(m_1, m_2, m_3)$.

Let $U := U_{\mathrm{grank}(m_1,m_2,m_3)}$ as defined in Definition 3.2. Then any $\mathcal{T} \in \mathbb{C}^{m_1 \times m_2 \times m_3} \backslash U$ is equal to some $\mathbf{f}_{\mathrm{grank}(m_1,m_2,m_3)}(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{\mathrm{grank}(m_1,m_2,m_3),3})$, i.e. $\mathcal{T}$ is of rank $\mathrm{grank}(m_1, m_2, m_3)$ at most. This proves *1*.

Let $V = U \cup Y_{\mathrm{grank}(m_1,m_2,m_3)-1}$. Then $\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}\backslash V$ has rank grank(m_1, m_2, m_3), i.e. *2* holds. Let $\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}\backslash V$. Then $\mathbf{f}^{-1}_{\mathrm{grank}(m_1,m_2,m_3)}(\mathcal{T})$ is a nonempty algebraic set of $(\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{\mathrm{grank}(m_1,m_2,m_3)}$. As stated in Appendix A.1., there exists a strict algebraic subset $W \subset \mathbb{C}^{m_1\times m_2\times m_3}$, which contains V, such that the first claim of *3* holds.

Recall that rank one tensor $\mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3}$ is a point in the manifold $(\mathbb{C}\backslash\{0\}) \times \mathbb{P}\mathbb{C}^{m_1-1} \times \mathbb{P}\mathbb{C}^{m_2-1} \times \mathbb{P}\mathbb{C}^{m_3-1}$ of dimension $m_1+m_2+m_3-2$. Hence $\mathbf{f}_k$ can be viewed as a map $\tilde{\mathbf{f}}_k : ((\mathbb{C}\backslash\{0\}) \times \mathbb{P}\mathbb{C}^{m_1} \times \mathbb{P}\mathbb{C}^{m_2} \times \mathbb{P}\mathbb{C}^{m_3})^k \to \mathbb{C}^{m_1\times m_2\times m_3}$. This interpretation of $\mathbf{f}_k$, combined with the first part of *3* yields the second part of *3*. $\square$

Definition 3.5

- The integer grank(m_1, m_2, m_3) is called the *generic rank* of $\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}$.
- k ($\le$ grank(m_1, m_2, m_3)) is called small if there is a rank k tensor $\mathcal{T}$ of the form (2.4) such that the Jacobian matrix at $\mathcal{T}$ has rank $k(m_1 + m_2 + m_3 - 2)$.
- k ($\ge$ grank(m_1, m_2, m_3)) is called big if there is a rank k tensor $\mathcal{T}$ of the form (2.4) such that the Jacobian matrix at $\mathcal{T}$ has rank equal to the maximal rank $m_1m_2m_3$.
- (m_1, m_2, m_3) is called perfect if $k = \frac{m_1m_2m_2}{m_1+m_2+m_3-2}$ is a small integer.

Corollary 3.6. brank $\mathcal{T} \le$ grank(m_1, m_2, m_3) *for any* $\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}$.

The generic rank grank(m_1, m_2, m_3) has the following interpretation. Assume that the entries of $\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}$ are independent random variables, with normal complex Gaussian distribution. Then with probability 1 the rank of $\mathcal{T}$ is grank(m_1, m_2, m_3). Furthermore, Proposition 3.3 yields that with probability 1 the border rank of $\mathcal{T}$ is also equal to grank(m_1, m_2, m_3).

Since the dimension of any algebraic variety is nonnegative the second part of *3* of Theorem 3.4 yields the well known result, e.g. [4, Chapter 20]:

Corollary 3.7. grank$(m_1, m_2, m_3) \ge \lceil \frac{m_1m_2m_3}{m_1+m_2+m_3-2} \rceil$.

The following result is known, e.g. [28, Proposition 2.3], and we give its proof for completeness.

Proposition 3.8. *Let* $m_1 \ge l_1, m_2 \ge l_2, m_3 \ge l_3$ *be positive integers. Then* grank$(m_1, m_2, m_3) \ge$ grank(l_1, l_2, l_3).

Proof. Since grank(m_1, m_2, m_3) is a symmetric function in m_1, m_2, m_3, it is enough to show that grank(m_1, m_2, m_3), $m_1 = 1, 2, \ldots$ is a nondecreasing sequence. Assume that $(T_{1,1}, \ldots, T_{l+1,1}) \in (\mathbb{C}^{m_2\times m_3})^{l+1}$ is a generic point. Then $(T_{1,1}, \ldots, T_{l,1}) \in (\mathbb{C}^{m_2\times m_3})^l$ is also a generic point. Theorem 2.4 implies that the minimal dimensions of subspaces spanned by rank one matrices containing span$(T_{1,1}, \ldots, T_{l+1,1})$, span$(T_{1,1}, \ldots, T_{l,1})$ are grank$(l + 1, m_2, m_3)$, grank(l, m_2, m_3). Hence grank$(l + 1, m_2, m_3) \ge$ grank(l, m_2, m_3). $\square$

Proposition 3.9. *Let* $l \ge 3, m \ge 4$ *be integers. Then* grank$(l, m, m) \ge m + 2$.

Proof. Fix $m \ge 4$ and let $\phi(t) = \frac{tm^2}{t+2m-2}$ be a function of $t > 0$. Then $\phi(t)$ is increasing. Hence for $t \ge 3$

$$\phi(t) \ge \phi(3) = \frac{3m^2}{2m+1} > m + 1 \text{ for } m \ge 4.$$

Therefore for $l \ge 3$, $m \ge 4$ grank$(l, m, m) \ge m + 2$. $\square$

As $\mathrm{grank}(3,3,3) = 5$, see (5.4) it follows that $\mathrm{grank}(l,m,m) \ge m+2$ for $l, m \ge 3$, which was shown in [33].

4. Maximal rank

Theorem 4.1. *Let m_1, m_2, m_3, k be three positive integers and assume that $\mathbf{f}_k$ is given by (3.4). Let $\mathrm{mrank}(m_1,m_2,m_3)$ be the smallest integer k such that equality holds in (3.8). I.e.*

$$\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = \mathbf{f}_{k+1}((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k+1}) \tag{4.1}$$

for $k = \mathrm{mrank}(m_1,m_2,m_3)$, and

$$\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) \subsetneq \mathbf{f}_{k+1}((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k+1}), \tag{4.2}$$

for $k < \mathrm{mrank}(m_1,m_2,m_3)$. Then the maximal rank of all 3-tensors in $\mathbb{C}^{m_1\times m_2\times m_3}$ is $\mathrm{mrank}(m_1,m_2,m_3)$, and

$$\mathrm{grank}(m_1,m_2,m_3) \le \mathrm{mrank}(m_1,m_2,m_3). \tag{4.3}$$

For each integer $k \in [1, \mathrm{mrank}(m_1,m_2,m_3)]$ the set of all tensors of rank k is a nonempty constructible algebraic set $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) \backslash \mathbf{f}_{k-1}((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k)$, ($\mathbf{f}_0((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^0) := \{\mathbf{0}\}$). If strict inequality in (4.3) holds then the set of all 3-tensors in $\mathbb{C}^{m_1\times m_2\times m_3}$ of rank greater than $\mathrm{grank}(m_1,m_2,m_3)$ is a constructible algebraic set of $\mathbb{C}^{m_1\times m_2\times m_3}$ of dimension $m_1m_2m_3 - 1$ at most. Furthermore for each nonnegative integer $k < \mathrm{grank}(m_1,m_2,m_3)$ the following holds:

$$\dim \mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) < \dim \mathbf{f}_{k+1}((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k+1}). \tag{4.4}$$

In particular for $k \le \mathrm{grank}(m_1,m_2,m_3)$ the dimension of the constructible algebraic set of all 3-tensor of rank k is

$$\dim \mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = r(k,m_1,m_2,m_3), \tag{4.5}$$

which is the rank of the Jacobian matrix $\mathrm{D}\mathbf{f}_k$ at the generic point $(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{k,3}) \in \mathbb{C}^{m_1\times m_2\times m_3}$, (which is also the maximal rank of $\mathrm{D}\mathbf{f}_k(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{k,3})$).

Proof. Assume the notation of Definition 3.2 for $k \ge 0$, where $Y_0 := \{\mathbf{0}\} \in \mathbb{C}^{m_1\times m_2\times m_3}$, $U_0 = \emptyset$. Suppose that (4.1) holds for $k = p$. Then any tensor of the form $\sum_{l=1}^{p+1} \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3}$ is of the form $\sum_{l=1}^{p} \mathbf{y}_{l,1} \otimes \mathbf{y}_{l,2} \otimes \mathbf{y}_{l,3}$. Hence the rank of any tensor is p at most. Thus (4.1) holds for any $k \ge p$. The second part of (3.8) yields $\mathrm{mrank}(m_1,m_2,m_3) \le m_1m_2m_3$, and $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = \mathbb{C}^{m_1\times m_2\times m_3}$ for $k = \mathrm{mrank}(m_1,m_2,m_3)$. Thus the rank of any 3-tensor is at most $\mathrm{mrank}(m_1,m_2,m_3)$. From the definition of $\mathrm{mrank}(m_1,m_2,m_3)$ we deduce (4.2). That is for each integer $k \in [1, \mathrm{mrank}(m_1,m_2,m_3)]$, $Z_k := (Y_k \backslash U_k) \backslash (Y_{k-1} \backslash U_{k-1})$ is the nonempty constructible algebraic set of rank k tensors.

From the definition of $q := \mathrm{grank}(m_1,m_2,m_3)$ we deduce that $Y_k = \mathbb{C}^{m_1\times m_2\times m_3}$ for $k \ge q$. Hence $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = \mathbb{C}^{m_1\times m_2\times m_3} \backslash U_k$, $k \ge q$, where each U_k for $k \ge q$ is a constructible algebraic set satisfying

$$U_q \supsetneq U_{q+1} \supsetneq \ldots \supsetneq U_{\mathrm{mrank}(m_1,m_2,m_3)} = \emptyset.$$

(Note that $U_k = \emptyset$ for $k > \mathrm{mrank}(m_1,m_2,m_3)$.)

We now show (4.4) for $k < q$. Definition 3.2 implies the equality (4.5). Assume to the contrary that $r(k,m_1,m_2,m_3) = r(k+1,m_1,m_2,m_3)$ for some integer $k \in [1, q-1]$. Let s be the smallest positive integer satisfying this condition. Then there exists an algebraic set $X_s \subsetneq (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^s$ such that rank $\mathrm{D}\mathbf{f}_s(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{s,3}) = r(s,m_1,m_2,m_3)$ for any $(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{s,3}) \in (\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^s\backslash X_s$. I.e., $s(m_1+m_2+m_3)$ tensors given in (3.5) span $r(s,m_1,m_2,m_3)$ dimensional subspace in $\mathbb{C}^{m_1\times m_2\times m_3}$ for any $(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{s,3}) \in (\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^s\backslash X_s$.

Let $(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{s+1,3}) \in (\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^{s+1}$. Then

$$\mathrm{rank}\ \mathrm{D}\mathbf{f}_{s+1}(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{s+1,3}) \le r(s+1,m_1,m_2,m_3) = r(s,m_1,m_2,m_3).$$

I.e., $(s+1)(m_1+m_2+m_3)$ tensor given in (3.5) span at most $r(s,m_1,m_2,m_3)$ dimensional subspace in $\mathbb{C}^{m_1\times m_2\times m_3}$. Assume that $(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{s,3}) \in (\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^s\backslash X_s$. Then $(s+1)(m_1+m_2+m_3)$ tensor given in (3.5) span exactly $r(s,m_1,m_2,m_3)$ dimensional subspace in $\mathbb{C}^{m_1\times m_2\times m_3}$. Moreover, the $s(m_1+m_2+m_3)$ tensor given by (3.5) for $k=s$ span the above subspace. Hence the tensors

$$\begin{aligned}&\mathbf{e}_{i_1,1}\otimes\mathbf{x}_{s+1,2}\otimes\mathbf{x}_{s+1,3},\quad \mathbf{x}_{s+1,1}\otimes\mathbf{e}_{i_2,2}\otimes\mathbf{x}_{s+1,3},\\ &\mathbf{x}_{s+1,1}\otimes\mathbf{x}_{s+1,2}\otimes\mathbf{e}_{i_3,3},\qquad i_j=1,\ldots,m_j,\ j=1,2,3,\end{aligned}$$

are spanned by $s(m_1+m_2+m_3)$ tensor given by (3.5) for $k=s$.

Let $k>s+1$ and consider rank $\mathrm{D}\mathbf{f}_k(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{k,3})$, which is equal to the dimension of the subspace spanned by $k(m_1+m_2+m_3)$ tensor given by (3.5). Let $(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{s,3}) \in (\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^s\backslash X_s$. Then the above arguments show that rank $\mathrm{D}\mathbf{f}_k(\mathbf{x}_{1,1},\ldots,\mathbf{x}_{k,3}) = r(s,m_1,m_2,m_3)$. Since $X_s\times(\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^{k-s}$ is an algebraic set of $(\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^k$ it follows that $r(k,m_1,m_2,m_3) = r(s,m_1,m_2,m_3)$. This is impossible, since $r(s,m_1,m_2,m_3) < m_1m_2m_3 = r(q,m_1,m_2,m_3)$. Hence (4.4) holds for $k<q$. $\square$

Combine the arguments of the proof of Theorem 3.4 with the results in Appendix A.1. to obtain.

Theorem 4.2. *Let m_1, m_2, m_3, k be three positive integers and assume that $\mathbf{f}_k$ is given by (3.4). Suppose that $k \le \mathrm{grank}(m_1,m_2,m_3)$. Let $\mathcal{T} \in \mathbb{C}^{m_1\times m_2\times m_3}$ be a generic tensor of rank k, i.e. a generic point in $\mathbf{f}_k((\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^k) \subset \mathbb{C}^{m_1\times m_2\times m_3}$. Then the set of all possible decompositions of $\mathcal{T}$ as a sum of k rank one tensors is a disjoint union of $\deg f_k$ varieties of dimension $k(m_1+m_2+m_3-2)-r(k,m_1,m_2,m_3)$. In particular, if $r(k,m_1,m_2,m_3) = k(m_1+m_2+m_3-2)$, i.e. k is small, then $\mathcal{T}$ can be decomposed as a sum of k-rank tensors in a finite number of ways given by a number $N(k,m_1,m_2,m_3) = \deg \mathbf{f}_k$.*

We remark that in the case $r(k,m_1,m_2,m_3) = k(m_1+m_2+m_3-2)$ the positive integer $N(k,m_1,m_2,m_3)$ is divisible by $k!$, since we can permute the k summands in (2.4). If $N(k,m_1,m_2,m_3) = k!$, this means that a generic rank k tensor $\mathcal{T}$ has a unique decomposition to k factors. As we can see later,the numerical evidence points out that the equality $r(k,m_1,m_2,m_3) = k(m_1+m_2+m_3-2)$ occurs for many $k < \mathrm{grank}(m_1,m_2,m_3)$.

5. Known theoretical results

The following results are known. See the references below.

$$\mathrm{grank}(m_1,m_2,m_3) = \min(m_3, m_1m_2) \text{ if } m_3 \ge (m_1-1)(m_2-1)+1, \tag{5.1}$$

$$\text{in particular } \mathrm{grank}(2,m_2,m_3) = \min(m_3, 2m_2) \text{ if } 2 \le m_2 \le m_3, \tag{5.2}$$

$$\mathrm{grank}(3,2p,2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \text{ and } \left\lfloor \frac{12p^2}{4p+1} \right\rfloor \text{ is small,} \tag{5.3}$$

$$\mathrm{grank}(3,2p+1,2p+1) = \left\lceil \frac{3(2p+1)^2}{4p+3} \right\rceil + 1, \tag{5.4}$$

$$(n,n,n+2) \text{ is perfect for } n \ne 2 \ (\mathrm{mod}\ 3), \tag{5.5}$$

$$(n-1, n, n) \text{ is perfect for } n = 0 \,(\mathrm{mod}\ 3), \tag{5.6}$$

$$\mathrm{grank}(4, m, m) = \left\lceil \frac{4m^2}{2m+2} \right\rceil, \tag{5.7}$$

$$\mathrm{grank}(n, n, n) = \left\lceil \frac{n^3}{3n-2} \right\rceil \text{ and } \left\lfloor \frac{n^3}{3n-2} \right\rfloor \text{ is small for } n \neq 3, \tag{5.8}$$

$$(m_1, 2m_2', 2m_3') \text{ perfect if } \frac{2m_1 m_2'}{m_1 + 2m_2' + 2m_3' - 2} \text{ is integer }, \tag{5.9}$$

where (2.2) holds.

See [6] for (5.1), [28] for (5.3–5.6), [4] for (5.7), [23] and [1, Theorem 5.3] for (5.8–5.9). Note that in view of (5.1)

$$(m_1, m_2, (m_1-1)(m_2-1)+1) \text{ is perfect.} \tag{5.10}$$

We bring another proof of (5.1) using matrices in Section 6. It was conjectured in [10].

Conjecture 5.1. Let $3 \le m_1 \le m_2 \le m_3 \le (m_1-1)(m_2-1)$ and $(m_1, m_2, m_3) \neq (3, 2p+1, 2p+1)$, $p \in \mathbb{N}$. Then $\mathrm{grank}(m_1, m_2, m_3) = \lceil \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rceil$.

Combine Corollary 3.7, Proposition 3.8 and (5.10) to deduce.

$$\mathrm{grank}(m_1, m_2, m_3) = \left\lceil \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \right\rceil = \mathrm{grank}(m_1, m_2, m_3+1) \quad \text{for } m_3 = (m_1-1)(m_2-1) \text{ and } 3 \le m_1, m_2. \tag{5.11}$$

I.e., the above conjecture holds for $m_3 = (m_1-1)(m_2-1)$. A more precise version of Conjecture 5.1 is

Conjecture 5.2. . Let the assumptions of Conjecture 5.1 hold. Then any integer $k \in [2, \lceil \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rceil - 1]$ is small.

We call (m_1, m_2, m_3) *regular* if (m_1, m_2, m_3) satisfies Conjecture 5.1 and $\lfloor \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rfloor$ is small.

We verified numerically[1] the above two conjectures for $m_1 \le m_2 \le m_3 \le 14$ as follows. We chose at random $k \in [2, \lceil \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rceil]$ vectors $\mathbf{x}_{l,i} \in (\mathbb{Z} \cap [-99, 99])^{m_i}$, $i = 1, 2, 3$, $l = 1, \ldots, k$ such that the rank of the Jacobian matrix at the corresponding rank k tensor

$$\mathcal{T} = \sum_{l=1}^{k} \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3} \tag{5.12}$$

was $\min(k(m_1+m_2+m_3-2), m_1 m_2 m_3)$. See also [7] for numerical results.

The values of $\mathrm{mrank}(m_1, m_2, m_3)$ are much harder to compute. The following results are known. First, [21, p. 10], (see also [18]),

$$\mathrm{mrank}(2, m, n) = m + \min\left(m, \left\lfloor \frac{n}{2} \right\rfloor\right) \text{ for } 2 \le m \le n. \tag{5.13}$$

Second, it is claimed in [25] that

$$\mathrm{mrank}(3, 3, 3) = 5 \tag{5.14}$$

1 I thank M. Tamura for programming the software to compute $\mathrm{grank}(m_1, m_2, m_3)$ and $r(k, m_1, m_2, m_3)$.

6. Matrices and the rank of 3-tensors

In this section we use known results for matrices to find estimates on the generic and maximal rank of tensors.

Proposition 6.1. *Let* $\mathbf{U}_i$ *be* m_i*-dimensional vector space over* $\mathbb{F}$*, for* $i = 1, 2, 3$*. Then* $\mathrm{mrank}(m_1, m_2, m_3) = m_1 m_2$ *for* $m_1 m_2 \le m_3$*. More precisely, let* $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$ *be given by (2.3). Let* R_1, R_2, R_3 *be defined as in Proposition 2.2. Assume that* $R_3 = R_1 R_2$*. Then* $\mathrm{rank}\, \tau = R_1 R_2$.

Proof. Since $\mathbb{F}^{m_1 \times m_2}$ is spanned by $m_1 m_2$ rank one matrices, Theorem 2.4 yields that $\mathrm{mrank}(m_1, m_2, m_3) \le m_1 m_2$. Choose τ represented by (2.3), such that $T_{1,3}, \dots, T_{m_3,3} \in \mathbb{F}^{m_1 \times m_2}$ span $\mathbb{F}^{m_1 \times m_2}$. Theorem 2.4 yields that $\mathrm{rank}\, \tau = m_1 m_2$, i.e. $\mathrm{mrank}(m_1, m_2, m_3) = m_1 m_2$. The second part of the proposition follows from Proposition 2.2. $\square$

(The above results in this section hold for any field $\mathbb{F}$. We remind the reader that from now and on $\mathbb{F} = \mathbb{R}, \mathbb{C}$.) We now show how to deduce (5.1) using matrices. For a finite dimensional vector space $\mathbf{U}$ over $\mathbb{F}$ of dimension N denote by $\mathrm{Gr}(k, \mathbf{U})$, the k-Grassmannian, the manifold of all k dimensional subspaces of $\mathbf{U}$. ($k \in [0, N]$.) Note that $\mathrm{Gr}(1, \mathbb{F}^{m \times n})$ can be identified with $\mathbb{PF}^{mn-1}$, a the projective space of dimension $mn - 1$. Equivalently, if $0_{m \times n} \ne A \in \mathbb{F}^{m \times n}$, then $\hat{A} \in \mathrm{Gr}(1, \mathbb{F}^{m \times n})$ corresponds to all points tA, $t \in \mathbb{F} \backslash \{0\}$. Note that $\mathrm{rank}\, A = \mathrm{rank}\, tA$ for any $t \in \mathbb{F} \backslash \{0\}$. Thus we define $\mathrm{rank}\, \hat{A} := \mathrm{rank}\, A$. Usually we will identify $\hat{A} \in \mathrm{Gr}(1, \mathbb{F}^{m \times n})$ with one of $tA \in \mathbb{F}^{m \times n} \backslash \{0\}$ and no ambiguity will arise.

Let $L \subseteq \mathbb{F}^{m \times n}$ be a subspace of dimension $d \ge 1$. Then $\mathrm{proj}\, L \subset \mathrm{Gr}(1, \mathbb{F}^{m \times n})$, the set of all one dimensional subspaces in L. The dimension of $\mathrm{proj}\, L$ is $d - 1$ and $\mathrm{proj}\, L$ can be identifies with $\mathbb{PF}^{d-1}$. $\mathrm{proj}\, L$ is called a *linear space* in $\mathrm{proj}\, \mathbb{F}^{m \times n}$. The following result is known [16,11].

Theorem 6.2. *Let* $\mathrm{U}_{k,m,n}(\mathbb{F}) \subseteq \mathbb{F}^{m \times n}$ *be the set of all* $m \times n$ *matrices of rank* k *at most. Then* $\mathrm{U}_{k,m,n}(\mathbb{F})$ *is an irreducible variety of dimension* $k(m+n-k)$*. Furthermore,* $\mathrm{U}_{k,m,n}(\mathbb{F}) \backslash \mathrm{U}_{k-1,m,n}(\mathbb{F})$ *is quasi-projective variety of all matrices of rank* k *exactly, which is a manifold of dimension* $k(m + n - k)$.

Any complex subspace of $L \subset \mathbb{C}^{m \times n}$ *of dimension* $(m - k)(n - k) + 1$ *contains a nonzero matrix of rank* k *at most. More precisely, for a generic subspace* $L \subset \mathbb{C}^{m \times n}$ *of dimension* $(m - k)(n - k) + 1$*, the linear space* $\mathrm{proj}\, L$ *contains exactly*

$$\gamma_{k,m,n} := \prod_{j=0}^{n-k-1} \frac{\binom{m+j}{m-k}}{\binom{m-k+j}{m-k}} = \prod_{j=0}^{n-k-1} \frac{(m+j)!\, j!}{(k+j)!\, (m-k+j)!}, \tag{6.1}$$

distinct matrices of rank k *exactly.*

Theorem 6.3. *Let* $2 \le m, n$ *and* $d \in [(m - 1)(n - 1) + 1, mn - 1]$ *be fixed integers. Then a generic subspace* $L \subset \mathbb{C}^{m \times n}$ *of dimension* d *is spanned by rank one matrices.*

Proof. We first consider the case $d = (m - 1)(n - 1) + 1$. It is not difficult to check that $d \le \gamma_{1,m,n}$. Let L be a generic subspace L of dimension $(m-1)(n-1)+1$ Then $L \cap U_{k,m,n}(\mathbb{F}) = \{A_1, \dots, A_{\gamma_{1,m,n}}\}$ be a set of $\gamma_{1,m,n}$ distinct matrices. We show that for a generic L $A_1, \dots, A_d$ are linearly independent. Otherwise, for any subspace L of dimension d any d rank one matrices in L must be linearly dependent. (This follows from the fact that linear dependence of d matrices can be stated in terms of polynomial equations in the entries of $A_1, \dots, A_d$.) To show that the last condition does not always hold, choose d linearly independent rank one matrices, and let L be the subspace spanned by these matrices.

Assume now that L is a generic subspace of dimension $d \in [(m - 1)(n - 1) + 2, mn - 1]$. Then $L \cap U_{k,m,n}(\mathbb{F})$ is a variety of dimension $d - (m - 1)(n - 1) - 1$. Similar arguments show that any d generic matrices in $L \cap U_{k,m,n}(\mathbb{F})$ are linearly independent. $\square$

Corollary 6.4

(1) (5.1) holds.
(2) $\mathrm{grank}(m_1, m_2, (m_1-1)(m_2-1)) = (m_1-1)(m_2-1)+1$ *for* $m_1, m_2 \geq 2$*, i.e. (5.11) holds.*

Proof. In view of Proposition 6.1 we discuss first the case $m_3 \in [(m_1-1)(m_2-1)+1, m_1m_2-1]$. View a generic $\mathcal{T} = [t_{ijk}] \in \mathbb{C}^{m_1\times m_2\times m_3}$ as m_3 generic matrices $A_k = [t_{ijk}]_{i=j=1}^{m_1,m_2} \in \mathbb{C}^{m_1\times m_2}$ for $k = 1,\dots,m_3$. Hence $L = \mathrm{span}(A_1,\dots,A_{m_3})$ is a generic subspace of dimension m_3. Theorem 6.3 yields that L is spanned by rank one matrices. Theorem 2.4 yields that $\mathrm{grank}(m_1, m_2, m_3) = m_3$.

Assume now that $m_3 = (m_1-1)(m_2-1)$ and $\mathcal{T} = [t_{ijk}] \in \mathbb{C}^{m_1\times m_2\times m_3}$ be a generic tensor. Let $L \subset \mathbb{C}^{m_1\times m_2}$ be the generic subspace defined as above. Theorem 6.2 yields that L is not spanned by rank one matrices. Hence the minimal dimension of a subspace spanned by rank one matrices containing L is at least m_3+1. Let $X \in \mathbb{C}^{m_1\times m_2}$ be a generic matrix. Then $L_1 = \mathrm{span}(L, X)$ is a generic subspace of dimension $(m_1-1)(m_2-1)+1$. Hence L_1 is spanned by rank one matrices. Therefore $\mathrm{rank}\ \mathcal{T} = m_3+1$. $\square$

Corollary 6.5. $\mathrm{grank}(2, m_2, m_3) = \min(m_3, 2m_2)$ *for* $2 \leq m_2 \leq m_3$.

We now show how to apply the above results to obtain upper estimates of $\mathrm{grank}(m_1, m_2, m_3)$ and $\mathrm{mrank}(m_1, m_2, m_3)$. Let us start with the case $m_2 = m_3 \geq 3$.

Theorem 6.6. *Let* $m, n \geq 3$ *be integers. Then*

$$\mathrm{grank}(n, m, m) \leq \left\lfloor \frac{n}{2} \right\rfloor m + \left(n - 2\left\lfloor\frac{n}{2}\right\rfloor\right)(m - \lfloor\sqrt{n-1}\rfloor) \text{ if } m \geq 2\lfloor\sqrt{n-1}\rfloor \tag{6.2}$$

$$\mathrm{grank}(n, m, m) \leq n(m - \lfloor\sqrt{n-1}\rfloor) \text{ if } m < 2\lfloor\sqrt{n-1}\rfloor < 2(m-1), \tag{6.3}$$

$$\mathrm{grank}(n, m, m) = \min(n, m^2) \text{ if } n \geq (m-1)^2+1, \tag{6.4}$$

$$\mathrm{mrank}(n, m, m) \leq \sum_{i=1}^{\lfloor\sqrt{n-1}\rfloor} (2i-1)(m-i+1) + (m - \lfloor\sqrt{n-1}\rfloor^2)(m - \lfloor\sqrt{n-1}\rfloor). \tag{6.5}$$

Proof. We first discuss the $\mathrm{grank}(n, m, m)$. Clearly, (6.4) is implied by Corollary 6.4.

Assume now that $n < (m-1)^2+1$, i.e. $2\lfloor\sqrt{n-1}\rfloor < 2(m-1)$. Let $\tau \in \mathbb{C}^{n\times m\times m}$ be a tensor of the form (2.3). Assume that $(T_{1,1} = [t_{1jk}],\dots,T_{n,1} = [t_{njk}]) \in (\mathbb{C}^{m\times m})^n$ is a generic point. Let $l = \lfloor\sqrt{n-1}\rfloor$. So $n \geq l^2+1$. Theorem 6.2 yields that $\mathrm{span}(T_{1,1},\dots,T_{n,1})$ contains at least $\gamma_{m-l,m,m}$ distinct matrices of rank $m-l$. It is straightforward to show that $\gamma_{m-l,m,m} \geq n$. Since $(T_{1,1},\dots,T_{n,1})$ was a generic point we may assume $\mathrm{span}(T_{1,1},\dots,T_{n,1})$ contain n linearly independent rank $m-l$ matrices $Q_1,\dots,Q_n$. (See the proof of Theorem 6.3.) This gives the inequality (6.3) for all $n < (m-1)^2+1$.

Since $T_{1,1},\dots,T_{n,1}$ are generic, we can assume that $T_{2i-1,1}$ is invertible and $T_{2i-1,1}^{-1}T_{2i,1}$ is diagonable. Hence $T_{2i-1,1}, T_{2i,1}$ are contained in a subspace spanned by m rank one matrices. If n is even we obtain that $\mathrm{span}(T_{1,1},\dots,T_{n,1})$ are contained in $\frac{n}{2}m$ dimensional subspace spanned by rank one matrices. Theorem 2.4 yields the inequality (6.2). If n is odd, we can assume that $Q_1 = T_{n,1} - \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} T_{i,1}$ has at rank $m - \lfloor\sqrt{n-1}\rfloor$. Hence, we deduce (6.2) in this case too.

We now prove the inequality (6.5). We assume the worst case which will give the upper bound. So it is enough to consider the case where $T_{1,1}, T_{2,1},\dots,T_{n,1}$ linearly independent. Now we choose a new base $S_1,\dots,S_n$ in $\mathrm{span}(T_{1,1},\dots,T_{n,1})$ such that $\mathrm{rank}\ S_1 \geq \mathrm{rank}\ S_2 \geq \dots \geq \mathrm{rank}\ S_n$. So the worst case is $\mathrm{rank}\ S_1 = m$. Since any 2 dimensional space contains a singular matrix we can assume that $\mathrm{rank}\ S_i \leq m-1$ for $i = 2, 3, 4$. According to Theorem 6.2 any 5 dimensional vector space contains a nonzero matrix of rank $m-2$ at most. Hence $\mathrm{rank}\ S_i \leq m-2$ for $i = 5, 6, 7, 8, 9$. Theorem 6.2 implies that any subspace of dimension 10 contains a nonzero matrix of rank $m-3$. Hence rank $S_i \le m-3$ for $i = 10, \dots,$. Continuing the use of Theorem 6.2, and combing it with Theorem 2.4 we deduce (6.5). $\square$

Use Corollary 3.7, Proposition 3.8 and the above theorem to deduce:

Corollary 6.7

$$\begin{aligned}
4 \le \mathrm{grank}(3,3,3) \le 5 = 1\cdot 3+2, &\quad \mathrm{mrank}(3,3,3) \le 7 = 3+2+2,\\
\mathrm{grank}(4,3,3) = 5\ (4 = (3-1)^2), &\quad \mathrm{mrank}(4,3,3) \le 9 = 3+2+2+2,\\
\mathrm{grank}(5,3,3) = 5\ (5 > (3-1)^2), &\quad \mathrm{mrank}(5,3,3) \le 10 = 3+2+2+2+1,\\
6 \le \mathrm{grank}(3,4,4) \le 7 = 1\cdot 4+3, &\quad \mathrm{mrank}(3,4,4) \le 10 = 4+3+3,\\
7 \le \mathrm{grank}(4,4,4) \le 8 = 2\cdot 4, &\quad \mathrm{mrank}(4,4,4) \le 13 = 4+3+3+3,\\
8 \le \mathrm{grank}(5,4,4) \le 10 = 2\cdot 4+2, &\quad \mathrm{mrank}(5,4,4) \le 15 = 4+3+3+3+2,\\
7 \le \mathrm{grank}(3,5,5) \le 9 = 1\cdot 5+4, &\quad \mathrm{mrank}(3,5,5) \le 13 = 5+4+4,\\
9 \le \mathrm{grank}(4,5,5) \le 10 = 2\cdot 5, &\quad \mathrm{mrank}(4,5,5) \le 17 = 5+4+4+4,\\
10 \le \mathrm{grank}(5,5,5) \le 13 = 2\cdot 5+3, &\quad \mathrm{mrank}(5,5,5) \le 20 = 5+4+4+4+3.
\end{aligned}$$

Recall that in all the examples of $\mathrm{grank}(n, m, m)$ given by Corollary 6.7 we know that $\mathrm{grank}(3, 3, 3) = 5$, $\mathrm{grank}(3, 5, 5,) = 8$, while all other values of $\mathrm{grank}(n, m, m)$ are given by the lower bound. It is claimed that $\mathrm{mrank}(3, 3, 3) = 5$ [25].

Note that if n is even and $m \gg n$ then the upper bound (6.2) combined with Corollary 3.7 implies that $\mathrm{grank}(n, m, m)$ is of order $\frac{nm}{2}$. However if $n = O(m^{1+a})$ for $a \in (0, 1]$ then the upper bounds (6.2) and (6.3) are not of the right order, (which is m^2).

7. Typical ranks of real 3-tensors

The study of the rank of a real 3-tensor is closely related to the real semi-algebraic geometry. See Appendix A.2. for the results in semi-algebraic geometry needed here.

Theorem 7.1. *The space $\mathbb{R}^{m_1\times m_2\times m_3}$, $m_1, m_2, m_3 \in \mathbb{N}$, contains a finite number of open connected disjoint semi-algebraic sets $O_1, \dots, O_M$ satisfying the following properties.*

(1) $\mathbb{R}^{m_1\times m_2\times m_3} \setminus \cup_{i=1}^M O_i$ *is a closed semi-algebraic set* $\mathbb{R}^{m_1\times m_2\times m_3}$ *of dimension strictly less than* $m_1m_2m_3$.
(2) Each $\mathcal{T} \in O_i$ *has rank* r_i *for* $i = 1, \dots, M$.
(3) $\min(r_1, \dots, r_M) = \mathrm{grank}(m_1, m_2, m_3)$.
(4) $\mathrm{mtrank}(m_1, m_2, m_3) := \max(r_1, \dots, r_M)$ *is the minimal* $k \in \mathbb{N}$ *such that the closure of* $\mathbf{f}_k((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^k)$ *is equal to* $\mathbb{R}^{m_1\times m_2\times m_3}$.
(5) For each integer $r \in [\mathrm{grank}(m_1, m_2, m_3), \mathrm{mtrank}(m_1, m_2, m_3)]$ *there exists* $r_i = r$ *for some integer* $i \in [1, M]$.

Proof. Consider the polynomial map $\mathbf{f}_k : (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k \to \mathbb{C}^{m_1\times m_2\times m_3}$ be given by (3.4). Note that $\mathbf{f}_k : (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^k \to \mathbb{R}^{m_1\times m_2\times m_3}$. Denote by Y_k and Q_k the closure of $\mathbf{f}_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k)$ and $Z_k := \mathbf{f}_k((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^k)$, respectively. Clearly,

$$Y_i \subseteq Y_{i+1},\ Q_i \subseteq Q_{i+1} \text{ for } i \in \mathbb{N},\ Y_{m_1m_2m_3} = \mathbb{C}^{m_1\times m_2\times m_3},\ Q_{m_1m_2m_3} = \mathbb{R}^{m_1\times m_2\times m_3}.$$

Let $\mathrm{mtrank}(m_1, m_2, m_3)$ be the smallest k such that $Q_k = \mathbb{R}^{m_1\times m_2\times m_3}$.

Let $q = \mathrm{grank}(m_1, m_2, m_3)$. Then Y_{q-1} is a strict complex subvariety of $\mathbb{C}^{m_1 \times m_2 \times m_3}$. (See Definition 3.2.) In particular $Y_{q-1}^{\mathbb{R}} = Y_{q-1} \cap \mathbb{R}^{m_1 \times m_2 \times m_3}$ is a strict real subvariety of $\mathbb{R}^{m_1 \times m_2 \times m_3}$. Hence $Q_{q-1} \subseteq Y_{q-1}^{\mathbb{R}}$ is a semi-algebraic of dimension $\dim Y_{q-1}$ at most, which is strictly less than $m_1 m_2 m_3 - 1$. In particular

$$\mathrm{mtrank}(m_1, m_2, m_3) \geq \mathrm{grank}(m_1, m_2, m_3). \tag{7.1}$$

From the proof of Theorem 4.1 it follows that there exists an algebraic subset $X_q \subset (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^q$ such that rank $\mathrm{D}\mathbf{f}_q$ is $m_1 m_2 m_3$ at each point of $(\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^q \backslash X_q$. Then $X_q^{\mathbb{R}} = X_q \cap (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^q$ is a real algebraic set of $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^q$. Thus the Jacobian of the real map $\mathbf{f}_q : (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^k \to \mathbb{R}^{m_1 \times m_2 \times m_3}$ has rank $m_1 m_2 m_3$ at each point of the open semi-algebraic set $P_q := (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^q \backslash X_q^{\mathbb{R}}$. Hence $\mathbf{f}_q(P_q)$ is an open semi-algebraic set in $\mathbb{R}^{m_1 \times m_2 \times m_3}$. Therefore $\mathbf{f}_q(P_q) \backslash Y_{q-1}^{\mathbb{R}}$ is an open semi-algebraic set in $\mathbb{R}^{m_1 \times m_2 \times m_3}$. Clearly $Q_q \backslash Q_{q-1} \supseteq Q_q \backslash Y_{q-1}^{\mathbb{R}} \supseteq \mathbf{f}_q(P_q) \backslash Y_{q-1}^{\mathbb{R}}$. Hence the interior of $Q_q \backslash Q_{q-1}$, denoted as int $(Q_q \backslash Q_{q-1})$ is an open semi-algebraic set, which consists of tensors of rank q exactly. The theory of semi-algebraic sets implies that int $(Q_q \setminus Q_{q-1}) = \cup_{i=1}^{M_1} O_i$, where each O_i is an open semi-algebraic set. Observe next that the semi-algebraic set $(Q_q \setminus Q_{q-1}) \setminus$ int $(Q_q \setminus Q_{q-1})$ has dimension $m_1 m_2 m_3 - 1$ at most. Since $\dim Q_{q-1} \leq m_1 m_2 m_3 - 1$ we deduce that

$$\dim Q_q \setminus \mathrm{Closure}(\cup_{i=1}^{M_1} O_i) \leq m_1 m_2 m_3 - 1. \tag{7.2}$$

Suppose $Q_q = \mathbb{R}^{m_1 \times m_2 \times m_3}$, i.e. equality holds in (7.1), so $M = M_1$. We claim that $W_q := \mathbb{R}^{m_1 \times m_2 \times m_3} \setminus \mathrm{Closure}(\cup_{i=1}^{M_1} O_i)$ is an empty set. Otherwise W_q is a nonempty open semi-algebraic set. Hence $\dim W_q = m_1 m_2 m_3$ which contradicts (7.2). The proof of the theorem is completed in this case.

Assume now that $Q_q \subsetneq \mathbb{R}^{m_1 \times m_2 \times m_3}$. Recall that $\dim \mathrm{Closure}(S) \setminus S < \dim S$ for any semi-algebraic set. Hence $\dim Q_{q+1} = \dim Z_{q+1}$. We claim that $\dim(Z_{q+1} \setminus Q_q) = m_1 m_2 m_3$, i.e. the interior of $Z_{q+1} \setminus Q_q$ contains an open set. Assume to the contrary that $\dim(Z_{q+1} \setminus Q_q) < m_1 m_2 m_3$. Hence $\dim(Z_{q+1} \setminus Z_q) < m_1 m_2 m_3$. ($\dim Q_q \setminus Z_q < \dim Z_q = m_1 m_2 m_3$.) So a sum of generic $q + 1$ real rank one tensors is a sum of generic q real rank one tensors. Hence a sum of generic $m_1 m_3 m_3$ rank one tensors is a sum of q generic rank one tensors. So $Q_q = \mathbb{R}^{m_1 \times m_2 \times m_3}$, which contradicts our assumption. Thus, the interior of $Q_{q+1} \setminus Q_q$ is an open semi-algebraic set, which is a union of disjoint open connected semi-algebraic sets $O_{M_1+1}, \ldots, O_{M_2}$. Note that the rank $\mathcal{T} \in O_j$ is $\mathrm{grank}(m_1, m_2, m_3) + 1$ for $j = M_1 + 1, \ldots, M_2$. Continue in this manner we deduce the rest of the theorem. □

Definition 7.2. Let r be a positive integer. $\mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ has a border rank r, denoted as brank $\mathcal{T}$, if $\mathcal{T} \in \mathrm{Closure}\ \mathbf{f}_r((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^r) \setminus \mathrm{Closure}\ \mathbf{f}_{r-1}((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^{r-1})$. ($\mathbf{f}_0((\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})^0 = \{0\}$.) r is called an (m_1, m_2, m_3) *typical* rank, or simply typical rank, if $r \in [\mathrm{grank}(m_1, m_2, m_3), \mathrm{mtrank}(m_1, m_2, m_3)]$.

The proof of Theorem 7.1 yields.

Corollary 7.3. *Assume that the entries of* $\mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ *are independent random variables with standard normal Gaussian distribution. Then the probability that* rank $\mathcal{T} = r$ *is positive if and only if* r *is a typical rank. Assume that* r *is a typical rank. Then the probability that* rank $\mathcal{T} >$ brank $\mathcal{T}$, *provided that* (rank $\mathcal{T} - r$)(brank $\mathcal{T} - r$) $= 0$, *is* 0. *In particular, the probability that* rank $\mathcal{T} = \mathrm{grank}(m_1, m_2, m_3)$ *is positive.*

The last part of this Corollary is shown in [33, Appendix B] for $m_1 = m_2 = 4, m_3 = 3$. For $l = 2 \leq m \leq n$ the following is known: $\mathrm{mtrank}(2, m, m) = \mathrm{grank}(2, m, m) + 1 = m + 1$ [29] and $\mathrm{mtrank}(2, m, n) = \mathrm{grank}(2, m, n) = \min(n, 2m)$ for $m < n$ [32]. [25] claims that $\mathrm{mtrank}(3, 3, 3) = \mathrm{grank}(3, 3, 3) = 5$. It is shown in [31] that $\mathrm{mtrank}(3, 3, 5) = \mathrm{grank}(3, 3, 5) + 1 = 6$. For other additional known results for typical rank see [7]. In particular, $\mathrm{mtrank}(4, 4, 12) = \mathrm{grank}(4, 4, 12) +$

$1 = 12$ [7, Table I]. We now give additional examples, where a strict inequality holds in (7.1). All of them, except the above mentioned examples, are new.

Theorem 7.4. *In the following cases* $\mathrm{mtrank}(m_1, m_2, m_3) > \mathrm{grank}(m_1, m_2, m_3)$.

(1) $m_1 = m_2 = m \geq 2, m_3 = (m-1)^2 + 1$.
(2) $m_1 = m_2 = 4, m_3 = 11, 12$.

We do not know if $\mathrm{mtrank}(m, m, (m-1)^2 + 1) = \mathrm{grank}(m, m, (m-1)^2 + 1) + 1$ for $m \geq 4$. To prove Theorem 7.4 we need a few auxiliary results. The following result is known, e.g. [11, Proposition 5.2].

Proposition 7.5. *Let* $\mathbb{F} = \mathbb{C}, \mathbb{R}$, $n \geq 2$, $p \geq 1$ *be integers and assume that* $p \leq \lfloor \frac{n}{2} \rfloor$. *Let* $\mathrm{proj}\ \mathrm{A}_n(\mathbb{F}) \supseteq \mathrm{proj}\ \mathrm{W}_{2p,n}(\mathbb{F})$ *be the projective variety of all (nonzero) skew symmetric matrices and the projective subvariety of all skew symmetric matrices of rank* $2p$ *at most, respectively. Then* $\mathrm{proj}\ \mathrm{W}_{2p,n}(\mathbb{F})$ *is an irreducible projective variety in* $\mathrm{proj}\ \mathrm{A}_n(\mathbb{F})$ *of codimension* $\binom{n-2p}{2}$. *The variety of its singular points is* $\mathrm{proj}\ \mathrm{W}_{2(p-1),n}(\mathbb{F})$.

Corollary 7.6. *A generic subspace L of the linear space of* $n \times n$ *skew symmetric matrices* $\mathrm{A}_n(\mathbb{F}) \subset \mathbb{F}^{n \times n}$ *of dimension* $\binom{n-2p}{2}$ *does not contain a nonzero matrix of rank* $2p$ *at most. In particular, for each generic point* $\mathbf{T} := (T_1, \ldots, T_{\binom{n-2p}{2}}) \in \mathrm{A}_n(\mathbb{F})^{\binom{n-2p}{2}}$, *there exists an open neighborhood of* $O \subset \mathrm{A}_n(\mathbb{F})^{\binom{n-2p}{2}}$ *such that for each* $\mathbf{X} := (X_1, \ldots, X_{\binom{n-2p}{2}}) \in O$, $L(\mathbf{X}) := \mathrm{span}(X_1, \ldots, X_{\binom{n-2p}{2}})$ *is a subspace of dimension of* $\binom{n-2p}{2}$ *which does not contain a nonzero matrix of rank* $2p$ *at most.*

Proof. A subspace $L \subset \mathrm{A}_n(\mathbb{F})$ of dimension d induces a linear space $\mathrm{proj}\ L$ of dimension $d-1$ in the projective space $\mathrm{proj}\ \mathrm{A}_n(\mathbb{F})$. Hence the dimension count implies that $\mathrm{proj}\ L \cap \mathbb{P}\mathrm{W}_{2p,n}(\mathbb{F}) = \emptyset$ for a generic subspace L of dimension $\binom{n-2p}{2}$. Hence L does not contain a nonzero matrix of rank $2p$ at most.

A generic point $\mathbf{T} \in \mathrm{A}_n(\mathbb{F})^{\binom{n-2p}{2}}$ generates a generic subspace $L(\mathbf{T})$ of dimension $\binom{n-2p}{2}$. Hence $\mathrm{proj}\ L(\mathbf{T}) \cap \mathrm{proj}\ \mathrm{W}_{2p,n}(\mathbb{F}) = \emptyset$. For a small enough open neighborhood O of $\mathbf{T}$, for any $\mathbf{X} \in O$, the subspace $L(\mathbf{X})$ is a perturbation of $L(\mathbf{T})$. Hence $\mathrm{proj}\ L(\mathbf{X}) \cap \mathrm{proj}\ \mathrm{W}_{2p,n}(\mathbb{F}) = \emptyset$. $\square$

It is well known that for $\mathbb{F} = \mathbb{R}$ the above corollary can be improved for certain values of n, p. See [11] and the references therein. We now bring a well known improvement of the above corollary for $n = 4, p = 1$.

Proposition 7.7. *There exists an neighborhood* O *of* $\mathbf{T} = (T_1, \ldots, T_l) \in \mathrm{A}_4(\mathbb{R})^l$ *such that for any* $\mathbf{X} = (X_1, \ldots, X_l) \in \mathrm{A}_4(\mathbb{R})^l$ *the subspace* $L(\mathbf{X})$ *does not contain a matrix of rank* 2 *for* $l = 2, 3$.

Proof. Let $l = 3$ and

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{T} = (T_1, T_2, T_3)$. Note that any nonzero matrix $B \in L(\mathbf{T})$ is a multiple of an orthogonal matrix. Hence $\mathrm{rank}\ B = 4$ and $\dim L = 3$. Thus $\mathrm{proj}\ L(\mathbf{T}) \cap \mathrm{proj}\ \mathrm{W}_{2,4}(\mathbb{R}) = \emptyset$. Therefore, there exists a small open neighborhood O of $\mathbf{T}$ such that for any $\mathbf{X} = (X_1, X_2, X_3) \in O$ $\mathrm{proj}\ L(\mathbf{X}) \cap \mathrm{proj}\ \mathrm{W}_{2,4}(\mathbb{R}) = \emptyset$.

Similar results hold for $l = 2$ if we let $\mathbf{T} = (T_1, T_2)$. $\square$

The next result appears in [12].

Proposition 7.8. *Let* $\mathrm{S}_{n,0} \subset \mathbb{R}^{n\times n}$ *be the subspace of real symmetric matrices of trace zero. Then* $\mathrm{S}_{n,0}$ *is an* $\frac{(n+1)n}{2} - 1$ *dimensional subspace which does not contain a rank one matrix.*

Proof. Clearly, $\dim \mathrm{S}_{n,0} = \frac{(n+1)n}{2} - 1$. Assume to the contrary that a rank one matrix B is in $\mathrm{S}_{n,0}$. Since B is symmetric $B = \pm\mathbf{x}\mathbf{x}^\top$, where $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$. Then trace $B = \pm\mathbf{x}^\top\mathbf{x} = 0$. So $\mathbf{x} = \mathbf{0}$, contradicting our assumption. □

Proof of Theorem 7.4. We first begin with the case $(m, m, l = (m-1)^2 + 1)$. Assume first $m = 2, 3$. Note that $\dim \mathrm{S}_m = l$. Choose a basis $T_1, \ldots, T_l$ in S_m. Let $\mathbf{T} = (T_1, \ldots, T_l)$. Proposition 7.8 yields that proj $\mathrm{S}_{n,0} \cap$ proj $U_{1,m,m} = \emptyset$. The arguments of the proof of Corollary 7.6 yield that there exists an open neighborhood O of $\mathbf{T} \in (\mathbb{R}^{m\times m})^l$ so that for each $\mathbf{X} = (X_1, \ldots, X_l) \in (\mathbb{R}^{m\times m})^l$ we have proj $L(\mathbf{X}) \cap$ proj $U_{1,m,m} = \emptyset$. Hence $L(\mathbf{X})$ is not spanned by rank one matrices.

Let $\mathcal{T} = [t_{ijk}] \in \mathbb{R}^{m\times m\times l}$ be the set of $C \subset \mathbb{R}^{m\times m\times l}$ of all 3-tensors such that $\mathbf{X} \in O$, where $X_k := [t_{ijk}]_{i=j=1}^m$ for $k = 1, \ldots, l$. Clearly, C is open. Theorem 2.4 implies that the $\mathrm{rank}_{\mathbb{R}}\mathcal{T} > l$ for each $\mathcal{T} \in C$. In view of Theorem 7.1, C has a nontrivial intersection with at least one O_i. Hence $r_i > l = \mathrm{grank}(m, m, l)$. Assume now that $m > 3$. Let $L_1 \subset \mathrm{A}_m(\mathbb{R})$ be a generic subspace of dimension $\binom{m-2}{2}$. Then L_1 does not contain a matrix of rank 2. Clearly $\mathrm{S}_{m,0} \cap L_1 = \{0_{m\times m}\}$. Then $L = S_{m,0} + L_1$ is $l = (m-1)^2 + 1$ dimensional subspace of trace zero matrices. Observe that if $B \in L$ then $B^\top \in L$. We claim that L does not contain a rank one matrix $B \in \mathbb{R}^{m\times m}$. Assume to the contrary that $B \in L$ is a rank one matrix. Proposition 7.8 implies that $B \notin \mathrm{S}_{m,0}$. So

$$B = B_1 + B_2, \quad B_1 = \frac{1}{2}(B + B^\top) \in \mathrm{S}_{m,0}, \quad B_2 = \frac{1}{2}(B - B^\top) \in L_1.$$

Since B is a rank one nonsymmetric matrix B_2 is a skew symmetric matrix of rank 2. This contradicts our assumption. Hence proj $L \cap$ proj $U_{1,m,m} = \emptyset$. The above arguments show that $\mathrm{mtrank}(m, m, l) > l = \mathrm{grank}(m, m, l)$.

Assume finally that $m = 4$ and $l = 11, 12$. Repeat the above arguments where L_1 has dimension 2 or 3, as given in Proposition 7.7. □

Appendix A. Complex and real algebraic geometry

In this section we give basic facts in complex and real algebraic geometry needed for this paper. The emphasize is on simplicity and intuitive understanding. We supply references for completeness. Our basic references are [24,26,15] for complex algebraic geometry, and [3] for real algebraic geometry.

We first start with some general definitions which hold for general field $\mathbb{F}$. Denote by $\mathbb{F}[x_1, \ldots, x_n]$, $\mathbb{F}(x_1, \ldots, x_n)$ the ring of polynomials and its field of rational functions in n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{F}$, respectively. We will identify $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \ldots, x_n]$, $\mathbb{F}(\mathbf{x}) = \mathbb{F}(x_1, \ldots, x_n)$, where $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$. For $p_1, \ldots, p_m \in \mathbb{F}[\mathbf{x}]$ denote by $Z(p_1, \ldots, p_m) = \{\mathbf{y} \in \mathbb{F}^n,\ p_i(\mathbf{y}) = 0, i = 1, \ldots, m\}$. Equivalently let $\mathbf{P} = (p_1, \ldots, p_m)^\top$ be a polynomial map $\mathbf{P} : \mathbb{F}^n \to \mathbb{F}^m$. Then $Z(p_1, \ldots, p_m) = \mathbf{P}^{-1}(\mathbf{0})$. $V \subset \mathbb{F}^n$ is called an *algebraic set*, if $V = Z(p_1, \ldots, p_m)$ for some $p_1, \ldots, p_m \in \mathbb{F}[\mathbf{x}]$. Note that $\emptyset$ and $\mathbb{F}^n$ algebraic sets.

Recall that $\mathbb{PF}^n$, the n-dimensional projective space over $\mathbb{F}$, is identified with one dimensional subspaces of $\mathbb{F}^{n+1}$, i.e. lines through the origin in $\mathbb{F}^{n+1}$. So $\mathbb{F}^n$ is viewed as a subset of $\mathbb{PF}^n$ where each $\mathbf{x} = (x_1, \ldots, x_n)^\top$ is identified with a one dimensional subspace spanned by $\hat{x} = (x_1, \ldots, x_n, 1)^\top$. $\mathbb{PF}^n$ can be viewed as the union of two disjoint sets $\mathbb{F}^n$ and $\mathbb{PF}^{n-1}$, where $\mathbb{PF}^{n-1}$ is all one dimensional subspaces in $\mathbb{F}^{n+1}$ spanned by nonzero $\mathbf{y} = (y_1, \ldots, y_n, 0)^\top$.

Denote by $\mathbb{F}_h[\mathbf{y}]$, $\mathbf{y} = (y_1, \ldots, y_{n+1})^\top$, the set of homogeneous polynomials in $y_1, \ldots, y_{n+1}$. Let $q_1, \ldots, q_m \in \mathbb{F}_h[\mathbf{y}]$. Consider the variety $Z(q_1, \ldots, q_m) \subset \mathbb{F}^{n+1}$. If $\mathbf{0} \neq \mathbf{y} \in Z(q_1, \ldots, q_m)$

then $\mathrm{span}(\mathbf{y}) \subset Z(q_1, \dots, q_m)$. Hence $Z(q_1, \dots, q_m)$ induces a subset $\tilde{Z}(q_1, \dots, q_m) \subset \mathbb{PF}^n$. (If $Z(q_1, \dots, q_m) = \{\mathbf{0}\}$ then $\tilde{Z}(q_1, \dots, q_m) = \emptyset$.) $V \subseteq \mathbb{PF}^{n+1}$ is called a *projective algebraic set* if $V = \tilde{Z}(q_1, \dots, q_m)$ for some $q_1, \dots, q_m \in \mathbb{F}_h[\mathbf{y}]$. It is easy to show that an intersection and union of two affine or projective algebraic sets is an affine or projective algebraic. An affine or projective algebraic set is called irreducible if it cannot be written as the union of two proper algebraic subsets. An irreducible affine or projective algebraic set is called an affine or projective variety, respectively. (An affine variety will be referred sometimes as variety.) Let V be a projective variety in $\mathbb{PF}^n$, and $W \subsetneq V$ a projective algebraic set. Then $V \setminus W$ is called a *quasi-projective* variety. Note that an affine variety $Z(p_1, \dots, p_m)$ can be viewed as a quasi projective variety. First homogenize $p_1, \dots, p_m$ to $\hat{p}_1, \dots, \hat{p}_m \in \mathbb{F}_h[\mathbf{y}]$. Let $W \subset \mathbb{PF}^n$ to be the zero set of $y_{n+1} = 0$. Then $Z(p_1, \dots, p_m)$ can be identified with $\tilde{Z}(\hat{p}_1, \dots, \hat{p}_m) \backslash W$.

A.1. Complex algebraic sets and polynomial maps

In this section $\mathbb{F} = \mathbb{C}$. Let $\mathbf{P} = (p_1, \dots, p_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map. Denote by $\mathrm{D}\mathbf{P}(x)$, the derivative of $\mathbf{P}$ or the Jacobian matrix of $\mathbf{P}$, the matrix $[\frac{\partial p_i}{\partial x_j}]_{i=j=1}^{m,n}$. For any $U \subseteq \mathbb{C}^n$ denote $\mathrm{rank}_U \mathrm{D}\mathbf{P} = \max_{\mathbf{x} \in U} \mathrm{rank}\ \mathrm{D}\mathbf{P}(\mathbf{x})$. Assume that U is a variety. Note that the set $\mathrm{Sing}\ U = \{\mathbf{x} \in U,\ \mathrm{rank}\ \mathrm{D}\mathbf{P}(\mathbf{x}) < \mathrm{rank}_U \mathrm{D}\mathbf{P}\}$ is a strict algebraic subset of U. (Observe that $\mathbf{x} \in \mathrm{Sing}\ U$ if and only if all minors of $\mathrm{D}\mathbf{P}(\mathbf{x})$, $\mathbf{x} \in U$ of order $\mathrm{rank}_U \mathrm{D}\mathbf{P}$ vanish.) $\mathrm{Sing}\ U$ is called the set of singular points of U. Let $V = Z(p_1, \dots, p_m)$ be a variety. The dimension of V, denoted by $\dim V$, equals to $n - \mathrm{rank}_V \mathrm{D}\mathbf{P}$. Then $V \backslash \mathrm{Sing}\ V$, the set of regular (smooth) points of V, is a quasi-projective variety, and a complex manifold of dimension $\dim V$. See [24, Section 1A]. For any variety V and a strict algebraic subset W in V, the quasi-projective variety $V \setminus W$ is connected [24, Corollary 4.16], and its dimension equal to the dimension of the complex manifold $V \setminus (W \cup \mathrm{Sing}\ V)$, which is $\dim V$. We say that a given property holds *generically* in V, if it holds for each $\mathbf{x} \in V \setminus W$, for some strict algebraic subset W of V, where W depends on the given property.

Hilbert basis theorem, (Nullstellensatz), claims that a countable intersection of algebraic sets is an algebraic set [26, p. 17]. An algebraic set $U \subset \mathbb{C}^n$ is a union of finitely many pairwise distinct varieties $U_1, \dots, U_k$, and this decomposition is unique [26, Theorems I.3.1 and I.3.2]. We define $\dim U = \max \dim U_i$. A product of two irreducible varieties is an irreducible variety [26, Theorem I.3.3]. Similar results holds for projective algebraic sets.

A set $V \subset \mathbb{C}^n$ is called a *constructible* algebraic set of dimension d if it can be represented as $V \setminus W$ were V is an algebraic set of dimension d and W is a constructible algebraic set of dimension $d - 1$ at most [15]. Note that a constructible algebraic set of dimension 0 is a set consisting of a finite number of points. It is easy to show that a finite union and a finite intersection of constructible algebraic sets is a constructible algebraic set. Finally if $V, W \subset \mathbb{C}^n$ are constructible algebraic sets then $V \setminus W$ is constructible algebraic.

Let $\mathbf{P}$ be a polynomial map as above. From the definition of an algebraic set we deduce that for any algebraic set $W \subset \mathbb{C}^m$ the set $\mathbf{P}^{-1}(W)$ is an algebraic set of $\mathbb{C}^n$. Denote $\mathrm{rank}\ \mathrm{D}\mathbf{P} = \mathrm{rank}_{\mathbb{C}^n} \mathrm{D}\mathbf{P}$. Then $V = \mathrm{Closure}\ \mathbf{P}(\mathbb{C}^n)$ is a variety, of dimension $\mathrm{rank}\ \mathrm{D}\mathbf{P}$. (Here the closure is in the standard topology in $\mathbb{C}^n$ or $\mathbb{R}^n$.) Moreover, $\mathrm{Sing}\ \mathbf{P} = \{\mathbf{x} \in \mathbb{C}^n,\ \mathrm{rank}\ \mathrm{D}\mathbf{P}(x) < \mathrm{rank}\ \mathrm{D}\mathbf{P}\}$ is a strict algebraic subset of $\mathbb{C}^n$. Hence $\mathbf{P}(\mathbb{C}^n \backslash \mathrm{Sing}\ \mathbf{P})$ is a constructive algebraic variety in $\mathbb{C}^m$ of dimension $\mathrm{rank}\ \mathrm{D}\mathbf{P}$ [15]. Furthermore, there exists a strict algebraic set $W \subsetneq V$, such that for each $\mathbf{z} \in V \setminus W$ the algebraic set $\mathbf{P}^{-1}(\mathbf{z})$ is a disjoint union of k varieties $U_1(z), \dots, U_k(z) \subset \mathbb{C}^n$, each of dimension $n - \mathrm{rank}\ \mathrm{D}\mathbf{P}$. The integer k is independent of $\mathbf{z} \in V \setminus W$, and is called the *degree* of $\mathbf{P}$ [24, Corollaries 3.15 and 3.16].

More general, let $U \subset \mathbb{C}^n$ be a constructible algebraic set. Then $\mathbf{P}(U) \subset \mathbb{C}^m$ is a constructible algebraic set of dimension $\mathrm{rank}_U \mathrm{D}\mathbf{P}$. This applies in particular to a projections $\mathbf{P}$, where $\mathbf{P}(\mathbf{x})$ obtained from $\mathbf{x}$ be deleting a number of coordinates. See [26, Sections 3 and 4].

A.2. Real semi-algebraic sets and polynomial maps

In this section the topology on $\mathbb{R}^n$ is assumed to be the standard topology: open sets, closed sets, the interior and the closure of sets are in the standard topology of $\mathbb{R}^n$. A real algebraic set in $\mathbb{R}^n$ is the zero set of m polynomials $p_1, \ldots, p_m \in \mathbb{R}[\mathbf{x}]$, and is denoted by $Z^{\mathbb{R}}(p_1, \ldots, p_m) \subset \mathbb{R}^n$. We can view $p_1, \ldots, p_m$ as polynomials with complex variables $\mathbf{z} = (z_1, \ldots, z_n)^\top \in \mathbb{C}^n$ with real coefficients. Then $U = Z(p_1, \ldots, p_m) = \{\mathbf{z} \in \mathbb{C}^n, \ p_1(\mathbf{z}) = \ldots = p_m(\mathbf{z}) = 0\}$ and $U^{\mathbb{R}} = Z^{\mathbb{R}}(p_1, \ldots, p_m) = U \cap \mathbb{R}^n$. $Z^{\mathbb{R}}(p_1, \ldots, p_m)$ is called irreducible, if $Z(p_1, \ldots, p_m)$ is irreducible. Since any algebraic set $U \subset \mathbb{C}^n$ is a finite union of pairwise distinct irreducible varieties $V_1, \ldots, V_k$ it follows that any real algebraic set is a finite union of irreducible real algebraic sets. A set S is called *semi-algebraic* if S is a finite union of subsets $S_1, \ldots, S_k$, where each S_i is of the following form. There exists an algebraic set $V_i^{\mathbb{R}} \subset \mathbb{R}^n$ and a finite number of polynomials $g_{1,i}, \ldots g_{n_i,i} \in \mathbb{R}[\mathbf{x}]$ such that $S_i = \{\mathbf{x} \in V_i^{\mathbb{R}}, \ g_{j,i}(\mathbf{x}) > 0, \ j = 1, \ldots, n_i\}$ for $i = 1, \ldots, k$. Here each $n_i \geq 0$. So if $n_i = 0$ then $S_i = V_i^{\mathbb{R}}$. (Algebraic set is semi-algebraic.) Since each algebraic set is a finite union of irreducible real varieties we may assume that in the definition of semi-algebraic set S each $V_i^{\mathbb{R}}$ is irreducible. Furthermore, without loss of generality, we may assume that each $S_i \subset V_i^{\mathbb{R}}$ is relative open, i.e. S_i is a nonempty intersection of an open set in $\mathbb{R}^n$ and $V_i^{\mathbb{R}}$. Hence $\dim S_i = \dim V_i^{\mathbb{R}}$, and $\dim S = \max \dim S_i$. See [3, Section 2.8].

Semi-algebraic sets are stable under finite union, finite intersection, taking complements and closures [3, Section 2.2]. (I.e. all the above operations on semi-algebraic sets yield semi-algebraic sets.) Hence if S, T are semi-algebraic subsets of $\mathbb{R}^n$ then $A \setminus B = A \cap (\mathbb{R}^n \setminus B)$ is a semi-algebraic set. For any semi-algebraic set S the following inequality holds $\dim \mathrm{Closure}(S) \setminus S < \dim S$ [3, Proposition 2.8.13].

A projection of semi-algebraic set is semi-algebraic [3, Theorem 2.2.1]. Hence the image of a semi-algebraic set by a polynomial map is semi-algebraic [3, Proposition 2.2.7]. The closure and the interior of semi-algebraic set are semi-algebraic [3, Proposition 2.2.2]. Every open semi-algebraic subset S of $\mathbb{R}^n$ is a finite union of disjoint open connected semi-algebraic sets in $\mathbb{R}^n$. For more general statement see [3, Theorem 2.4.4].

References

[1] H. Abo, G. Ottaviani, C. Peterson, Induction for secant varieties of Segre varieties, Trans. Amer. Math. Soc. 361 (2009) 767–792.
[2] D. Bini, M. Capovani, G. Lotti, F. Romani, $O(n^{2.7799})$ complexity for matrix multiplication, Inform. Process. Lett. 8 (1979) 234–235.
[3] J. Bochnak, M. Coste, M.F. Roy, Real Algebraic Geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1998. pp. 36.
[4] P. Bürgisser, M. Clausen, M.A. Shokrollahi, Algebraic Complexity Theory, with the Collaboration of Thomas Lickteig, Grundlehren der Mathematischen Wissenschaften, 315, Springer-Verlag, Berlin, 1997. xxiv + 618 pp.
[5] J. Carrol, J. Chang, Analysis of individual differences in multidimensional scaling via N-way generalization of Eckhart–Young decomposition, Psychometrika 9 (1970) 267–283.
[6] M.V. Catalisano, A.V. Geramita, A. Gimigliano, Ranks of tensors, secant varieties of Segre varieties and fat points, Linear Algebra Appl. 355 (2002) 263–285.
[7] P. Comon, J.M.F. Ten Berge, L. De Lathauwer, J. Castaing, Generic and typical ranks of multi-way arrays, Linear Algebra Appl. 430 (2009) 2997–3007.
[8] R. Coppi, S. Bolasco (Eds.), Multiway Data Analysis, Elsevier Science Publishers (North-Holland), 1989.
[9] V. De Silva, L.-H. Lim, Tensor rank and the ill-posedness of the best low-rank approximation problem, SIAM J. Matrix Anal. Appl. 30 (2008) 1084–1127.
[10] S. Friedland, 3-Tensors: ranks and approximations, in: Workshop on Algorithms for Modern Massive Data Sets, Stanford-Yahoo, June 21–24, 2006. Available from: <http://www.math.uic.edu/∼friedlan/yahostan06.pdf>.
[11] S. Friedland, C. Krattenthaler, 2-Adic valuations of certain ratios of products of factorials and applications, Linear Algebra Appl. 426 (2007) 159–189. arXiv.math.NT/0508498v2.
[12] S. Friedland, R. Loewy, Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue, Pacific J. Math. 62 (1976) 389–399.
[13] S. Friedland, V. Mehrmann, A. Miedlar, M. Nkengla, Fast low rank approximations of matrices and tensors, 2008. Available from: <http//www.matheon.de/preprints/4903>.
[15] J. Harris, Algebraic Geometry: A First Course, Springer, 1992.
[16] J. Harris, L.W. Tu, On symmetric and skew-symmetric determinantal varieties, Topology 23 (1984) 71–84.
[17] R. Harshman, Foundations of the PARAFAC procedure: models and conditions for an explanatory multi-modal factor analysis, UCLA Working Pap. Phonetics 16 (1970) 1–84.
[18] J. Ja'Ja', Optimal evaluation of a pair of bilinear forms, SIAM J. Comput. 8 (1979) 281–293.
[19] T.G. Kolda, B.W. Bader, Tensor decompositions and applications, SIAM Rev. 51 (2009) 455–500.
[20] J.B. Kruskal, Three-way arrays: rank and uniqueness of trilinear decompositions, with applications to arithmetic complexity and statistics, Linear Algebra Appl. 18 (1977) 95–138.
[21] J.B. Kruskal, Rank, decomposition, and uniqueness for 3-way and N-way arrays, Multi-way Data Analysis, vol. 7–18, North-Holland, Amsterdam, 1989.

[22] L. de Lathauwer, B. de Moor, J. Vandewalle, Computation of the canonical decomposition by means of a simultaneous generalized Schur decomposition, SIAM J. Matrix Anal. Appl. 26 (2004) 295–327.
[23] T. Lickteig, Typical tensorial rank, Linear Algebra Appl. 69 (1985) 95–120.
[24] D. Mumford, Algebraic Geometry I: Complex Projective Varieties, Springer-Verlag, Berlin, 1976.
[25] R. Rocci, Unpublished notes, 1993.
[26] I.R. Shafarevich, Basic Algebraic Geometry, Springer-Verlag, Berlin, 1977.
[27] A. Stegeman, Degeneracy in CANDECOMP/PARAFAC explained for $p \times p \times 2$ arrays of rank $p + 1$ or higher, Psychometrika 71 (2006) 483–501.
[28] V. Strassen, Rank and optimal computation of generic tensors, Linear Algebra Appl. 52/53 (1983) 645–685.
[29] J.M. F ten Berge, Kruskal's polynomial for $2 \times 2 \times 2$ arrays and a generalization $2 \times n \times n$ arrays, Psychometrika 56 (1991) 631–636.
[30] J.M.F. Ten Berge, The typical rank of tall three-way arrays, Psychometrika 65 (2000) 525–532.
[31] J.M.F. Ten Berge, Partial uniqueness in Candecomp/Parafac, J. Chemometrics 18 (2004) 12–16.
[32] J.M.F. ten Berge, HAL Kiers, Simplicity of core arrays in three-way principal component analysis and the typical rank of $p \times q \times 2$ arrays, Linear Algebra Appl. 294 (1999) 169–179.
[33] J.M.F. ten Berge, A. Stegeman, Symmetry transformations for squared sliced three-way arrays, with applications to their typical rank, Linear Algebra Appl. 418 (2006) 215–224.
[34] A. Terracini, Sulla rappresentazione delle forme quaternarie mediante somme di potenze di forme lineari, Atti Rc. Accad. delle Scienze di Torino, vol. 51, 1915–16.

Further reading

[14] F.R. Gantmacher, The Theory of Matrices, vol. 2, Chelsea Pub. Company, New York, 1964.
[35] L.R. Tucker, Some mathematical notes of three-mode factor analysis, Psychometrika 31 (1966) 279–311.